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Vector bundles of degree zero over an elliptic curve

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Abstract In this Note we study indecomposable vector bundles of degree zero over an elliptic curve. We show that each bundle generates a ring and a Tannakian category, such that the ring and the group scheme associated to the Tannakian category are of the same dimension. Furthermore we show that the result does not extend to curves of genus 2. ***To cite this article: S. Lekaus, C. R. Acad. Sci. Paris, Ser. I 335 (2002) 351–354.***

Fibrés vectoriels de degré zéro sur une courbe elliptique

Résumé Dans cette Note, nous étudions les fibrés vectoriels indécomposables de degré zéro sur une courbe elliptique. Nous montrons que chaque fibré engendre un anneau et une catégorie tannakienne tels que l'anneau et le schéma en groupes associé à la catégorie soient de la même dimension. De plus, nous montrons que ce résultat ne s'étend pas aux courbes de genre 2. ***Pour citer cet article : S. Lekaus, C. R. Acad. Sci. Paris, Ser. I 335 (2002) 351–354.***

1. Introduction and notations

Let X be a complete, connected, reduced scheme over a perfect field k. We define $\mathrm{Vect}(X)$ to be the set of isomorphism classes $[V]$ of vector bundles V over X and the ring $K(X)$ to be the Grothendieck group associated to the additive monoid $\mathrm{Vect}(X)$, endowed with the multiplication induced by the tensor product of vector bundles. The indecomposable vector bundles over X form a $\mathbb{Z}$-basis of $K(X)$. Since $\mathrm{H}^0(X, \mathrm{End}(V))$ is finite dimensional, the Krull–Schmidt theorem [1] holds on X. This means that a decomposition of a vector bundle into indecomposable components is unique up to isomorphism.

We want to generalize a theorem by M. Nori on finite vector bundles. A vector bundle V over X is called finite, if the set $S(V)$ of all indecomposable components of $V^{\otimes n}$ for all integers $n \in \mathbb{Z}$ is finite, where $V^{\otimes n} := (V^\vee)^{\otimes(-n)}$ for $n < 0$.

In the following, we denote by R(V) the $\mathbb{Q}$-subalgebra of $K(X) \otimes_{\mathbb{Z}} \mathbb{Q}$ generated by the set $S(V)$. If V is a finite vector bundle, the $\mathbb{Q}$-algebra $R(V)$ is of Krull dimension zero, since a finite vector bundle is integral over $\mathbb{Q}$ (*see* [9], Lemma (3.1)).

In [9], Nori proves the following:

For every finite vector bundle V over X there exists a finite group scheme G_V and a principal G_V-bundle $\pi : P \to X$, such that $\pi^* V$ is trivial over P. In particular, the equality $\dim R(V) = \dim G_V$ $(= 0)$ holds.

The group scheme G_V is the group scheme associated to a Tannakian category $\mathcal{C}_V$, generated by V as subcategory of $SS(X)$, where $SS(X)$ denotes the full subcategory of the category of quasi-coherent sheaves

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on X, whose objects are the vector bundles that are semistable of degree zero when restricted to any curve in X.

As every (arbitrary) vector bundle V over X of rank r trivializes over its associated principal GL(r)-bundle, we can look for a group scheme G of smallest dimension such that there exists a principal G-bundle over which the pullback of the vector bundle V is trivial. We might also compare the dimension of the group scheme to $\dim R(V)$.

In this Note, we consider the family of vector bundles of degree zero over an elliptic curve defined over an algebraically closed field of characteristic zero and prove that they trivialize over a principal G-bundle with a group scheme G of smallest possible dimension equal to 1. As in Nori's theorem, this dimension turns out to be equal to the dimension of the ring $R(V)$, and the group scheme is the one associated to the Tannakian category generated by the vector bundle. Furthermore we show that this result does not extend to vector bundles over a curve of higher genus: we construct a stable vector bundle E of degree zero over a curve of genus 2, whose ring $R(E)$ is of dimension 1, whereas the group scheme associated to its Tannakian category is 3-dimensional.

2. Dimension relation

Let X be an elliptic curve over an algebraically closed field k of characteristic zero. We consider vector bundles of degree zero over X; such bundles were classified by Atiyah [2]. By $\mathcal{E}(r,0)$ we denote the set of indecomposable vector bundles of rank r and degree zero over X.

THEOREM 2.1 (Atiyah [2]). – *There exists a vector bundle $F_r \in \mathcal{E}(r,0)$, unique up to isomorphism, with $\Gamma(X, F_r) \neq 0$. Moreover we have exact sequences*

$$0 \to \mathcal{O}_X \to F_r \to F_{r-1} \to 0. \tag{1}$$

If $E \in \mathcal{E}(r,0)$, then $E \cong L \otimes F_r$, where L is a line bundle of degree zero, unique up to isomorphism.

PROPOSITION 2.2. –

(1) *The $\mathbb{Q}$-subalgebra $R(F_r)$ of $K(X) \otimes_{\mathbb{Z}} \mathbb{Q}$ generated by $S(F_r)$ is $\mathbb{Q}[x]$, with $x = [F_2]$, if r is even, and $x = [F_3]$, if r is odd.*

(2) *There exists a principal $\mathbb{G}_a$-bundle $\pi : P \to X$ such that $\pi^*(F_r)$ is trivial for all $r \geqslant 2$. There is no finite group scheme G, such that $\pi^*(F_r)$ trivializes over a principal G-bundle.*

Remark. – As in the case of finite bundles we have a correspondence of dimensions $\dim R(F_r) = \dim \mathbb{G}_a$ $(= 1)$.

Proof. – The vector bundles fulfill the multiplication formula $F_r \otimes F_s \cong F_{r+s-1} \oplus \cdots \oplus F_{r-s+1}$ for $2 \leqslant s \leqslant r$ (*see* [2], Lemma 21), which is the Clebsch–Gordan formula for the symmetric product of a vector bundle of rank 2, since $\det(F_2) \cong \mathcal{O}_X$ and $F_r \cong S^{r-1}(F_2)$ for all $r \geqslant 1$ (*see* [2], Theorem 9). This implies that there exist $a_i(n) \in \mathbb{Z}$ such that $F_r^{\otimes n} \cong a_1(n)\mathcal{O}_X \oplus a_2(n)F_2 \oplus \cdots \oplus a_{(r-1)n-1}F_{(r-1)n-1} \oplus F_{(r-1)n+1}$, where $a_i(n) = 0$ for odd i, if r is even and n is odd, and $a_i(n) = 0$ for even i, if either r is odd or if r and n are both even. Else $a_i(n) > 0$. Therefore $S(F_r) = \{F_i \mid i \in \mathbb{N}\}$, if r is even, and $S(F_r) = \{F_i \mid i \geqslant 1,\ i \text{ odd}\}$, if r is odd. In particular every bundle F_i, $i \geqslant 1$, appears as subbundle of a tensor power of F_2, and every F_i, i odd, as subbundle of a tensor power of F_3, and the computation of $R(F_r)$ follows.

By definition, F_2 is defined by an element of $\mathrm{Ext}^1(\mathcal{O}_X, \mathcal{O}_X) = \mathrm{H}^1(X, \mathcal{O}_X)$, hence defines a principal $\mathbb{G}_a$-bundle, over which F_2 and all $F_r \cong S^{r-1}(F_2)$ trivialize. Every finite, unramified covering Y of X is an elliptic curve, and the pullback yields an isomorphism of $\mathrm{H}^1(X, \mathcal{O}_X)$ and $\mathrm{H}^1(Y, \mathcal{O}_Y)$. Therefore F_2, and inductively F_r, cannot trivialize over such a covering. □

There is a similar statement for $E \cong L \otimes F_r$ with L a line bundle of degree zero (see Theorem 2.1):

PROPOSITION 2.3. –

(1) *If L is not torsion, the ring $R(E)$ is isomorphic to $\mathbb{Q}[x, x^{-1}] \otimes \mathbb{Q}[y]$ and E trivializes over a principal $\mathbb{G}_m \times \mathbb{G}_a$-bundle.*

(2) *If L is torsion, let $n \in \mathbb{N}$, $n \geqslant 1$, be the minimal number such that $L^{\otimes n} \cong \mathcal{O}_X$. If n and r are both even, the ring $R(E)$ is isomorphic to $\mathbb{Q}[x]/\langle x^{n/2} - 1\rangle \otimes \mathbb{Q}[y]$, and E trivializes over a principal $\mu_n \times \mathbb{G}_a$-bundle. There is no principal $\mu_{n/2} \times \mathbb{G}_a$-bundle over which E is trivial.*
If n and r are not both even, the ring $R(E)$ is isomorphic to $\mathbb{Q}[x]/\langle x^n - 1\rangle \otimes \mathbb{Q}[y]$, and E trivializes over a principal $\mu_n \times \mathbb{G}_a$-bundle.

In all cases the trivializing principal bundle is $P_L \times_X P$, where P is the principal $\mathbb{G}_a$-bundle from Proposition 2.2 *and P_L is a principal bundle over which L is trivial.*

Proof. – The computation of the rings $R(E)$ follows from the multiplication formula for the bundles F_r, $r \geqslant 2$ ([2], Lemma 21). If P_L is the principal bundle associated to L via the transition functions of L, it is clear that every $L \otimes F_r$ trivializes over $P \times_X P_L$, therefore it only remains to prove that, for an n-torsion bundle L, $L \otimes F_r$ is not trivial over a $\mu_m \times \mathbb{G}_a$-bundle $P_m \times_X P$ with $m < n$. Since every finite unramified covering P_m of X is again an elliptic curve, we assume without loss of generality that $X = P_m$ and show that $L \otimes F_r$ cannot be trivial over P, if L is non-trivial. Else we have for arbitrary large N that $\mathcal{O}_{\mathbb{P}(F_2)} \hookrightarrow \pi^*(L \otimes F_r)(N\infty)$, where $\pi : \mathbb{P}(F_2) \to X$ is the projection and ∞ denotes the hyperplane $\mathbb{P}(\mathcal{O}_X) \subset \mathbb{P}(F_2)$. The projection formula and [5], II, (7.11) imply that $\mathcal{O}_X \hookrightarrow (L \otimes F_r) \otimes \pi_* \mathcal{O}_{\mathbb{P}(F_2)}(N\infty) = (L \otimes F_r) \otimes S^N(F_2) = (L \otimes F_r) \otimes F_{N+1}$, and hence that $\mathcal{O}_X \hookrightarrow L \otimes (F_{N+2-r} \oplus F_{N+4-r} \oplus \cdots \oplus F_{N+r})$ (*see* Proposition 2.2). Thus one of the direct summands must have a non-trivial global section. By the uniqueness of the F_i, $i \in \mathbb{N}$, this implies that L must be trivial, which contradicts our assumption. □

Remark 1. – The correspondence between the dimension of the "minimal" group scheme and the dimension of the ring $R(E)$ also occurs in the case of vector bundles over the projective line. This follows immediately from the fact that every vector bundle over $\mathbb{P}^1$ splits into a direct sum of line bundles.

3. Tannakian category associated to a vector bundle

PROPOSITION 3.1. – *Every indecomposable vector bundle of degree zero over an elliptic curve X is semistable.*

Proof. – This follows inductively from the exact sequence in Theorem 2.1 and the fact that every line bundle of degree zero is semistable. □

If E is an indecomposable vector bundle of degree zero over X, it generates a category $\mathcal{C}_E$, which is the full subcategory of $SS(X)$ with set of objects $\overline{S(E)}$, where $\overline{S(E)}$ denotes the set of vector bundles that are isomorphic to a bundle V_2/V_1, where V_1 and V_2 are objects of $SS(X)$ such that $V_1 \subset V_2 \subset \bigoplus_{i=1}^t P_i$ for some $P_i \in S(E)$, $1 \leqslant i \leqslant t$, and with $S(E)$ as defined in the introduction. As X is an elliptic curve, here $SS(X)$ just denotes the category of semistable vector bundles of degree zero.

For example, since $S(F_r) = \{F_i \mid i \in \mathbb{N}\}$, if r is even, and $S(F_r) = \{F_i \mid i \geqslant 1,\ i \text{ odd}\}$, if r is odd, and $F_i \subset F_{i+1}$ for all $i \geqslant 1$, we obtain that $\mathcal{C}_{F_r} = \mathcal{C}_{F_2}$ for all $r \geqslant 2$, with the objects being subquotients of finite direct sums of the bundles F_i, $i \geqslant 1$.

Since $SS(X)$ is Abelian, all objects of $\overline{S(E)}$ are objects of $SS(X)$. By construction, $\mathcal{C}_E$ is abelian for every indecomposable vector bundle E over X (*see* [9], §1).

PROPOSITION 3.2 (compare [9]). – *For every $E \in \mathcal{E}(r, 0)$, $r \in \mathbb{N}$, the category $\mathcal{C}_E$ together with the fibre functor x^*, mapping an object of $\mathcal{C}_E$ to its fibre in a k-rational point x, is a neutralized Tannakian category.*

For the formalism of Tannakian categories see [10] and [3]. By a theorem of Saavedra [10] for any neutralized Tannakian category $(\mathcal{C}, \omega)$ there exists an affine group scheme G such that $\mathcal{C}$ is equivalent to G-mod, the category of finite-dimensional representations of G.

PROPOSITION 3.3. – *Let $E \in \mathcal{E}(r, 0)$, i.e. $E \cong L \otimes F_r$ for some line bundle L of degree zero. Then $\mathcal{C}_E \cong (\mathbb{G}_m \times \mathbb{G}_a)$*-mod, *if L is not a torsion bundle, and $\mathcal{C}_E \cong (\mu_n \times \mathbb{G}_a)$*-mod, *if L is an n-torsion bundle with $n \geqslant 1$ the minimal number such that $L^{\otimes n} \cong \mathcal{O}_X$.*

Proof. – First we note that for all $r \geqslant 2$, $\mathcal{C}_{L\otimes F_r} = \mathcal{C}_{L\otimes F_2}$. By [3], Prop. 2.8, there is a functorial isomorphism between the group scheme G, corresponding to $\mathcal{C}_{L\otimes F_2}$, and $\mathrm{Aut}^{\otimes}(x^*)$ which is a functor assigning to every k-algebra R a set of families $\{\alpha(V)\}$, $V \in \mathrm{Obj}\, \mathcal{C}_{L\otimes F_2}$, of R-linear endomorphisms of $x^*(V) \otimes_k R$, compatible with the tensor product and morphisms in $\mathcal{C}_{L\otimes F_2}$. Then $\alpha(L \otimes F_2) = \alpha(L) \otimes \alpha(F_2)$

is in $(\mathbb{G}_m \times \mathbb{G}_a)(R)$ or in $(\mu_n \times \mathbb{G}_a)(R)$, depending on whether L is a torsion bundle. Every $\alpha(V)$, $V \in \mathrm{Obj}\ \mathcal{C}_E$, is uniquely determined by $\alpha(L \otimes F_2)$, because of the compatibility properties of the family and the fact that every V is a subquotient of a finite direct sum of bundles $(L \otimes F_2)^{\otimes a} \otimes ((L \otimes F_2)^{\vee})^{\otimes b}$, $a, b \in \mathbb{Z}, \geqslant 0$. Conversely, every element of $(\mathbb{G}_m \times \mathbb{G}_a)(R)$ or $(\mu_n \times \mathbb{G}_a)(R)$ defines a family in this way. □

4. Counter-example on a curve of genus 2

Let X be a curve of genus $g = 2$ over $\mathbb{C}$, and let x be a $\mathbb{C}$-rational point of X.

We want to show that there exists a stable bundle E over X, generating a Tannakian category $\mathcal{C}_E$, such that the ring $R(E)$ is of smaller dimension than the group scheme associated to $\mathcal{C}_E$. For this let $\rho : \pi_1(X, x) \to \mathrm{GL}_2(\mathbb{C})$ be the irreducible, unitary representation defined by $\rho(a) := A$, $\rho(b) := B$, $\rho(c) := B$, $\rho(d) := A$, with generators a, b, c, and d of $\pi_1(X, x)$, satisfying $[a, b][c, d] = 1$, and

$$A := \begin{pmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{pmatrix}, \qquad B = \begin{pmatrix} \cos(2\pi\Phi) & -\sin(2\pi\Phi) \\ \sin(2\pi\Phi) & \cos(2\pi\Phi) \end{pmatrix}, \tag{2}$$

where $\lambda = \mathrm{e}^{2\pi\,\mathrm{i}\Phi}$ for an irrational $\Phi \in [0, 1]$.

The Zariski-closure of the image of ρ in $\mathrm{GL}_2(\mathbb{C})$ is $\mathrm{SL}_2(\mathbb{C})$, hence by [6], (1.2.2), the Tannakian category generated by ρ, whose objects are subquotients of finite direct sums of tensor powers of ρ and $\rho^{\vee}$, is equivalent to the category $\mathrm{SL}_2(\mathbb{C})$-mod.

Let E be the stable vector bundle of degree zero corresponding to ρ via the Narasimhan–Seshadri correspondence ([8], §12). All objects of $\mathcal{C}_E$, as constructed in the previous paragraph, are polystable vector bundles of degree zero (i.e. direct sums of stable vector bundles of degree zero [7]): any semistable bundle V of degree zero has, by Jordan–Hölder-filtration, a stable subbundle W of degree zero. Hence, if V is a subbundle of a finite direct sum of objects of $S(E)$ (which are all stable), W must be isomorphic to one of the direct summands and must split from V. Inductively, we obtain that V is polystable. By the Narasimhan–Seshadri correspondence there is therefore a 1–1 correspondence between the objects of $\mathcal{C}_E$ and of $\mathcal{C}_\rho$, so we obtain an equivalence of categories $\mathcal{C}_E \cong \mathcal{C}_\rho \cong \mathrm{SL}_2(\mathbb{C})$-mod. Since $\mathrm{SL}_2(\mathbb{C})$ is connected and simply connected, its representations are in 1–1 correspondence with those of the Lie algebra $sl_2(\mathbb{C})$ (*see* [4], Part II, §8.1). As $sl_2(\mathbb{C})$ is completely reducible ([4], Theorem 9.19), this implies that the indecomposable vector bundles in $\mathcal{C}_E$ cannot have any proper subbundles. Hence the definition of $\mathcal{C}_E$ implies that the elements of $S(E)$ are the only indecomposable bundles in $\mathcal{C}_E$. Therefore the elements of $S(E)$ are in 1–1 correspondence with the irreducible (=indecomposable) representations of $\mathrm{SL}_2(\mathbb{C})$, and it follows that the ring $R(E)$ is isomorphic to the representation ring of $\mathrm{SL}_2(\mathbb{C})$ over $\mathbb{Q}$. As all representations of $sl_2(\mathbb{C})$ are symmetric powers of the standard representation, we conclude that $R(E) \cong \mathbb{Q}[x]$.

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