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Γ-convergence of nonlinear functionals in thin reticulated structures

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Abstract We study the Γ-convergence of nonlinear functionals considered in nonperiodic 2D lattice-like structures. The Γ-limit functional is obtained in the explicit form. ***To cite this article: L. Pankratov, C. R. Acad. Sci. Paris, Ser. I 335 (2002) 315–320.*** © 2002 Académie des sciences/Éditions scientifiques et médicales Elsevier SAS

Γ-convergence des fonctionelles non linéaires dans des structures réticulées de faible épaisseur

Résumé On étudie la Γ-convergence de fonctionelles non linéaires considérées dans des structures non périodiques de type de grille dans l'espace $\mathbf{R}^2$. La fonctionelle Γ-limite est obtenue sous forme explicite. ***Pour citer cet article : L. Pankratov, C. R. Acad. Sci. Paris, Ser. I 335 (2002) 315–320.*** © 2002 Académie des sciences/Éditions scientifiques et médicales Elsevier SAS

Version française abrégée

On étudie la Γ-convergence de fonctionelles $J^{(\varepsilon)}:\mathrm{W}^{1,m}(\Omega^{(\varepsilon)}) \to \mathbf{R}$ définies par (1)–(4) dans des domaines $\Omega^{(\varepsilon)}$ ayant la forme d'une grille. De façon précise on considère une grille non périodique $\Omega^{(\varepsilon)}$, $\Omega^{(\varepsilon)} \subset \Omega \equiv (0, H)^2 \subset \mathbf{R}^2$, constituée de deux systèmes de bandes minces dont les axes sont paralleles aux axes de coordonnées. On suppose que la distance entre les axes de bandes est égale à ε. On note par x_1^r et x_2^q les points d'intersection des axes de bandes avec les axes de coordonnées Ox_2 et Ox_1. On définit alors les épaisseurs de bandes parallèles aux axes Ox_1 et Ox_2 par $d_\varepsilon\psi_1(x_2^q)$ et $d_\varepsilon\psi_2(x_1^r)$, respectivement. Ici $d_\varepsilon = \mathrm{o}(\varepsilon)$ lorsque $\varepsilon \to 0$, $\psi_i(t)$ $(i = 1, 2)$ sont des fonctions lisses à valeurs dans l'espace $\mathbf{R}$. Les problèmes elliptiques linéaires dans les grilles minces périodiques ont été considérés dans [1,4,12].

On introduit ensuite la notion de la Γ-convergence pour les fonctionelles définies dans $\mathrm{W}^{1,m}(\Omega^{(\varepsilon)})$, où $\mathrm{meas}(\Omega^{(\varepsilon)}) \to 0$ lorsque $\varepsilon \to 0$ (*voir* Définition 2, Paragraphe 2 de la version anglaise). Cette définition est très proche de la définition de la Γ-convergence pour des fonctionelles définies dans $\mathrm{W}^{1,m}(\Omega)$ (*voir* [6,13]) et aussi de la définition pour des fonctionelles définies dans $\mathrm{W}^{1,m}(\Omega^{(\varepsilon)})$, où $\Omega^{(\varepsilon)}$ est un domaine perforé

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tel que meas$(\Omega^{(\varepsilon)}) \geqslant a_0 > 0$ (*voir* [8]). Finalement, en utilisant des idées des articles [5,7,10] on obtient le résultat fondamental de cette Note.

THÉORÈME 1. – *Soit* $J^{(\varepsilon)}[u^\varepsilon]$ *la fonctionelle définie par* (1)–(4). *Alors la suite des fonctionelles* $J^{(\varepsilon)}: \mathrm{W}^{1,m}(\Omega^{(\varepsilon)}) \to \mathbf{R}$, Γ*-converge vers la fonctionelle* $J_{\mathrm{hom}}: \mathrm{W}^{1,m}(\Omega) \to \mathbf{R}$ *définie par* (5), (6).

Le schéma de la démonstration du Théorème 1 est présenté dans le Paragraphe 3 (*voir* la version anglaise). On considère le problème variationnel (7) correspondant à la fonctionelle (1)–(4). On calcule alors les caractéristiques géométriques (8), (9) du domaine $\Omega^{(\varepsilon)}$ et la caractéristique locale non linéaire (10) de $\Omega^{(\varepsilon)}$ est attachée au problème (7). Cette dernière est calculée à partir des fonctions $\hat{v}^\varepsilon(x)$ définies par (11). La démonstration du Théorème 1 se décompose alors en trois étapes.

Étape 1. Soit $u^\varepsilon(x)$ la solution du problème variationnel (7). En utilisant la fonction test $w^\varepsilon(x)$ définie par (13) on obtient (14).

Étape 2. Pour toute suite $\{z^\varepsilon(x)\} \subset \mathcal{A}(u)$ (*voir* Définition 2 de la version anglaise) on montre (15). En particulier, cette inégalité est vérifiée pour la solution $u^\varepsilon(x)$ du problème (7). Dans ce cas la fonction $u(x)$ dans (15) est la solution du problème variationnel pour la fonctionelle (5), (6).

Étape 3. En utilisant alors la forme explicite (13) de la fonction $w^\varepsilon(x)$, on construit les fonctions $\tilde{u}^\varepsilon(x)$ de l'ensemble $\mathcal{A}(u)$ qui vérifient (16).

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1. Introduction

The paper is devoted to the homogenization of nonlinear variational problems in nonperiodic 2D lattice-like domains $\Omega^{(\varepsilon)}$, where $\varepsilon > 0$ is a parameter characterizing the scale of the microstructure. Notice that there is a considerable number of papers devoted to the homogenization of PDE considered in strongly perforated domains or those with strongly oscillating coefficients (*see*, e.g., [1–4,11,13] containing extensive bibliography).

The structure of the lattice (*see* Section 2) implies that meas$(\Omega^{(\varepsilon)}) \to 0$ as $\varepsilon \to 0$. The linear elliptic problems in 2D rectangular lattice-like structures having this property were studied in [1,4] and in domains of degenerating measure without any periodicity condition in [12] (for the definitions, *see* also [5]). In [12] the convergence result is given in terms of the D-convergence. Let us recall this notion.

DEFINITION 1. – A sequence of functions $\{u^\varepsilon\} \subset \mathrm{W}^{1,m}(\Omega^{(\varepsilon)})$ is said to D-converge in $\mathrm{W}^{1,m}(\Omega^{(\varepsilon)})$ to a function $u \in \mathrm{W}^{1,m}(\Omega)$ if there exists an approximating sequence of functions $\{u_M \in \mathrm{Lip}_1(C_M, \Omega), M = 1, 2, \ldots\}$ that converges in $\mathrm{W}^{1,m}(\Omega)$ to u, and

$$\lim_{M\to\infty} \varlimsup_{\varepsilon\to 0} \frac{1}{\mathrm{meas}(\Omega^{(\varepsilon)})} \left\| u^\varepsilon - u_M \right\|^m_{1,\Omega^{(\varepsilon)}} = 0.$$

Here $\mathrm{Lip}_1(C_M, \Omega) = \{u \in \mathrm{C}^1(\Omega): |D^\alpha u(x)| \leqslant M,\ |\alpha| \leqslant 1;\ |D^\alpha u(x) - D^\alpha u(y)| \leqslant M|x-y|,\ |\alpha| = 1;\ x, y \in \Omega\}$; $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_n)$, $|\alpha| = \sum_{i=1}^n \alpha_i$; $\|\cdot\|_{1,\mathcal{O}}$ is the norm in $\mathrm{W}^{1,m}(\mathcal{O})$.

The results of [12] are extentended in [5] to nonlinear variational problems. In [5] we also develop the method of discontinuous approximations which allow us to obtain explicitly the homogenized model for 2D lattice-like periodic structures in the case $m = 4$.

The main goal of the present paper is to extend the methods developed [5] to nonperiodic 2D lattice-like structures and arbitrary $m \geqslant 2$. The homogenization result is formulated in terms of the Γ-convergence (*see* Definition 2 below).

2. Homogenization result

Let $\Omega = (0, H)^2$ be a square in $\mathbf{R}^2$ with edge lengths H, and $\mathcal{L}^{(\varepsilon)} \subset \Omega$ a 2D rectangular lattice-like structure consisting of two systems of thin strips oriented in the coordinate directions. The axes of the strips form a periodic lattice in $\mathbf{R}^2$ with the period ε. The widths of the strips are defined as follows. Denote the points of intersection of the axes of the strips and the coordinate axes Ox_2 and Ox_1 by x_1^r and x_2^q, respectively. Let $\psi_1(t)$ and $\psi_2(t)$ be smooth real functions. Then we assume that the width of the strip parallel to axis Ox_1 (Ox_2) is equal to $d_\varepsilon \psi_1(x_2^q)$ ($d_\varepsilon \psi_2(x_1^r)$, respectively), where $d_\varepsilon = \mathrm{o}(\varepsilon)$ as $\varepsilon \to 0$. We set $\Omega^{(\varepsilon)} = \Omega \cap \mathcal{L}^{(\varepsilon)}$; then $\mathrm{meas}(\Omega^{(\varepsilon)}) \to 0$ as $\varepsilon \to 0$.

Consider the functionals $J^{(\varepsilon)} : \mathrm{W}^{1,m}(\Omega^{(\varepsilon)}) \to \mathbf{R}$ defined by:

$$J^{(\varepsilon)}[u^\varepsilon] = \mu^\varepsilon \int_{\Omega^{(\varepsilon)}} \left\{ |\nabla u^\varepsilon|^m + F(x, u^\varepsilon) \right\} \mathrm{d}x, \tag{1}$$

$$\mu^\varepsilon = H^2 \left(\mathrm{meas}\left(\Omega^{(\varepsilon)}\right)\right)^{-1}. \tag{2}$$

Here $F(x,u)$ is a continuous function, $F(x,u) \in \mathrm{C}(\overline{\Omega}, \mathbf{R})$, having the partial derivative F_u, $F_u(x,u) \in \mathrm{C}(\overline{\Omega}, \mathbf{R})$, and satisfying the following conditions:

$$\left| F(x,u) - F(x,v) \right| \leqslant A_1 \left(1 + |u| + |v|\right)^{m-1} |u - v|, \tag{3}$$

$$F(x,u) \geqslant A_2 \left(|u|^m - 1\right). \tag{4}$$

To formulate the main result of the paper, introduce the notion of the Γ-convergence for functionals $J^{(\varepsilon)} : \mathrm{W}^{1,m}(\Omega^{(\varepsilon)}) \to \mathbf{R}$. This notion is similar to that for functionals defined in $\mathrm{W}^{1,m}(\Omega)$ (*see*, e.g. [3,6]), and also in $\mathrm{W}^{1,m}(\Omega^{(\varepsilon)})$, where $\Omega^{(\varepsilon)}$ is a sequence of perforated domains such that $\mathrm{meas}(\Omega^{(\varepsilon)}) \geqslant a_0 > 0$ (*see*, e.g. [8]).

For any $u \in \mathrm{W}^{1,m}(\Omega)$ we denote by $\mathcal{A}(u)$ the set of all sequences $\{u^\varepsilon(x)\}$ such that
(a) $u^\varepsilon \in \mathrm{W}^{1,m}(\Omega^{(\varepsilon)})$ for any $\varepsilon > 0$,
(b) $u^\varepsilon(x)$ D-converges in $\mathrm{L}^m(\Omega^{(\varepsilon)})$ to a function $u \in \mathrm{W}^{1,m}(\Omega)$,
(c) $\sup_\varepsilon \mu^\varepsilon \|u^\varepsilon\|^m_{1,\Omega^{(\varepsilon)}} < \infty$.

DEFINITION 2. – A sequence of functionals $J^{(\varepsilon)} : \mathrm{W}^{1,m}(\Omega^{(\varepsilon)}) \to \mathbf{R}$ is said to Γ-converge to a functional $J : \mathrm{W}^{1,m}(\Omega) \to \mathbf{R}$, if, for any $u \in \mathrm{W}^{1,m}(\Omega)$, $\overline{\lim}_{\varepsilon \to 0} J^{(\varepsilon)}[u^\varepsilon] \geqslant J[u]$ for any $\{u^\varepsilon(x)\} \subset \mathcal{A}(u)$, and there exists a sequence $\{w^\varepsilon(x)\} \subset \mathcal{A}(u)$ such that $\lim_{\varepsilon \to 0} J^{(\varepsilon)}[w^\varepsilon] = J[u]$.

THEOREM 1. – *Let* $J^{(\varepsilon)} : \mathrm{W}^{1,m}(\Omega^{(\varepsilon)}) \to \mathbf{R}$ *be the functional defined by* (1)–(4). *Then* $J^{(\varepsilon)}$ Γ*-converges to*

$$J_{\mathrm{hom}}[u] = \gamma \int_\Omega \left\{ \psi_1(x_2) u_{x_1}^m + \psi_2(x_1) u_{x_2}^m + \left(\psi_1(x_2) + \psi_2(x_1)\right) F(x,u) \right\} \mathrm{d}x, \tag{5}$$

where $x = \{x_1, x_2\}$ *and*

$$\gamma^{-1} = H^{-1} \int_0^H \left(\psi_1(t) + \psi_2(t)\right) \mathrm{d}t. \tag{6}$$

The proof of Theorem 1 is based on the ideas of [5,7,10]. The sketch of the proof is given in the following section.

3. Sketch of the proof

Consider the variational problem for the functional (1)–(4):

$$J^{(\varepsilon)}[u^\varepsilon] \to \inf, \quad u^\varepsilon \in \mathrm{W}^{1,m}\left(\Omega^{(\varepsilon)}\right), \tag{7}$$

in the 2D nonperiodic lattice-like domains $\Omega^{(\varepsilon)}$. Notice that no problems arise from the complicated structure of the domain $\Omega^{(\varepsilon)}$ in the proof of the existence of a solution of the variational problem (7) (*see*, e.g. [9], Chapter 5).

First we obtain certain local characteristics of $\Omega^{(\varepsilon)}$. The geometric characteristics of $\Omega^{(\varepsilon)}$ are the following:

$$\mu^\varepsilon \sim \gamma\varepsilon(d_\varepsilon)^{-1}, \tag{8}$$

$$\lim_{h\to 0}\lim_{\varepsilon\to 0}\mu^\varepsilon h^{-2}\,\mathrm{meas}\big[K_y^h\cap\Omega^{(\varepsilon)}\big] = \gamma\big(\psi_1(y_2)+\psi_2(y_1)\big), \tag{9}$$

where K_y^h is the square centered at $y\in\Omega$ with edge lengths h, $1\gg h\gg\varepsilon>0$, and γ is defined by (6).

Introduce the nonlinear local characteristic of $\Omega^{(\varepsilon)}$ related to the variational problem (7). Consider the functional

$$C(y,\varepsilon,h;b) = \inf_v \mu^\varepsilon \int_{K_y^h\cap\Omega^{(\varepsilon)}} \big\{|\nabla v|^m + h^{-m-\tau}\big|v-(x-y,b)\big|^m\big\}\,\mathrm{d}x \equiv \inf_v I_b^{(\varepsilon)}[v], \tag{10}$$

where $m>\tau>0$, $(\cdot,\cdot)$ is the scalar product in $\mathbf{R}^2$, $b=\{b_1,b_2\}$; the infimum in (10) is taken over $v\in \mathrm{W}^{1,m}(K_y^h\cap\Omega^{(\varepsilon)})$. To calculate $\lim_{h\to 0}\lim_{\varepsilon\to 0}h^{-2}C(y,\varepsilon,h;b)$ we make use of the method of discontinuous approximations (*see* [5]). Namely, let $P_{1qq'}^{(\varepsilon)}$ and $P_{2rr'}^{(\varepsilon)}$ be the rectangles parallel to the axes Ox_1 and Ox_2, respectively, and $Q_{rq}^{(\varepsilon)}$ be the rectangles formed by the intersections of the strips. Here q and r are the numbers of the strips and p', q' are the numbers of the rectangles in the corresponding strips. We denote by x_1^r and x_2^q the points of intersections of the axes of the strips and the coordinate axes and by x^{rq} the centers of the squares.

Denote by $v^\varepsilon(x)$ the minimiser of the functional (10). Define the functions $\hat v^\varepsilon(x)$ approximating $v^\varepsilon(x)$ as follows:

$$\hat v^\varepsilon(x) = \begin{cases} b_2x_2-(y,b)+b_1x_1^r, & x\in P_{2rr'}^{(\varepsilon)}\cap K_y^h,\\ b_1x_1-(y,b)+b_2x_2^q, & x\in P_{1qq'}^{(\varepsilon)}\cap K_y^h,\\ C_1x_1+C_2x_2+C_3, & x\in Q_{rq}^{(\varepsilon)}\cap K_y^h,\end{cases} \tag{11}$$

where the coefficients $C_1,\ C_2$ satisfy the system of equations $(C_1^2+C_2^2)^{(m-2)/2}C_i = b_i|b_i|^{m-2}$ and C_3 is defined by: $\hat v^\varepsilon(x^{rq}) = (x^{rq}-y,b)$. Then $\hat v^\varepsilon(x)$ has the following properties:

(1) $\hat v^\varepsilon(x)$ is a 'good' approximation of the function $(x-y,b)$ in any subdomain $P_{1qq'}^{(\varepsilon)}$, $P_{2rr'}^{(\varepsilon)}$, and $Q_{rq}^{(\varepsilon)}$ of $\Omega^{(\varepsilon)}$, i.e. $|\hat v^\varepsilon(x)-(x-y,b)| = \mathrm{O}(d_\varepsilon)$;

(2) the normal derivative of $\hat v^\varepsilon(x)$ vanishes on $\partial\Omega^{(\varepsilon)}$, and $|\nabla\hat v^{\alpha(\varepsilon)}|^{m-2}\partial\hat v^{\alpha(\varepsilon)}/\partial\nu$ is continuous across the inner boundaries of $P_{1qq'}^{(\varepsilon)}$, $P_{2rr'}^{(\varepsilon)}$, and $Q_{rq}^{(\varepsilon)}$;

(3) $\sum_{i=1}^2 \frac{\partial}{\partial x_i}(|\nabla\hat v^\varepsilon|^{m-2}\partial\hat v^\varepsilon/\partial x_i) = 0$ in $P_{1qq'}^{(\varepsilon)}$, $P_{2rr'}^{(\varepsilon)}$, and $Q_{rq}^{(\varepsilon)}$;

(4) the jumps of $\hat v^\varepsilon(x)$ across the inner boundaries of $P_{1qq'}^{(\varepsilon)}$, $P_{2rr'}^{(\varepsilon)}$, and $Q_{rq}^{(\varepsilon)}$ are $\mathrm{O}(d_\varepsilon)$.

Using the explicit form of $\hat v^\varepsilon(x)$ and the inequality:

$$F(u+v)\geqslant F(u)+\theta F(v)+F_{x_i}v_{x_i}+F_u v,$$

where $F(u)=|\nabla u|^m+|u|^m$, $0<\theta\leqslant 1$, one can show that the residual $w^\varepsilon(x)=v^\varepsilon(x)-\hat v^\varepsilon(x)$ gives a vanishing contribution (as $\varepsilon\to 0$, $h\to 0$) to the functional (10). Finally, we obtain:

$$\lim_{h\to 0}\lim_{\varepsilon\to 0}h^{-2}C(y,\varepsilon,h;b) = \lim_{h\to 0}\lim_{\varepsilon\to 0}h^{-2}I_b^{(\varepsilon)}[\hat v^\varepsilon] = \gamma\big(b_1^m\psi_1(y_2)+b_2^m\psi_2(y_1)\big). \tag{12}$$

The proof of Theorem 1 uses the local characteristics (8), (9), (12) and consists of three main steps.

Step 1. Let $\{x^\alpha\}$ be a set of points in Ω forming a space lattice with a period $h-r$, where $r = h^{1+\tau/m}$. Cover Ω by the squares $K_\alpha^h = K(x^\alpha, h)$ centered at x^α with edge lengths $h \gg \varepsilon > 0$. Associate, with this covering, a partition of unity $\{\varphi_\alpha(x)\}: 0 \leqslant \varphi_\alpha(x) \leqslant 1$; $\varphi_\alpha(x) = 0$ for $x \notin K_\alpha^h$; $\varphi_\alpha(x) = 1$ for $x \in K_\alpha^h \setminus \bigcup_{\beta\neq\alpha} K_\beta^h$; $\sum_\alpha \varphi_\alpha(x) = 1$ for $x \in \Omega$; $|\nabla\varphi_\alpha(x)| \leqslant C h^{-1-\tau/m}$.

Let $w(x)$ be an arbitrary smooth function in Ω. Define in $\Omega^{(\varepsilon)}$ the function:

$$w^\varepsilon(x) = \sum_\alpha \bigl\{ w(x) + v^{\alpha(\varepsilon)}(x) - \bigl(x - x^\alpha, \nabla w\bigl(x^\alpha\bigr)\bigr)\bigr\}\varphi_\alpha(x), \tag{13}$$

where $v^{\alpha(\varepsilon)}(x)$ is the minimiser of the functional (10) in K_α^h. Let $u^\varepsilon(x)$ be a solution of the variational problem (7). Then we show that

$$\varlimsup_{\varepsilon\to 0} J^{(\varepsilon)}[u^\varepsilon] \leqslant \varlimsup_{\varepsilon\to 0} J^{(\varepsilon)}[w^\varepsilon] \leqslant J_{\mathrm{hom}}[w] \tag{14}$$

for any $w \in \mathrm{W}^{1,m}(\Omega)$.

Step 2. We prove that

$$\varliminf_{\varepsilon\to 0} J^{(\varepsilon)}[z^\varepsilon] \geqslant J_{\mathrm{hom}}[u] \tag{15}$$

for any sequence $\{z^\varepsilon(x)\} \subset \mathcal{A}(u)$. In particular, this estimate is valid for the sequence of solutions $\{u^\varepsilon(x)\}$ of the variational problem (7). In this case, the function $u(x)$ in (15) is the solution of the variational problem for the functional (5), (6).

Step 3. Using the explicit form of the test function $w^\varepsilon(x)$ in (13), where $w(x)$ is taken to be $u(x)$, the solution of the variational problem for the functional (5), (6), we obtain $\tilde{u}^\varepsilon(x) \in \mathcal{A}(u)$ such that

$$\lim_{\varepsilon\to 0} J^{(\varepsilon)}[\tilde{u}^\varepsilon] = J_{\mathrm{hom}}[u], \tag{16}$$

that completes the proof.

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