Interpolation orbits in couples of $L_p$ spaces

Vladimir I. Ovchinnikov

Voronezh State University, Universitetskaja pl., 1, Voronezh, 394693, Russia

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Abstract

We consider linear operators $T$ mapping a couple of weighted $L_p$ spaces $(L_{p_0}(U_0), L_{p_1}(U_1))$ into $(L_{q_0}(V_0), L_{q_1}(V_1))$ for any $1 \leq p_0, p_1, q_0, q_1 \leq \infty$, and describe the interpolation orbit of any $a \in L_{p_0}(U_0) + L_{p_1}(U_1)$ that is we describe a space of all $\{T \alpha\}$, where $\alpha$ runs over all linear bounded mappings from $(L_{p_0}(U_0), L_{p_1}(U_1))$ into $(L_{q_0}(V_0), L_{q_1}(V_1))$. We show that interpolation orbit is obtained by the Lions–Peetre method of means with functional parameter as well as by the $K$-method with a weighted Orlicz space as a parameter. To cite this article: V.I. Ovchinnikov, C. R. Acad. Sci. Paris, Ser. I 334 (2002) 881–884. © 2002 Académie des sciences/Éditions scientifiques et médicales Elsevier SAS

Orbites d’interpolation pour les couples d’espaces $L_p$

Résumé

Nous considérons les opérateurs $T$ partant d’un couple d’espaces $L_p$ à poids $(L_{p_0}(U_0), L_{p_1}(U_1))$ à valeurs dans $(L_{q_0}(V_0), L_{q_1}(V_1))$, où $1 \leq p_0, p_1, q_0, q_1 \leq \infty$, et donnons une description de l’orbite d’interpolation de tout élément $a \in L_{p_0}(U_0) + L_{p_1}(U_1)$; autrement dit nous décrivons l’espace de toutes les images $(T \alpha)$, où $\alpha$ parcourt l’espace des opérateurs linéaires bornés de $(L_{p_0}(U_0), L_{p_1}(U_1))$ dans $(L_{q_0}(V_0), L_{q_1}(V_1))$. Nous montrons que l’orbite d’interpolation est obtenue par la méthode des moyennes de Lions–Peetre avec un paramètre fonctionnel, et aussi par la $K$-méthode avec un espace d’Orlicz à poids comme paramètre fonctionnel. Pour citer cet article : V.I. Ovchinnikov, C. R. Acad. Sci. Paris, Ser. I 334 (2002) 881–884. © 2002 Académie des sciences/Éditions scientifiques et médicales Elsevier SAS

This paper is devoted to description of interpolation orbits with respect to linear operators mapping an arbitrary couple of $L_p$ spaces with weights $(L_{p_0}(U_0), L_{p_1}(U_1))$ into an arbitrary couple $(L_{q_0}(V_0), L_{q_1}(V_1))$, where $1 \leq p_0, p_1, q_0, q_1 \leq \infty$. By $L_p(U)$ we denote the space of measurable functions $f$ on a measure space $\mathcal{M}$ such that $\|f\|_{L_p(U)} = \|f\|_{L_p}$.

Let $(X_0, X_1)$ and $(Y_0, Y_1)$ be two Banach couples, $a \in X_0 + X_1$. The space $\text{Orb}(a, [X_0, X_1] \rightarrow [Y_0, Y_1])$ is a Banach space of $y \in Y_0 + Y_1$ such that $y = Ta$, where $T$ is a linear operator mapping the couple $[X_0, X_1]$ into the couple $[Y_0, Y_1]$. This space is called an interpolation orbit of the element $a$.

We are going to describe the spaces $\text{Orb}(a, [L_{p_0}(U_0), L_{p_1}(U_1)] \rightarrow [L_{q_0}(V_0), L_{q_1}(V_1)])$ for any $a$, any $1 \leq p_0, p_1, q_0, q_1 \leq \infty$ and any positive weights $U_0, U_1, V_0, V_1$.

Fundamental results on description of these spaces in separate cases are well known since 1965. The key role was played by the J. Peetre $K$-functional.

E-mail address: vio@func.vsu.ru (V.I. Ovchinnikov).

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Let \( [X_0, X_1] \) be a Banach couple, \( x \in X_0 + X_1, s > 0, t > 0 \). Denote by
\[
K(s, t, x; [X_0, X_1]) = \inf_{x = x_0 + x_1} s\|x_0\|_0 + t\|x_1\|_1,
\]
where infimum is taken over all representations of \( x \) as a sum of \( x_0 \in X_0 \) and \( x_1 \in X_1 \). The function \( K(s, t) \) is concave and is uniquely defined by the function \( K(1, t, x; [X_0, X_1]) \) which is also denoted by \( K(t, x; [X_0, X_1]) \).

If \( 1 \leq p_0 \leq q_0 \leq \infty, 1 \leq p_1 \leq q_1 \leq \infty \), the orbits \( \text{Orb}(a, [L_{p_0}(U_0), L_{p_1}(U_1)]) \rightarrow [L_{q_0}(V_0), L_{q_1}(V_1)] \) were described as the generalized Marcinkiewicz spaces, i.e.,
\[
\text{Orb}(a, [L_{p_0}(U_0), L_{p_1}(U_1)]) \rightarrow [L_{q_0}(V_0), L_{q_1}(V_1)] = \left\{ y : \sup_{s, t} \frac{K(s, t, y; [L_{q_0}(V_0), L_{q_1}(V_1)])}{K(s, t, a; [L_{p_0}(U_0), L_{p_1}(U_1)])} < \infty \right\}
\]
for any \( a \in L_{p_0}(U_0) + L_{p_1}(U_1) \). The decisive steps were done by Sparr in [10,11] and Dmitriev in [3]. In particular Sparr showed that if
\[
K(s, t, y; [L_{p_0}(V_0), L_{p_1}(V_1)]) \leq CK(s, t, a; [L_{p_0}(U_0), L_{p_1}(U_1)]),
\]
then there exists a linear operator \( T : [L_{p_0}(U_0), L_{p_1}(U_1)] \rightarrow [L_{q_0}(V_0), L_{q_1}(V_1)] \) such that \( y = Ta \).

Dmitriev in [3] had also found a description of orbits in the case of arbitrary \( 1 \leq p_0, p_1 \leq \infty \) and \( q_0 = q_1 = 1 \) as well as in the case of arbitrary \( 1 \leq p_1, q_0 \leq \infty \) and \( p_0 = q_1 = 1 \).

The result we are going to present here goes up to the paper [6] where some optimal interpolation theorems were found. Developing this approach the following hypothesis was formulated in [7]. Roughly speaking it states that the space \( \text{Orb}(a, [L_{p_0}(U_0), L_{p_1}(U_1)]) \rightarrow [L_{q_0}(V_0), L_{q_1}(V_1)] \) is situated between \( L_{q_0}(V_0) \) and \( L_{q_1}(V_1) \) exactly in the same place as the Calderon–Lozanovskii space \( L_{r_1}(W_1) \) between \( L_{q_1}(W_0) \) and \( L_{r_1}(W_1) \), where \( \varphi(s, t) = K(s, t, a; [L_{p_0}(U_0), L_{p_1}(U_1)]) \) and \( r_0^{-1} = (q_0^{-1} - p_0^{-1})_+ \), \( r_1^{-1} = (q_1^{-1} - p_1^{-1})_+ \). This hypothesis was partially confirmed in [8]. Now we show that hypothesis from [7] is true for any \( a \in L_{p_0}(U_0) + L_{p_1}(U_1) \). We also present a slightly modified description of interpolation orbits which resembles Dmitriev’s description from [3].

1. **The method of means for any quasi-concave functional parameter**

Let \( \varphi(s, t) \) be interpolation function, that is \( \rho(t) = \varphi(1, t) \) be quasi-concave and \( \varphi(s, t) \) be homogeneous of the degree one. Assume that \( \varphi \in \Phi_0 \) which means that \( \varphi(1, t) \rightarrow 0 \) and \( \varphi(t, 1) \rightarrow 0 \) as \( t \rightarrow 0 \). Denote by \( \{t_n\} \) the sequence invented by K. Oskolkov and introduced to interpolation by S. Janson. The sequence is constructed by induction \( \min(\rho(t_{n+1})/\rho(t_n), t_{n+1}/t_n, \rho(t_{n+1})/\rho(t_n)) = q > 1 \). (For simplicity in the sequel we suppose that \( \{t_n\} \) is two-sided.)

The main property of this sequence is the following
\[
K(s, t, \{\rho(t_n)\}, \{I_{p_0}, I_{p_1}(t_n^{-1})\}) \asymp \varphi(s, t) \tag{1}
\]
for any \( 1 \leq p_0, p_1 \leq \infty \).

**Definition 1.** Let \( [X_0, X_1] \) be any Banach couple, \( \rho(t) \) is a quasi-concave function such that \( \varphi \in \Phi_0 \) and \( 1 \leq p_0, p_1 \leq \infty \). Denote by \( \varphi(X_0, X_1)_{p_0, p_1} \) the space of \( x \in X_0 + X_1 \) such that
\[
x = \sum_{n \in \mathbb{Z}} \rho(t_n) w_n \quad \text{(convergence in } X_0 + X_1), \tag{2}
\]
where \( w_n \in X_0 \cap X_1 \) and \( \{\|w_n\|_{X_0}\} \in I_{p_0}, \{t_n\|w_n\|_{X_1}\} \in I_{p_1} \).
The norm in $\psi(X_0, X_1)_{p_0, p_1}$ is naturally defined. In the case of $\varphi(s, t) = s^{1-\theta}t^\theta$, where $0 < \theta < 1$, these spaces were introduced by Lions and Peetre in [5] and were called the spaces of means.

Note that $\psi(X_0, X_1)_{\infty, \infty}$ coincides with the generalized Marcinkiewicz space $M_\varphi(X_0, X_1)$ as well as with the space $(X_0, X_1)_{p, \infty}$ (see, for instance, [9]).

Let $\{X_0, X_1\}$ be a couple of Banach lattices. Recall that $\psi(X_0, X_1)$ is the space of all elements from $X_0 + X_1$ such that $|x| = \varphi(|x_0|, |x_1|)$, where $x_0 \in X_0$, $x_1 \in X_1$.

**Lemma 1.** Let $1 \leq p_0, p_1 < \infty$, then $\psi(L_{p_0}(U_0), L_{p_1}(U_1)) = \psi(L_{p_0}(U_0), L_{p_1}(U_1))_{p_0, p_1}$.

(Not that if $U_0 = 1$ and $U_1 = 1$, then $\psi(L_{p_0}, L_{p_1})$ is an Orlicz space.)

Recall that interpolation function $\varphi$ is called non-degenerate if the ranges of the functions $\varphi(t, 1)$ and $\varphi(1, t)$ where $t > 0$ coincide with $(0, \infty)$.

**Lemma 2.** If $\varphi$ is non-degenerate, then for any Banach couple the space $\psi(X_0, X_1)_{p_0, p_1}$ consists of $x \in X_0 + X_1$ for which $\{K(\mu_m, x, [X_0, X_1]) \in \varphi(l_{p_0}, l_{p_1}(u_m^{-1}))\$, where $[u_m]$ is the Oskolkov–Janson sequence for the function $K(t, x, [X_0, X_1])$.

We omit the proof. Note however that the proof is based on the $K$-divisibility (see [2]) and Lemma 1. With the help of $K$-divisibility for the couple $[l_{p_0}, l_{p_1}(u_m^{-1})]$ the expansion (2) of $x \in \psi(X_0, X_1)_{p_0, p_1}$ in the couple $\{X_0, X_1\}$ generates the analogous expansion of the sequence $\{K(\mu_m, x, [X_0, X_1]) \in \varphi(l_{p_0}, l_{p_1}(u_m^{-1}))_{p_0, p_1} = \varphi(l_{p_0}, l_{p_1}(u_m^{-1}))$ in the couple $[l_{p_0}, l_{p_1}(u_m^{-1})]$, and vice versa.

**Remark.** Note that the spaces $\psi(X_0, X_1)_{p, p}$ coincide with the space $(X_0, X_1)_{p, p}$ introduced by Janson (see [4]). Lemma 2 gives us a new description of these spaces as well.

2. The main theorem

**Theorem.** Let $\{L_{p_0}(U_0), L_{p_1}(U_1)\}$ and $\{L_{q_0}(V_0), L_{q_1}(V_1)\}$ be two Banach couples, where $1 \leq p_0$, $p_1$, $q_0$, $q_1$, $a \in L_{p_0}(U_0) + L_{p_1}(U_1)$ such that $\varphi(s, t) = K(s, t, a, [L_{p_0}(U_0), L_{p_1}(U_1)]) \in \Phi_0$, then

$$\text{Orb}(a, \{L_{p_0}(U_0), L_{p_1}(U_1)\}) = \{L_{q_0}(V_0), L_{q_1}(V_1)\} = \varphi(L_{q_0}(V_0), L_{q_1}(V_1))_{r_0, r_1},$$

where $r_0 = (q_0^{-1} - p_0^{-1})^+$ and $r_1 = (q_1^{-1} - p_1^{-1})^+$. (As usual $x^+$ denotes the positive part of $x$.)

The rest cases $\varphi(s, t) \notin \Phi_0$ can be easily reduced to $\varphi(s, t) \notin \Phi_0$ as it was done in [9] where the analogous situation takes place for $p_0 < q_0$ and $p_1 \leq q_1$.

The proof is a combination of the following propositions.

**Proposition 1.** For any $1 \leq p_0$, $p_1$, $q_0$, $q_1$, $a \in L_{p_0}(U_0) + L_{p_1}(U_1)$

$$\text{Orb}(a, \{L_{p_0}(U_0), L_{p_1}(U_1)\}) \subset \varphi(L_{q_0}(V_0), L_{q_1}(V_1))_{r_0, r_1},$$

**Proof.** Let $b = Ta$, where $T : \{L_{p_0}(U_0), L_{p_1}(U_1)\} \rightarrow \{L_{q_0}(V_0), L_{q_1}(V_1)\}$. Recall that $\rho(t) = \varphi(1, t)$. Denote $a_\rho = \{\rho(t_n)\}$, $\psi(a) = K(a, b, [L_{q_0}(V_0), L_{q_1}(V_1)])$ and $b_\psi = \{\psi(u_m)\}$, where $u_m$ is the Oskolkov–Janson sequence for $\psi(a)$. The span theorem implies that there exists a linear operator $S : [l_{p_0}, l_{p_1}(u_m^{-1})] \rightarrow [l_{q_0}, l_{q_1}(u_m^{-1})]$ such that $S(a_\rho) = b_\psi$.

We consider the embedding $[l_{q_0}, l_{q_1}(u_m^{-1})] \subset [l_{\infty}, l_{\infty}(u_m^{-1})]$. It is known that the embedding $l_{q_1} \subset l_\infty$ are $(1, q_1)$-summing operators (by the Karl–Bennett theorem, see [11]). Hence if $q_0 < p_0$, then the image of the standard basis sequence in $l_{p_0}$ with respect to $S : l_{p_0} \rightarrow l_{\infty}$ is $l_{r_0}$-sequence, that is $\|S(e_n)\|_{l_\infty} \in l_{r_0}$, where $r_0 = q_0^{-1} - p_0^{-1}$. Analogously $\{l_{r_0}S(e_n)\}_{l_{\infty}(u_m^{-1})} \in l_{r_1}$, where $r_1 = q_1^{-1} - p_1^{-1}$. Hence in any case we have $\|S(e_n)\|_{l_{\infty}} \in l_{r_0}$, and $\{l_{r_0}S(e_n)\}_{l_{\infty}(u_m^{-1})} \in l_{r_1}$, where $r_0 = q_0^{-1} - p_0^{-1}$.
Theorem 2. Let \( \{ \psi(u_m) \} \in \varphi(l_{p_\alpha}(U_0), l_{p_\beta}(U_1)) \), then there exist sequences \( \{ \beta_m^0 \} \in \varphi(l_{p_\alpha}, l_{p_\beta}) \) and \( \{ \beta_m^1 \} \in \varphi(l_{p_\alpha}, l_{p_\beta}) \), such that \( K(s, t, a_{p_\alpha}) \subset Orb(a_{p_\alpha}, l_{p_\alpha}(U_0), l_{p_\beta}(U_1)) \rightarrow \{ \psi(u_m) \} \).

Proof. Without loss of generality we assume that \( p_0 \geq q_0, p_1 \geq q_1 \). By Proposition 2 we can find \( \beta^0 \in l_{p_\alpha} \) and \( \beta^1 \in l_{p_\beta} \). Consider the embedding

\[
\{ l_1(1/\beta^0_m), l_1(1/\beta^1_mu_m) \} \subset \{ l_{p_0}(1/\beta^0_m), l_{p_1}(1/\beta^1_mu_m) \} \subset \{ l_{q_0}, l_{q_1}(u_m^{-1}) \}
\]

and the element \( b_{p_\beta} \). By Proposition 2 we have \( K(s, t, b_{p_\beta}, \{ l_{p_0}(1/\beta^0_m), l_{p_1}(1/\beta^1_mu_m) \}) \subset C \psi(s, t) \). Since \( \psi(s, t) \simeq K(s, t, a_{p_\alpha}, \{ l_{p_0}, l_{p_1}(t_m^{-1}) \}) \), by the Sparre theorem there exists an operator \( S : \{ l_{p_0}, l_{p_1}(t_m^{-1}) \} \rightarrow \{ l_{q_0}, l_{q_1}(u_m^{-1}) \} \) mapping \( a_{p_\alpha} \) into \( b_{p_\beta} \).

The composition of \( S \) and the right-hand side embedding in (3) is the desired mapping. Thus proposition and theorem are proved.

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References