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A result on the $\widehat{A}$ and elliptic genera on non-spin manifolds with circle actions

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Abstract We prove the vanishing of the $\widehat{A}$-genus of compact smooth manifolds with finite second homotopy group and endowed with smooth S^1 actions. These manifolds are not necessarily spin, hence, this vanishing extends that of Atiyah and Hirzebruch on spin manifolds with S^1 actions. The proof is accomplished by proving a rigidity theorem under circle actions of the elliptic genus on these manifolds. ***To cite this article: H. Herrera, R. Herrera, C. R. Acad. Sci. Paris, Ser. I 335 (2002) 371–374.***

Un résultat sur les variétés non-spinorielles de genres $\widehat{A}$ et elliptique munies d'actions de S^1

Résumé On montre que le $\widehat{A}$-genre d'une variété lisse, compacte munie d'un second groupe d'homotopie fini et dotée d'une action de S^1 est égal à zéro. Ces variétés ne sont pas nécessairement spinorielles de sorte que ce théorème d'annulation étend le résultat d'Atiyah–Hirzebruch établi pour des variétés spinorielles avec actions de S^1. La démonstration est faite à partir d'un théorème de rigidité sous des actions de S^1 de genre elliptique sur ces variétés. ***Pour citer cet article : H. Herrera, R. Herrera, C. R. Acad. Sci. Paris, Ser. I 335 (2002) 371–374.***

1. Introduction

The main results of this paper are the following vanishing theorem and a rigidity theorem stated below (Theorem 2.1).

THEOREM 1.1. – *Let M be a $2n$-dimensional, oriented, compact, connected, smooth manifold with finite second homotopy group, and endowed with a smooth S^1 action. Then*

$$\widehat{A}(M) = 0.$$

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This vanishing theorem is new since it *does not* follow from results on spin (nor spin^c, nor spin^h) manifolds, and the manifolds under consideration are not necessarily of this type. Hence, Theorem 1.1 extends that of Atiyah and Hirzebruch on spin manifolds endowed with smooth S^1 actions [1]. Furthermore, such manifolds may admit no spin structure and, therefore, have neither spin bundle, nor spinors, nor Dirac operator. This means that the characteristic number $\widehat{A}(M)$ is, a priori, a rational number and that we cannot estimate $\widehat{A}(M)$ in the usual index-theoretical way. We prove this theorem by means of the "rigidity under S^1 actions" of the elliptic genus of Ochanine [8] on these manifolds.

The Note is organized as follows. In Section 2 we define the elliptic genus, state the Rigidity Theorem 2.1, and outline the proof of Theorem 1.1. In Section 3 we outline the proof of Theorem 2.1 along the lines of Bott and Taubes [2]. In Section 4 we give some applications. The full proofs of the theorems and further applications will appear in [4].

2. Rigidity and vanishing theorems

Let $\bigwedge_c^{\pm}$ be the even and odd complex forms, respectively, on an oriented $2n$-dimensional smooth Riemannian manifold M, with respect to the Hodge star operator $*$. The signature operator $d_s : \bigwedge_c^+ \to \bigwedge_c^-$ acting on forms is elliptic, and the virtual dimension of its index equals the signature $\tau(M)$. By means of a connection on a complex vector bundle W on M one can *twist* the signature operator to forms with values in W, $d_s \otimes W : \bigwedge_c^+(W) \to \bigwedge_c^-(W)$. This operator is also elliptic and the virtual dimension of its index is denoted by $\tau(M, W)$.

Let $T = TM \otimes \mathbb{C}$ be the complexified tangent bundle of M. Let R_i be the sequence of (representations) bundles defined by the formal power series

$$R(q, T) = \sum_{i=0}^{\infty} R_i q^i = \bigotimes_{i=1}^{\infty} \bigwedge\nolimits_{q_i} T \otimes \bigotimes\nolimits_{i=1}^{\infty} S_{q^i} T,$$

where $S_a T = \sum_{j=0}^{\infty} a^j S^j T$, $\bigwedge_a T = \sum_{j=0}^{\infty} a^j \bigwedge^j T$, and $S^j T$, $\bigwedge^j T$ denote the j-th symmetric and exterior tensor powers of T respectively. We refer the reader to [5] for an introduction to the subject.

DEFINITION 2.1. – The *elliptic genus* of M is defined by the following power series

$$\tau_q(M) = \sum_{i=0}^{\infty} \tau(M, R_i) q^i.$$

If M is endowed with an S^1 action, the equivariant genus $\tau_q(M)_g$ is defined by the analogous q-series using equivariant twisted signatures as coefficients, for any $g \in S^1$.

THEOREM 2.1 (Rigidity theorem). – *Let M be an $2n$-dimensional, oriented, compact, smooth manifold with finite second homotopy group and endowed with a smooth S^1 action. Then we have*

$$\tau_q(M) = \tau_q(M)_g \tag{1}$$

for every $g \in S^1$.

Sketch of proof of Theorem 1.1. – Assume $\dim(M) = 2n \equiv 0 \bmod 4$. According to Theorem 2.1, the value of $\tau_q(M)_g$ does not depend on g. Following [6] we apply the Atiyah–Segal G-signature theorem: $\tau_q(M)_g$ can be expressed in terms of the fixed point set M^g of g and the action of g in the normal bundle of $M^g \subset M$. Let $g = -1 \in S^1$. The self-intersection $M^g \circ M^g$ is oriented and smooth. In [6], Hirzebruch and Slodowy showed that $\tau_q(M)_g = \tau_q(\mathcal{M}^g \circ M^g)$, and by Eq. (1), $\tau_q(M) = \tau_q(\mathcal{M}^g \circ M^g)$.

The expansion of $\tau_q(M)$ at the other cusp [5] can be written as the following power series,

$$\tilde{\tau}_q(M) = \frac{1}{q^{n/4}} \sum_{j=0}^{\infty} \widehat{A}(M, R'_j) q^j,$$

where R'_j is a sequence of representations in terms of T starting with $R'_0 = 1$, $R'_1 = -T$, etc. By rigidity, it also satisfies

$$\tilde{\tau}_q(M) = \tilde{\tau}_q\big(\mathcal{M}^g \circ M^g\big). \tag{2}$$

The codimension of M^g is positive and even, therefore the right-hand side of (2) has a pole of order less than $n/4$, which implies that the coefficient of $q^{n/4}$ on the left-hand side of (2) must vanish, i.e. $\widehat{A}(M) = 0$. □

3. Sketch of the proof of the Rigidity theorem

The proof of Theorem 2.1 is along the lines of [2], to which we refer the reader for the notation. The fixed points of an S^1 action on M fall into connected components $\{P\}$ which are oriented smooth submanifolds. Let P be a component of the fixed point set, so that T splits as $T|_P = TP \oplus \bigoplus E_i^{\#}$, where $E_i^{\#}$ denotes the underlying real bundle of the complex bundle E_i on which S^1 acts by sending ξ to ξ^{m_i}. Applying the Atiyah–Segal equivariant index theorem we obtain the following localization formula

$$\tau_q(M) = \sum_P \mu(P).$$

The contribution $\mu(P)$ of P to $\tau_q(M)$ is the index of the signature operator on P twisted by an appropiate power series in the E_i's [2]. Both $\mu(P)$ and $\tau_q(M)$ are meromorphic functions on $T_{q^2} = \mathbb{C}^*/q^2$ (the non-zero complex numbers modulo the multiplicative group generated by $q^2 \neq 0$). The proof of Theorem 2.1 depends on proving that $\tau_q(M)$ has no poles at all on T_{q^2}, which implies that $\tau_q(M)$ is a constant.

This will follow from carrying out localizations to intermediate (auxiliary) submanifolds. They are the submanifolds M_k of fixed points under the action of the subgroup $\mathbb{Z}_k \subset S^1$. The argument of the proof follows as in [2] with the exception of two technical points due to the (possible) non-spin nature of the manifold: (i) we have to prove that the submanifolds M_k are orientable; (ii) we have to prove that it is possible to choose an orientation of M_k compatible with M and all the components P contained in M_k. The following lemma addresses (i), and is the analogue of [2, Lemma 10.1] in our set-up.

LEMMA 3.1. – *Let M be an oriented $2n$-dimensional smooth manifold endowed with a smooth S^1 action. Consider $\mathbb{Z}_k \subset S^1$ and its corresponding action on M. If k is odd then the fixed point set M_k of the $\mathbb{Z}_k$ action is orientable. If k is even and M_k contains a fixed point of the S^1 action, then M_k is also orientable.*

Point (ii) is the content of [2, Lemma 9.3] and is equivalent to the verification of the even parity of the first Chern class of the tangent bundle TM evaluated on S^1-invariant 2-spheres which contain exactly two S^1-fixed points from disjoint connected components of $\{P\}$. In our non-spin set-up, this is proved by applying [3, Theorem V], which implies that such a number is identically zero.

4. Application

An oriented, connected, irreducible, Riemannian $4n$-manifold M is called a *quaternion-Kähler manifold*, $n \geqslant 2$, if its linear holonomy is contained in $\mathrm{Sp}(n)\,\mathrm{Sp}(1)$. We shall call M *positive* if its metric is complete and has positive scalar curvature. It is known that if a positive quaternion-Kähler manifold M is not the complex Grassmannian $\mathbb{G}r_2(\mathbb{C}^{n+2})$, then $\pi_2(M)$ is finite [7], and that the $8m+4$-dimensional positive quaternion-Kähler manifolds are *not spin* in general.

COROLLARY 4.1 ([4]). – *Let M be a positive quaternion-Kähler manifold different from $\mathbb{G}r_2(\mathbb{C}^{n+2})$ which admits S^1 actions. Then we have*

$$\widehat{A}(M) = 0.$$

This corollary turns out to be the key in the classification of such manifolds in 12 dimensions (*cf.* [4]).

Examples. – The real Grassmannian $\mathbb{G}r_4(\mathbb{R}^{2m+5})$ is not spin, its isometry group is $\mathrm{SO}(2m+5)$ and satisfies the hypotheses of Theorem 1.1. Therefore $\widehat{A}(\mathbb{G}r_4(\mathbb{R}^{2m+5})) = 0$. Furthermore, one can also check that its elliptic genus vanishes $\tau_q(\mathbb{G}r_4(\mathbb{R}^{2m+5})) = 0$.

In contrast, we have the complex Grassmannian $\mathbb{G}r_2(\mathbb{C}^{2m+3})$ which is neither spin nor has finite second homotopy group. In fact $\pi_2(\mathbb{G}r_2(\mathbb{C}^{2m+3})) = \mathbb{Z}$. Direct calculation shows that $\widehat{A}(\mathbb{G}r_2(\mathbb{C}^{2m+3})) \neq 0$.

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