On a class of anisotropic asymptotically periodic Hamiltonians

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Received 25 January 2002; accepted 4 February 2002
Note presented by Jean-Michel Bony.

Abstract
We construct a $C^*$-algebra $C$ proper to an anisotropic asymptotically periodic quantum system and we compute its quotient by the algebra of compact operators. We describe then the self-adjoint operators affiliated to $C$ and their essential spectrum.

Résumé
Nous construisons une $C^*$-algèbre $C$ adaptée au traitement des systèmes quantiques anisotropes asymptotiquement périodiques et nous calculons son quotient par l’algèbre des opérateurs compacts. Nous décrivons alors les opérateurs auto-adjoints affiliés à $C$ et leurs spectres essentiels.

Version française abrégée
Considérons l’opérateur auto-adjoint $H = -\Delta + V$ dans $\mathcal{H} = L^2(\mathbb{R})$, où $V$ est l’opérateur de multiplication par une fonction asymptotiquement périodique avec des périodes différentes à $+\infty$ et $-\infty$. Les résultats obtenus par Georgescu et Ifimovici dans [6] nous ont suggéré l’étude, par des méthodes nouvelles, de cette classe particulière d’hamiltoniens. L’idée générale consiste à construire une $C^*$-algèbre $C$ dont le quotient par l’algèbre des opérateurs compacts puisse être calculé, et telle que les opérateurs que l’on veut étudier lui soient affiliés. Dans le cas présent, cette $C^*$-algèbre $C$ est obtenue grâce à la notion de produit croisé à partir d’une $C^*$-algèbre $C$ suggérée par la classe de fonctions $V$. L’objectif est de préciser la classe la plus large d’hamiltoniens $H$ affiliés à $C$. Ceci permet l’étude du spectre essentiel ou l’estimation de Mourre d’une manière unifiée. Introduisons à présent la $C^*$-algèbre commutative $\mathbb{C}$ suivante suggérée par une situation anisotrope:

$$C = \left\{ f \in C_{bu}(\mathbb{R}) \mid \lim_{n \to \pm\infty} f(x + na_{\pm}) \text{ existent, } \forall x \in \mathbb{R} \right\},$$

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où $C_{ba}(\mathbb{R}) = \{f : \mathbb{R} \to \mathbb{C} \mid f$ est bornée, uniformément continue$\}$. $n \in \mathbb{Z}$ et $(a_+, a_-) \in \mathbb{R}_+^* \times \mathbb{R}_+^*$. On montre que l’on a un plongement canonique : $C(\mathbb{R}) \subset C(\mathbb{R}/a_+ \mathbb{Z}) \oplus C(\mathbb{R}/a_- \mathbb{Z})$. Notons $\mathcal{C}$ la $C^*$-algèbre $C \times \mathbb{R}$ (produit croisé de $C$ par l’action du groupe additif $\mathbb{R}$). On a alors

$$\mathcal{C}/K(L^2(\mathbb{R})) \cong (C \times \mathbb{R})/(C_0(\mathbb{R}) \times \mathbb{R}) \cong (C/C_0(\mathbb{R})) \times \mathbb{R}$$

si bien que $\mathcal{C}/K(L^2(\mathbb{R})) \subset \mathcal{C}_+ \oplus \mathcal{C}_-$ où $\mathcal{C}_\pm = C_\pm \times \mathbb{R}$ et $\mathcal{C}_\pm = C(\mathbb{R}/a_\pm \mathbb{Z})$.

**Théorème 1.** — Soit $\mathcal{C}$ l’ensemble des opérateurs $T \in B(L^2(\mathbb{R}))$ tels que :

(i) il existe $T_+ \in \mathcal{C}_+$ tel que $\|\chi(Q > r)(T - T_+)^{(s)}\| \to 0$ si $r \to \infty$ ;

(ii) il existe $T_- \in \mathcal{C}_-$ tel que $\|\chi(Q < -r)(T - T_-)^{(s)}\| \to 0$ si $r \to \infty$ ;

(iii) $\|(e^{itP} - 1)T^{(s)}\| \to 0$ si $x \to 0$.

Alors $\mathcal{C}$ est une $C^*$-algèbre canoniquement isomorphe à $\mathcal{C}$.

**Théorème 2.** — Soient $H$ un opérateur auto-adjoint dans $\mathcal{H} = L^2(\mathbb{R})$ et $H_{\pm}$ un couple d’opérateurs auto-adjoints affiliés à $\mathcal{C}_\pm$ tels que $D(H_{\pm}) = D(H)$. Alors $H$ est affilié à $\mathcal{C}$ si

$$\|\theta_+(\epsilon Q)(H - H_+)\|_{D(H) \to \mathcal{H}} \to 0 \quad \text{et} \quad \|\theta_-(\epsilon Q)(H - H_-)\|_{D(H) \to \mathcal{H}} \to 0 \quad \text{si} \ \epsilon \to 0.$$

The study of the Schrödinger operator with periodic potential is now a classical subject many times explored since the article of Bloch [1] published in 1928. This operator gives a description of the motion of a particle in a crystal. It is well known that the spectrum of this operator has a band structure. Gel’fand [4] and Titchmarsh [8] were among the first to study rigorously the periodic one dimensional Schrödinger operator $H = -\Delta + V$ in $\mathcal{H} = L^2(\mathbb{R})$ (where the Laplacian $\Delta$, free Hamiltonian, is the quantization of the kinetic energy and $V$ is the operator of multiplication by a periodic potential function). More recently Davies and Simon have studied in [3] the scattering theory for systems with asymptotic spatial behaviour different on the right and the left. Also Roberts develops in [7] the quantum scattering for impurities in potentials that tend to a periodic function in one direction and to a constant one in the other. The references on this topic are multiple.

The new results of Georgescu and Iftimovici in [6] suggested the study to us, with their new methods, of a class of anisotropic Hamiltonians: the periodic Schrödinger operator with different behaviors at $\pm \infty$. The general idea is to construct a $C^*$-algebra $\mathcal{C}$, to study its properties and to point out a class $H$ of Hamiltonians affiliated to $\mathcal{C}$; this allows one to describe the essential spectrum. This $C^*$-algebra $\mathcal{C}$ is obtained by the notion of crossed product from a $C^*$-algebra $\mathcal{C}$ suggested by the class of functions $V$. One of the main goals is to determine the largest class of Hamiltonians $H$ affiliated to $\mathcal{C}$.

Let $\mathcal{H}$ be a Hilbert space and $H$ a self-adjoint operator in $\mathcal{H}$. Recall that the formula $\sigma(H) = \{\lambda \in \mathbb{R} \mid \phi \in C_0(\mathbb{R}) \text{ and } \phi(\lambda) \neq 0 \Rightarrow \phi(H) \neq 0\}$ gives a description of the spectrum of $H$. Here $C_0(\mathbb{R}) = \{\phi : \mathbb{R} \to \mathbb{C} \mid \phi \text{ is continuous and convergent to zero at infinity}\}$. Recall also that the essential spectrum $\sigma_{\text{ess}}(H)$ of $H$ is the set of $\lambda \in \sigma(H)$ such that either $\lambda$ is not isolated from the rest of the spectrum or it is an eigenvalue of infinite multiplicity. We have:

$$\sigma_{\text{ess}}(H) = \{\lambda \in \mathbb{R} \mid \phi \in C_0(\mathbb{R}) \text{ and } \phi(\lambda) \neq 0 \Rightarrow \phi(H) \notin K(\mathcal{H})\},$$

where $K(\mathcal{H})$ is the $C^*$-algebra of compact operators in $\mathcal{H}$ (closed self-adjoint ideal in the $C^*$-algebra $B(\mathcal{H})$ of bounded linear operators in $\mathcal{H}$). Let $C(\mathcal{H}) = B(\mathcal{H})/K(\mathcal{H})$ be the Calkin algebra (it is also a $C^*$-algebra). Denote by $H \mapsto \tilde{H}$ the canonical surjection of $B(\mathcal{H})$ onto $C(\mathcal{H})$. It is easy to see that $\sigma_{\text{ess}}(H) = \sigma(\tilde{H})$. If $\mathcal{C}$ is a $C^*$-subalgebra of $B(\mathcal{H})$, we say that $H$ is affiliated to $\mathcal{C}$ if its associated functional calculus is in $\mathcal{C}$, i.e., if $\forall \phi \in C_0(\mathbb{R})$, $\phi(H) \in \mathcal{C}$. In fact, it suffices to verify for a complex $z \notin \sigma(H)$ that $(H - z)^{-1} \in \mathcal{C}$. In [2] and [6] several classes of operators related to interesting physical situations are shown to be affiliated to
Let us consider the $C^*$-algebra $C_{bu}(\mathbb{R}) = \{ \varphi : \mathbb{R} \to C \mid \varphi \text{ is bounded, uniformly continuous} \}$ and the following commutative $C^*$-algebra suggested by an anisotropic situation:

$$C = \left\{ \varphi \in C_{bu}(\mathbb{R}) \mid \lim_{n \to \pm \infty} \tau_{na_n} \varphi(x) = I_{\pm}(x), \forall x \in \mathbb{R} \right\},$$

where $\tau_{na_n}(x) = \varphi(x + na_n)$, $n \in \mathbb{Z}$, and $(a_+, a_-) \in \mathbb{R}_+^2 \times \mathbb{R}_+^2$. Then $C$ is a $C^*$-subalgebra of $C_{bu}(\mathbb{R})$ which contains $C_0(\mathbb{R})$ and is stable by translations. The functions $I_{\pm}$ are continuous on $\mathbb{R}$ and clearly periodic of periods $a_+$ and $a_-$, respectively. We describe now the quotient $C^*$-algebra $C / C_0(\mathbb{R})$.

**PROPOSITION 1.** $C / C_0(\mathbb{R}) \subset C(\mathbb{R} / a_+ \mathbb{Z} \oplus C(\mathbb{R} / a_- \mathbb{Z})$.

Recall that the self-adjoint operators of $L^2(\mathbb{R})$, $Q$ (position observable) and $P$ (momentum observable), are defined by $(Qf)(x) = xf(x)$ and $(Pf)(x) = -i\hbar \frac{d}{dx} f(x)$. Then $\varphi(Q)$ is the operator of multiplication by the Borel function $\varphi$ in $L^2(\mathbb{R})$ and $\varphi(P) = \mathcal{F}^*\varphi(Q)\mathcal{F}$ where $\mathcal{F} : L^2(\mathbb{R}) \to L^2(\mathbb{R})$ is the Fourier transformation.

An elementary computation gives $e^{i\hbar Q} \varphi(Q) e^{-i\hbar Q} = \varphi(Q + x)$ and $e^{i\hbar Q} \varphi(P) e^{-i\hbar Q} = \varphi(P - k)$. If $\mathfrak{A}$, $\mathfrak{B}$ are subspaces of an algebra $\mathfrak{D}$ then we denote by $\mathfrak{A} \cdot \mathfrak{B}$ the linear subspace of $\mathfrak{D}$ generated by the elements of the form $AB$ with $A \in \mathfrak{A}$ and $B \in \mathfrak{B}$. If $\mathfrak{D}$ is a $C^*$-algebra then $\mathfrak{A} \cdot \mathfrak{B}$ is the norm closure of $\mathfrak{A} \cdot \mathfrak{B}$ in $\mathfrak{D}$. Since $C$ is a $C^*$-subalgebra of $C_{bu}(\mathbb{R})$ stable under translations, $C$ is provided with a continuous action of the additive group $\mathbb{R}$. Then the crossed product $C \rtimes \mathbb{R}$ is well defined and denoted by $C$. It is shown in [6] that $C$ is isomorphic to the (norm) closed linear subspace of $B(L^2(\mathbb{R}))$ generated by operators of the form $\varphi(Q)\psi(P)$ with $\varphi \in C$ and $\psi \in C_0(\mathbb{R})$. Below we denote $C_0(\mathbb{R}^*)$ the set of operators $\psi(P)$ with $\psi \in C_0(\mathbb{R})$. We thus may write $C = [C \cdot C_0(\mathbb{R}^*)]$. One also have $K(L^2(\mathbb{R})) = \{ C_0(\mathbb{R}) \cdot C_0(\mathbb{R}^*) \} \subset C_0(\mathbb{R}) \rtimes \mathbb{R}$ is isomorphic to $K(L^2(\mathbb{R}))$. Then

$$C / K(L^2(\mathbb{R})) \cong (C \rtimes \mathbb{R}) / (C_0(\mathbb{R}) \rtimes \mathbb{R}) \cong (C / C_0(\mathbb{R})) \rtimes \mathbb{R}$$

so that $C / K(L^2(\mathbb{R})) \subset C_+ \oplus C_-$ where $C_\pm = C_\pm \rtimes \mathbb{R}$ and $C_\pm = C(\mathbb{R} / a_\pm \mathbb{Z})$. Recall again that $C_\pm = \{ e^{ix\hbar \varphi} : \varphi \in C_0(\mathbb{R}^*) \}$.

Now we give a new characterization of $C$. If a symbol like $T^{(\star)}$ is used then the relation must hold both for the operator $T$ and for its adjoint $T^*$. We denote $\chi(A > r)$ the spectral projection of a self-adjoint operator $A$ associated to the interval $(r, \infty)$. The symbol $\chi(A < r)$ has a similar meaning.

**THEOREM 2.** $C$ coincides with the set of the operators $T \in B(L^2(\mathbb{R}))$ such that

1. there is some $T_+ \in C_+ \subset C_+ \subset C_+ \subset C_-$ such that $\| \chi(Q > r)(T - T_+)^{(\star)} \| \to 0$ if $r \to \infty$;
2. there is some $T_- \in C_- \subset C_- \subset C_- \subset C_+$ such that $\| \chi(Q < r)(T - T_-)^{(\star)} \| \to 0$ if $r \to \infty$;
3. $\| (e^{i\hbar T} - 1)^{(\star)} \| \to 0$ if $x \to 0$.

**Proof.** Let $C$ be the set of the operators $T_\pm$ satisfying (i)–(iii); we first show that $C \subset C$. Since $C$ is isomorphic to the (norm) closed linear subspace of $B(L^2(\mathbb{R}))$ generated by operators of the form $\varphi(Q)\psi(P)$ with $\varphi \in C$ and $\psi \in C_0(\mathbb{R}^*)$, our task reduces to show that these operators belong to $C$. So let $T = \varphi(Q)\psi(P)$ and set $T_\pm = I_{\pm}(Q)\psi(P)$. It suffices to show (i) and (ii) for $r = na_+$. Then

$$\| \chi(Q > na_+)(T - T_+) \| + \| e^{i\hbar \varphi_1} \chi(Q > 0) e^{i\hbar \varphi_1} (\varphi(Q)\psi(P) - I_{\pm}(Q)\psi(P)) \|$$

But $e^{i\hbar \varphi_1} I_{\pm}(Q)\psi(P) e^{-i\hbar \varphi_1} (Q + na_+)\psi(P) = I_{\pm}(Q)\psi(P)$ since $I_{\pm}$ is periodic of period $a_+$. Similarly $e^{i\hbar \varphi_1} \varphi(Q)\psi(P) e^{-i\hbar \varphi_1} = \varphi(Q + na_+)\psi(P)$. Then

$$\| \chi(Q > na_+)(T - T_+) \| = \| \chi(Q > 0) e^{i\hbar \varphi_1} (\varphi(Q)\psi(P) - I_{\pm}(Q)\psi(P)) \|$$

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We get to prove that $\theta(Q)$ is included in the form domain of a locally compact $\sigma(H)$, and such that $\lim_{Q \to \pm \infty} \theta(Q) = T_{\pm}$ if $\lim_{r \to \infty} (\|\chi([P > r) T]\| + \|T \chi([P > r)]\|) = 0$.

Reciprocally, let $T \in \mathcal{E}$. Let $\theta_+ \in C^\infty(\mathbb{R})$, $0 \leq \theta_+ \leq 1$, such that $\theta_+(x) = 0$ if $x < 1$ and $\theta_+(x) = 1$ if $x > 2$. Set $\theta_-(x) = \theta_+(-x)$, $\theta_0 = 1 - \theta_+ - \theta_-$, $\theta_+ = \theta_+(\varepsilon Q)$ and $\theta_0 = \theta_0(\varepsilon Q)$. (i) and (ii) are equivalent to $\|\theta_+^x(T - T_\pm)(\varepsilon)\| \to 0$ if $\varepsilon \to 0$ respectively. Since $T = \theta_0^xT + \theta_+^xT_+ + \theta_-^xT_- + \theta_+^x(T - T_+) + \theta_-^x(T - T_-)$, the last two terms tend to zero in norm when $\varepsilon \to 0$. Also, $\theta_+^x \in C$ yields $\theta_+^xT_\pm \in \mathcal{E}$. Finally we have to prove that $\theta_0^xT \in K(L^2(\mathbb{R})) \subset \mathcal{E}$. By a characterization of $K(L^2(\mathbb{R}))$ given in [6], $\theta_0^xT \in K(L^2(\mathbb{R}))$ if $\|\theta^x(T - T_\pm)(\varepsilon)\| \to 0$ as $\varepsilon \to 0$ and $\|\theta_0^x(T - T_\pm)(\varepsilon)\| \to 0$ as $k \to 0$. But $\|\theta^x(T - T_\pm)(\varepsilon)\| \to 0$ as $\varepsilon \to 0$, thus $\theta_0^xT \in K(L^2(\mathbb{R}))$, which finishes the proof of the theorem. □

We shall point out now an affiliation criterion to $\mathcal{E}$ for a self-adjoint operator. We first introduce some useful definitions (see [5]).

**Definition 3.** An operator $T \in B(L^2(\mathbb{R}))$ will be called *semi-compact* if for all $\theta \in C_0(\mathbb{R})$ the operators $\theta(Q)T$ and $T\theta(Q)$ are compact. The self-adjoint operators affiliated to $SK(L^2(\mathbb{R}))$ are called *locally compact*.

Remark that the set $SK(L^2(\mathbb{R}))$ of semi-compact operators is a $C^*$-subalgebra of $B(L^2(\mathbb{R}))$ and that $SK(L^2(\mathbb{R})) \subset SK(L^2(\mathbb{R}))$ where

$$HK(L^2(\mathbb{R})) = \left\{ T \in B(L^2(\mathbb{R})) \mid \lim_{r \to \infty} \left( \|\chi([P > r] T\| + \|T \chi([P > r)]\| \right) = 0 \right\}.$$  

**Definition 4.** We say that $T \in B(L^2(\mathbb{R}))$ has limit $T_{\pm}$ at $Q = \pm \infty$ (and set $\lim_{Q \to \pm \infty} T = T_{\pm}$) if $\lim_{x \to \infty} \|\|\theta_0^x(\varepsilon Q) - T_{\pm}\|\| = \|\theta_0^x(\varepsilon Q) - T_{\pm}\| = 0$.

**Proposition 5.** $\mathcal{E} = \{ T \in SK(L^2(\mathbb{R})) \mid \lim_{Q \to \pm \infty} T \text{ exist and belong to } \mathcal{E}_\pm \}$.

**Corollary 6.** The following assertions are equivalent:

(i) $H$ self-adjoint operator affiliated to $\mathcal{E}$.

(ii) $H$ is locally compact and $\exists z \in \mathcal{E} \setminus \mathbb{C}$ such that $\lim_{Q \to \pm \infty} (H - z)^{-1}$ exist and are in $\mathcal{E}_\pm$.

The next proposition gives a method for checking local compactness.

**Proposition 7.** Assume that $K$ is a Banach space continuously embedded in $H$ and such that $\theta(Q) \in K(\mathcal{H}, H)$ for each $\theta \in C^\infty(\mathbb{R})$. If $H$ is a self-adjoint operator in $\mathcal{H}$ and $(H + i)^{-1} \mathcal{H} \subset \mathcal{K}$, then $H$ is locally compact. In particular, if the domain of $H$ is included in the form domain of a locally compact operator, then $H$ is locally compact.

Our second important result states as follows:

**Theorem 8.** Let $H$ be a self-adjoint operator in $\mathcal{H} = L^2(\mathbb{R})$ and $H_\pm$ a pair of self-adjoint operators affiliated to $\mathcal{E}_\pm$ respectively such that $D(H_\pm) = D(H)$. Assume that

$$\|\theta_+(\varepsilon Q) (H - H_+)\|_{D(H) \to \mathcal{H}} \to 0 \quad \text{and} \quad \|\theta_- (\varepsilon Q) (H - H_-)\|_{D(H) \to \mathcal{H}} \to 0 \quad \text{if } \varepsilon \to 0.$$ 

Then $H$ is affiliated to $\mathcal{E}$. In particular $\sigma_{eq}(H) = \sigma(H_+) \cup \sigma(H_-)$.  

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Proof. – Since $H_{\pm}$ are affiliated to $C_{\pm} \subset SK(H)$, $H_{\pm}$ are locally compact. $H$ is locally compact because of $D(H_{\pm}) = D(H)$ and of Proposition 7. So, by Corollary 6, it suffices to prove that $\lim_{Q \to \pm \infty} (H - z)^{-1} = (H_{\pm} - z)^{-1}$. We denote $R(z) = (H - z)^{-1}$ and $R_{\pm}(z) = (H_{\pm} - z)^{-1}$. Then

$$\left\| \theta_{\pm}(\varepsilon Q)(R - R_{\pm}) \right\| = \left\| \theta_{\pm}(\varepsilon Q) R_{\pm} (H - H_{\pm}) R \right\| \leq \left\| \left[ \theta_{\pm}(\varepsilon Q), R_{\pm} \right] \right\| \cdot \left\| (H - H_{\pm}) R \right\| + \left\| R_{\pm} \right\| \cdot \left\| \theta_{\pm}(\varepsilon Q) (H - H_{\pm}) R \right\|.

But $\left\| \theta_{\pm}(\varepsilon Q) (H - H_{\pm}) R \right\| \to 0$ when $\varepsilon \to 0$ because of the hypothesis. On the other hand, $\left\| \left[ \theta_{\pm}(\varepsilon Q), R_{\pm} \right] \right\| \leq (\varepsilon/\sqrt{2\pi}) \cdot \left\| \theta_{\pm}' \right\|_{L^1(\mathbb{R})} \cdot \left\| R_{\pm}' \right\|_{L^\infty(\mathbb{R})}$, thus $\left\| \theta_{\pm}(\varepsilon Q) (R - R_{\pm}) \right\| \to 0$ when $\varepsilon \to 0$. 

We give now an example. Let $h : \mathbb{R} \to \mathbb{R}$ be a continuous function such that $c^{-1}|x|^s \leq |h(x)| \leq c|x|^s$ for some real $s > 0$, a constant $c > 0$ and large $x$. Set $H_0 = h(P)$, its domain being the Sobolev space $H^s(\mathbb{R})$. Let $V$ be a real function such that $V(Q) \in B(H^s, H)$ with $H_0$-bound less than 1 and assume that there are $a_{\pm}$-periodic functions $V_{\pm}$ such that $\left\| \theta_{\pm}(\varepsilon Q) (V - V_{\pm}) \right\|_{H^s \to H} \to 0$ if $\varepsilon \to 0$. Then $H = H_0 + V(Q)$ is affiliated to $C$ and $H_{\pm} = H_0 + V_{\pm}(Q)$, so $\sigma_{ess}(H) = \sigma(H_+) \cup \sigma(H_-)$.

Remark. – If $E$ is a Hilbert space, then one can replace above $C$ by $C \otimes K(L^2(E))$. Thus our results cover the case of one-dimensional Dirac operators for example (then $E$ is finite dimensional).

Acknowledgments. I express my gratitude to V. Georgescu, A. Iftimovici and S. Golénia for their helpful advice.

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