Symplectic capacities of toric manifolds and combinatorial inequalities

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Abstract
We shall give concrete estimations for the Gromov symplectic width of toric manifolds in combinatorial data. As by-products some combinatorial inequalities in the polytope theory are obtained. To cite this article: G. Lu, C. R. Acad. Sci. Paris, Ser. I 334 (2002) 889–892.
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Capacités symplectiques de variétés toriques et inégalités combinatoires

Résumé

The toric manifolds are a very beautiful family of Kähler manifolds. Since they admit a combinatorial description it is very interesting to estimate their (pseudo) symplectic capacities in terms of combinatorial data. Recall that the Gromov symplectic width $W_G(M,\omega)$ of a $2n$-dimensional symplectic manifold $(M,\omega)$ is defined by the supremum of all numbers $\pi r^2$ for which there exists a symplectic embedding from a ball $B^{2n}(r)$ in $(\mathbb{R}^{2n},\omega_0)$ of radius $r$ into $(M,\omega)$. It is the first symplectic capacity. Recently, the author introduced the notion of pseudo symplectic capacity [6]. Let us begin by briefly recalling it. For its properties and applications the reader refer to [6]. Given a connected symplectic manifold $(M,\omega)$ of dimension $2n$ and a smooth function $H$ on it let $X_H$ denote the symplectic gradient of $H$. An isolated critical point $p$ of $H$ is called admissible if the spectrum of the linear transformation $DX_H(p):T_pM \rightarrow T_pM$ is contained in $\mathbb{C}\setminus\{\lambda i | 2\pi \leq \pm \lambda < +\infty\}$. For two given nonzero homology classes $\alpha_0,\alpha_\infty \in H_*(M)$ we denote by $\mathcal{H}_{ad}(M,\omega;\alpha_0,\alpha_\infty)$ (resp. $\tilde{\mathcal{H}}_{ad}(M,\omega;\alpha_0,\alpha_\infty)$) the set of all smooth functions on $M$ for which there exist two smooth compact submanifolds $P$ and $Q$ of $M$ with connected smooth boundaries and of codimension zero such that the following condition groups (a)–(f) (resp. (a)–(e), (g)) are satisfied:

(a) $P \subset \text{Int}(Q)$ and $Q \subset \text{Int}(M)$;
(b) $H|_P = 0$ and $H|_{M\setminus \text{Int}(Q)} = \max H$;
(c) $0 \leq H \leq \max H$;
(d) There exist chain representatives of $\alpha_0$ and $\alpha_\infty$, still denoted by $\alpha_0,\alpha_\infty$, such that $\text{supp}(\alpha_0) \subset \text{Int}(P)$ and $\text{supp}(\alpha_\infty) \subset M \setminus Q$;

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(e) $H$ has only finitely many critical points in $\text{Int}(Q) \setminus P$ and each of them is admissible in the above sense;
(f) The Hamiltonian system $\dot{x} = X_H(x)$ on $M$ has no nonconstant periodic solutions of period less than 1;
(g) The Hamiltonian system $\dot{x} = X_H(x)$ on $M$ has no nonconstant contractible periodic solutions of period less than 1.

If $\alpha_0 \in H_0(M)$ can be represented by a point we allow $P$ to be an empty set. If $M$ is a closed manifold and $\alpha_\infty \in H_0(M)$ is represented by a point, we also allow $Q = M$.

The pseudo symplectic capacities of Hofer–Zehnder type are defined by
\[
\begin{align*}
\tilde{C}^{(2)}_{HZ}(M, \omega; \alpha_0, \alpha_\infty) &:= \sup \{ \max H | H \in \mathcal{H}_{ad}(M, \omega; \alpha_0, \alpha_\infty) \}, \\
\tilde{C}^{(2)}_{HZ}(M, \omega; \alpha_0, \alpha_\infty) &:= \sup \{ \max H | H \in \hat{\mathcal{H}}_{ad}(M, \omega; \alpha_0, \alpha_\infty) \}.
\end{align*}
\]
In this Note we denote by $pt$ the generator of $H_0(M)$ represented by a point, and always make the convention that $\sup \emptyset = 0$ and $\inf \emptyset = +\infty$.

1. The pseudo symplectic capacity of toric manifolds

For the following related knowledge on the toric manifolds the reader may refer to [1,2,5]. Let $\Sigma$ be a complete regular fan in $\mathbb{R}^n$ and $G(\Sigma) = \{u_1, \ldots, u_d\}$ the set of all generators of 1-dimensional cones in $\Sigma$. Denote by $P_\Sigma$ the toric manifold associated with $\Sigma$. It is well known that every Kähler form on $P_\Sigma$ can be represented by a strictly convex support function $\varphi$ for $\Sigma$ and that every strictly convex support function for $\Sigma$ represents a Kähler form on $P_\Sigma$. Therefore, in this Note we shall use the same letter to denote a Kähler form on $P_\Sigma$ and the corresponding strictly convex support function for $\Sigma$ when the context makes our meaning clear. In the following we denote by $\mathbb{Z}_{\geq 0}$ the set of all nonnegative integers.

**Theorem 1.** Under the assumptions above let $\omega$ be a strictly convex support function for $\Sigma$. Then it holds that
\[
\Gamma(\Sigma, \omega) := \frac{1}{2\pi} \inf \left\{ \sum_{k=1}^d \omega(u_k) a_k > 0 \left| \sum_{k=1}^d a_k u_k = 0, \ a_k \in \mathbb{Z}_{\geq 0}, \ k = 1, \ldots, d \right. \right\} > 0,
\]
and that for every $n \geq 2$,
\[
\mathcal{W}_G(P_\Sigma, \omega) \leq C_{HZ}(P_\Sigma, \omega; pt, PD(\{\omega_0\})) \leq 2\pi \cdot \Gamma(\Sigma, \omega).
\]

In particular, let us consider a Delzant polytope in $(\mathbb{R}^n)^*$
\[
\Delta = \bigcap_{k=1}^d \{ x \in (\mathbb{R}^n)^* | l_k(x) := x(u_k) - \lambda_k \geq 0 \}
\]
(cf. [1,5]), where $d$ is the number of the $(n - 1)$-dimensional faces of $\Delta$, $u_k$ is a uniquely primitive element of the lattice $\mathbb{Z}^n \subset \mathbb{R}^n$ (the inward-pointing normal to the $k$-th face of $\Delta$), and $\lambda_k$ is a real number. Denote by $X_\Delta$ the toric manifold associated with the fan generated by $\Delta$, and by $\omega_\Delta$ the canonical symplectic form on it.

**Theorem 2.** Under the assumptions above, it holds that
\[
\Gamma(\Delta) := \inf \left\{ - \sum_{k=1}^d \lambda_k a_k > 0 \left| \sum_{k=1}^d a_k u_k = 0, \ a_k \in \mathbb{Z}_{\geq 0}, \ k = 1, \ldots, d \right. \right\} > 0,
\]
and that for any $n \geq 2$,
\[
\mathcal{W}_G(X_\Delta, \omega_\Delta) \leq C_{HZ}(X_\Delta, \omega_\Delta; pt, PD(\{\omega_\Delta\})) \leq 2\pi \cdot \Gamma(\Delta).
\]
Moreover, if $\text{Vert}(\Delta)$ denotes the set of all vertices of $\Delta$ and $E_p(\Delta)$ is the shortest distance from the vertex $p$ to the adjacent $n$ vertices, then for any capacity function $c$,
\[
2\pi \cdot \max_{p \in \text{Vert}(\Delta)} E_p(\Delta) \leq c(X_\Delta, \omega_\Delta).
\]
Remark 3. – For the $n$-simplex $\Delta = \Delta_n$ in $(\mathbb{R}^n)^*$ spanned by the origin and the dual basis $e_1^*, \ldots, e_n^*$ the associated toric manifold $(X_{\Delta_n}, \omega_{\Delta_n})$ is $(\mathbb{CP}^n, 2\omega_{FS})$ with $\int_{\mathbb{CP}^n} \omega_{FS} = \pi$. It is easily seen that $\Upsilon(\Delta_n) = 1$. Thus the estimates in (6) are optimal. In particular, it follows from the proof of Theorem 2 that

$$\mathcal{W}_G(\Delta^n(1) \times \mathbb{T}^n, \omega_{\Delta_n}) \leq \mathcal{W}_G(\Delta^n(1) \times \mathbb{T}^n, \omega_{can}) \leq 2\pi,$$

where $\Delta^n(a) = \{(x_1, \ldots, x_n) \in \mathbb{R}^n \mid \sum_{k=1}^n x_k < a\} \subset \mathbb{R}^n$ and $\mathbb{T}^n = \{(\theta_1, \ldots, \theta_n) \in \mathbb{R}^n \mid 0 < \theta_k < a, \forall 1 \leq k \leq n\}$ for any $a > 0$. But from Theorem 5.1 in [10] one can only get $\mathcal{W}_G(\Delta^n(1) \times \mathbb{T}^n, \omega_0) \leq 8n\pi$.

Examples. – (i) Let $e_1, e_2, e_3$ be the standard basis of $\mathbb{R}^3$ and $u_1 = e_1, u_2 = -e_1, u_3 = e_2, u_4 = e_3, u_5 = -e_2 - e_3 - 2e_1$. Consider a fan $\Sigma \subset \mathbb{R}^3$ in which $G(\Sigma) = \{u_1, u_2, u_3, u_4, u_5\}$ is the set of all generators of 1-dimensional cones and whose set of primitive collections is $\{[u_1, u_2], [u_3, u_4, u_5]\}$. It is easily checked that this fan is complete and regular. Its associated toric manifold $P_{\Sigma}$ is the Fano threefold $P(\mathbb{CP}^2(\mathbb{O}_2) \oplus 1)$. Note that each strictly convex support function for $\Sigma$ can be determined by its values at points $u_i, i = 1, \ldots, 5$. Let $\omega$ be a $\Sigma$-piecewise linear function such that $\omega(u_i) = 1, i = 1, \ldots, 5$. It is easy to prove that it is a strictly convex support function for $\Sigma$ and that $\Upsilon(\Sigma, \omega) = 1/\pi$. Thus by Theorem 1 we get $\mathcal{W}_G(P_{\Sigma}, \omega) \leq C_HZ(P_{\Sigma}, \omega; pt., PD(\omega_0)) \leq 2$.

(ii) Consider a Delzant polytope $\Delta \subset (\mathbb{R}^3)^*$ with vertices $v_0 = 0, v_1 = e_1^*, v_2 = e_2^*, v_3 = (1-a)e_1^* + ae_3^*, v_4 = ae_2^*, v_5 = (1-a)e_1^* + ae_3^*$. Here $0 < a < 1$ and $e_1^*, e_2^*, e_3^*$ are the dual basis of the standard basis $e_1, e_2, e_3$ in $\mathbb{R}^3$. It is easy to see that the normal vectors to the 2-dimensional faces are $u_1 = e_1^*, u_2 = e_2^*, u_3 = e_3^*, u_4 = -e_2^*, u_5 = -e_1^* - e_2^* - e_3^*$. Furthermore, $\Delta$ can be expressed as the intersection of the half spaces $(x, u_j) \geq 0, j = 1, 2, 3$, and $(x, u_4) \geq -a, (x, u_5) \geq -1$. Thus $\Upsilon(\Delta) = a$ and it follows from Theorem 2 that the associated toric manifold $(X_{\Delta}, \omega_{\Delta})$ has the capacities

$$\mathcal{W}_G(X_{\Delta}, \omega_{\Delta}) \leq C_HZ(X_{\Delta}, \omega_{\Delta}; pt., PD(\omega_0)) \leq 2\pi a.$$

Notice that the toric manifold $(X_{\Delta}, \omega_{\Delta})$ is exactly the blow-up of $(\mathbb{CP}^3, 2\omega_{FS})$ of weight $2(1-a)$ at a point. That is, it is obtained by removing the interior of a symplectic embedding ball $(B^3(\sqrt{2}(1-a)), \omega_0)$ of radius $\sqrt{2}(1-a)$ in $(\mathbb{CP}^3, 2\omega_{FS})$ and collapsing the bounding sphere to the exceptional divisor by the Hopf map.

2. Seshadri constants

For a compact complex manifold $(M, J)$ of dimension $n$, and an ample line bundle $L \to M$ Demailly [4] defined the Seshadri constant of $L$ at a point $x \in M$ to be the nonnegative real number

$$\epsilon(L, x) := \inf_{C \in L} \frac{\int_{C} c_1(L)}{\text{mult}_x C},$$

where the infimum is taken over all irreducible curves passing through the point $x$, and $\text{mult}_x C$ is the multiplicity of $C$ at $x$. The global Seshadri constant is defined by

$$\epsilon(L) := \inf_{x \in M} \epsilon(L, x).$$

For more details the reader should refer to [4,3] and the references therein.

Let the toric manifold $P_{\Sigma}$ be as in Theorem 1 and $L_k = L_k(\Sigma) \to P_{\Sigma}$ the corresponding line bundles to the standard toric divisors $D_k(\Sigma), k = 1, \ldots, d$. It is well known that the Chern class $c_1(L_k)$ is Poincaré dual to $[D_k] \in H^2(P_{\Sigma}, \mathbb{Z})$ for each $k$. For $m = (m_1, \ldots, m_d) \in \mathbb{Z}^d$ consider the line bundle $L = L_1^{m_1} \otimes \cdots \otimes L_d^{m_d}$. By the toric manifold theory it is ample if and only if the $\Sigma$-piecewise linear function $\varphi_L = \varphi_{(m_1, \ldots, m_d)} \in PL(\Sigma)$ determined by $\varphi_L(u_k) = m_k, k = 1, \ldots, d$, is a strictly convex support function.

**Theorem 4.** Let $P_{\Sigma}$ be the toric manifold associated with a complete regular fan $\Sigma$ in $\mathbb{R}^n$ and $L \to P_{\Sigma}$ an ample line bundle on it. If $\varphi_L$ is any strictly convex support function in $PL(\Sigma)$ representing the class $c_1(L)$ then

$$\epsilon(L) \leq 2\pi \cdot \Upsilon(\Sigma, \varphi_L).$$
Furthermore, if \( m = (m_1, \ldots, m_d) \in \mathbb{Z}^d \) is such that the \( \Sigma \)-piecewise linear function \( \varphi_{(m_1, \ldots, m_d)} \) in (10) is a strictly convex support function, then

\[
\varepsilon \left( L_1^{m_1} \otimes \cdots \otimes L_d^{m_d} \right) \leq \inf \left\{ \sum_{k=1}^{d} m_k a_k > 0 \mid \sum_{k=1}^{d} a_k u_k = 0, \ a_k \in \mathbb{Z}_{\geq 0}, \ k = 1, \ldots, d \right\}.
\]

3. The strategies of proof of the main results

We only outline the proof of Theorem 1. For a closed symplectic manifold \((M, \omega)\), by Proposition 1.8, Theorem 1.12 and Remark 1.13 in [6] we know that if there exist homology classes \( A \in H_2(M, \mathbb{Z}) \) and \( a_k \in H_4(M, \mathbb{Q}), \ i = 1, \ldots, m, \) such that the Gromov–Witten invariant \( \Psi_{A,0,m+1}^{(M, \omega)}(pt; pt, a_1, \ldots, a_m) \neq 0 \) then \( \mathcal{W}G(M, \omega) \leq \mathcal{C}H_{2}(M, \omega; pt, P D([\omega])) \leq \omega(A). \) Since such \( A \) has always the representatives of rational curves it follows from the Gromov compactness theorem that the infimum \( GW_0(M, \omega; pt, P D([\omega])) \) of all \( \omega(A) \) when \( A \) taking over such classes is more than zero. If \( GW_0(M, \omega; pt, P D([\omega])) \) is finite the symplectic manifold \((M, \omega)\) is called strong 0-symplectic uniruled in Definition 1.16 of [6]. Batyrev’s computation for the quantum cohomology rings of toric manifolds [2] (cf. [8] for a rigorous explanation) showed that the toric manifolds are strong 0-symplectic uniruled. Precisely, under the assumptions of Theorem 1 let us denote by \( R(\varepsilon \Sigma_A) \) the inverse map of the isomorphism. It was proved in [2] that for every \( A = \varepsilon \Sigma_A(a) \in \varepsilon \Sigma_A(\mathbb{Z}^d \cap R(\Sigma)) \subset H_2(P_\Sigma, \mathbb{Z}) \) and any Kähler form \( \omega \) on \( P_\Sigma \) the Gromov–Witten invariant \( \Psi_{A,0,m+1}^{(P_\Sigma, \omega)}(pt; pt, P D([\omega]), \ldots, P D([\omega])) = 1, \) where \( m = 1 + \sum_{k=1}^{d} a_k \) and \( c_k \in H^2(P_\Sigma, \mathbb{Z}) \) are the Poincaré dual of \([D_k(\Sigma)], k = 1, \ldots, d. \) On the other hand each Kähler form \( \omega \) on \( P_\Sigma \) may be represented by a strictly convex support function for \( \Sigma, \) also denoted by \( \omega. \) By the arguments in §3 of [2] we have \( \omega(A) = ([\omega], A) = \sum_{k=1}^{d} \omega(u_k) a_k. \) Now Theorem 1 may be derived from these arguments.

Theorem 2 may be derived from Theorem 1, the main result in [9] and Lemma 3.11 in [7]. The proof of Theorem 4 may be completed by using Proposition 6.3 in [3] and Theorem 1.39 in [6].

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