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Géométrie différentielle/*Differential Geometry*

The effect of perturbations on the first eigenvalue of the p-Laplacian

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Abstract Let Ω be a domain with Lipschitzian boundary of a compact Riemannian manifold (M, g) and $p > 1$. We prove that we can make the volume of M arbitrarily close to the volume of (Ω, g) while the first eigenvalue of the p-Laplacian on M remains uniformly bounded from below in terms of the the first eigenvalue of the Neumann problem for the p-Laplacian on (Ω, g). ***To cite this article: A.-M. Matei, C. R. Acad. Sci. Paris, Ser. I 335 (2002) 255–258.***

L'éffet des pérturbations sur la première valeur propre du p-Laplacien

Résumé Soit Ω un domaine à bord Lipschitz d'une variété riemannienne compacte (M, g) et $p > 1$. Nous montrons qu'on peut rendre le volume de M arbitrairement proche du volume de (Ω, g) tout en gardant la première valeur propre du p-Laplacien sur M uniformement minorée en termes de la première valeur propre du problème de Neumann pour le p-Laplacien sur (Ω, g). ***Pour citer cet article : A.-M. Matei, C. R. Acad. Sci. Paris, Ser. I 335 (2002) 255–258.***

1. Preliminaries and main result

Let (M, g) be a compact Riemannian manifold. The p-Laplacian on (M, g) is defined by $\Delta_p f := \delta(|\mathrm{d}f|^{p-2}\,\mathrm{d}f)$, where $\delta = -\mathrm{div}_g$ is the opposite of the divergence. This operator may be seen as a natural extension of the Laplace–Beltrami operator which corresponds to $p = 2$.

The real constants λ for which the equation $\Delta_p f = \lambda |f|^{p-2} f$ has nontrivial solutions are the *eigenvalues* of Δ_p and the associated solutions are the *eigenfunctions*.

It was proved that the set of the nonzero eigenvalues is a nonempty, unbounded subset of $]0, \infty[$ [2]. Its infimum $\lambda_{1,p}(M, g)$ is itself a positive eigenvalue called the *first eigenvalue of* Δ_p and has the following variational caracterisation [4]:

$$\lambda_{1,p}(M, g) = \inf\left\{ \frac{\int_M |\mathrm{d}f|^p \nu_g}{\int_M |f|^p \nu_g};\ f \in \mathrm{W}^{1,p}(M) \setminus \{0\},\ \int_M |f|^{p-2} f \nu_g = 0 \right\}.$$

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Let Ω be a domain in M and consider the Dirichlet problem associated to Δ_p on Ω:

$$\begin{cases} \Delta_p f = \mu |f|^{p-2} f & \text{in } \Omega, \\ f = 0 & \text{on } \partial\Omega. \end{cases}$$

The first eigenvalue for this problem $\mu_{1,p}(\Omega, g)$, has the variational characterization

$$\mu_{1,p}(\Omega, g) = \inf\left\{ \frac{\int_\Omega |\mathrm{d}f|^p \nu_g}{\int_\Omega |f|^p \nu_g};\ f \in \mathrm{W}_0^{1,p}(\Omega, g) \setminus \{0\} \right\}.$$

We did not find in the literature the corresponding characterization for the Neumann problem for Δ_p. By analogy with the linear case consider

$$\lambda_{1,p}^N(\Omega, g) := \inf\left\{ \frac{\int_\Omega |\mathrm{d}f|^p \nu_g}{\int_\Omega |f|^p \nu_g};\ f \in \mathrm{W}^{1,p}(\Omega, g) \setminus \{0\},\ \int_\Omega |f|^{p-2} f \nu_g = 0 \right\}.$$

Mimiking the proof of the closed case [4] one obtain that $\lambda_{1,p}^N(\Omega, g)$ is the first nonzero eigenvalue for the Neumann type problem

$$\begin{cases} \Delta_p f = |f|^{p-2} f & \text{in } \Omega, \\ \mathrm{d}f(\eta) = 0 & \text{on } \partial\Omega, \end{cases}$$

where η denotes the exterior unit normal vector field to $\partial\Omega$.

A more general regularity result [3] says that the eigenfunctions for these problems are locally $\mathrm{C}^{1,\alpha}$.

The main result of this Note is the following.

THEOREM 1. – *Let (M, g) be a compact Riemannian manifold, $p > 1$ and $\Omega \subset M$ a domain with Lipschitzian boundary. Then for any $\delta > 0$, there exists a metric $\tilde{g}$ on M such that*:

(i) $g_{|\Omega} = \tilde{g}_{|\Omega}$;
(ii) $\lambda_{1,p}(M, \tilde{g}) > \lambda_{1,p}^N(\Omega, g) - \delta$;
(iii) $|\mathrm{Vol}(M, \tilde{g}) - \mathrm{Vol}(\Omega, g)| < \delta$.

For $p = 2$ and $m = \dim M \geqslant 3$, this result is contained in Theorem III.1 of [1]; we use the same type of metrics but the arguments from the linear case do not apply to the nonlinear case.

The proof of Theorem 1 is based only on functional analysis in Sobolev spaces without using deeper properties of the eigenfunctions and allows us to consider the case of surfaces.

As a consequence of Theorem 1. we have

COROLLARY 2. – *If $1 < p < m$ then Theorem 1 remains true if we replace* (ii) *by* (ii′) $|\lambda_{1,p}(M, \tilde{g}) - \lambda_{1,p}^N(\Omega, g)| < \delta$.

2. Proof of Theorem 1

Part I: singular metrics. – For $\varepsilon > 0$ denote by $\varphi_\varepsilon = 1 \cdot \chi_\Omega + \varepsilon \cdot \chi_{M \setminus \Omega}$ and by

$$\lambda_{1,p}(\varepsilon) = \inf\left\{ R_\varepsilon(f) := \frac{\int_M |\mathrm{d}f|^p (\varphi_\varepsilon)^{(m-p)/2} \nu_g}{\int_M |f|^p (\varphi_\varepsilon)^{m/2} \nu_g} \,\middle|\, f \in \mathrm{W}^{1,p}(M, g) \setminus \{0\},\ \int_M |f|^{p-2} f (\varphi_\varepsilon)^{m/2} \nu_g = 0 \right\}.$$

Let $\delta > 0$. The aim of this first part is to prove that there exists ε small enough such that $\lambda_{1,p}(\varepsilon) \geqslant \lambda_{1,p}^N(\Omega, g) - \frac{\delta}{2}$ and $\varepsilon^{m/2} \mathrm{Vol}(M \setminus \Omega, g) < \frac{\delta}{2}$.

We may assume that $\lambda_{1,p}(\varepsilon)$ is bounded from above by a constant c_0 as $\varepsilon \to 0$. Since $\partial\Omega$ is Lipschitzian, classical density arguments implies that there exists $f_\varepsilon \in \mathrm{W}^{1,p}(M,g) \setminus \{0\}$ with $\int_M |f_\varepsilon|^{p-2} f_\varepsilon\, (\varphi_\varepsilon)^{m/2} \nu_g = 0$ such that $\lambda_{1,p}(\varepsilon) = R_\varepsilon(f_\varepsilon)$. Let A_ε, B_ε be the nodal domains of f_ε. Then

$$\lambda_{1,p}(\varepsilon) = \frac{\int_{A_\varepsilon} |\mathrm{d}f_\varepsilon|^p (\varphi_\varepsilon)^{(m-p)/2} \nu_g}{\int_{A_\varepsilon} |f_\varepsilon|^p (\varphi_\varepsilon)^{m/2} \nu_g} = \frac{\int_{B_\varepsilon} |\mathrm{d}f_\varepsilon|^p\, (\varphi_\varepsilon)^{(m-p)/2} \nu_g}{\int_{B_\varepsilon} |f_\varepsilon|^p (\varphi_\varepsilon)^{m/2} \nu_g}.$$

We claim that for ε small enough, f_ε change sign on Ω. Indeed if for instance $A_\varepsilon \cap \Omega = \emptyset$, then

$$c_0 \geqslant \lambda_{1,p}(\varepsilon) = \frac{\int_{A_\varepsilon} |\mathrm{d}f_\varepsilon|^p (\varphi_{\varepsilon_n})^{(m-p)/2} \nu_g}{\int_{A_\varepsilon} |f_\varepsilon|^p (\varphi_\varepsilon)^{m/2} \nu_g} = \frac{\varepsilon^{(m-p)/2} \int_{A_\varepsilon} |\mathrm{d}f_\varepsilon|^p \nu_g}{\varepsilon^{m/2} \int_{A_\varepsilon} |f_\varepsilon|^p \nu_g}$$
$$\geqslant \varepsilon^{-p/2} \mu_{1,p}(A_\varepsilon, g) \geqslant \varepsilon^{-p/2} \mu_{1,p}(M \setminus \Omega, g),$$

where the last inequality is due to the fact that $A_\varepsilon \subset M \setminus \Omega$.

It follows that for ε small enough, $A_\varepsilon \cap \Omega \neq \emptyset$ and $B_\varepsilon \cap \Omega \neq \emptyset$. Hence, there exists a constant c_ε such that the map $\tilde{f}_\varepsilon = f_\varepsilon^+ + c_\varepsilon f_\varepsilon^-$ satisfies the orthogonality condition on Ω: $\int_\Omega |\tilde{f}_\varepsilon|^{p-2} \tilde{f}_\varepsilon \nu_g = 0$. Moreover $R_\varepsilon(f_\varepsilon) = R_\varepsilon(\tilde{f}_\varepsilon)$ and therefore

$$\lambda_{1,p}(\varepsilon) = R_\varepsilon\big(\tilde{f}_\varepsilon\big) = \frac{\int_\Omega |\mathrm{d}\tilde{f}_\varepsilon|^p \nu_g + \varepsilon^{(m-p)/2} \int_{M\setminus\Omega} |\mathrm{d}\tilde{f}_\varepsilon|^p \nu_g}{\int_\Omega |\tilde{f}_\varepsilon|^p \nu_g + \varepsilon^{m/2} \int_{M\setminus\Omega} |\tilde{f}_\varepsilon|^p \nu_g}. \tag{1}$$

We may normalise $\tilde{f}_\varepsilon$ to have $\int_M |\tilde{f}_\varepsilon|^p \nu_g = 1$. Then since $\int_\Omega |\mathrm{d}\tilde{f}_\varepsilon|^p \nu_g \leqslant \int_M |\mathrm{d}\tilde{f}_\varepsilon|^p\, (\varphi_\varepsilon)^{(m-p)/2} \nu_g \leqslant c_0$, we have that $(\tilde{f}_\varepsilon)$ is bounded in $\mathrm{W}^{1,p}(\Omega, g)$ as $\varepsilon \to 0$. Since $\partial\Omega$ is Lipschitzian, we may extract a sequence $\varepsilon_n \to 0$, such that there exists $\tilde{f} \in \mathrm{W}^{1,p}(\Omega, g)$ with $\tilde{f}_{\varepsilon_n} \to \tilde{f}$ strongly in $\mathrm{L}^p(\Omega, g)$ and weakly in $\mathrm{W}^{1,p}(\Omega, g)$. The strong convergence gives $\int_\Omega |\tilde{f}|^p \nu_g = \lim_{n\to\infty} \int_\Omega |\tilde{f}_{\varepsilon_n}|^p \nu_g$ and the orthogonality condition for $\tilde{f}: \int_\Omega |\tilde{f}|^{p-2} \tilde{f} \nu_g = \lim_{n\to\infty} \int_\Omega |\tilde{f}_{\varepsilon_n}|^{p-2} \tilde{f}_{\varepsilon_n} \nu_g = 0$. The weak convergence implies $\int_\Omega |\mathrm{d}\tilde{f}|^p \nu_g \leqslant \liminf_{n\to\infty} \int_\Omega |\mathrm{d}\tilde{f}_{\varepsilon_n}|^p \nu_g$.

There are two possibilities:

- $\tilde{f} \neq 0$. We may pass to the limit in (1) and obtain from the discussion above

$$\liminf_{n\to\infty} \lambda_{1,p}(\varepsilon_n) \geqslant \liminf_{n\to\infty} \frac{\int_\Omega |\mathrm{d}\tilde{f}_{\varepsilon_n}|^p \nu_g}{\int_\Omega |\tilde{f}_{\varepsilon_n}|^p \nu_g + \varepsilon_n^{m/2}} \geqslant \frac{\int_\Omega |\mathrm{d}\tilde{f}|^p \nu_g}{\int_\Omega |\tilde{f}|^p \nu_g} \geqslant \lambda^N_{1,p}(\Omega, g). \tag{2}$$

- $\tilde{f} = 0$. From (1) we have

$$\lambda_{1,p}(\varepsilon_n) \geqslant \min\left\{ \frac{\int_\Omega |\mathrm{d}\tilde{f}_{\varepsilon_n}|^p \nu_g}{\int_\Omega |\tilde{f}_{\varepsilon_n}|^p \nu_g}, \frac{\varepsilon_n^{(m-p)/2} \int_{M\setminus\Omega} |\mathrm{d}\tilde{f}_{\varepsilon_n}| \nu_g}{\varepsilon_n^{m/2} \int_{M\setminus\Omega} |\tilde{f}_{\varepsilon_n}|^p\, \nu_g} \right\}$$
$$\geqslant \min\left\{ \lambda^N_{1,p}(\Omega, g),\ \varepsilon_n^{-p/2} \int_{M\setminus\Omega} \big|\mathrm{d}\tilde{f}_{\varepsilon_n}\big|^p \nu_g \right\}. \tag{3}$$

Now if $\limsup_{n\to\infty} \int_{M\setminus\Omega} |\mathrm{d}\tilde{f}_{\varepsilon_n}|^p \nu_g = 0$, then since $\tilde{f}_{\varepsilon_n}$ is also bounded in $\mathrm{L}^p(M, g)$ and in $\mathrm{W}^{1,p}(\Omega, g)$ we have that $\tilde{f}_{\varepsilon_n}$ is bounded in $\mathrm{W}^{1,p}(M, g)$. Quite to extract a subsequence again, there exists $\widetilde{F} \in \mathrm{W}^{1,p}(M, g)$ such that $\tilde{f}_{\varepsilon_n}$ converges to $\widetilde{F}$ strongly in $\mathrm{L}^p(M, g)$ and weakly in $\mathrm{W}^{1,p}(M, g)$. The unicity of the limit implies that $\widetilde{F} = \tilde{f} = 0$ in Ω. On the other hand, the same type of arguments implies that quite to extract a subsequence again we have weak convergence to $\widetilde{F}$ in $\mathrm{W}^{1,p}(M \setminus \Omega, g)$

and therefore

$$\int_M \left|d\widetilde{F}\right|^p \nu_g = \int_{M\setminus\Omega} \left|d\widetilde{F}\right|^p \nu_g \leqslant \liminf_{n\to\infty} \int_{M\setminus\Omega} \left|d\tilde{f}_{\varepsilon_n}\right|^p \nu_g = 0 \quad \Rightarrow \quad \widetilde{F} = 0 \text{ on } M.$$

But this contradicts $\int_M |\widetilde{F}|^p \nu_g = \lim_{n\to\infty} \int_M |\tilde{f}_n|^p \nu_g = 1$.

It follows that we must have $\limsup_{n\to\infty} \int_{M\setminus\Omega} |d\tilde{f}_{\varepsilon_n}|^p \nu_g > 0$ and therefore when passing to the limit in (3) we obtain

$$\limsup_{n\to\infty} \lambda_{1,p}(\varepsilon_n) \geqslant \lambda^N_{1,p}(\Omega, g). \tag{4}$$

Inequalities (2), (4) yield the desired result.

Part II: smooth metrics. – Let $\delta > 0$ and take ε small enough such that $\lambda_{1,p}(\varepsilon) > \lambda^N_{1,p}(\Omega, g) - \frac{\delta}{2}$ and $\varepsilon^{m/2}\mathrm{Vol}(M \setminus \Omega, g) < \frac{\delta}{2}$.

Let φ_n be a sequence of $C^\infty(M)$ functions such that φ_n converges to $\varphi_\varepsilon = 1\cdot\chi_\Omega + \varepsilon\cdot\chi_{M\setminus\Omega}$ and $\varphi_n = 1$ on Ω, $\varepsilon \leqslant \varphi_n \leqslant 1$ on $M \setminus \Omega$. Consider the family of metrics on M: $g_n = \varphi_n g$.

Our aim in this part is to prove that for n big enough, $\lambda_{1,p}(M, g_n) \geqslant \lambda_{1,p}(\varepsilon) - \frac{\delta}{2}$.

It suffices to consider the case where $\lambda_{1,p}(M, g_n)$ is bounded from above by a positive constant K_0.

Let f_n be an eigenfunction for $\lambda_{1,p}(M, g_n)$ such that $\int_M |f_n|^p \nu_{g_n} = 1$. Then $\int_M |df_n|^p \nu_{g_n} \leqslant K_0$ and the sequence f_n is bounded in $W^{1,p}(M, g)$; indeed we have $\int_M |f_n|^p \nu_g \leqslant \varepsilon^{-m/2} \int_M |f_n|^p \nu_{g_n} = \varepsilon^{-m/2}$ and $\int_M |df_n|^p \nu_g \leqslant \max\{1, \varepsilon^{(p-m)/2}\} \int_M |df_n|^p \nu_{g_n} \leqslant K_0 \max\{1, \varepsilon^{(p-m)/2}\}$.

Quite to extract a subsequence there exists $f_0 \in W^{1,p}(M, g)$ such that $f_n \to f_0$ strongly in $L^p(M, g)$ and weakly in $W^{1,p}(M, g)$. The strong convergence implies

$$\int_M |f_0|^p \varphi_\varepsilon^{m/2} \nu_g = \lim_{n\to\infty} \int_M |f_n|^p \nu_{g_n} = 1, \qquad \int_M |f_0|^{p-2} f_0\, \varphi_\varepsilon^{m/2} \nu_g = \lim_{n\to\infty} \int_M |f_n|^{p-2} f_n \nu_{g_n} = 0,$$

while the weak convergence gives $\liminf_{n\to\infty} \int_M |df_n|^p \nu_{g_n} \geqslant \int_M |df_0|^p \varphi_\varepsilon^{(m-p)/2} \nu_g$. Hence

$$\liminf_{n\to\infty} \big(\lambda_{1,p}(M, g_n)\big) = \liminf_{n\to\infty} \frac{\int_M |df_n|^p \nu_{g_n}}{\int_M |f_n|^p \nu_{g_n}} \geqslant \frac{\int_M |df_0|^p \varphi_\varepsilon^{(m-p)/2} \nu_g}{\int_M |f_0|^p\, \varphi_\varepsilon^{m/2} \nu_g} \geqslant \lambda_{1,p}(\varepsilon).$$

Choosing now n big enough and $\tilde{g} = g_n$ we have $\lambda_{1,p}(M, \tilde{g}) \geqslant \lambda_{1,p}(\varepsilon) - \frac{\delta}{2} \geqslant \lambda^N_{1,p}(\Omega, g) - \delta$ and $|\mathrm{Vol}(M, \tilde{g}) - \mathrm{Vol}(\Omega, g)| = \mathrm{Vol}(M \setminus \Omega, \tilde{g}) < \mathrm{Vol}(M \setminus \Omega, g) + \frac{\delta}{2} < \delta$.

Remark 3. – The proof of Corollary 2 follows from the same type of arguments as above.

References

[1] Y. Colin de Verdière, Sur la multiplicité de la première valeur propre non nulle du Laplacien, Comment. Math. Helv. 61 (1986) 254–270.
[2] J.P. Garzia Azorero, I. Peral Alonso, Existence and nonuniqueness for the p-Laplacian eigenvalues, Comm. Partial Differential Equations 12 (1987) 1389–1430.
[3] P. Tolksdorf, Regularity for a more general class of quasilinear elliptic equations, J. Differential Equations 51 (1984) 126–150.
[4] L. Veron, Some existence and uniqueness results for solution of some quasilinear elliptic equations on compact riemannian manifolds, Colloq. Math. Soc. Janos Bolyai 62 (1991) 317–352.