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Équations aux dérivées partielles/*Partial Differential Equations*

Existence and uniqueness results for nonlinear elliptic problems with a lower order term and measure datum

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Abstract In this Note we consider a class of noncoercive nonlinear problems whose prototype is

$$-\triangle_p u + b(x)|\nabla u|^{\lambda} = \mu \quad \text{in } \Omega, \qquad u = 0 \quad \text{on } \partial\Omega,$$

where Ω is a bounded open subset of $\mathbb{R}^N$ ($N \geqslant 2$), $\triangle_p$ is the so called p-Laplace operator ($1 < p < N$) or a variant of it, μ is a Radon measure with bounded variation on Ω or a function in $\mathrm{L}^1(\Omega)$, $\lambda \geqslant 0$ and b belongs to the Lorentz space $\mathrm{L}^{N,1}(\Omega)$ or to the Lebesgue space $\mathrm{L}^{\infty}(\Omega)$. We prove existence and uniqueness of renormalized solutions. ***To cite this article: M.F. Betta et al., C. R. Acad. Sci. Paris, Ser. I 334 (2002) 757–762.*** © 2002 Académie des sciences/Éditions scientifiques et médicales Elsevier SAS

Existence et unicité de solutions renormalisées d'équations elliptiques non linéaires avec des termes d'ordre inférieur et données mesures

Résumé Dans cette Note nous considérons une classe de problèmes non linéaires et non coercifs dont le prototype est

$$-\triangle_p u + b(x)|\nabla u|^{\lambda} = \mu \quad \text{dans } \Omega, \qquad u = 0 \quad \text{sur } \partial\Omega,$$

où Ω est un ouvert borné de $\mathbb{R}^N$ ($N \geqslant 2$), $\triangle_p$ est le p-Laplacien ($1 < p < N$) ou une variante de cet opérateur, μ est une mesure de Radon bornée ou une function de $\mathrm{L}^1(\Omega)$, $\lambda \geqslant 0$ et b appartient à l'espace de Lorentz $\mathrm{L}^{N,1}(\Omega)$ ou à l'espace de Lebesgue $\mathrm{L}^{\infty}(\Omega)$. Nous démontrons l'existence et l'unicité de solutions renormalisées de ce problème. ***Pour citer cet article : M.F. Betta et al., C. R. Acad. Sci. Paris, Ser. I 334 (2002) 757–762.*** © 2002 Académie des sciences/Éditions scientifiques et médicales Elsevier SAS

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Dans cette Note, qui annonce [2] et [3], nous considérons un problème elliptique non linéaire qui peut être écrit formellement sous la forme $-\operatorname{div}(a(x,u,\nabla u)) + H(x,u,\nabla u) = \mu$ dans Ω, $u = 0$ sur $\partial\Omega$ (équation (3) de la version anglaise), où Ω est un ouvert borné de $\mathbb{R}^N$, $N \geqslant 2$, p est un réel tel que $1 < p < N$, $a : \Omega \times \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}^N$ est une fonction de Carathéodory qui définit un opérateur pseudomonotone coercif continu sur $\mathrm{W}_0^{1,p}(\Omega)$ (*voir* hypothèses (4)–(6) de la version anglaise) et où $H : \Omega \times \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}$ est une fonction de Carathéodory qui vérifie la condition de croissance $|H(x,s,\xi)| \leqslant b_0(x)|\xi|^{p-1} + b_1(x)$, avec $b_0 \in \mathrm{L}^{N,1}(\Omega)$ et $b_1 \in \mathrm{L}^1(\Omega)$; enfin μ est une mesure à variation bornée, qui est décomposée (*voir* [8]) sous la forme $\mu = f - \operatorname{div}(g) + \mu_s^+ - \mu_s^-$, avec $f \in \mathrm{L}^1(\Omega)$, $g \in (\mathrm{L}^{p'}(\Omega))^N$, et μ_s^+ et μ_s^- des mesures à variation bornée non négatives qui sont concentrées sur deux ensembles disjoints E^+ et E^- de p-capacité nulle.

Sous les hypothèses qui précédent, nous démontrons l'existence d'une solution renormalisée de (3) (*voir* Théorème 2.1 de la version anglaise). Par solution renormalisée nous entendons une fonction u mesurable, finie presque partout, telle que $T_k(u) \in \mathrm{W}_0^{1,p}(\Omega)$ pour tout $k > 0$, et qui vérifie (10)–(12) et (13) (*voir* Définition 2.1 dans la version anglaise). On notera en particulier que (13) correspond formellement à utiliser la fonction test $h(u)v$ dans (3), avec $h \in \mathrm{W}^{1,\infty}(\mathbb{R})$ à support compact et $v \in \mathrm{W}^{1,p}(\Omega) \cap \mathrm{L}^\infty(\Omega)$.

D'autre part, nous démontrons l'unicité d'une solution renormalisée de (3) (*voir* Théorème 3.1 de la version anglaise), sous des hypothèses un peu différentes de celles sous lesquelles nous avons obtenu l'existence. Pour démontrer l'unicité nous supposons en effet que $a(x,s,\xi)$ et $H(x,s,\xi)$ ne dépendent pas de s, que a est fortement monotone, que μ n'est plus une mesure à variation bornée générale mais qu'elle appartient à $\mathrm{L}^1(\Omega) + \mathrm{W}^{-1,p'}(\Omega)$ (*voir* hypothèses (14), (15) et (17) de la version anglaise), et que H ne vérifie plus la condition de croissance (7) mais qu'elle est localement lipschitzienne en ξ, et plus précisément qu'elle vérifie $|H(x,\xi) - H(x,\eta)| \leqslant b(x)\,(1 + |\xi| + |\eta|)^\sigma\,|\xi - \eta|$, avec $b \in \mathrm{L}^\infty(\Omega)$ et $0 \leqslant \sigma < \sigma_0(N,p)$, où $\sigma_0(N,p)$ est précisé dans le Théorème 3.1 de la version anglaise (*voir* (19), (20) et (21)). Notons que, lorsque $H(x,s,\xi) = (1+|\xi|^2)^{\lambda/2}$, les intervalles $[0, p-1]$ et $[0, 1+\sigma_0(N,p)[$, pour lesquels nous démontrons d'une part l'existence et d'autre part l'unicité, ne coincident pas en général.

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1. Introduction

In this Note, which announces [2] and [3], we consider a class of problems whose prototype is

$$-\triangle_p u + b(x)|\nabla u|^\lambda = \mu \quad \text{in } \Omega, \qquad u = 0 \quad \text{on } \partial\Omega, \tag{1}$$

where Ω is a bounded open subset of $\mathbb{R}^N$ $(N \geqslant 2)$, $\triangle_p$ is the so called p-Laplace operator $(1 < p < N)$ or a variant of it, μ is a Radon measure with bounded variation on Ω or a function in $\mathrm{L}^1(\Omega)$, $\lambda \geqslant 0$ and the coefficient b belongs to the Lorentz space $\mathrm{L}^{N,1}(\Omega)$ or to the Lebesgue space $\mathrm{L}^\infty(\Omega)$. We are interested in proving existence and uniqueness results.

As far as existence results are concerned, there are two main difficulties in this problem: on the one hand, the right-hand side is a measure; on the other hand, the operator is in general not coercive when the norm of b in the Lorentz space $\mathrm{L}^{N,1}(\Omega)$ or in $\mathrm{L}^\infty(\Omega)$ is not small. For what concerns uniqueness, a new difficulty, namely the fact that one has to deal with an "$\mathrm{H}^1(\Omega)-$ space with weight" and not to the $\mathrm{W}_0^{1,p}(\Omega)$ space, is added to the previous ones.

Let us explain how we face those difficulties.

The first one is the fact that the right-hand side of (1) is a measure, even in the case where $b = 0$. In the linear case (where $p = 2$), Stampacchia [14] defined a notion of solution of (1) by duality, for which he proved existence and uniqueness. The nonlinear case was firstly studied in [5] (and in [9] in the case where $b \neq 0$) and existence of a solution which satisfies the equation in the distributional sense was proven when $p > 2 - 1/N$. There are however two difficulties when one considers this type of solution for equation (1).

When p is close to 1, i.e., $p \leqslant 2 - 1/N$, simple examples show that the solution of (1) does not in general belong to the space $W^{1,1}_{loc}(\Omega)$. On the other hand a classical counterexample [13] shows that such a solution is, in general, not unique. To overcome these difficulties two equivalent notions of solutions have been introduced, the notion of entropy solution in [1,6] and the notion of renormalized solution in [10–12], in the case where the measure μ belongs to $L^1(\Omega)$ or to $L^1(\Omega) + W^{-1,p'}(\Omega)$; in these papers the authors proved the existence and uniqueness of such solutions. Similar results (existence and partial uniqueness) have been obtained in [8] in the case of a general measure with bounded total variation.

The second difficulty is due to the non coerciveness of the operator $-\triangle_p u + b(x)|\nabla u|^\lambda$ when b is not small. We overcome this difficulty by using the technique of Bottaro and Marina [7], which consists in some sense in splitting the problem in a finite number of problems with b small, which are therefore coercive.

Therefore we are able to prove (*see* Theorem 2.1) the existence of a renormalized solution of (1) when $0 \leqslant \lambda \leqslant p-1$, when b belongs to the Lorentz space $L^{N,1}(\Omega)$ and when μ is a general measure with bounded variation.

For what concerns uniqueness, we consider a class of problems whose prototype is a variation of (1), namely

$$-\operatorname{div}\big(a(x,\nabla u)\big) + b(x)\big(1+|\nabla u|^2\big)^{\lambda/2} = f - \operatorname{div}(g) \quad \text{in } \Omega, \qquad u = 0 \quad \text{on } \partial\Omega; \tag{2}$$

in this problem $f \in L^1(\Omega)$ and $g \in (L^{p'}(\Omega))^N$, so that the right-hand side is no more a general measure as in the existence setting; the coefficient b is now assumed to belong to the Lebesgue space $L^\infty(\Omega)$ (in our forthcoming paper [3], we study the case where b belongs to some Lebesgue space $L^r(\Omega)$); finally, when $p \geqslant 2$, we assume that the operator $-\operatorname{div}(a(x,\nabla u))$ is a nondegenerated variation of the p-Laplace operator, i.e., is of the type $-\operatorname{div}(a(x)(1+|\nabla u|^2)^{(p-2)/2}\nabla u)$, where $a \in L^\infty(\Omega)$, $a(x) \geqslant \alpha_0 > 0$ (such a nondegeneracy restriction is not necessary when $p < 2$).

Under those assumptions we prove the uniqueness of the renormalized solution of (2) for certain values of $\lambda \geqslant 0$. Observe that the set of values of λ for which we prove uniqueness does not coincide with the set $0 \leqslant \lambda \leqslant p-1$ for which we prove existence.

2. Existence result

Let us consider a nonlinear elliptic problem which can formally be written as

$$-\operatorname{div}\big(a(x,u,\nabla u)\big) + H(x,u,\nabla u) = \mu \quad \text{in } \Omega, \qquad u = 0 \quad \text{on } \partial\Omega. \tag{3}$$

Here Ω is a bounded open subset of $\mathbb{R}^N$, $N \geqslant 2$, p is a real number with $1 < p < N$, and $a : \Omega \times \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}^N$ and $H : \Omega \times \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}$ are Carathéodory functions satisfying

$$a(x,s,\xi)\xi \geqslant \alpha|\xi|^p, \quad \alpha > 0, \tag{4}$$

$$\big|a(x,s,\xi)\big| \leqslant c\big[|\xi|^{p-1} + |s|^{p-1} + a_0(x)\big], \quad a_0(x) \in L^{p'}(\Omega),\ c > 0, \tag{5}$$

$$\big(a(x,s,\xi) - a(x,s,\eta), \xi - \eta\big) > 0, \quad \xi \neq \eta, \tag{6}$$

$$\big|H(x,s,\xi)\big| \leqslant b_0(x)|\xi|^{p-1} + b_1(x), \quad b_0 \in L^{N,1}(\Omega),\ b_1 \in L^1(\Omega), \tag{7}$$

for almost every $x \in \Omega$, for every $s \in \mathbb{R}$ and $\xi \in \mathbb{R}^N$, $\eta \in \mathbb{R}^N$. Finally we assume that μ is a general measure with bounded variation, which is decomposed as

$$\mu \in M_b(\Omega), \quad \mu = f - \operatorname{div}(g) + \mu_s^+ - \mu_s^-, \quad f \in L^1(\Omega), \quad g \in \big(L^{p'}(\Omega)\big)^N, \tag{8}$$

where the singular parts μ_s^+ and μ_s^- are nonnegative measures in $M_b(\Omega)$, which are concentrated on two disjoint subsets E^+ and E^- of zero p-capacity (such a decomposition is always possible, *see* [8]).

DEFINITION 2.1. – We say that u is a renormalized solution of (3) if it satisfies the following conditions:

$$u \text{ is measurable on } \Omega, \text{ almost everywhere finite and such that } T_k(u) \in W_0^{1,p}(\Omega), \ \forall k > 0, \tag{9}$$

where $T_k(u) = \max\{-k, \min\{k, u\}\}$;

$$|u|^{p-1} \in L^{N/(N-p),\infty}(\Omega) \quad \text{and} \quad |\nabla u|^{p-1} \in L^{N',\infty}(\Omega), \tag{10}$$

where ∇u is defined as in [1,10–12] by $\nabla T_k(u) = \chi_{\{|u|<k\}} \nabla u$ a.e. in Ω;

$$\frac{1}{n}\int_{n \leqslant u < 2n} a(x, u, \nabla u)\nabla u\, \varphi \to \int_\Omega \varphi\, d\mu_s^+, \quad n \to +\infty, \tag{11}$$

$$\frac{1}{n}\int_{-2n < u \leqslant -n} a(x, u, \nabla u)\nabla u\, \varphi \to \int_\Omega \varphi\, d\mu_s^-, \quad n \to +\infty, \tag{12}$$

for every $\varphi \in C_b^0(\Omega)$; and finally

$$\begin{aligned} &\int_\Omega a(x, u, \nabla u) \cdot \nabla u\, h'(u)v + \int_\Omega a(x, u, \nabla u) \cdot \nabla v\, h(u) + \int_\Omega H(x, u, \nabla u)h(u)v \\ &\quad = \int_\Omega f h(u)v + \int_\Omega g \cdot \nabla u\, h'(u)v + \int_\Omega g \cdot \nabla v\, h(u), \end{aligned} \tag{13}$$

for every $v \in W^{1,p}(\Omega) \cap L^\infty(\Omega)$, for all $h \in W^{1,\infty}(\mathbb{R})$ with compact support in $\mathbb{R}$, which are such that $h(u)v \in W_0^{1,p}(\Omega)$.

Remark 1. – Definition 2.1 is a variant of the definition of renormalized solution introduced in [8]. This definition is now known to be the right one for nonlinear monotone problems with right-hand side a measure, at least when the singular parts μ_s^+ and μ_s^- of the measure are zero, since in this case the problem is well posed in the sense of Hadamard (*see* [1,10–12]).

Remark 2. – Equation (13) corresponds to take $h(u)v$ as test function in (3). Note that in (13) every term is well defined due to (9) and to the fact that h has compact support, which implies that in (13) u can be replaced by $T_k(u)$ for k such that $\operatorname{supp} h \subset [-k, k]$.

THEOREM 2.1. – *Under assumptions* (4)–(8), *there exists at least one renormalized solution u of* (3).

Let us sketch the proof in the case where $g = 0$. The idea is to consider first the case where $\|b_0\|_{L^{N,1}}$ is small; in this case the operator is coercive. Hence, using the truncation $T_k(u)$ as test function in (3), we formally obtain that $\alpha\|\nabla T_k(u)\|_{L^p}^p \leqslant Mk$ for every $k > 0$, where $M = \|\mu\|_{M_b} + \|b_1\|_{L^1} + \|b_0|\nabla u|^{p-1}\|_{L^1} \leqslant \|\mu\|_{M_b} + \|b_1\|_{L^1} + \|b_0\|_{L^{N,1}}\||\nabla u|^{p-1}\|_{L^{N',\infty}}$. A result of [1] proves that, when the truncations $T_k(v)$ of a function v satisfy $\alpha\|\nabla T_k(v)\|_{L^p}^p \leqslant Mk$ for all $k > 0$, then v satisfies $\||\nabla v|^{p-1}\|_{L^{N',\infty}} \leqslant C_0 M/\alpha$. Therefore, when $\|b_0\|_{L^{N,1}}$ is small, we obtain an a priori estimate for $\||\nabla u|^{p-1}\|_{L^{N',\infty}}$, and therefore for $\|H(x, u, \nabla u)\|_{L^1}$; this allows us to prove the existence result.

In the case where $\|b_0\|_{L^{N,1}}$ is not small, we use the technique of Bottaro and Marina [7], which in some sense allows one to reduce the problem to a finite sequence of problems with $\|b_0\|_{L^{N,1}}$ small and to prove the existence of a solution. For the details of the proof, see [2].

3. Uniqueness results

In order to prove the uniqueness of the renormalized solution of (3) we make some assumptions on the data which are a little bit different (but not always stronger) of the assumptions we made to prove the existence. First we assume that $a(x, s, \xi)$ and $H(x, s, \xi)$ are Carathéodory functions which do not depend on s, i.e.,

$$a(x, s, \xi) = a(x, \xi), \qquad H(x, s, \xi) = H(x, \xi), \tag{14}$$

for almost every $x \in \Omega$, for every $s \in \mathbb{R}$ and $\xi \in \mathbb{R}^N$. Second we assume that $a(x,\xi)$ satifies the conditions (4) and (5) and, moreover, is strongly monotone, i.e.,

$$\big(a(x,\xi) - a(x,\eta), \xi - \eta\big) \geqslant \beta\big(1 + |\xi| + |\eta|\big)^{p-2}|\xi - \eta|^2, \quad \beta > 0, \tag{15}$$

for almost every $x \in \Omega$ and for every $\xi \in \mathbb{R}^N$, $\eta \in \mathbb{R}^N$. Third we assume that $H(x,0) \in \mathrm{L}^1(\Omega)$ and that $H(x,\xi)$ is locally Lipschitz continuous with respect to ξ, i.e.,

$$\big|H(x,\xi) - H(x,\eta)\big| \leqslant b(x)\big(1 + |\xi| + |\eta|\big)^{\sigma}|\xi - \eta|, \quad b \in \mathrm{L}^\infty(\Omega),\ \sigma \geqslant 0, \tag{16}$$

for almost every $x \in \Omega$ and for every $\xi \in \mathbb{R}^N$, $\eta \in \mathbb{R}^N$, where σ is a constant to be specified in the statement of Theorem 3.1. Finally we assume that μ is no more a general measure, but satisfies

$$\mu = f - \mathrm{div}(g), \quad f \in \mathrm{L}^1(\Omega), \quad g \in \big(\mathrm{L}^{p'}(\Omega)\big)^N. \tag{17}$$

THEOREM 3.1. – *Assume that* (4), (5) *and* (14)–(17) *are satisfied with*

$$0 \leqslant \sigma < \sigma_0(N,p), \tag{18}$$

where $\sigma_0(N,p)$ *is defined by*

$$\sigma_0(N,p) = \frac{1 - N(2-p)}{N-1} \quad \textit{if } 2 - \frac{1}{N} < p \leqslant 2, \tag{19}$$

$$\sigma_0(N,p) = \frac{N(p-2)+p}{2(N-1)} \quad \textit{if } 2 < p < \frac{2N-4}{N-3}, \tag{20}$$

$$\sigma_0(N,p) = \frac{2(p-1)}{N-1} \quad \textit{if } \frac{2N-4}{N-3} \leqslant p < N. \tag{21}$$

If u_1 *and* u_2 *are two renormalized solutions of* (3), *then* $u_1 = u_2$.

Remark 3. – Observe that p is assumed to satisfy $p < N$, which is a further restriction on p in (19) and (20) when $N = 2$ or 3.

Observe also that definitions (19) and (20) [(20) and (21)] coincide in the "borderline case" $p = 2$ [in the "borderline case" $p = (2N-4)/(N-3)$].

Remark 4. – Let us consider the model case $H(x,\xi) = (1 + |\xi|^2)^{\lambda/2}$. Theorem 2.1 implies the existence of a renormalized solution when $0 \leqslant \lambda \leqslant p - 1$, while Theorem 3.1 implies the uniqueness of the renormalized solution, if it exists, when $0 \leqslant \sigma < \sigma_0(N,p)$, therefore when $0 \leqslant \lambda < 1 + \sigma_0(N,p)$. But the intervals $0 \leqslant \lambda \leqslant p - 1$ and $0 \leqslant \lambda < 1 + \sigma_0(N,p)$ do not coincide, and therefore for some values of p and λ, we have proved both existence and uniqueness, for some other values, existence but not uniqueness, and finally for some other values, uniqueness but not existence. Observe that we do not have any uniqueness result when $p \leqslant 2 - 1/N$.

Remark 5. – In [3] we prove Theorem 3.1 under (a little bit) more general assumptions. In particular, we allow the function b which appears in (16) to belongs to some space $\mathrm{L}^r(\Omega)$, with r sufficiently large. Moreover we consider the case where in (15), $1 + |\xi| + |\eta|$ is replaced by $A(x) + |\xi| + |\eta|$, with $A \in \mathrm{L}^{N',\infty}(\Omega)$ and $A(x) \geqslant A_0 \geqslant 0$, and where in (16), $(1 + |\xi| + |\eta|)^\sigma$ is replaced by $(A_0 + |\xi| + |\eta|)^{\sigma_{A_0}}$ $(B(x) + |\xi| + |\eta|)^{\sigma_B}$ with $|B|^{p-1} \in \mathrm{L}^{N',\infty}(\Omega)$; when $2 - 1/N < p \leqslant 2$, we can take $A_0 = 0$, while we assume $A_0 > 0$ when $2 < p < N$.

In [4] we treat the (simpler) case where the right-hand side is an element of $\mathrm{W}^{-1,p'}(\Omega)$ and where usual weak solutions $u \in \mathrm{W}_0^{1,p}(\Omega)$ are concerned.

The idea of the proof of Theorem 3.1 is the following (*see* [3] for the details). We first consider the case where b is small in $\mathrm{L}^\infty(\Omega)$, or more exactly in some $\mathrm{L}^r(\Omega)$, with r very large depending on σ and p. We formally use $T_k(u_1 - u_2)$ as test function in the difference of the two renormalized solutions u_1 and u_2

of (3). The fact that we use renormalized solutions allows us to make this correctly; here the fact that the measure satisfies (17), i.e., has not the singular parts μ_s^+ and μ_s^- is crucial. Using (15) and (16), we obtain

$$\begin{aligned}\beta \int_\Omega \big(1+|\nabla u_1|+|\nabla u_2|\big)^{p-2}\big|\nabla T_k(u_1-u_2)\big|^2 \\ \leqslant \int_\Omega b(x)\big(1+|\nabla u_1|+|\nabla u_2|\big)^{\sigma}|\nabla u_1-\nabla u_2|\big|T_k(u_1-u_2)\big| \\ \leqslant k\int_\Omega b(x)\big(1+|\nabla u_1|+|\nabla u_2|\big)^{\sigma}|\nabla u_1-\nabla u_2| = Mk,\end{aligned}$$

where $M=\int_\Omega B|\nabla u_1-\nabla u_2|$ with $B=b(1+|\nabla u_1|+|\nabla u_2|)^\sigma$. Let us now give a simplified idea of the next step of the proof, which is in fact much more technical. Using the same result of [1] (now with $p=2$) we invoked in the existence proof, we deduce from $\beta\int_\Omega|\nabla T_k(u_1-u_2)|^2\leqslant Mk$, for all $k>0$, that $\|\nabla u_1-\nabla u_2\|_{\mathrm{L}^{N',\infty}}\leqslant C_0M/\beta$. Using Hölder inequality in the definition of M, we conclude that $\nabla u_1-\nabla u_2=0$ when B is sufficiently small in a convenient space, which follows from b sufficiently small in $\mathrm{L}^r(\Omega)$. Actually the proof of this step is much more technical; when we tried to optimize the various parameters which enter in the proof, we were led to the three definitions (19), (20) and (21) of $\sigma_0(N,p)$.

When b is not small, we use as test function $T_k(S_m(u_1-u_2))$, where $S_m(u)=u-T_m(u)$, so that b can be replaced by $b\chi_{\{|u_1-u_2|\geqslant m\}}$. This allows us to prove that $S_m(u_1-u_2)=0$ for m sufficiently large, and then to conclude that $u_1-u_2=0$.

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