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Existence of a solution for an unsteady elasticity problem in large displacement and small perturbation

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Abstract In this Note we present a model for an unsteady pure traction problem in large displacement and small perturbation for an elastic body in dimension 2, and we show the existence of a solution to the associated problem. The weak formulation of this nonlinear problem involves test-functions depending on the solution, which is not standard. We then study the dynamic of the translation, of the rotation, and of the perturbation associated to the deformation of the body. We prove the existence of a weak solution using a Galerkin method. ***To cite this article: C. Grandmont et al., C. R. Acad. Sci. Paris, Ser. I 334 (2002) 521–526.*** © 2002 Académie des sciences/Éditions scientifiques et médicales Elsevier SAS

Existence d'une solution pour un modèle d'élasticité instationnaire en grands déplacements et petites perturbations

Résumé Nous présentons dans cette Note la modélisation et l'analyse d'un problème d'élasticité instationnaire en grands déplacements et petites perturbations pour un corps non-encastré en dimension 2. La formulation faible de ce problème non-linéaire utilise des fonctions-tests dépendant de la solution. Nous étudions alors la dynamique de la translation, de la rotation et de la perturbation associées à la déformation du corps élastique. Nous montrons l'existence d'une solution faible au problème par une méthode de Galerkin. ***Pour citer cet article : C. Grandmont et al., C. R. Acad. Sci. Paris, Ser. I 334 (2002) 521–526.*** © 2002 Académie des sciences/Éditions scientifiques et médicales Elsevier SAS

Version française abrégée

Nous présentons dans cette Note la modélisation et l'analyse d'un problème d'élasticité instationnaire en grands déplacements et petites perturbations pour un corps non-encastré en dimension 2. La configuration de référence de ce solide déformable est un ouvert borné de $\mathbb{R}^2$ à frontière lipschitzienne, noté Ω ; son centre de gravité est noté G. Nous recherchons la déformation ϕ du corps sous la forme $\phi = \tau + R_\theta(\overrightarrow{G\xi} + \mathbf{d}(\xi))$

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pour $\xi \in \Omega$, où τ est une translation de $\mathbb{R}^2$, $R = R_\theta = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$ est une rotation d'angle $\theta \in \mathbb{R}$, et $\mathbf{d}$ est une application de Ω dans $\mathbb{R}^2$ appelée perturbation. Nous justifions en premier lieu cette décomposition. Pour cela nous introduisons, pour $s \geqslant 0$, les ensembles présentés en (1), et $\mathbf{Z}_s = \mathbb{R}^2 \times \mathbb{R}/2\pi\mathbb{Z} \times \mathbf{Y}_s$. Dans ce qui suit, nous ne distinguerons pas les espaces de Sobolev $H^s(\Omega)$ à valeurs dans $\mathbb{R}$ de ceux à valeurs dans $\mathbb{R}^2$. Énonçons maintenant le résultat préliminaire suivant :

PROPOSITION 0.1. – (i) *L'application*

$$\mathcal{A}: \quad \mathbb{R}^2 \times \mathbb{R} \times \mathrm{L}^2(\Omega) \longrightarrow \mathrm{L}^2(\Omega), \qquad (\tau, \theta, \mathbf{d}) \longmapsto \phi = \tau + R_\theta\big(\overrightarrow{G\xi} + \mathbf{d}\big)$$

est une bijection de $\mathbf{Z}_s$ *sur* $\mathbf{X}_s$ *de classe* C^∞.

(ii) *De plus* $\mathcal{A}$ *est un* C^1*-difféomorphisme. Soient* $(\tau, \theta, \mathbf{d}) \in \mathbf{Z}_s$, $\phi = \mathcal{A}(\tau, \theta, \mathbf{d}) \in \mathbf{X}_s$ *et* $\mathbf{v} \in \mathrm{H}^s(\Omega)$, *alors nous avons* $[D(\mathcal{A}^{-1})](\phi) \cdot \mathbf{v} = (\overline{\tau}, \overline{\theta}, \overline{\mathbf{d}})^{\mathrm{T}}$ *avec les expressions de* $\overline{\tau}$, $\overline{\theta}$ *et* $\overline{\mathbf{d}}$ *données en* (2) *et* (3).

Soit maintenant un temps $T > 0$. Nous considérons la déformation instationnaire $\phi(\cdot, t)$ d'un corps hyperélastique non-encastré, dont la configuration de référence $\overline{\Omega}$ est un état naturel. D'après la Proposition 0.1, la déformation peut s'écrire de manière unique selon (4) pour presque tout $t \in (0, T)$. Nous considérons par ailleurs le déplacement $\mathbf{u} = \phi - \mathrm{id}_{\mathbb{R}^2}$. En partant d'un modèle élastique de Saint Venant–Kirchhoff valable dans le cas de grands déplacements et de petites déformations, et en remarquant que l'énergie mécanique associée ne dépend que de $\mathbf{d}$, nous obtenons, après linéarisation au premier ordre en $\nabla\mathbf{d}$, la formulation faible (6) du problème. Dans cette formulation, les fonctions-tests dépendent de la solution du problème, ce qui est non-standard.

Énonçons le résultat principal d'existence suivant :

THÉORÈME 0.2. – *Soient* $\mathbf{u}_0 \in \mathrm{H}^1(\Omega)$ *tel que* $\mathbf{u}_0 + \overrightarrow{O\xi} \in \mathbf{X}_1$, $\mathbf{u}_1 \in \mathrm{L}^2(\Omega)$, *et les forces* $\mathbf{f} \in \mathrm{L}^2(0, T; \mathrm{L}^2(\Omega))$ *et* $\mathbf{g} \in \mathrm{H}^1(0, T; \mathrm{H}^{-1/2}(\partial\Omega)) \cap \mathrm{L}^2(0, T; \mathrm{L}^1(\partial\Omega))$. *Si ces données sont assez petites ou si le temps* T *est assez petit, alors il existe une solution* $\mathbf{u} \in \mathrm{H}^2(0, T; (\mathcal{E}_1)') \cap \mathrm{W}^{1,\infty}(0, T; \mathrm{L}^2(\Omega)) \cap \mathrm{L}^\infty(0, T; \mathrm{H}^1(\Omega))$ *au système* (7), *telle que* $\phi(\cdot, t) = \mathbf{u}(\cdot, t) + \mathrm{id}_{\mathbb{R}^2} \in \mathbf{X}_1$ *pour presque tout* $t \in (0, T)$, *et avec la perturbation* $\mathbf{d}$ *définie dans la Proposition* 0.1 *à partir de* ϕ.

De plus, si $\mathbf{f} \in \mathrm{L}^2(0, \infty; \mathrm{L}^2(\Omega))$ *et* $\mathbf{g} \in \mathrm{H}^1(0, \infty; \mathrm{H}^{-1/2}(\partial\Omega)) \cap \mathrm{L}^2(0, \infty; \mathrm{L}^1(\partial\Omega))$, *alors la solution* $\mathbf{u}$ *existe sur* $[0, T^*[$ *avec* $T^* = \sup\{t > 0;\ \mathbf{d}(\cdot, t) \in \mathbf{Y}_0\}$.

La démonstration repose sur la décomposition (9) de $\mathrm{H}^1(\Omega)$ et la projection de l'équation de la formulation faible sur chacun des sous-espaces de cette décomposition. Nous sommes alors ramenés à étudier la dynamique de la translation τ, de la rotation R_θ et de la perturbation $\mathbf{d}$ associées à la déformation ϕ. Nous utilisons ensuite une méthode de Galerkin pour montrer l'existence d'une solution faible satisfaisant cette formulation équivalente de notre problème.

In this Note we present a model for an unsteady pure traction problem in large displacement and small perturbation for an elastic body in dimension 2, and we show existence of a weak solution to the associated problem.

1. Notations. Model

The reference configuration of this elastic medium is a bounded open subset Ω of $\mathbb{R}^2$ with a Lispchitz-continuous boundary; its center of mass is denoted by G. We are looking for a deformation ϕ of the domain Ω under the form $\phi(\xi) = \tau + R_\theta(\overrightarrow{G\xi} + \mathbf{d}(\xi))$ for $\xi \in \Omega$, where τ is a translation of $\mathbb{R}^2$, $R = R_\theta = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$ is a rotation of angle $\theta \in \mathbb{R}$, and $\mathbf{d}$ is a map from Ω onto $\mathbb{R}^2$, called perturbation. To justify this decomposition, we introduce some notations and then present a preliminary result. For all real $s \geqslant 0$, we define the following sets:

$$\begin{aligned}
\mathbf{X}_s &= \left\{\varphi \in \mathrm{H}^s(\Omega);\ \left(\int_\Omega \varphi(\xi)\cdot\overrightarrow{G\xi}\,\mathrm{d}\xi\right)^2 + \left(\int_\Omega \varphi(\xi)\wedge\overrightarrow{G\xi}\,\mathrm{d}\xi\right)^2 \neq 0\right\},\\
\mathcal{E}_s &= \left\{\mathbf{b} \in \mathrm{H}^s(\Omega);\ \int_\Omega \mathbf{b}(\xi)\,\mathrm{d}\xi = 0,\ \int_\Omega \mathbf{b}(\xi)\wedge\overrightarrow{G\xi}\,\mathrm{d}\xi = 0\right\}, \qquad (1)\\
\mathbf{Y}_s &= \left\{\mathbf{b} \in \mathcal{E}_s;\ \int_\Omega \big(\overrightarrow{G\xi} + \mathbf{b}(\xi)\big)\cdot\overrightarrow{G\xi}\,\mathrm{d}\xi > 0\right\},
\end{aligned}$$

and $\mathbf{Z}_s = \mathbb{R}^2 \times \mathbb{R}/2\pi\mathbb{Z} \times \mathbf{Y}_s$, where $\wedge$ denotes the exterior product in $\mathbb{R}^2$. We also imbed the plane $\mathbb{R}^2 = \langle\overrightarrow{e_x}, \overrightarrow{e_y}\rangle$ in $\mathbb{R}^3 = \langle\overrightarrow{e_x}, \overrightarrow{e_y}, \overrightarrow{e_z}\rangle$ in order to define the vectorial operator $\overrightarrow{e_z}\wedge\cdot$. We have the following proposition:

PROPOSITION 1.1. – (i) *The application*

$$\begin{aligned}
\mathcal{A}:\quad &\mathbb{R}^2 \times \mathbb{R} \times \mathrm{L}^2(\Omega) \longrightarrow \mathrm{L}^2(\Omega),\\
&(\tau, \theta, \mathbf{d}) \longmapsto \phi = \tau + R_\theta(\overrightarrow{G\xi} + \mathbf{d})
\end{aligned}$$

is a C^∞*-one-to-one map from* $\mathbf{Z}_s$ *on* $\mathbf{X}_s$.

(ii) *Moreover* $\mathcal{A}$ *is a* C^1*-diffeomorphism. Let* $(\tau, \theta, \mathbf{d}) \in \mathbf{Z}_s$, $\phi = \mathcal{A}(\tau, \theta, \mathbf{d}) \in \mathbf{X}_s$ *and* $\mathbf{v} \in \mathbf{X}_s$, *then we have* $[D(\mathcal{A}^{-1})](\phi)\cdot\mathbf{v} = (\overline{\tau}, \overline{\theta}, \overline{\mathbf{d}})^{\mathrm{T}}$ *with the following expressions*:

$$\overline{\tau}(\mathbf{v}) = \frac{1}{|\Omega|}\int_\Omega \mathbf{v}, \qquad \overline{\theta}(\mathbf{v}) = \frac{1}{\int_\Omega(\overrightarrow{G\xi} + \mathbf{d})\cdot\overrightarrow{G\xi}}\int_\Omega \mathbf{v}\cdot\big(\overrightarrow{e_z}\wedge R_\theta\overrightarrow{G\xi}\big) \quad \textit{and} \qquad (2)$$

$$\begin{aligned}
\overline{\mathbf{d}}(\mathbf{v}) &= R_\theta^{\mathrm{T}}\big[\mathbf{v} - \overline{\tau}(\mathbf{v}) - \overline{\theta}(\mathbf{v})\overrightarrow{e_z}\wedge R_\theta\big(\overrightarrow{G\xi} + \mathbf{d}\big)\big]\\
&= R_\theta^{\mathrm{T}}\left[\mathbf{v} - \frac{1}{|\Omega|}\int_\Omega \mathbf{v} - \frac{1}{\int_\Omega(\overrightarrow{G\xi} + \mathbf{d})\cdot\overrightarrow{G\xi}}\left(\int_\Omega \mathbf{v}\cdot\big(\overrightarrow{e_z}\wedge R_\theta\overrightarrow{G\xi}\big)\right)\overrightarrow{e_z}\wedge R_\theta\big(\overrightarrow{G\xi} + \mathbf{d}\big)\right]. \qquad (3)
\end{aligned}$$

Let $T > 0$. From now on, we consider for almost every $t \in [0, T]$ the unsteady deformation $\phi(\cdot, t)$ of a body whose reference configuration $\overline{\Omega}$ is a natural state. Following Proposition 1.1, for almost every $t \in (0, T)$, if the deformation $\phi(\cdot, t)$ belongs to $\mathbf{X}_s$, then there exists a unique triplet $(\tau, \theta, \mathbf{d}) \in \mathbf{Z}_s$ such that:

$$\phi(\xi, t) = \tau(t) + R_{\theta(t)}\big(\overrightarrow{G\xi} + \mathbf{d}(\xi, t)\big). \qquad (4)$$

Let $\mathbf{u} = \phi - \mathrm{id}_{\mathbb{R}^2}$ be the associated displacement. We now present the main steps of the modeling leading to the equations satisfied by $\mathbf{u}$. We set $E_K(\mathbf{u}, \dot{\mathbf{u}}) = \rho_S\int_\Omega(\dot{\mathbf{u}})^2/2$ the kinetic energy, where ρ_S is the density of the body. Considering a pure traction problem, we introduce $P(\mathbf{u}, \dot{\mathbf{u}}) = \int_\Omega \mathbf{f}\cdot\mathbf{u} + \int_{\partial\Omega}\mathbf{g}\cdot\mathbf{u}$ the work of the exterior body surface $\mathbf{f}$ and of the exterior force $\mathbf{g}$ applied on the boundary. Since we refer to a large displacement-small strain model, we first consider a Saint Venant–Kirchhoff hyperelastic material whose stored energy function is given by (*see* [2], Theorem 4.4-3):

$$\check{W}(\mathbf{E}) = \frac{\lambda}{2}\big[\mathrm{tr}(\mathbf{E})\big]^2 + \mu\,\mathrm{tr}\big[\mathbf{E}^2\big],$$

where $\mathbf{E}(\mathbf{u}) = (\nabla\mathbf{u} + \nabla\mathbf{u}^{\mathrm{T}} + \nabla\mathbf{u}^{\mathrm{T}}\nabla\mathbf{u})/2$ is the Green–Saint Venant strain tensor, and λ and μ are the Lamé constants. We then remark that the strain tensor $\mathbf{E}(\mathbf{u})$ only depends on the perturbation $\mathbf{d}$, i.e., $\mathbf{E}(\mathbf{u}) = \mathbf{E}(\mathbf{d})$. Next considering small perturbations, we linearize $\mathbf{E}(\mathbf{d})$ at the first order in $\nabla\mathbf{d}$ by $\overline{\varepsilon}(\mathbf{d}) = (\nabla\mathbf{d} + \nabla\mathbf{d}^{\mathrm{T}})/2$. We finally obtain the linearized mechanical instantaneous energy of the displacement $\mathbf{u}$:

$$W(\mathbf{u}, \dot{\mathbf{u}}) = \int_\Omega \widehat{\mathcal{W}}(\mathbf{d}), \quad \text{where } \widehat{\mathcal{W}}(\mathbf{d}) = \check{W}\big(\overline{\varepsilon}(\mathbf{d})\big). \qquad (5)$$

Introducing the Lagrangian $\mathcal{L}(\mathbf{u},\dot{\mathbf{u}}) = E_K + P - W$, we then deduce from the Lagrange equations (*see* [1] for example) the weak formulation of the equations modeling the displacement $\mathbf{u}$ (associated to the perturbation $\mathbf{d}$):

$$\begin{cases} \forall \mathbf{v} \text{ regular enough, with } \mathbf{b} \text{ uniquely defined by (3),} \\ \rho_S \int_\Omega (\partial_{tt}\mathbf{u})\cdot\mathbf{v} + \int_\Omega \overline{\sigma}(\mathbf{d}) : \overline{\varepsilon}(\mathbf{b}) = \int_\Omega \mathbf{f}\cdot\mathbf{v} + \int_{\partial\Omega} \mathbf{g}\cdot\mathbf{v} \quad \text{in } \mathcal{D}'(0,T), \end{cases} \tag{6}$$

where $\overline{\sigma}(\mathbf{d}) = \lambda\,\mathrm{tr}(\overline{\varepsilon}(\mathbf{d}))I_2 + 2\mu\overline{\varepsilon}(\mathbf{d})$, and with the initial conditions $\mathbf{u}(\cdot,t=0) = \mathbf{u}_0$ and $\partial_t\mathbf{u}(\cdot,t=0) = \mathbf{u}_1$.

2. Main result

THEOREM 2.1. – *Let* $\mathbf{u}_0 \in \mathrm{H}^1(\Omega)$ *such that* $\mathbf{u}_0 + \overrightarrow{O\xi} \in \mathbf{X}_1$, $\mathbf{u}_1 \in \mathrm{L}^2(\Omega)$, *and the forces* $\mathbf{f} \in \mathrm{L}^2(0,T;\mathrm{L}^2(\Omega))$ *and* $\mathbf{g} \in \mathrm{H}^1(0,T;\mathrm{H}^{-1/2}(\partial\Omega)) \cap \mathrm{L}^2(0,T;\mathrm{L}^1(\partial\Omega))$. *If those data are small enough or the time* T *is small enough, then there exists a solution* $\mathbf{u} \in \mathrm{H}^2(0,T;(\mathcal{E}_1)') \cap \mathrm{W}^{1,\infty}(0,T;\mathrm{L}^2(\Omega)) \cap \mathrm{L}^\infty(0,T;\mathrm{H}^1(\Omega))$ *to the following system*:

$$\begin{cases} \forall \mathbf{v} \in \mathrm{H}^1(\Omega), \quad \rho_S \dfrac{\mathrm{d}}{\mathrm{d}t}\left(\int_\Omega \partial_t\mathbf{u}\cdot\mathbf{v}\right) + \int_\Omega \overline{\sigma}(\mathbf{d}) : \overline{\varepsilon}(\mathbf{b}) = \int_\Omega \mathbf{f}\cdot\mathbf{v} + \int_{\partial\Omega}\mathbf{g}\cdot\mathbf{v} \quad \textit{in} \quad \mathcal{D}'(0,T), \\ \quad \textit{with } \mathbf{b} \textit{ associated to } \mathbf{v} \textit{ thanks to } (3), \\ \mathbf{u}(\cdot,t=0) = \mathbf{u}_0, \quad \partial_t\mathbf{u}(\cdot,t=0) = \mathbf{u}_1, \end{cases} \tag{7}$$

such that $\phi(\cdot,t) = \mathbf{u}(\cdot,t) + \mathrm{id}_{\mathbb{R}^2} \in \mathbf{X}_1$, *for almost every* $t \in (0,T)$, *and with* $\mathbf{d}$ *defined in Proposition* 1.1 *from* ϕ. *Furthermore, if* $\mathbf{f} \in \mathrm{L}^2(0,\infty;\mathrm{L}^2(\Omega))$ *and* $\mathbf{g} \in \mathrm{H}^1(0,\infty;\mathrm{H}^{-1/2}(\partial\Omega)) \cap \mathrm{L}^2(0,\infty;\mathrm{L}^1(\partial\Omega))$, *then the solution* $\mathbf{u}$ *exists on the interval* $[0,T^*[$ *with* $T^* = \sup\{t>0;\, \mathbf{d}(\cdot,t) \in \mathbf{Y}_0\}$.

Remark 1. – In this formulation, the test-function $\mathbf{b}$ depends implicitly on the solution $\mathbf{u}$.

3. Sketch of the proof

3.1. Model study: energy estimates and equivalent formulation

We obtain classical energy estimates by taking $\mathbf{v} = \partial_t\mathbf{u} = \dot{\tau} + \dot{\theta}\,\overrightarrow{e_z} \wedge R_\theta(\overrightarrow{G\xi} + \mathbf{d}) + R_\theta\partial_t\mathbf{d}$ in formulation (6), where we denoted with an overlaying dot the total derivative operator with respect to time. If $\mathbf{f} \in \mathrm{L}^2(0,T;\mathrm{L}^2(\Omega))$ and $\mathbf{g} \in \mathrm{H}^1(0,T;\mathrm{H}^{-1/2}(\partial\Omega))$, we get in a first step that $\partial_t\mathbf{u} \in \mathrm{L}^\infty(0,T;\mathrm{L}^2(\Omega))$ and $\mathbf{d} \in \mathrm{L}^\infty(0,T;\mathrm{H}^1(\Omega))$. We derive further regularities in a second step if we assume, for $\delta > 0$:

$$\text{for almost every } t \in (0,T), \quad \mathbf{d}(\cdot,t) \in \mathbf{Y}_1^\delta = \left\{\mathbf{d} \in \mathcal{E}_1;\ \int_\Omega \left(\overrightarrow{G\xi} + \mathbf{d}\right)\cdot\overrightarrow{G\xi} \geqslant \delta > 0\right\}. \tag{8}$$

By using an explicit expression of $\theta = (\mathcal{A}^{-1}(\mathbf{u} + \mathrm{id}_{\mathbb{R}^2}))_2$, we get $\theta \in \mathrm{L}^\infty(0,T)$, which allows us, thanks to (2), (3), to conclude that τ, θ and $\mathbf{d}$ are respectively bounded in $\mathrm{W}^{1,\infty}(0,T)$, $\mathrm{W}^{1,\infty}(0,T)$ and $\mathrm{W}^{1,\infty}(0,T;\mathrm{L}^2(\Omega)) \cap \mathrm{L}^\infty(0,T;\mathrm{H}^1(\Omega))$.

By taking special test-functions in (6), we obtain an equivalent system to (6) concerning the dynamic of the underlying translation τ, rotation R_θ and perturbation $\mathbf{d}$. Let us remind that, from the definition of $\mathcal{E}_1$ (*see* (1)), we also have:

$$\mathrm{H}^1(\Omega) = \left\langle \overrightarrow{e_x}, \overrightarrow{e_y} \right\rangle \oplus \left\langle \overrightarrow{e_z} \wedge R\left(\overrightarrow{G\xi} + \mathbf{d}\right) \right\rangle \oplus \langle R\mathbf{b};\, \mathbf{b} \in \mathcal{E}_1 \rangle \tag{9}$$

because of $\int_\Omega (\overrightarrow{G\xi} + \mathbf{d})\cdot\overrightarrow{G\xi} \neq 0$. Thus taking in (6) respectively $\mathbf{v} \in \langle \overrightarrow{e_x}, \overrightarrow{e_y} \rangle$, $\mathbf{v} \in \langle \overrightarrow{e_z} \wedge R(\overrightarrow{G\xi} + \mathbf{d}) \rangle$ and $\mathbf{v} = R\mathbf{b}$ for $\mathbf{b} \in \mathcal{E}_1$, we obtain, at least formally:

$$m\ddot{\tau} = \int_\Omega \mathbf{f} + \int_{\partial\Omega} \mathbf{g}, \tag{10}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}(\dot{\theta} J) + \rho_S \int_\Omega \mathbf{d} \wedge \partial_{tt}\mathbf{d} = \int_\Omega R\big(\overrightarrow{G\xi} + \mathbf{d}\big) \wedge \mathbf{f} + \int_{\partial\Omega} R\big(\overrightarrow{G\xi} + \mathbf{d}\big) \wedge \mathbf{g}, \tag{11}$$

$$\forall \mathbf{b} \in \mathcal{E}_1, \quad \rho_S \left(\int_\Omega \partial_{tt}\mathbf{d} \cdot \mathbf{b} + 2\dot{\theta} \int_\Omega \partial_t \mathbf{d} \wedge \mathbf{b} + \ddot{\theta} \int_\Omega \mathbf{d} \wedge \mathbf{b} - (\dot{\theta})^2 \int_\Omega \big(\overrightarrow{G\xi} + \mathbf{d}\big) \cdot \mathbf{b} \right)$$
$$+ \int_\Omega \overline{\sigma}(\mathbf{d}) : \overline{\varepsilon}(\mathbf{b}) = \int_\Omega \mathbf{f} \cdot R\mathbf{b} + \int_{\partial\Omega} \mathbf{g} \cdot R\mathbf{b}, \tag{12}$$

where $J(t) = \rho_S \int_\Omega (\overrightarrow{G\xi} + \mathbf{d})^2$ is the inertia momentum of the body at time t. We also have initial conditions stemming from $\mathbf{u}_0 = \tau_0 + R_{\theta_0}(\overrightarrow{G\xi} + \mathbf{d}_0) - \overrightarrow{O\xi}$ and $\mathbf{u}_1 = \tau_1 + \theta_1 \overrightarrow{e_z} \wedge R_{\theta_0}(\overrightarrow{G\xi} + \mathbf{d}_0) + R_{\theta_0}\mathbf{d}_1$:

$$\begin{cases} \tau(0) = \tau_0, \quad \theta(0) = \theta_0, \quad \mathbf{d}(\cdot, t=0) = \mathbf{d}_0, \quad \text{with} \displaystyle\int_\Omega \big(\overrightarrow{G\xi} + \mathbf{d}_0\big) \cdot \overrightarrow{G\xi} > 0, \\ \dot{\tau}(0) = \tau_1, \quad \dot{\theta}(0) = \theta_1, \quad \partial_t \mathbf{d}(\cdot, t=0) = \mathbf{d}_1. \end{cases}$$

3.2. Choice of the test-functions

To study the existence of solutions for the previous problem, we use a Galerkin method. Let us first choose a Galerkin basis of the space $\mathcal{E}_1$. We remark that $\mathcal{E}_1 = \ker(\overline{\Sigma})^\perp$, where $\overline{\Sigma}$ stands for the operator of the linearized elasticity ($\overline{\Sigma} = -\operatorname{div}(\overline{\sigma})$) with homogeneous Neumann boundary condition. This operator is self-adjoint from $\mathrm{L}^2(\Omega)$ into $\mathrm{L}^2(\Omega)$, and its inverse is a bounded compact operator from $\mathcal{E}_0$ into $\mathcal{E}_0$. Denoting by ψ_i, $i \geqslant 1$, the eigenfunctions of $\overline{\Sigma}$ with positive eigenvalues, we have $\mathcal{E}_1 = \langle \psi_i \rangle_{i \geqslant 1}$. Thus we choose $\{\psi_i\}_{i \geqslant 1}$ as a Galerkin basis.

3.3. Finite dimensional approximation

We approach $\mathbf{d}$ by a combination of a finite number $N \in \mathbb{N}^*$ of eigenmodes, $\mathbf{d}_N = \sum_{i=1}^N \alpha_i(t)\psi_i$, with coefficients α_i functions of time. Substituting this expression in (11) and in (12) where we take successively $\mathbf{b} = \psi_i$ for $1 \leqslant i \leqslant N$, we obtain a finite dimensional system associated to the continuous problem (10)–(12). We have also to add the initial conditions. Denoting by $(\tau_N, \theta_N, \mathbf{d}_N)$ the solution of the discrete system, the initial conditions are $\tau_N(0) = \tau_0$, $\dot{\tau}_N(0) = \tau_1$, $\theta_N(0) = \theta_0$, $\dot{\theta}_N(0) = \theta_1$, and $\mathbf{d}_N(\cdot, t=0) = \mathbf{d}_N^0$, $\partial_t \mathbf{d}_N(\cdot, t=0) = \mathbf{d}_N^1$, where $\mathbf{d}_N^0$ and $\mathbf{d}_N^1$ are the $\mathrm{L}^2(\Omega)$-projections of $\mathbf{d}_0$ and $\mathbf{d}_1$ onto the space $\langle \psi_i \rangle_{1 \leqslant i \leqslant N}$; we have in particular $\mathbf{d}_N^0 \to \mathbf{d}_0$ in $\mathrm{H}^1(\Omega)$ and $\mathbf{d}_N^1 \to \mathbf{d}_1$ in $\mathrm{L}^2(\Omega)$ as $N \to \infty$.

The equation of the translation has a unique solution $\tau_N = \tau$ in $\mathrm{H}^2(0,T)$ as soon as $\mathbf{f} \in \mathrm{L}^2(0,T;\mathrm{L}^1(\Omega))$ and $\mathbf{g} \in \mathrm{L}^2(0,T;\mathrm{L}^1(\partial\Omega))$, and is independent of N.

The discrete system can anyway be rewritten under the form $A(t,X)\ddot{X} = h(t,X,\dot{X})$, which is a nonlinear second order ODE, with $A \in \mathcal{M}_{N+1}(\mathbb{R})$ a positive symmetric matrix. The matrix A is definite as long as $\int_\Omega (\overrightarrow{G\xi} + \mathbf{d}_N) \cdot \overrightarrow{G\xi} \neq 0$. Under this hypothesis, there exist a time $0 < T_N \leqslant T$ and a unique solution $(\tau, \theta, \alpha_1, \ldots, \alpha_N) \in [\mathrm{H}^2(0,T_N)]^{N+3}$ to the discrete system. By using now the second step of the *a priori* estimates study, we obtain:

$$\|\mathbf{d}_N\|_{\mathrm{L}^\infty(0,T_N;\mathrm{H}^1(\Omega))} \leqslant C_1^*, \tag{13}$$

with $C_1^* = C(\mathbf{f}, \mathbf{g}, \partial_t\mathbf{g}, \mathbf{d}_0, \mathbf{d}_1, \theta_0, \theta_1, \lambda, \mu, \rho_S)$ independent of N, and:

$$\big\|(\theta_N, \mathbf{d}_N)\big\|_{\mathrm{W}^{1,\infty}(0,T_N) \times (\mathrm{W}^{1,\infty}(0,T_N;\mathrm{L}^2(\Omega)) \cap \mathrm{L}^\infty(0,T_N;\mathrm{H}^1(\Omega)))} \leqslant C_2^*, \tag{14}$$

where $C_2^* = C(\mathbf{f}, \mathbf{g}, \partial_t\mathbf{g}, \mathbf{d}_0, \mathbf{d}_1, \theta_0, \theta_1, \lambda, \mu, \rho_S, \delta)$ is independent of N. So, for a fixed $\delta > 0$, if the data are small enough, then C_1^* is small enough. In that case or if the time T is small enough, we have $\mathbf{d}_N(t) \in \mathbf{Y}_1^\delta$ for almost every $t \leqslant T$, where T is independent of N. It follows that $T_N = T$.

3.4. Existence of a solution

In order to pass to the limit when N goes to infinity, we need extra energy estimates. By using the previous inequalities and the eigenfunctions properties, we may obtain by standard arguments (*see* [3], p. 74) that $\partial_{tt}\mathbf{d}_N$ and $\ddot{\theta}_N$ are respectively bounded in $\mathrm{L}^2(0,T;(\mathcal{E}_1)')$ and $\mathrm{L}^2(0,T)$ independently of N. Thanks to this result and to (13), (14), we pass to the limit in the equations of the discrete system using in particular the compact imbedding:

$$\mathrm{W}^{1,\infty}\big(0,T;\mathrm{L}^2(\Omega)\big)\cap\mathrm{L}^\infty\big(0,T;\mathrm{H}^1(\Omega)\big)\subset\mathrm{H}^1\big((0,T)\times\Omega\big)\overset{c}{\hookrightarrow}\mathrm{L}^2\big((0,T)\times\Omega\big).$$

So we obtain a solution provided that the data or the time T are small enough. We can next extend it on the interval $[0,T^*[$ with $T^*=\sup\{t>0;\ \mathbf{d}(\cdot,t)\in\mathbf{Y}_0\}$.

We then have the theorem:

THEOREM 3.1. – *Let* $\tau_0,\tau_1\in\mathbb{R}^2$, $\theta_0,\theta_1\in\mathbb{R}$, $\mathbf{d}_0\in\mathbf{Y}_1$, $\mathbf{d}_1\in\mathrm{L}^2(\Omega)$, $\mathbf{f}\in\mathrm{L}^2(0,T;\mathrm{L}^2(\Omega))$ *and* $\mathbf{g}\in\mathrm{H}^1(0,T;\mathrm{H}^{-1/2}(\partial\Omega))\cap\mathrm{L}^2(0,T;\mathrm{L}^1(\partial\Omega))$. *If the time T is small enough or those data are small enough, then there exists a solution* $(\tau,\theta,\mathbf{d})$ *to the following problem*:

$$\left\{\begin{array}{l}
\textit{Find } \tau\in\mathrm{H}^2(0,T),\ \theta\in\mathrm{H}^2(0,T),\\
\quad \textit{and } \mathbf{d}\in\mathrm{H}^2\big(0,T;(\mathcal{E}_1)'\big)\cap\mathrm{W}^{1,\infty}\big(0,T;\mathrm{L}^2(\Omega)\big)\cap\mathrm{L}^\infty\big(0,T;\mathrm{H}^1(\Omega)\big)\\
\quad \textit{such that for almost } t\in(0,T),\ \mathbf{d}(\cdot,t)\in\mathbf{Y}_1, \textit{satisfying}:\\
m\ddot{\tau}=\displaystyle\int_\Omega\mathbf{f}+\int_{\partial\Omega}\mathbf{g}\quad \textit{in } (0,T),\\
\dfrac{\mathrm{d}}{\mathrm{d}t}\left(\dot{\theta}J+\rho_S\displaystyle\int_\Omega\mathbf{d}\wedge\partial_t\mathbf{d}\right)=\displaystyle\int_\Omega R_\theta\big(\overrightarrow{G\xi}+\mathbf{d}\big)\wedge\mathbf{f}+\int_{\partial\Omega}R_\theta\big(\overrightarrow{G\xi}+\mathbf{d}\big)\wedge\mathbf{g}\quad \textit{in } \mathcal{D}'(0,T),\\
\rho_S\left(\displaystyle\int_\Omega\partial_{tt}\mathbf{d}\cdot\mathbf{b}+2\dot{\theta}\int_\Omega\partial_t\mathbf{d}\wedge\mathbf{b}+\ddot{\theta}\int_\Omega\mathbf{d}\wedge\mathbf{b}-(\dot{\theta})^2\int_\Omega\big(\overrightarrow{G\xi}+\mathbf{d}\big)\cdot\mathbf{b}\right)\\
\quad +\displaystyle\int_\Omega\overline{\sigma}(\mathbf{d}):\overline{\varepsilon}(\mathbf{b})=\int_\Omega\mathbf{f}\cdot R_\theta\mathbf{b}+\int_{\partial\Omega}\mathbf{g}\cdot R_\theta\mathbf{b}\quad \textit{in } \mathrm{L}^1(0,T)\ \textit{weak},\ \forall\mathbf{b}\in\mathcal{E}_1,\\
\textit{with } \tau(0)=\tau_0,\ \dot{\tau}(0)=\tau_1,\ \theta(0)=\theta_0,\ \dot{\theta}(0)=\theta_1,\ \mathbf{d}(\cdot,0)=\mathbf{d}_0 \textit{ and } \partial_t\mathbf{d}(\cdot,0)=\mathbf{d}_1.
\end{array}\right.\tag{15}$$

Furthermore, if $\mathbf{f}\in\mathrm{L}^2(0,\infty;\mathrm{L}^2(\Omega))$ *and* $\mathbf{g}\in\mathrm{H}^1(0,\infty;\mathrm{H}^{-1/2}(\partial\Omega))\cap\mathrm{L}^2(0,\infty;\mathrm{L}^1(\partial\Omega))$, *then the solution* $(\tau,\theta,\mathbf{d})$ *exists on the interval* $[0,T^*[$ *with* $T^*=\sup\{t>0;\mathbf{d}(\cdot,t)\in\mathbf{Y}_0\}$.

To such a triplet $(\tau,\theta,\mathbf{d})$ obtained in Theorem 3.1, we may associate a unique deformation ϕ such that for almost every $t\in(0,T)$, $\phi(\cdot,t)=\tau(t)+R_{\theta(t)}(\overrightarrow{G\xi}+\mathbf{d}(\cdot,t))\in\mathbf{X}_1$, and then the displacement $\mathbf{u}=\phi-\mathrm{id}_{\mathbb{R}^2}$. By algebraic calculations, we can return to a formulation with the displacement. We have then the result presented in Theorem 2.1.

Remark 2. – In a forthcoming work, we shall deal with the uniqueness of the solution, and the extension of the model and the results to dimension 3.

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