C. R. Acad. Sci. Paris, Ser. I 335 (2002) 329–332

Équations aux dérivées partielles/*Partial Differential Equations*

Homogenization and concentrated capacity in reticular almost disconnected structures

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Received 21 May 2002; accepted 3 June 2002

Note presented by Philippe G. Ciarlet.

Abstract We compute the homogenized-concentrated limit for a pair of non-linearly coupled diffusion equations in a perforated cylindric domain with coaxial cylindric holes periodically distributed along its axis. This problem arises from visual transduction. ***To cite this article: D. Andreucci et al., C. R. Acad. Sci. Paris, Ser. I 335 (2002) 329–332.***

Homogénéisation et capacité concentrée dans les structures réticulaires presque déconnectées

Résumé On calcule la limite homogénéisée-concentrée pour deux équations de diffusion couplées de façon non linéaire dans un domaine cylindrique avec une distribution périodique de cavités cylindriques coaxiales le long de son axe. Ce problème émane de la transduction visuelle. ***Pour citer cet article : D. Andreucci et al., C. R. Acad. Sci. Paris, Ser. I 335 (2002) 329–332.***

1. Introduction

A rod outer segment is a light-capturing device in vertebrates. It consists of a cylinder of height H and cross section a circle $D_{R+\sigma\varepsilon_o}$ containing a layered distribution of n_o equal, coaxial thin cylinders, C_i of thickness ε_o and radius R called *discs* [8]. In what follows, ε will denote a parameter ranging over $(0, \varepsilon_o]$. For $\varepsilon = \varepsilon_o$ Fig. 1 corresponds to the physical configuration of the rod outer segment.

Notation: D_R circle of radius R; $\Omega_\varepsilon = D_{R+\sigma\varepsilon} \times (0, H)$; $\Omega_o = D_R \times (0, H)$; $S_\varepsilon = \{D_{R+\sigma\varepsilon} - D_R\} \times (0, H)$ *outer shell*; $C^o = D_R \times (0, \varepsilon)$; $I^o = D_R \times (0, \nu\varepsilon)$; C_i equispaced, equiaxial cylinders congruent to C^o, called *discs* in phototransduction; I_i equispaced, equiaxial cylinders congruent to I^o *interdiscal spaces* between C_i and C_{i+1}; L_i lateral boundary of the discs C_i; Λ_i lateral boundary of the interdiscal spaces I_i; $F_i^\pm$ upper and lower faces of the discs C_i; $\partial^\pm I_i$ upper and lower faces of interdiscal spaces I_i; $\partial^+ I_i = F_{i+1}^-$, $\partial^- I_i = F_i^+$, $i = 1, 2, \ldots, (n-1)$; $\theta_o =$ (volume of the union of C_i)/(volume of Ω_o) $= 1/(1+\nu)$; $\widetilde{\Omega}_\varepsilon = \Omega_\varepsilon - \bigcup_{i=1}^n \overline{C}_i$ *space available for diffusion*.

The space variable is denoted by $x = (\overline{x}, z)$, where $\overline{x} = (x_1, x_2)$ and z is the vertical coordinate. Moreover, σ and ν denote fixed positive numbers.

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S1631-073X(02)02482-2/FLA

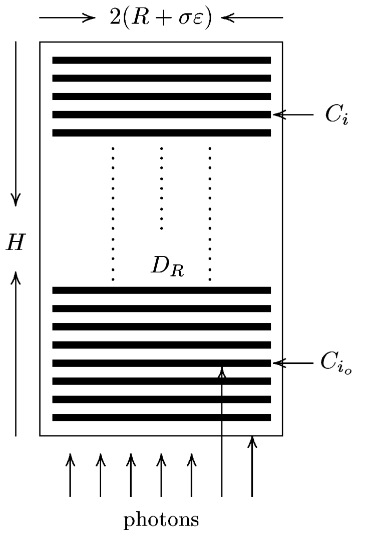

Figure 1. – The rod outer segment.

As photons hit the rod, they are captured by the discs and trigger a biochemical cascade whose net effect is the depletion of cyclic Guanilate Monophosphate (cGMP). Depletion of cGMP causes the closing of ionic channels located at the lateral boundary of the rod, thereby generating a variation of ionic current. Such variations are at the basis of the mechanism of vision. A precise description of the phototransduction cascade is in [1,8]. cGMP is depleted through a coupled diffusion process of cGMP and Calcium Ca^{2+} within $\widetilde{\Omega}_{\varepsilon_o}$. Denoting by u_{ε_o} and v_{ε_o} the volumic, dimensionless concentrations of cGMP and Ca^{2+},

$$u_{\varepsilon_o,t} - k_u \Delta u_{\varepsilon_o} = 0, \quad v_{\varepsilon_o,t} - k_v \Delta v_{\varepsilon_o} = 0 \quad \text{in } \widetilde{\Omega}_{\varepsilon_o,T} = \widetilde{\Omega}_{\varepsilon_o} \times (0,T]; \tag{1}$$

where k_u, k_v are given positive constants. Their non-linear coupling occurs through their fluxes on the faces on the discs C_i, i.e., [1]

$$k_u u_{\varepsilon_o,z}|_{F_i^{\pm}} = \frac{1}{2} v \varepsilon_o \bigl\{ \pm \gamma_o u_{\varepsilon_o} \mp f(u_{\varepsilon_o}, v_{\varepsilon_o} x, t) \bigr\}, \quad i = 1, 2, \ldots, n_o, \tag{2}$$

where γ_o is a given positive constant and f is a given, positive, bounded, smooth function of its arguments. Also cGMP does not penetrate the discs C_i through their lateral boundaries, nor it can exit the boundary of the rod. Calcium v_{ε_o} does not penetrate the discs C_i at all, nor outflows the rod through its bottom $x = 0$ or top $z = H$. However it can flow through the lateral boundary of the rod,

$$\{\text{influx of } v_{\varepsilon_o} \text{ through } |\overline{x}| = R + \sigma\varepsilon_o\} = -g_1(v_{\varepsilon_o}) + g_2(u_{\varepsilon_o}) \tag{3}$$

for given, positive, bounded, smooth functions $g_1(\cdot)$ and $g_2(\cdot)$. The derivation of the model (1)–(3) is in [1], where the various terms are discussed and justified.

1.1. Homogenization and concentrated capacity

Diffusion of cGMP and Ca^{2+} occurs in two thin compartments; the interdiscal spaces I_i and the outer shell S_{ε_o}. Since their thickness is of the order of $\varepsilon_o \ll R$ we regard ε_o as a parameter to be let go to zero. The process is carried so that as $\varepsilon_o \to 0$ the number of discs increases but the ratio θ_o between the volume occupied by the discs and the volume of the rod remains constant. As $\varepsilon_o \to 0$ the outer shell tends to a cylindrical surface S. Information on diffusion in thin domains is preserved by concentrating the capacities. Essentially the coefficients in the diffusion equation are scaled to compensate for the shrinking of the outer shell, so that the total mass remains stable in the limit. The rod outer segment tends to the cylinder Ω_o with no discs within it. The homogenized-concentrated limit problem is in Section 3.

1.2. Novelty and significance

Limits of concentrated-capacity or homogenized limits are extensively treated in the literature in separate settings [3–5,7]. A novelty of this investigation is their simultaneous occurrence. However the main mathematical significance is in the technical computation of the homogenized limit. In most of the homogenization literature the 'holes' to be removed are 'ball-like' and their shrinking to points does not disconnect ther ancestor domain. The cylinders C_i tend to disconnect the rod outer segment. The difficulty is overcome by establishing that the approximating solutions $\{u_\varepsilon\}$ and $\{v_\varepsilon\}$ satisfying ε-versions of (1)–(3) are equi-Hölder continuous away from the outer shell. Whence such a compactness has been established, the actual computation of the homogenized limit requires that the approximating solutions be extended in some fashion with regular functions defined in the whole $\mathbb{R}^3$. Such an extension is realized by the Kirzbraun–Pucci theorem valid for functions with concave modulus of continuity ([6], p. 197). This is also a novel approach to homogenization.

2. Pointwise formulation of the ε-problem

The functions in play are $u_\varepsilon, v_\varepsilon$ representing dimensionless approximations of cGMP and Ca^{2+} and defined in $\widetilde{\Omega}_\varepsilon$. It is convenient to distinguish them as x ranges over the interdiscal spaces or over the outer shell by denoting by $u_\varepsilon^{\text{int}}$, $v_\varepsilon^{\text{int}}$ respectively the restrictions of u_ε, v_ε to $\bigcup_{i=0}^n I_i$ and by $u_\varepsilon^{\text{ext}}$, $v_\varepsilon^{\text{ext}}$ their restrictions to S_ε. For a domain $A \subset \mathbb{R}^N$ we set $A_T = A \times (0, T]$ for some given $T > 0$.

2.1. Equations in the interdiscal spaces I_i

$$u_{\varepsilon,t} - k_u \Delta u_\varepsilon = 0, \quad v_{\varepsilon,t} - k_v \Delta v_\varepsilon = 0 \qquad \text{on } I_{i,T} \text{ for } i = 0, 1, \ldots, n; \tag{4}$$

$$u_\varepsilon(\cdot, 0) = u_{o,\varepsilon}, \quad v_\varepsilon(\cdot, 0) = v_{o,\varepsilon} \qquad \text{for } t = 0 \text{ and } x \in \widetilde{\Omega}_\varepsilon; \tag{5}$$

$$k_u u_{\varepsilon,z} = \mp \frac{1}{2} \nu\varepsilon \big(\gamma_o u_\varepsilon - f(u_\varepsilon, v_\varepsilon, x, t)\big) \qquad \text{on } \partial^\pm I_{i,T}; \tag{6}$$

$$u_{\varepsilon,z} = 0 \qquad \text{on } \partial^- I_{0,T} \text{ and } \partial^+ I_{n,T}; \tag{7}$$

$$v_{\varepsilon,z} = 0 \qquad \text{on } \partial^\pm I_{i,T} \text{ for } i = 0, 1, \ldots, n; \tag{8}$$

$$k_u \nabla u_\varepsilon^{\text{int}} \cdot \frac{\overline{x}}{R} = \frac{\varepsilon_o}{\varepsilon} k_u \nabla u_\varepsilon^{\text{ext}} \cdot \frac{\overline{x}}{R} \qquad \text{on } \Lambda_{i,T}; \tag{9}$$

$$k_v \nabla v_\varepsilon^{\text{int}} \cdot \frac{\overline{x}}{R} = \frac{\varepsilon_o}{\varepsilon} k_v \nabla v_\varepsilon^{\text{ext}} \cdot \frac{\overline{x}}{R} \qquad \text{on } \Lambda_{i,T}; \tag{10}$$

$$u_\varepsilon^{\text{int}} = u_\varepsilon^{\text{ext}}, \quad v_\varepsilon^{\text{int}} = v_\varepsilon^{\text{ext}} \qquad \text{on } \Lambda_{i,T}. \tag{11}$$

2.2. Equations in the outer shell

$$u_{\varepsilon,t} - k_u \Delta u_\varepsilon = 0, \quad v_{\varepsilon,t} - k_v \Delta v_\varepsilon = 0 \qquad \text{on } S_{\varepsilon,T}; \tag{12}$$

$$\nabla u_\varepsilon \cdot \frac{\overline{x}}{R} = 0, \quad \nabla v_\varepsilon \cdot \frac{\overline{x}}{R} = 0 \qquad \text{on } L_{i,T} \quad i = 1, 2, \ldots, n; \tag{13}$$

$$\nabla u_\varepsilon \cdot \frac{\overline{x}}{R + \sigma\varepsilon} = 0 \qquad \text{on } |\overline{x}| = R + \sigma\varepsilon; \tag{14}$$

$$\frac{\varepsilon_o}{\varepsilon} k_v \nabla v_\varepsilon \cdot \frac{\overline{x}}{R + \sigma\varepsilon} = -g_1(v_\varepsilon) + g_2(u_\varepsilon) \qquad \text{on } |\overline{x}| = R + \sigma\varepsilon; \tag{15}$$

$$u_{\varepsilon,z} = 0, \quad v_{\varepsilon,z} = 0 \qquad \text{for } z = 0 \text{ and } z = H. \tag{16}$$

The initial conditions are inherited from (5) for the portions S_ε of $\widetilde{\Omega}_\varepsilon$. Both u_ε and v_ε are continuous from within each of the interdiscal spaces I_i into S_ε through the cylindrical surface Λ_i, i.e., (11) continues to hold. The flux relations (9), (10) are also in force and form part of the boundary conditions to be associated with (12)–(16).

3. The homogenized-concentrated limit

As $\varepsilon \to 0$ the layered domain $\widetilde{\Omega}_\varepsilon$ tends to the cylinder Ω_o and the outer shell S_ε tends to the surface $S = \{|\overline{x}| = R\} \times (0, H)$. The family of solutions to problems (4)–(16) tends to two pairs of functions, i.e., $\{u, v\}$ defined in $\Omega_{o,T}$ called the interior limit and $\{\hat{u}, \hat{v}\}$ defined in S_T called the limit on the outer shell.

3.1. *The interior limit*: $u, v \in C(0, T; L^2(\Omega_o))$, $|\nabla_{\overline{x}}u|, |\nabla_{\overline{x}}v| \in L^2(\Omega_{o,T})$. –

$$\begin{cases} u_t - k_u \Delta_{\overline{x}} u = \{\gamma_o u - f(u, v, \overline{x}, t)\} \\ v_t - k_v \Delta_{\overline{x}} v = 0 \end{cases} \quad \text{in } \mathcal{D}'(\Omega_{o,T}). \tag{17}$$

These can be regarded as diffusion processes, parametrized with $z \in (0, H)$, taking place on the circle $\{|\overline{x}| < R\}$. The homogenized limit transforms the boundary fluxes in (6) into source terms holding in Ω_o.

Denote coordinates on the limit surface S by $\theta \in (0, 2\pi]$ and $z \in (0, H)$. The level z traces on S a circle $\ell_z = \{|\overline{x}| = R\} \times \{z\}$. The boundary limits $\hat{u}, \hat{v}$ are functions of θ, z, t.

3.2. *The limit in outer shell*: $\hat{u}, \hat{v} \in C(0, T; S)$, $|(\hat{u}_z, \hat{u}_\theta)|$, $|(\hat{v}_z, \hat{v}_\theta)| \in L^2(S_T)$. – These boundary limits $\hat{u}, \hat{v}$ are related to the interior limits u, v in two ways. First u and v have traces on $\{|\overline{x}| = R\}$ in $L^2(S_T)$ and,

$$\hat{u}(\theta, z, t) = u(\overline{x}, z, t)|_{|\overline{x}|=R}, \quad \hat{v}(\theta, z, t) = v(\overline{x}, z, t)|_{|\overline{x}|=R} \quad \text{in } L^2(S_T). \tag{18}$$

Second, denoting by Δ_S the Laplace–Beltrami operator on S,

$$\begin{cases} \hat{u}_t - k_u \Delta_S \hat{u} = -\frac{(1-\theta_o)k_u}{\sigma\varepsilon_o} u_\rho \Big|_{|\overline{x}|=R} \\ \hat{v} - k_v \Delta_S \hat{v} = -\dfrac{(1-\theta_o)k_v}{\sigma\varepsilon_o} v_\rho \Big|_{|\overline{x}|=R} + \dfrac{1}{\sigma\varepsilon_o}\{g_1(\hat{v}) - g_2(\hat{u})\} \end{cases} \quad \text{in } \mathcal{D}'(S_T). \tag{19}$$

The limit problem (17)–(19) consists of a system of diffusion equations taking place in different domains, which are coupled through the fluxes exchanged at the common boundaries.

The regularity of u, v does not insure that u_ρ, v_ρ have traces on S. In this sense (19) is formal. The limit problem (17)–(19) can be given the following rigorous weak form.

The functions $\{u, v\}$ and $\{\hat{u}, \hat{v}\}$ are in the stated regularity classes and satisfy the integral identities,

$$\begin{aligned} &(1-\theta_o)\left\{\int_{\Omega_{o,T}} \{-u\varphi_t + k_u \nabla_{\overline{x}} u \cdot \nabla_{\overline{x}}\varphi\}\,\mathrm{d}x\,\mathrm{d}t - \int_{\Omega_o} u_o\varphi(x,0)\,\mathrm{d}x + \int_{\Omega_{o,T}} \big(\gamma_o u - f_1(v)\big)\varphi\,\mathrm{d}x\,\mathrm{d}t\right\} \\ &\quad + \sigma\varepsilon_o\left\{\int_{S_T} \{-\hat{u}\varphi_t + k_u \nabla_S \hat{u} \cdot \nabla_S\varphi\}\,\mathrm{d}\eta\,\mathrm{d}t - \int_S \hat{u}_o\varphi(x,0)\,\mathrm{d}\eta\right\} = 0 \end{aligned} \tag{20}$$

for all testing functions $\varphi \in C^1(\overline{\Omega}_{o,T})$ vanishing for $t = T$;

$$\begin{aligned} &(1-\theta_o)\left\{\int_{\Omega_{o,T}} \{-v\psi_t + k_v \nabla_{\overline{x}} v \cdot \nabla_{\overline{x}}\psi\}\,\mathrm{d}x\,\mathrm{d}t - \int_{\Omega_o} v_o\psi(x,0)\,\mathrm{d}x\right\} \\ &\quad + \sigma\varepsilon_o\left\{\int_{S_T} \{-\hat{v}\psi_t + k_v \nabla_S \hat{v} \cdot \nabla_S\psi\}\,\mathrm{d}\eta\,\mathrm{d}t - \int_S \hat{v}_o\psi(x,0)\,\mathrm{d}x + \frac{1}{\sigma\varepsilon_o}\int_{S_T}\{g_1(\hat{v}) - g_2(\hat{u})\}\psi\,\mathrm{d}\eta\,\mathrm{d}t\right\} = 0 \end{aligned} \tag{21}$$

for all testing functions $\psi \in C^1(\overline{\Omega}_{o,T})$ vanishing for $t = T$. Proofs are in [2].

[1] It is assumed that the photons generate source terms distributed on all the faces $F_i^{\pm}$. One might visualize an idealized experiment where a single photon is captured by a disc say C_{i_o} thereby generating a depletion term located say on $F_{i_o}^{-}$. Mathematical modeling and analysis of this process are in [1,2].

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