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An interior–exterior Schwarz algorithm and its convergence

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Abstract In this work we study the solution of Laplace's equation in a domain with holes by an iteration consisting of splitting the problem in an exterior one, around the holes, plus an interior problem in the unholed domain. We show the existence of a decomposition of the solution when the exterior problem is represented by means of a single-layer protential. Also, for the three-dimensional case and with some adjustments for the two-dimensional case, we prove convergence of the method by writing the iteration as a Jacobi iteration for an operator equation and studying the spectrum of the iteration operator. ***To cite this article: R. Celorrio et al., C. R. Acad. Sci. Paris, Ser. I 334 (2002) 923–926.*** © 2002 Académie des sciences/Éditions scientifiques et médicales Elsevier SAS

Un algorithme de Schwarz intérieur–extérieur et sa convergence

Résumé Dans ce travail on étudie la résolution de l'équation de Laplace sur un domaine avec des trous par une méthode itérative consistente à diviser le problème en un problème extérieur, autour des trous, plus un problème intérieur dans le domaine complet. On montre l'existence d'une décomposition de la solution lorsque le problème extérieur est représenté par une potentiel de couche simple. En plus, pour le cas tridimensionnel et pour le bidimensionnel avec quelques modifications, on montre la convergence de la méthode en l'écrivant comme une itération de Jacobi pour une équation opérationnelle et en étudiant le spectre de l'opérateur d'itération. ***Pour citer cet article : R. Celorrio et al., C. R. Acad. Sci. Paris, Ser. I 334 (2002) 923–926.*** © 2002 Académie des sciences/Éditions scientifiques et médicales Elsevier SAS

1. Introduction

This work is concerned with the definition and proof of convergence of an iterative method (in the spirit of Schwarz algorithms [2]) for domains with interior holes. The basic idea consists of decoupling the solution as the sum of an interior solution (for the domain without the obstacles) plus an exterior one (radiating from the obstacles). As such, this method participates of the ideas of multiple scattering techniques, widely

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employed in physical sciences. Moreover, if we consider solving the interior problem with finite elements and the exterior one with boundary elements, we also see its relation with Chimera-type methods.

Consider a bounded domain $Q \subset \mathbb{R}^d$ ($d = 2$ or 3), with Lipschitz boundary Σ and strictly containing another domain D, whose Lipschitz boundary is denoted by Γ (all what follows can be straightforwardly extended to non-connected D; we can also include simple cracks as interior obstacles). The corresponding annular domain – interior to Σ and exterior to Γ – is denoted Ω. We consider the following problem in $\mathrm{H}^1(\Omega)$

$$\Delta w = 0 \quad \text{in } \Omega, \qquad \gamma_\Sigma w = g_\Sigma, \qquad \gamma_\Gamma w = g_\Gamma$$

for given $g_\Sigma \in \mathrm{H}^{1/2}(\Sigma)$, $g_\Gamma \in \mathrm{H}^{1/2}(\Gamma)$, being γ_Σ and γ_Γ the trace operators. To solve this problem we propose a Schwarz algorithm, solving in parallel interior problems in Q and exterior problems around Γ with a single-layer potential ansatz in this last case. Let us consider the single-layer potential (*cf.* [1])

$$\mathcal{S}_\Gamma \psi := \int_\Gamma \Phi(\cdot, y)\psi(y)\,\mathrm{d}\sigma_y : \mathbb{R}^d \to \mathbb{R},$$

where $\Phi(x, y) := 1/|x - y|$ in three dimensions and $\Phi(x, y) := -\log|x - y|$ in two. The iterative method we propose is as follows: take starting values $g_\Gamma^1 := g_\Gamma$ and $g_\Sigma^1 = g_\Sigma$; for $n \geqslant 1$, solve

$$\left|\begin{array}{l} \Delta u_n = 0 \quad \text{in } Q, \\ \\ \gamma_\Sigma u_n = g_\Sigma^n \end{array}\right. \qquad \left|\begin{array}{l} v_n := \mathcal{S}_\Gamma \psi_n, \quad \psi_n \in \mathrm{H}^{-1/2}(\Gamma) \\ \\ \gamma_\Gamma \mathcal{S}_\Gamma \psi_n = g_\Gamma^n \end{array}\right.$$

and interchange

$$g_\Sigma^{n+1} := g_\Sigma - \gamma_\Sigma v_n, \qquad g_\Gamma^{n+1} := g_\Gamma - \gamma_\Gamma u_n.$$

The sequence $u_n + v_n$, restricted to Ω will be taken as an approximation of w.

We will show the convergence of this method in three dimensions and some hypotheses and/or modifications guaranteeing convergence in two dimensions.

Let $V_\Gamma := \gamma_\Gamma \mathcal{S}_\Gamma : \mathrm{H}^{-1/2}(\Gamma) \to \mathrm{H}^{1/2}(\Gamma)$ be the single-layer operator. We recall (*cf.* [3]) that in two dimensions, there is a unique solution $(\theta_\Gamma, \mu_\Gamma) \in \mathrm{H}^{-1/2}(\Gamma) \times \mathbb{R}$ of the equation $V_\Gamma \theta_\Gamma - \mu_\Gamma = 0$ under the restriction $\mathcal{J}_\Gamma \theta_\Gamma := \langle \theta_\Gamma, 1_\Gamma \rangle = 1$ (1_Γ is the unit constant function on Γ and $\langle \cdot, \cdot \rangle$ is the $\mathrm{H}^{-1/2}(\Gamma) \times \mathrm{H}^{1/2}(\Gamma)$ duality product): θ_Γ is the equilibrium distribution and μ_Γ is the Robin constant; the quantity $C_\Gamma := \exp(-\mu_\Gamma)$ is called the logarithmic capacity of Γ.

2. Decomposition theorem and convergence of the method

Let $\mathcal{L}_\circ := \{u \in \mathrm{H}^1(\circ) \mid \Delta u = 0\}$ ($\circ \in \{Q, \Omega\}$) and $\mathcal{P}_\Gamma := \{\mathcal{S}_\Gamma \psi \mid \psi \in \mathrm{H}^{-1/2}(\Gamma)\}$.

We define the operators $K_{\Sigma\Gamma} := \gamma_\Sigma \mathcal{S}_\Gamma V_\Gamma^{-1} : \mathrm{H}^{1/2}(\Gamma) \to \mathrm{H}^{1/2}(\Sigma)$ and $K_{\Gamma\Sigma} : \mathrm{H}^{1/2}(\Sigma) \to \mathrm{H}^{1/2}(\Gamma)$ defined by $K_{\Gamma\Sigma} g = \gamma_\Gamma u$, where $u \in \mathcal{L}_Q$ satisfies $\gamma_\Sigma u = g$. We finally consider the operators $\mathcal{A}$ and $\mathcal{K}$:

$$\mathcal{A} := \mathcal{I} + \mathcal{K} := \begin{bmatrix} I & K_{\Sigma\Gamma} \\ K_{\Gamma\Sigma} & I \end{bmatrix} : \mathrm{H}^{1/2}(\Sigma) \times \mathrm{H}^{1/2}(\Gamma) \to \mathrm{H}^{1/2}(\Sigma) \times \mathrm{H}^{1/2}(\Gamma).$$

THEOREM 2.1. – *Under the assumption that $C_\Gamma \neq 1$ in two dimensions, and considering the elements of $\mathcal{L}_Q$ and $\mathcal{P}_\Gamma$ restricted to Ω we have that $\mathcal{L}_\Omega = \mathcal{L}_Q \oplus \mathcal{P}_\Gamma$.*

Proof. – It is clear that $\mathcal{L}_Q + \mathcal{P}_\Gamma \subset \mathcal{L}_\Omega$. The fact that $\mathcal{L}_Q \cap \mathcal{P}_\Gamma = 0$ follows from the jumps of the normal derivatives across Γ of the elements of $\mathcal{P}_\Gamma$.

Since $K_{\Sigma\Gamma}$ and $K_{\Gamma\Sigma}$ are compact, $\mathcal{A}$ is Fredholm of index zero. Besides, it is injective by the same argument given to show the zero intersection of both sets. Moreover, given $w \in \mathcal{L}_\Omega$, the separate Dirichlet data $(h_\Sigma, h_\Gamma) := \mathcal{A}^{-1}(\gamma_\Sigma w, \gamma_\Gamma w)$ determine the elements of the desired decomposition. □

The iterative method corresponds to the Jacobi iteration (also to the Neumann series iteration) for the problem $(\mathcal{I} + \mathcal{K})(h_\Sigma, h_\Gamma) = (g_\Sigma, g_\Gamma)$ where h_Σ and h_Γ are the respective traces of the unique

decomposition $w = u + v \in \mathcal{L}_Q \oplus \mathcal{P}_\Gamma$. Therefore, convergence is restricted to when the spectral radius of $\mathcal{K}$ is less than one. We will denote $\sigma(\mathcal{K})$ to the spectrum of $\mathcal{K}$.

Consider the operators $V_\Sigma := \gamma_\Sigma \mathcal{S}_\Sigma$ (with the obvious definition for $\mathcal{S}_\Sigma$), $V_{\Sigma\Gamma} := \gamma_\Sigma \mathcal{S}_\Gamma : \mathrm{H}^{-1/2}(\Gamma) \to \mathrm{H}^{1/2}(\Sigma)$, its adjoint $V_{\Gamma\Sigma} := \gamma_\Gamma \mathcal{S}_\Sigma$ and the selfadjoint operator

$$\mathcal{V} := \begin{bmatrix} V_\Sigma & V_{\Sigma\Gamma} \\ V_{\Gamma\Sigma} & V_\Gamma \end{bmatrix} : \mathrm{H}^{-1/2}(\Sigma) \times \mathrm{H}^{-1/2}(\Gamma) \to \mathrm{H}^{1/2}(\Sigma) \times \mathrm{H}^{1/2}(\Gamma).$$

THEOREM 2.2. – *Assuming that in two dimensions $C_\Sigma < 1$ (and without additional hypotheses in three dimensions), $\sigma(\mathcal{K})$ is a discrete set of real values in the interval $(-1, 1)$, converging to zero and symmetric with respect to the origin.*

Proof. – Since $\mathcal{K}$ is compact, all non-zero values in the spectrum of $\mathcal{K}$ are eigenvalues. Elementary linear algebra shows that eigenvalues of $\mathcal{K}$ are the square roots (real and complex) of those of $K_{\Gamma\Sigma} K_{\Sigma\Gamma} : \mathrm{H}^{1/2}(\Gamma) \to \mathrm{H}^{1/2}(\Gamma)$.

The hypotheses imply that $\mathcal{V}$ is elliptic. Moreover, since $K_{\Gamma\Sigma} K_{\Sigma\Gamma} = V_{\Gamma\Sigma} V_\Sigma^{-1} V_{\Sigma\Gamma} V_\Gamma^{-1}$, eigenvalues of $K_{\Gamma\Sigma} K_{\Sigma\Gamma}$ are solutions of the generalized eigenvalue problem $V_{\Gamma\Sigma} V_\Sigma^{-1} V_{\Sigma\Gamma} \psi = \lambda V_\Gamma \psi$. They are, therefore, strictly positive and since $V_\Gamma - V_{\Gamma\Sigma} V_\Sigma^{-1} V_{\Sigma\Gamma}$ is positive definite (because of the ellipticity of $\mathcal{V}$), they are less than one, which proves the result. □

The symmetry of the spectrum of $\mathcal{K}$ proves also that no gain is obtained by relaxing the Jacobi iteration. It is also straightforward that convergence of a related Gauss–Seidel iteration depends on the spectral radius of $K_{\Gamma\Sigma} K_{\Sigma\Gamma}$ and is therefore equivalent to that of the Jacobi method.

3. The two-dimensional case

In case $C_\Sigma > 1$, there are two practical ways of dealing with the problem in a way that leads to a convergent situation: (a) rescaling the space variables to decrease the logarithmic capacity; (b) changing the fundamental solution as follows. We consider $\Phi_\varepsilon(x, y) := -\log|x - y| + \varepsilon$. If we use the modified potentials (*cf.* [1]) $\mathcal{S}_\Gamma^\varepsilon \psi = \mathcal{S}_\Gamma \psi + \varepsilon \mathcal{J}_\Gamma \psi$, with $\varepsilon > -\mu_\Sigma$, the whole of the proofs above can be shown to hold.

However, we are going to further investigate the behaviour of the spectrum of $\mathcal{K}$ when $C_\Sigma > 1$ (we are hence still using the original fundamental solution for the single-layer potentials). To do that we introduce the operators $V_\Gamma^\varepsilon := V_\Gamma + \varepsilon \mathcal{J}_\Gamma(\cdot)1_\Gamma$, $V_\Sigma^\varepsilon := V_\Sigma + \varepsilon \mathcal{J}_\Sigma(\cdot)1_\Sigma$, $V_{\Sigma\Gamma}^\varepsilon := V_{\Sigma\Gamma} + \varepsilon \mathcal{J}_\Gamma(\cdot)1_\Sigma$ and $V_{\Gamma\Sigma}^\varepsilon$ defined likewise. We will only consider values of $\varepsilon > -\mu_\Sigma$, so that the related operator $\mathcal{V}^\varepsilon$ is elliptic. Again, we only have to study the eigenvalues of $K_{\Gamma\Sigma} K_{\Sigma\Gamma}$.

A great deal of the analysis to come hinges on the following technical result, giving conditions on the existence of negative eigenvalues of rank one definite negative perturbations of definite positive operators.

LEMMA 3.1. – *Let $V \subset H \subset V'$ be a triad of Hilbert spaces with dense compact injections. Let $A : V' \to V$ be bounded, selfadjoint and positive definite and consider $a \in V$. Then, the operator $A - \langle \cdot, a \rangle a : V' \to V$ has at most a single negative eigenvalue with a single associated eigenfunction. Furthermore, if A is elliptic, the operator is positive definite if and only if $\langle A^{-1} a, a \rangle < 1$.*

Proof. – We restrict ourselves to the case of A elliptic. We are interested in non-trivial solutions to $(A + \lambda I)\phi = \langle \phi, a \rangle a$ for $\lambda \geqslant 0$. Denoting equally A to the restriction of A to the subspace $V \subset V'$ (as such A is compact), it is clear that for all $\lambda > 0$, $A + \lambda I : V \to V$ is an injective Fredholm operator of index zero and therefore has a bounded inverse. Then $\phi = \langle \phi, a \rangle (A + \lambda I)^{-1} a$ and hence $\langle \phi, a \rangle \neq 0$ and eigenfunction spaces are one-dimensional. Moreover, the problem has a nontrivial solution if and only if $f(\lambda) := \langle (A + \lambda I)^{-1} a, a \rangle = 1$. This is also valid for $\lambda = 0$.

The function $f : [0, \infty) \to \mathbb{R}$ is positive, continuous on $[0, \infty)$, converges to zero as $\lambda \to \infty$ and is strictly decreasing since $f'(\lambda) < 0$ for all $\lambda > 0$. Then $f(\lambda) = 1$ has a (single) root if and only if $f(0) \geqslant 1$ and has no roots otherwise.

In case A is simply positive definite, the proof is essentially valid, but for the fact that f is continuous on $(0,\infty)$. □

THEOREM 3.2. – *The eigenvalues of $K_{\Gamma\Sigma}K_{\Sigma\Gamma}$ lie in the interval $(-\infty,1)$. Moreover, there is at most a single negative eigenvalue and $\sigma(K_{\Gamma\Sigma}K_{\Sigma\Gamma})\subset(-1,1)$ if and only if the function $g(\varepsilon):=\varepsilon\langle(K_{\Gamma\Sigma}V_{\Sigma\Gamma}^{\varepsilon}+V_\Gamma)^{-1}1_\Gamma,1_\Gamma\rangle$ is less than one for some $\varepsilon>-\mu_\Sigma$.*

Proof. – As before, the problem of studying the spectrum of $K_{\Gamma\Sigma}K_{\Sigma\Gamma}$ is equivalent to studying solutions of $K_{\Gamma\Sigma}K_{\Sigma\Gamma}V_\Gamma\psi=\lambda V_\Gamma\psi$. We remark that by its definition, it can be easily seen that $K_{\Gamma\Sigma}1_\Sigma=1_\Gamma$. Moreover, for all $\varepsilon\neq-\mu_\Sigma$, $K_{\Gamma\Sigma}=V_{\Gamma\Sigma}^{\varepsilon}(V_\Sigma^{\varepsilon})^{-1}$. Then

$$K_{\Gamma\Sigma}K_{\Sigma\Gamma}V_\Gamma=V_{\Gamma\Sigma}^{\varepsilon}\big(V_\Sigma^{\varepsilon}\big)^{-1}V_{\Sigma\Gamma}=V_{\Gamma\Sigma}^{\varepsilon}\big(V_\Sigma^{\varepsilon}\big)^{-1}V_{\Sigma\Gamma}^{\varepsilon}-\varepsilon\mathcal{J}_\Gamma(\cdot)1_\Gamma=V_{\Gamma\Sigma}^{\varepsilon}\big(V_\Sigma^{\varepsilon}\big)^{-1}V_{\Sigma\Gamma}^{\varepsilon}-V_\Gamma^{\varepsilon}+V_\Gamma,$$

and therefore the eigenvalues are real; since $V_\Gamma^{\varepsilon}-V_{\Gamma\Sigma}^{\varepsilon}(V_\Sigma^{\varepsilon})^{-1}V_{\Sigma\Gamma}^{\varepsilon}$ is positive definite, they are less than one. By taking $A:=V_{\Gamma\Sigma}^{\varepsilon}(V_\Sigma^{\varepsilon})^{-1}V_{\Sigma\Gamma}^{\varepsilon}$ and $a:=\varepsilon^{1/2}1_\Gamma$ in Lemma 3.1, we see that there is at most a negative eigenvalue. Showing that this eigenvalue is greater than -1 is equivalent to showing that $K_{\Gamma\Sigma}K_{\Sigma\Gamma}V_\Gamma+V_\Gamma$ is positive definite and this is accomplished by taking $A:=V_{\Gamma\Sigma}^{\varepsilon}(V_\Sigma^{\varepsilon})^{-1}V_{\Sigma\Gamma}^{\varepsilon}+V_\Gamma$ and a as before in Lemma 3.1, which gives the necessary and sufficient condition for this. □

Condition $g(\varepsilon)<1$ for some $\varepsilon>-\mu_\Sigma$ is not easy to verify. By differentiating in its definition, it can be seen that $g'(\varepsilon)=g(\varepsilon)(1-g(\varepsilon))/\varepsilon$. The solutions to this differential equation are $g(\varepsilon)\equiv0$ and the uniparametric family $\varepsilon/(\varepsilon-C)$ for arbitrary $C\in\mathbb{R}$. It is clear that the constant C depends on Γ and Σ, and that the sign of $g(\varepsilon)-1$ remains unchanged, so the condition holds or does not hold for all ε in the interval considered.

PROPOSITION 3.3. – *If $C_\Gamma C_\Sigma<1$, $\sigma(K_{\Gamma\Sigma}K_{\Sigma\Gamma})\subset(-1,1)$.*

Proof. – Notice that we have to prove that the operator $K_{\Gamma\Sigma}K_{\Sigma\Gamma}V_\Gamma+V_\Gamma=V_{\Gamma\Sigma}^{\varepsilon}(V_\Sigma^{\varepsilon})^{-1}V_{\Sigma\Gamma}^{\varepsilon}+V_\Gamma-\varepsilon\mathcal{J}_\Gamma(\cdot)1_\Gamma$ is positive definite. Applying Lemma 3.1 to $V_\Gamma-\varepsilon\mathcal{J}_\Gamma(\cdot)1_\Gamma$, it is clear that this operator is positive definite if and only if $1>\varepsilon\langle V_\Gamma^{-1}1_\Gamma,1_\Gamma\rangle=\varepsilon/\mu_\Gamma$. Therefore, if there exists $\varepsilon>-\mu_\Sigma$ such that $\varepsilon/\mu_\Gamma<1$, then we have positive-definiteness of $V_\Gamma-\varepsilon\mathcal{J}_\Gamma(\cdot)1_\Gamma$ and hence of our original operator. This fact is possible when $\mu_\Sigma+\mu_\Gamma>1$, which is equivalent to $C_\Gamma C_\Sigma<1$. □

In the simple case of two concentric circles of radii $r<R$, the function $g(\varepsilon)$ can be computed exactly: $g(\varepsilon)=\varepsilon/(\varepsilon-\log(r\,R))$. Therefore $\sigma(K_{\Gamma\Sigma}K_{\Sigma\Gamma})\subset(-1,1)$ if and only if $r\,R<1$ (recall that the logarithmic capacity of a circle is its radius. Numerical experiments show that the constant C in the general expression $g(\varepsilon)=\varepsilon/(\varepsilon-C)$ depends strongly on the sizes of the boundaries and on their relative positions.

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