Rational homotopy groups and Koszul algebras

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Abstract

Let $X$ and $Y$ be finite-type CW-spaces ($X$ connected, $Y$ simply connected), such that the ring $H^*(Y, \mathbb{Q})$ is a $k$-rescaling of $H^*(X, \mathbb{Q})$. If $H^*(X, \mathbb{Q})$ is a Koszul algebra, then the graded Lie algebra $\pi_*(\Omega Y) \otimes \mathbb{Q}$ is the $k$-rescaling of $\text{gr}_*(\pi_1 X) \otimes \mathbb{Q}$. If $Y$ is a formal space, then the converse holds, and $Y$ is coformal. Furthermore, if $X$ is formal, with Koszul cohomology algebra, there exist filtered group isomorphisms between the Malcev completion of $\pi_1 X$, the completion of $[\Omega^{2k+1}, \Omega Y]$, and the Milnor–Moore group of coalgebra maps from $H_*(\Omega^{2k+1}, \mathbb{Q})$ to $H_*(\Omega Y, \mathbb{Q})$. To cite this article: S. Papadima, A.I. Suciu, C. R. Acad. Sci. Paris, Ser. I 335 (2002) 53–58. © 2002 Académie des sciences/Éditions scientifiques et médicales Elsevier SAS

Note fran\c{c}aise abr\é\g\é\e

Cette Note est un résumé des résultats de [9]. Commençons par définir la notion de recalibrage d’un espace topologique (ayant le type d’homotopie d’un CW-complexe connexe de type fini).

Soient $k$ un entier positif, et $A$ une algèbre graduée. On définit l’algèbre graduée $A[k]$ par $A[k]^{p(2k+1)} = A^g$ et $A[k]^p = 0$ si $2k + 1 \mid p$, avec la multiplication héritée de $A$. On dit qu’un espace $X$ est un $k$-recalibrage de $X$ si $\pi_1 X = 0$ et $H^*(Y, \mathbb{Q}) = H^*(X, \mathbb{Q})[k]$, en tant qu’algèbres graduées. Un tel espace $Y$ peut être construit à partir du modèle minimal de l’algèbre $H^*(X, \mathbb{Q})[k]$, munie de la différentielle nulle. Cette construction donne un espace formel, mais $X$ peut bien avoir des recalibrages non-formels. D’autre part, si $\dim_{\mathbb{Q}} H^*(X, \mathbb{Q}) < \infty$, alors $X$ a un $k$-recalibrage unique (à $\mathbb{Q}$-équivalence près), pour tout $k \gg 1$.

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Soit $L_x$ un espace vectoriel gradué, muni d’un crochet de Lie de degré 0. On définit l’algèbre de Lie graduée $L[k]$ par $L[k]_{2k} = L_k$ et $L[k]_p = 0$ si $2k \not\equiv p$, avec le crochet hérité de $L$.

**Théorème 1.** — Soit $Y$ un $k$-recalibrage d’un espace $X$. Soit $\text{gr}_k(\pi_1 X) \otimes \mathbb{Q}$ l’espace vectoriel gradué associé à la suite centrale descendante de $\pi_1 X$ (muni du crochet induit par le commutateur du groupe), et soit $\pi_*(\Omega Y) \otimes \mathbb{Q}$ l’algèbre de Lie d’homotopie de $Y$ (muni du crochet de Samelson).

(a) Si $A^* = H^*(X, \mathbb{Q})$ est une algèbre de Koszul (c’est-à-dire, si $\text{Tor}_p^{A^*}(\mathbb{Q}, \mathbb{Q}) = 0$, pour tous $p \neq q$), alors il existe un isomorphisme d’algèbres de Lie graduées

$$\pi_*(\Omega Y) \otimes \mathbb{Q} \cong \text{gr}_k(\pi_1 X) \otimes \mathbb{Q}[k].$$

(b) De plus, $\prod_{i \geq 1} (1 - t^{(2k+1)i})^{\text{rank } \pi_{2i}(\Omega Y)} = P_X(-2k+1)$.

(c) Si $Y$ est formel et $\pi_*(\Omega Y) \otimes \mathbb{Q} \cong \text{gr}_k(\pi_1 X) \otimes \mathbb{Q}[k]$, en tant qu’espaces vectoriels gradués, alors $H^*(X, \mathbb{Q})$ est une algèbre de Koszul. De plus, $Y$ est coformel.

La première partie du théorème nous permet de décrire le type d’homotopie rationnelle de l’espace de lacets de $Y$, uniquement à partir du polynôme de Poincaré de $X$. En particulier, $P_{\Omega Y}(t) = P_X(-2k)^{-1}$.

**Théorème 2.** — Soit $Y$ un $k$-recalibrage d’un espace $X$, tel que $H^*(X, \mathbb{Q})$ soit une algèbre de Koszul. L’espace $X$ est formel si et seulement s’il existe des isomorphismes de groupes filtrés

$$\text{Hom}^{\text{algèbre}}(H_*(\Omega S^{2k+1}, \mathbb{Q}), H_*(\Omega Y, \mathbb{Q})) \cong [\Omega S^{2k+1}, \Omega Y] \cong \pi_1 X \otimes \mathbb{Q}.$$

Les groupes ci-dessus—le groupe de Milnor–Moore de morphismes de cogèbres entre $H_*(\Omega S^{2k+1}, \mathbb{Q})$ et $H_*(\Omega Y, \mathbb{Q})$, le complété du groupe de classes d’homotopie pointées entre $\Omega S^{2k+1}$ et $\Omega Y$, et le complété de Malcev du groupe fondamental de $X$—sont tous pourvus de filtrations canoniques de limite inverse.

En passant aux gradués associés, l’isomorphisme de groupes filtrés $[\Omega S^{2k+1}, \Omega Y] \cong \pi_1 X \otimes \mathbb{Q}$ donne l’isomorphisme d’algèbres de Lie graduées (1).

Parmi les espaces admettant des recalibrages intégraux, on trouve les compléments d’arrangements d’hyperplans dans $C^k$, et les compléments d’entrelacs de cercles dans $S^3$. Le $k$-recalibrage (formel) d’un tel espace $X$ est fourni par le complément $Y$ d’un certain arrangement de sous-espaces de codimension $k + 1$ dans $C^{(k+1)i}$, et par le complément d’un certain entrelacs de $(2k + 1)$-sphères dans $S^{4k+3}$, respectivement.

La formule (1) est vraie pour les arrangements supersolvables (un résultat de [2]), ainsi que pour les entrelacs ayant un graphe d’enlacement connexe. Pour les arrangements génériques, la formule (1) n’est plus vraie en général (à cause de la non-coformalité de $Y$, détectée par les produits de Whitehead d’ordre supérieur). Dans le cas des entrelacs, la formule (2) n’est pas toujours vraie (à cause de la non-formalité de $X$, détectée par les invariants de Campbell–Hausdorff), même si la formule (1) est valable.

### 1. Rescaling operations

This Note is an announcement of [9]. We refer to that paper for full details, and complete proofs.

Let $A^*$ be a graded algebra over a ring $R$. For each integer $k \geq 1$, the $k$-rescaling of $A$ is the graded algebra $A[k]$ with $A[k]_{2k+1} = A^g$, and $A[k]_p = 0$ otherwise, and with multiplication rescaled accordingly.

Let $X$ be a connected space. A simply-connected space $Y$ is called a $k$-rescaling of $X$ (over $R$) if the cohomology algebra $H^*(Y, R)$ is the $k$-rescaling of $H^*(X, R)$. For example, the sphere $S^{2k+1}$ is a $k$-rescaling of $S^3$, the wedge $\bigvee^n S^{2k+1}$ is a $k$-rescaling of $\bigvee^n S^1$, and the connected sum $#_k S^{2k+1} \times S^{2k+1}$ is a $k$-rescaling of a genus $g$ orientable surface. Though here, and most throughout, the rescaling holds over $R = \mathbb{Z}$, the theory works best over $R = \mathbb{Q}$, and so this will be our default coefficients ring.

Using Sullivan’s minimal models [14], it is easy to see that any connected CW-space of finite type, $X$, admits a rational $k$-rescaling, for each $k \geq 1$. Indeed, $(H^*(X, \mathbb{Q})[k], d = 0)$ is a 1-connected, finite-
type differential graded algebra, with minimal model \( \mathcal{M} \). Hence, there exists a finite-type, 1-connected CW-space \( Y \) such that \( \mathcal{M}(Y) = \mathcal{M} \). In particular, \( H^*(Y, \mathbb{Q}) = H^*(X, \mathbb{Q})[k] \).

By construction, the space \( Y \) is formal, i.e., its rational homotopy type is a formal consequence of its rational cohomology algebra. Hence, \( Y \) is uniquely determined (up to rational homotopy equivalence) among spaces with the same cohomology ring. But there may be other, non-formal rescalings of \( X \). For example, take \( X = S^1 \vee S^1 \vee S^{2k+2} \). Clearly, the formal k-rescaling is \( Y = S^{2k+1} \vee S^{2k+1} \vee S^2 \bullet \). A non-formal rescaling is \( Z = (S^1 \vee S^1 \vee S^{2k+1}) \cup_{\alpha} e^{(2k+1)(2k+2)} \), where \( \alpha \) is the iterated Whitehead product \( ad^{2k+2}(x) = [x, [\cdots [x, y]]] \). Even so, if \( H^*(X, \mathbb{Q}) = 0 \), then \( X \) has a unique k-rescaling (up to \( \mathbb{Q} \)-equivalence), for all \( k > (d-1)/2 \); see [13].

A graded \( \mathbb{Q} \)-vector space \( L_\ast \), endowed with a bilinear operation \([, ] : L_p \otimes L_q \to L_{p+q} \) is called a Lie algebra with grading if the bracket satisfies the anti-commutativity and Jacobi identities. If the Lie identities are satisfied only up to sign (following the Koszul convention), then \( L_\ast \) is called a graded Lie algebra.

We are interested in two main examples. The associated graded Lie algebra of a formally finite group \( G \) is the Lie algebra with grading \( \text{gr}_*(G) \otimes \mathbb{Q} := \bigoplus_{r \geq 1} (\Gamma^r G/\Gamma_1 G) \otimes \mathbb{Q} \), where \( \{\Gamma^r G\}_{r \geq 1} \) is the lower central series of \( G \), and the bracket is induced by the group commutator. The homotopy Lie algebra \( \pi_\ast(\Omega Y) \otimes \mathbb{Q} := \bigoplus_{r \geq 1} \pi_r(\Omega Y) \otimes \mathbb{Q} \), where \( \Omega Y \) is the loop space of \( Y \), and the bracket is the Samelson product, obtained from the Whitehead product on \( \pi_r(\Omega Y) \) via the boundary map in the path fibration over \( Y \). A Lie algebra with grading \( L_\ast \), and a positive integer \( k \), the \( k \)-rescaling of \( L \) is the graded Lie algebra \( L[k] \), with \( L[k]_{2kq} = L_q \) and \( L[k]_{p} = 0 \) otherwise, and with Lie bracket rescaled accordingly.

### 2. The Rescaling Formula

Let \( A^* \) be a graded algebra over \( \mathbb{Q} \). By definition, \( A \) is a Koszul algebra if \( \text{Tor}^A_{k,q}(\mathbb{Q}, \mathbb{Q}) = 0 \), for all \( p \neq q \). A necessary condition is that \( A \) be the quotient of a free algebra on generators in degree 1 by an ideal \( I \) generated in degree 2. A sufficient condition is that \( I \) admits a quadratic Gröbner basis. A topological interpretation of Koszulness is as follows. Let \( X \) be a formal space. Then, \( H^*(X, \mathbb{Q}) \) is a Koszul algebra if and only if the (Bousfield–Kan) rationalization \( X_\mathbb{Q} \) is aspherical; see [10].

From now on, all spaces will be assumed to be connected, well-pointed, and homotopy equivalent to some finite-type CW-complex. Recall that a \( k \)-rescaling of a space \( X \) is a simply-connected space \( Y \) with \( H^*(Y, \mathbb{Q}) = H^*(X, \mathbb{Q})[k] \), as graded algebras. Our first result shows that, under a Koszulness assumption, this homological rescaling passes to a homotopical rescaling.

**THEOREM 2.1.** Let \( Y \) be a \( k \)-rescaling of a space \( X \). If \( H^*(X, \mathbb{Q}) \) is a Koszul algebra, then the following Rescaling Formula holds:

\[
\pi_\ast(\Omega Y) \otimes \mathbb{Q} \cong \text{gr}_\ast(\pi_\ast X) \otimes \mathbb{Q}[k], \quad \text{as graded Lie algebras.}
\]

We sketch the proof in the particular case when both \( X \) and \( Y \) are formal.

Let \( A^* = H^*(X, \mathbb{Q}) \), and let \( \mathcal{H}_\ast(A) = L^*(A_1)/\langle \text{im } \nabla \rangle \) be its holonomy Lie algebra, defined as the quotient of the free Lie algebra on the dual of \( A^1 \) by the Lie ideal generated by the image of the comultiplication map, \( \nabla : A_2 \to A_1 \wedge A_1 = L^2(A_1) \), and with grading given by bracket length. Since \( X \) is formal, \( \text{gr}_\ast(\pi_\ast X) \otimes \mathbb{Q} \cong \mathcal{H}_\ast(A) \), as Lie algebras with grading (see for instance [6]).

Let \( B^* = H^*(Y, \mathbb{Q}) = A^*[k] \), and let \( \mathcal{L}(B,0) = (\mathcal{L}(\pi^{-1}(\mathbb{Q}^0)), \partial) \) be the corresponding Quillen differential graded Lie algebra, defined as the free Lie algebra on the desuspension of the dual of the augmentation ideal of \( B \), with differential \( \partial \) arising from the dual of the multipication map. Since \( Y \) is formal, \( \pi_\ast(\Omega Y) \otimes \mathbb{Q} \cong \mathcal{H}_\ast(\mathcal{L}(B,0)) \), as graded Lie algebras (see [11]).

Now define a morphism of graded Lie algebras, \( \lambda : \mathcal{L}(B,0) \to (\mathcal{H}_\ast(A))[k], 0 \), by sending \( A_1 \) identically to \( A_1 \) (in degree \( 2k \)) and \( A_{i+1} \) to zero. It is readily checked that \( \lambda \) commutes with the differentials and induces a surjection in homology. Since the algebra \( A \) is Koszul, results from [11] and [10] insure that the induced map, \( \lambda_* : \pi_\ast(\Omega Y) \otimes \mathbb{Q} \to \text{gr}_\ast(\pi_\ast X) \otimes \mathbb{Q}[k] \), is in fact an isomorphism (of graded Lie algebras).
Our next result shows that the Rescaling Formula (even at the level of graded vector spaces) is strong enough to imply—under a formality assumption—the Koszulness of $H^\ast(X, \mathbb{Q})$.

**Theorem 2.2.**—Let $Y$ be a formal $k$-rescaling of a space $X$. If $\text{Hilb}(\pi_\ast(\Omega Y) \otimes \mathbb{Q}, t)$ equals $\text{Hilb}(\text{gr}_\ast(\pi_1 X) \otimes \mathbb{Q}, t^2k)$, then $H^\ast(X, \mathbb{Q})$ is a Koszul algebra. Moreover, $Y$ is a coformal space (i.e., its rational homotopy type is determined by its homotopy Lie algebra).

Let $P_X(t) = \text{Hilb}(H^\ast(X; \mathbb{Q}), t)$ be the Poincaré series of $X$, and set $\Phi_r := \text{rank gr}_r(\pi_1 X)$. The following *Lower Central Series formula* has received considerable attention: $\prod_{i=1}^{\infty} (1 - t^{(2k+1)i})^{2} = P_X(-t)$. This formula was established for classifying spaces of pure braid groups by Kohno, and then for complements of arbitrary fiber-type arrangements by Falk and Randell. The LCS formula was related to Koszul duality in [12], and extended to formal spaces $X$ with Koszul cohomology algebra in [10]. Our next result gives an LCS-type formula for the rational homotopy groups of a rescaling of $X$ (under no formality assumptions).

**Theorem 2.3.**—Let $Y$ be a simply-connected CW-space of finite type. Assume $H^\ast(Y, \mathbb{Q})$ is the $k$-rescaling of a Koszul algebra. Set $\Phi_r := \text{rank} \pi_r(\Omega Y)$. Then $\Phi_r = 0$, if $r$ is not a multiple of $2k$, and the following homotopy LCS formula holds:

$$\prod_{i=1}^{\infty} (1 - t^{(2k+1)i})^{2^{\Phi_{2i}}} = P_Y(-t).$$

Consequently, $\Omega Y \simeq \prod_{i=1}^{\infty} K(\mathbb{Q}, 2ki)^{2^{\Phi_{2i}}}$. If $H^\ast(Y, \mathbb{Q}) = H^\ast(X, \mathbb{Q})[k]$, and $H^\ast(X, \mathbb{Q})$ is a Koszul algebra, it follows that the rational homotopy type of $\Omega Y$ is determined by the Poincaré polynomial of $X$ in particular, the Poincaré series of $\Omega Y$ is given by $P_{\Omega Y}(t) = P_X(-t^{2k})^{-1}$. In fact, by Milnor-Moore [7], $H_r(\Omega Y, \mathbb{Q}) \cong \bigwedge_t \text{gr}_r(\pi_t X) \otimes \mathbb{Q}[K]$, as Hopf algebras.

We illustrate these results with some simple examples. In each case, $X$ is a formal space, with $H^\ast(X, \mathbb{Q})$ a Koszul algebra, and $Y$ is the unique up to $\mathbb{Q}$-equivalence formal $k$-rescaling of $X$.

- $X = S^1$, $Y = S^{2k+1}$. We have $\pi_1 X = \mathbb{Z}$, and so $\pi_\ast(\Omega Y) \otimes \mathbb{Q} = L^\ast(x)$, the free Lie algebra on a generator $x$ in degree 2. Thus, $\pi_\ast(\Omega Y) \otimes \mathbb{Q} = K(\mathbb{Q}, 2k)$, a result that goes back to Serre's thesis.

- $X = \bigvee^n S^1$, $Y = \bigvee^n S^{2k+1}$. The associated graded of $\pi_1 X$ was computed by Magnus. We obtain: $\pi_\ast(\Omega Y) \otimes \mathbb{Q} = L^\ast(x_1, \ldots, x_n)$. Hence, $\Phi_r = 0$ if $2k \nmid r$, and $\prod_{i=1}^{\infty} (1 - t^{(2k+1)i})^{2} = 1 - nt^{2k+1}$. Thus, $P_{\Omega Y}(t) = (1 - nt^{2k+1})^{-1}$, a result that goes back to Bott and Samelson.

- $X = \#^n S^1 \times S^1$, $Y = \#^n S^{2k+1} \times S^{2k+1}$. The associated graded of $\pi_1 X$ was computed by Labute. We obtain: $\pi_\ast(\Omega Y) \otimes \mathbb{Q} = L^\ast(x_1, x_2)/([x_1, x_2] + \cdots + [x_{2^g-1}, x_{2^g}] = 0)$. Hence, $\Phi_r = 0$ if $2k \nmid r$, and $\prod_{i=1}^{\infty} (1 - t^{(2k+1)i})^{2} = 1 - 2gr^{2k+1} + r^{4k+2}$. Thus, $P_{\Omega Y}(t) = (1 - 2gr^{2k} + r^{4k})^{-1}$.

### 3. Malcev completions and Milnor-Moore groups

Let $K$ be a connected, finite-type CW-complex $K$, with base-point $\ast$. Fix an increasing, exhaustive filtration of $K$ by connected, finite subcomplexes, $\{K_r\}_{r \geq 0}$, starting with $K_0 = \ast$. Let $Y$ be a based, simply-connected CW-space of finite type, and denote by $[K, \Omega Y]$ the group (under composition of loops) of based homotopy classes of based maps. Since $K_\ast$ is a finite complex, $[K_r, \Omega Y]$ is a finitely-generated nilpotent group. Define the completion $[K, \Omega Y] \hat{\ast} := \text{lim} ([K_{r-1}, \Omega Y] \otimes \mathbb{Q})$, and endow it with the inverse limit filtration, $\{F_r[K, \Omega Y] \hat{\ast} := \ker([K_{r-1}, \Omega Y] \otimes \mathbb{Q}) \subset [K_{r-1}, \Omega Y] \otimes \mathbb{Q}\}$.

For example, $K = \Omega S^m(m \geq 2)$ has a cell decomposition with one cell of dimension $(m-1)$, for each $r \geq 0$. Setting $K_r$ equal to the $(r-1)$-th skeleton, we obtain the filtered group $[\Omega S^m, \Omega Y] \hat{\ast}$. Now let $G$ be an arbitrary finitely-generated group. Then $G$ has a Malcev completion, defined as $G \otimes \mathbb{Q} := \text{lim}_r (G/F_rG) \otimes \mathbb{Q}$. This group comes equipped with the inverse limit filtration; see [11].

The next theorem lifts the Rescaling Formula (3) from the level of associated graded Lie algebras to the level of filtered groups.

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THEOREM 3.1. – Let $Y$ be a $k$-rescaling of a space $X$. Assume that $H^*(X, \mathbb{Q})$ is a Koszul algebra. Then, $X$ is formal if and only if the following Malcev Formula holds:

$$\Omega S^{2k+1}, \Omega Y \cong \pi_1 X \otimes \mathbb{Q}, \quad \text{as filtered groups.} \quad (5)$$

Let us sketch the proof of the forward implication. By the Campbell–Hausdorff formula; see [11]. For such arrangements, the Rescaling Formula was first established in [2], as a functorial way, a filtered group, called the exponential group of $L$. The underlying set of $\exp(L)$ is just $L$, while the group law is given by the Campbell–Hausdorff formula; see [11]. We then have:

$$\Omega S^{2k+1}, \Omega Y \cong \exp(\text{Hom}(H_{-1}(\Omega S^{2k+1}, \mathbb{Q}), \pi_\ast(\Omega Y) \otimes \mathbb{Q}))$$

$$\cong \exp(\pi_\ast(\Omega Y) \otimes \mathbb{Q}[2k + 1]) \cong \exp(\text{gr}_\ast(\pi_1 X) \otimes \mathbb{Q}) \cong \pi_1 X \otimes \mathbb{Q}.$$

The key is the first isomorphism, which follows from a theorem of H. Baues [1]. The second isomorphism requires a “rebracketing” of the homotopy Lie algebra. The third one is provided by Theorem 2.1, while the last one uses the formality of $X$ (see [14,6]).

Consider now the Milnor–Moore group of degree 0 coalgebra maps from $H_\ast(K, \mathbb{Q})$ to $H_\ast(\Omega Y, \mathbb{Q})$, as defined in [7]. There is a natural filtration on $\text{Hom}^\text{coalg}(H_\ast(K, \mathbb{Q}), H_\ast(\Omega Y, \mathbb{Q}))$, with $r$-th term equal to the kernel of the map induced by the inclusion $K_{r-1} \rightarrow K$; see [3]. Using results of Hilton–Mislin–Roitberg and Scheerer, we show that $\text{Hom}^\text{coalg}(H_\ast(K, \mathbb{Q}), H_\ast(\Omega Y, \mathbb{Q})) \cong [K, \Omega Y]^\sim$. Combined with Theorem 3.1, this proves the following theorem.

THEOREM 3.2. – Let $Y$ be a $k$-rescaling of a formal space $X$. If $H^*(X, \mathbb{Q})$ is a Koszul algebra, then $\text{Hom}^\text{coalg}(H_\ast(\Omega S^{2k+1}, \mathbb{Q}), H_\ast(\Omega Y, \mathbb{Q})) \cong \pi_1 X \otimes \mathbb{Q}, \quad \text{as filtered groups.} \quad (6)$

As noted by Cohen and Gitler in [3], the filtered group $[\Omega S^2, \Omega Y]$ is a particularly interesting object. As a set, it equals $\prod_{j \geq 1} \pi_\ast(\Omega Y)$, thus reassembling all the homotopy groups of $Y$ into a single group, called the “group of homotopy groups” of $Y$. In this context, we obtain a result similar to Theorem 3.2, with $\Omega S^{2k+1}$ replaced by $\Omega S^2$. In the case when $X$ is the configuration space of $\ell$ distinct points in $\mathbb{C}$, and $Y$ is the configuration space of $\ell$ distinct points in $\mathbb{C}^{k+1}$, this result answers a question posed by Cohen and Gitler. In the case when $X = \bigvee^n S^1$ and $Y = \bigvee^n S^{2k+1}$, we recover a result of Sato (see [4]).

4. Rescaling hyperplane arrangements

Let $\mathcal{A} = \{H_1, \ldots, H_n\}$ be an arrangement of hyperplanes in $\mathbb{C}^\ell$, with complement $X = M(\mathcal{A})$. For each $k \geq 1$, let $\mathcal{A}^k := \{H_1^k, \ldots, H_n^k\}$ be the corresponding redundant arrangement of codimension $k$ subspaces in $\mathbb{C}^\ell$. Then, as shown by Cohen, Cohen and Xicoténcatl [2], the complement $Y = M(\mathcal{A}^{k+1})$ is an integral $k$-rescaling of $X$. By work of Brieskorn and Yuzvinsky, respectively, it is known that both $X$ and $Y$ are formal spaces.

By Theorems 2.1 and 2.2, the Rescaling Formula (3) holds precisely for the class of arrangements for which $H^*(X, \mathbb{Q})$ is a Koszul algebra. In this case, $Y$ is coformal, and the Malcev Formula (5) also applies.

Presently, the only arrangements which are known to be Koszul are the fiber-type (or, supersolvable) arrangements; see [12]. For such arrangements, the Rescaling Formula was first established in [2], as a generalization of previous results of Fadell–Husseini and Cohen–Gitler on configuration spaces.

The Poincaré polynomial of the complement of a fiber-type arrangement $\mathcal{A}$, with exponents $d_1, \ldots, d_\ell$, factors as $P_X(t) = \prod_{j=1}^\ell (1 + d_j t)$. From Theorem 2.3, we see that $\Phi_\tau = \text{rank} \pi_\tau(\Omega Y)$ vanishes if $2k \nmid \tau$, and $\prod_{j \geq 1} (1 - r(2k+1)^k_\tau) \pi_{2k+1}(1 - d_j t^{2k+1})$. As a consequence, the rational homotopy type of $\Omega Y$ is determined solely by the exponents of $\mathcal{A}$. In particular, $\pi_{2k+1}(t) = \prod_{j=1}^\ell (1 - d_j t^{2k+1})^{-1}$.

If $\mathcal{A}$ is an affine, generic arrangement of $n$ hyperplanes in $\mathbb{C}^{n-1}$ ($n > 2$), then the Rescaling Formula fails for $X = M(\mathcal{A})$, due to the non-coformality of $Y = M(\mathcal{A}^{k+1})$. The absence of coformality is detected by higher-order Whitehead products, which also account for the deviation from equality in (3) and (4). For example, $\Phi_\ell(2k+1)n-2 = 1$, whereas $\text{gr}_{\ell-1}(\pi_1 X) = 0$.
5. Rescaling links in $S^3$

Let $K = (K_1, \ldots, K_n)$ be a link of oriented circles in $S^3$. For each $k \geq 1$, we define the $k$-rescaling $K \otimes_k$ to be the link of $(2k + 1)$-spheres in $S^{4k+3}$ obtained by taking the iterated join (in the sense of Koschorke and Rolfsen [5]) of the link $K$ with $k$ copies of the $n$-component Hopf link.

Let $X$ and $Y$ be the complements of $K$ and $K \otimes_k$. Clearly, $\pi_1(Y) = 0$. Interpreting cup products in $X$ and $Y$ in terms of linking numbers, we show that $Y$ is an integral $k$-rescaling of $X$. Since $H^{\geq 2}(X, \mathbb{Z}) = 0$, this rescaling is unique (up to $\mathbb{Q}$-equivalence), and so $Y$ is a formal space.

Associated to $K$ there is a linking graph, $G_K$, with vertices corresponding to the components $K_i$, and edges connecting pairs of distinct vertices for which $lk(K_i, K_j) \neq 0$. It is known that $G_K$ is connected if and only if $H^*(X, \mathbb{Q})$ is Koszul; see [6]. Examples of links with complete (hence, connected) linking graphs include algebraic links and singularity links of central arrangements of transverse planes in $\mathbb{R}^4$.

The Rescaling Formula (3) holds for $X$ and $Y$ if and only if $G_K$ is connected. In that case, $Y$ is also coformal. Its homotopy Lie algebra is a semidirect product of free Lie algebras generated in degree $2k$, with non-zero ranks given by $\prod_{i \geq 1} (1 - t^{(2k+1)i})^{\mathbb{Z}} = (1 - t^{2k+1})(1 - (n-1)t^{2k+1})$.

On the other hand, the Malcev Formula (5) may fail (due to the non-formality of $X$), even when the Rescaling Formula does hold. To illustrate this phenomenon, we use the Campbell–Hausdorff invariants of links, introduced in [8]. If $K_0$ and $K$ are two links with the same connected weighted linking graph, and if both link complements are formal, we show that $p^r(K_0) = p^r(K)$, for all $r \geq 1$.

Now take $K_0$ to be the $n$-component Hopf link ($n \geq 4$), and add the Borromean braid on three of its strands to get $K$. Then $K_0$ and $K$ have the same weighted linking graph (the complete graph on $n$ vertices, with all linking numbers equal to 1), but $p^r(K_0) \neq p^r(K)$. Since $X_0$ is obviously formal, $X$ must be nonformal. Hence, if $Y$ is the complement of the $k$-rescaling of $K$, then $\pi_1(\Omega Y) \otimes \mathbb{Q} \cong \mathfrak{g}_r(\pi_1 X) \otimes \mathbb{Q}[k]$, as graded Lie algebras, but $[\Omega S^{2k+1}, \Omega Y] \not\cong \pi_1 X \otimes \mathbb{Q}$, as filtered groups.

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References