The effect of perturbations on the first eigenvalue of the $p$-Laplacian

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Abstract
Let $\Omega$ be a domain with Lipschitzian boundary of a compact Riemannian manifold $(M, g)$ and $p > 1$. We prove that we can make the volume of $M$ arbitrarily close to the volume of $(\Omega, g)$ while the first eigenvalue of the $p$-Laplacian on $M$ remains uniformly bounded from below in terms of the the first eigenvalue of the Neumann problem for the $p$-Laplacian on $(\Omega, g)$. To cite this article: A.-M. Matei, C. R. Acad. Sci. Paris, Ser. I 335 (2002) 255–258. © 2002 Académie des sciences/Éditions scientifiques et médicales Elsevier SAS

1. Preliminaries and main result

Let $(M, g)$ be a compact Riemannian manifold. The $p$-Laplacian on $(M, g)$ is defined by $\Delta_p f := \delta(\|df\|^{p-2} df)$, where $\delta = -\text{div}_g$ is the opposite of the divergence. This operator may be seen as a natural extension of the Laplace–Beltrami operator which corresponds to $p = 2$.

The real constants $\lambda$ for which the equation $\Delta_p f = \lambda |f|^{p-2} f$ has nontrivial solutions are the eigenvalues of $\Delta_p$ and the associated solutions are the eigenfunctions.

It was proved that the set of the nonzero eigenvalues is a nonempty, unbounded subset of $]0, \infty[$. Its infimum $\lambda_{1,p}(M, g)$ is itself a positive eigenvalue called the first eigenvalue of $\Delta_p$ and has the following variational characterisation [4]:

$$\lambda_{1,p}(M, g) = \inf \left\{ \int_M |df|^{p} v_g, \ f \in W^{1,p}(M) \setminus \{0\}, \int_M |f|^{p-2} f v_g = 0 \right\}.$$
Let $\Omega$ be a domain in $M$ and consider the Dirichlet problem associated to $\Delta_p$ on $\Omega$:
\[
\begin{aligned}
\Delta_p f &= \mu |f|^{p-2} f \quad \text{in } \Omega, \\
f &= 0 \quad \text{on } \partial \Omega.
\end{aligned}
\]
The first eigenvalue for this problem $\mu_{1,p}(\Omega, g)$, has the variational characterization
\[
\mu_{1,p}(\Omega, g) = \inf \left\{ \int_{\Omega} |d f|^{p} \nu_g; \quad f \in W^{1,p}_0(\Omega, g) \setminus \{0\} \right\}.
\]

We did not find in the literature the corresponding characterization for the Neumann problem for $\Delta_p$. By analogy with the linear case, consider
\[
\lambda_{1,p}(\Omega, g) := \inf \left\{ \int_{\Omega} |d f|^{p} \nu_g; \quad f \in W^{1,p}(\Omega, g) \setminus \{0\}, \quad \int_{\Omega} |f|^{p-2} f \nu_g = 0 \right\}.
\]

Mimicking the proof of the closed case [4] one obtain that $\lambda_{1,p}(\Omega, g)$ is the first nonzero eigenvalue for the Neumann type problem
\[
\begin{aligned}
\Delta_p f &= |f|^{p-2} f \quad \text{in } \Omega, \\
d f(\eta) &= 0 \quad \text{on } \partial \Omega,
\end{aligned}
\]

where $\eta$ denotes the exterior unit normal vector field to $\partial \Omega$.

A more general regularity result [3] says that the eigenfunctions for these problems are locally $C^{1,\alpha}$.

The main result of this Note is the following.

**Theorem 1.** – Let $(M, g)$ be a compact Riemannian manifold, $p > 1$ and $\Omega \subset M$ a domain with Lipschitzian boundary. Then for any $\delta > 0$, there exists a metric $\tilde{g}$ on $M$ such that:

(i) $g|_{\Omega} = \tilde{g}|_{\Omega}$;

(ii) $\lambda_{1,p}(M, \tilde{g}) > \lambda_{1,p}(\Omega, g) - \delta$;

(iii) $|\operatorname{Vol}(M, \tilde{g}) - \operatorname{Vol}(\Omega, g)| < \delta$.

For $p = 2$ and $m = \dim M \geq 3$, this result is contained in Theorem III.1 of [1]; we use the same type of metrics but the arguments from the linear case do not apply to the nonlinear case.

The proof of Theorem 1 is based only on functional analysis in Sobolev spaces without using deeper properties of the eigenfunctions and allows us to consider the case of surfaces.

As a consequence of Theorem 1, we have

**Corollary 2.** – If $1 < p < m$ then Theorem 1 remains true if we replace (ii) by (ii') $|\lambda_{1,p}(M, \tilde{g}) - \lambda_{1,p}(\Omega, g)| < \delta$.

2. Proof of Theorem 1

**Part I: singular metrics.** – For $\varepsilon > 0$ denote by $\varphi_{\varepsilon} = 1 \cdot \chi_{\Omega} + \varepsilon \cdot \chi_{M \setminus \Omega}$ and by
\[
\lambda_{1,p}(\varepsilon) = \inf \left\{ R_{\varepsilon}(f) := \frac{\int_M |d f|^{p}(\varphi_{\varepsilon})^{(m-p)/2} \nu_g}{\int_M |f|^{p}(\varphi_{\varepsilon})^{m/2} \nu_g}; \quad f \in W^{1,p}(M, g) \setminus \{0\}, \quad \int_M |f|^{p-2} f(\varphi_{\varepsilon})^{m/2} \nu_g = 0 \right\}.
\]

Let $\delta > 0$. The aim of this first part is to prove that there exists $\varepsilon$ small enough such that $\lambda_{1,p}(\varepsilon) \geq \lambda_{1,p}(\Omega, g) - \frac{\delta}{2}$ and $\varepsilon^{m/2} \operatorname{Vol}(M \setminus \Omega, g) < \frac{\delta}{2}$.
We may assume that $\lambda_{1,p}(\epsilon)$ is bounded from above by a constant $c_0$ as $\epsilon \to 0$. Since $\partial\Omega$ is Lipschitzian, classical density arguments implies that there exists $f_\epsilon \in W^{1,p}(M, g) \setminus \{0\}$ with $\int_M |f_\epsilon|^{p-2} f_\epsilon \varphi g = 0$ such that $\lambda_{1,p}(\epsilon) = R_\epsilon(f_\epsilon)$. Let $A_\epsilon$, $B_\epsilon$ be the nodal domains of $f_\epsilon$. Then

$$\lambda_{1,p}(\epsilon) = \frac{\int_{A_\epsilon} |d f_\epsilon|^p \varphi_\epsilon g^{(m-p)/2} g}{\int_{A_\epsilon} |f_\epsilon|^{p(n-2)/2} g} = \frac{\int_{B_\epsilon} |d f_\epsilon|^p \varphi_\epsilon g^{(m-p)/2} g}{\int_{B_\epsilon} |f_\epsilon|^{p(n-2)/2} g}.$$  

We claim that for $\epsilon$ small enough, $f_\epsilon$ change sign on $\Omega$. Indeed if for instance $A_\epsilon \cap \Omega = \emptyset$, then

$$c_0 \geq \lambda_{1,p}(\epsilon) = \frac{\int_{A_\epsilon} |d f_\epsilon|^p \varphi_\epsilon g^{(m-p)/2} g}{\int_{A_\epsilon} |f_\epsilon|^{p(n-2)/2} g} = \frac{\epsilon^{(m-p)/2} \int_{A_\epsilon} |d f_\epsilon|^p g^{m/2} g}{\epsilon^{m/2} \int_{A_\epsilon} |f_\epsilon|^p g} \geq \epsilon^{-(m/2)^2} \mu_{1,p}(A_\epsilon, g) \geq \epsilon^{-p/2} \mu_{1,p}(M \setminus \Omega, g),$$

where the last inequality is due to the fact that $A_\epsilon \subset M \setminus \Omega$.

It follows that for $\epsilon$ small enough, $A_\epsilon \cap \Omega \neq \emptyset$ and $B_\epsilon \cap \Omega \neq \emptyset$. Hence, there exists a constant $c_\epsilon$ such that the map $\tilde{f}_\epsilon = f_\epsilon^+ + c_\epsilon f_\epsilon^-$ satisfies the orthogonality condition on $\Omega$: $\int \tilde{f}_\epsilon (p-2) \tilde{f}_\epsilon g = 0$. Moreover $R_\epsilon(f_\epsilon) = R_\epsilon(\tilde{f}_\epsilon)$ and therefore

$$\lambda_{1,p}(\epsilon) = R_\epsilon(\tilde{f}_\epsilon) = \frac{\int_\Omega |d \tilde{f}_\epsilon|^p \varphi g + \epsilon^{(m-p)/2} \int_{M \setminus \Omega} |d \tilde{f}_\epsilon|^p \varphi g}{\int_\Omega |\tilde{f}_\epsilon|^p \varphi g + \epsilon^{m/2} \int_{M \setminus \Omega} |\tilde{f}_\epsilon|^p \varphi g}.$$  

We may normalise $\tilde{f}_\epsilon$ to have $\int_\Omega |\tilde{f}_\epsilon|^p \varphi g = 1$. Then since $\int_\Omega |d \tilde{f}_\epsilon|^p \varphi g \leq \int_\Omega |d \tilde{f}_\epsilon|^p \varphi_\epsilon g^{(m-p)/2} g \leq c_0$, we have that $(\tilde{f}_\epsilon)$ is bounded in $W^{1,p}(\Omega, g)$ as $\epsilon \to 0$. Since $\partial\Omega$ is Lipschitzian, we may extract a sequence $\epsilon_n \to 0$, such that there exists $\tilde{f} \in W^{1,p}(\Omega, g)$ with $\tilde{f}_n \to \tilde{f}$ strongly in $L^p(\Omega, g)$ and weakly in $W^{1,p}(\Omega, g)$. The strong convergence gives $\int_\Omega |\tilde{f}|^p \varphi g = \lim_{n \to \infty} \int_\Omega |\tilde{f}_n|^p \varphi g$ and the orthogonality condition for $\tilde{f}$: $\int_\Omega |\tilde{f}|^{p-2} \tilde{f} g = \lim_{n \to \infty} \int_\Omega |\tilde{f}_n|^{p-2} \tilde{f}_n g = 0$. The weak convergence implies $\int_\Omega |d \tilde{f}|^p \varphi g \leq \liminf_{n \to \infty} \int_\Omega |d \tilde{f}_n|^p \varphi g$.

There are two possibilities:

- $\tilde{f} \neq 0$. We may pass to the limit in (1) and obtain from the discussion above

$$\liminf_{n \to \infty} \lambda_{1,p}(\epsilon_n) \geq \liminf_{n \to \infty} \frac{\int_\Omega |d \tilde{f}_n|^p \varphi g}{\int_\Omega |\tilde{f}_n|^p \varphi g + \epsilon_n^{m/2} \int_{M \setminus \Omega} |\tilde{f}_n|^p \varphi g} = \liminf_{n \to \infty} \int_\Omega |\tilde{f}|^p \varphi g \geq \lambda^N_{1,p}(\Omega, g).$$  

- $\tilde{f} = 0$. From (1) we have

$$\lambda_{1,p}(\epsilon_n) \geq \min \left\{ \frac{\int_\Omega |d \tilde{f}_n|^p \varphi g}{\int_\Omega |\tilde{f}_n|^p \varphi g + \epsilon_n^{m/2} \int_{M \setminus \Omega} |\tilde{f}_n|^p \varphi g}, \frac{\epsilon_n^{(m-p)/2} \int_{M \setminus \Omega} |d \tilde{f}_n|^p \varphi g}{\epsilon_n^{m/2} \int_{M \setminus \Omega} |\tilde{f}_n|^p \varphi g} \right\} \geq \min \left\{ \lambda^N_{1,p}(\Omega, g), \epsilon_n^{-p/2} \int_{M \setminus \Omega} |\tilde{f}_n|^p \varphi g \right\}.$$  

Now if $\limsup_{n \to \infty} \int_{M \setminus \Omega} |d \tilde{f}_n|^p \varphi g = 0$, then since $\tilde{f}_n$ is also bounded in $L^p(M, g)$ and in $W^{1,p}(\Omega, g)$ we have that $\tilde{f}_n$ is bounded in $W^{1,p}(M, g)$. Quite to extract a subsequence again, there exists $\tilde{F} \in W^{1,p}(M, g)$ such that $\tilde{f}_n$ converges to $\tilde{F}$ strongly in $L^p(M, g)$ and weakly in $W^{1,p}(M, g)$. The unicity of the limit implies that $\tilde{F} = \tilde{f} = 0$ in $\Omega$. On the other hand, the same type of arguments implies that quite to extract a subsequence again we have weak convergence to $\tilde{F}$ in $W^{1,p}(M \setminus \Omega, g)$.
and therefore
\[\int_M |d\tilde{F}|^p v_\varepsilon = \int_{M \setminus \Omega} |d\tilde{F}|^p v_\varepsilon \leq \liminf_{n \to \infty} \int_{M \setminus \Omega} |d\tilde{f}_n|^p v_\varepsilon = 0 \quad \Rightarrow \quad \tilde{F} = 0 \text{ on } M.
\]

But this contradicts \(\int_M |\tilde{F}|^p v_\varepsilon = \lim_{n \to \infty} \int_M |\tilde{f}_n|^p v_\varepsilon = 1\).

It follows that we must have \(\limsup_{n \to \infty} \int_{M \setminus \Omega} |d\tilde{f}_n|^p v_\varepsilon > 0\) and therefore when passing to the limit in (3) we obtain
\[\limsup_{n \to \infty} \lambda_{1,p}(\varepsilon_n) \geq \lambda^N_{1,p}(\Omega, g).
\]

Inequalities (2), (4) yield the desired result.

Part II: Smooth metrics. – Let \(\delta > 0\) and take \(\varepsilon\) small enough such that \(\lambda_{1,p}^{M}(\varepsilon) > \lambda_{1,p}^{N}(\Omega, g) - \frac{\delta}{2}\) and \(\varepsilon^{m/2}\Vol(M \setminus \Omega, g) < \frac{\delta}{2}\).

Let \(\varphi_n\) be a sequence of \(C^\infty(M)\) functions such that \(\varphi_n\) converges to \(\varphi = 1 \cdot \chi_\Omega + \varepsilon \cdot \chi_{M \setminus \Omega}\) and \(\varphi_n = 1\) on \(\Omega\), \(\varepsilon \leq \varphi_n \leq 1\) on \(M \setminus \Omega\). Consider the family of metrics on \(M\): \(g_n = \varphi_n g\).

Our aim in this part is to prove that for \(n\) big enough, \(\lambda_{1,p}(M, g_n) \geq \lambda_{1,p}(\varepsilon) - \frac{\delta}{2}\).

It suffices to consider the case where \(\lambda_{1,p}(M, g_n)\) is bounded from above by a positive constant \(K_0\).

Let \(f_n\) be an eigenfunction for \(\lambda_{1,p}(M, g_n)\) such that \(\int_M |f_n|^p v_{g_n} = 1\). Then \(\int_M |d f_n|^p v_{g_n} \leq K_0\) and the sequence \(f_n\) is bounded in \(W^{1,p}(M, g)\); indeed we have \(\int_M |f_n|^p v_{g_n} \leq \varepsilon^{-m/2} \int_M |f_n|^p v_{g_n} = \varepsilon^{-m/2}\) and \(\int_M |d f_n|^p v_{g_n} \leq \max\{1, \varepsilon^{(p-m)/2}\} \int_M |d f_n|^p v_{g_n} \leq K_0 \max\{1, \varepsilon^{(p-m)/2}\}\).

Quite to extract a subsequence there exists \(f_0 \in W^{1,p}(M, g)\) such that \(f_n \to f_0\) strongly in \(L^p(M, g)\) and weakly in \(W^{1,p}(M, g)\). The strong convergence implies
\[\int_M |f_0|^p v_{\varepsilon}^{m/2} v_\varepsilon = \lim_{n \to \infty} \int_M |f_n|^p v_{g_n}^\varphi = 1, \quad \int_M |f_0|^{p-2} f_0 v_{\varepsilon}^{m/2} v_\varepsilon = \lim_{n \to \infty} \int_M |f_n|^{p-2} f_n v_{g_n} = 0,
\]

while the weak convergence gives \(\liminf_{n \to \infty} \int_M |d f_n|^p v_{g_n} \geq \int_M |d f_0|^p v_{\varepsilon}^{(p-m)/2} v_\varepsilon\). Hence
\[\liminf_{n \to \infty} (\lambda_{1,p}(M, g_n)) = \liminf_{n \to \infty} \frac{\int_M |d f_n|^p v_{g_n}^\varphi}{\int_M |d f_n|^p v_{g_n}} \geq \frac{\int_M |d f_0|^p v_{\varepsilon}^{(p-m)/2} v_\varepsilon}{\int_M |d f_0|^p v_{\varepsilon}^{m/2} v_\varepsilon} \geq \lambda_{1,p}(\varepsilon).
\]

Choosing now \(n\) big enough and \(g = g_n\) we have \(\lambda_{1,p}(M, g) \geq \lambda_{1,p}(\varepsilon) - \frac{\delta}{2} \geq \lambda_{1,p}^{N}(\Omega, g) - \delta\) and \(|\Vol(M, g) - \Vol(\Omega, g)| = |\Vol(M \setminus \Omega, g)| < \Vol(M \setminus \Omega, g) + \frac{\delta}{2} < \delta|.

Remark 3. – The proof of Corollary 2 follows from the same type of arguments as above.

References