Normal solvability of linear elliptic problems

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Abstract

The paper is devoted to general linear elliptic problems in Hölder spaces. We consider unbounded domains and define limiting problems at infinity. We give a necessary and sufficient condition of normal solvability through uniqueness of solutions of limiting problems. We study a structure of spaces dual to Hölder spaces and specify the subspace of functionals, which provide the condition of normal solvability. This allows us to prove that for Fredholm operators all limiting operators are invertible. To cite this article: V. Volpert, A. Volpert, C. R. Acad. Sci. Paris, Ser. I 334 (2002) 457–462. © 2002 Académie des sciences/Éditions scientifiques et médicales Elsevier SAS

Résolubilité normale des problèmes elliptiques linéaires

Résumé


Version française abrégée

Nous considérons le problème

\[ A_i u = \sum_{k=1}^{p} \sum_{|\beta|_{j} \leq \beta_{ik}} a_{j}^{\beta} (x) D^{\beta} u_k \quad (i = 1, \ldots, p), \quad x \in \Omega \subset \mathbb{R}^n, \]

\[ B_i u = \sum_{k=1}^{p} \sum_{|\beta|_{j} \leq \gamma_{ik}} b_{j}^{\beta} (x) D^{\beta} u_k \quad (i = 1, \ldots, r), \quad x \in \partial \Omega. \]

Nous supposons qu’il est uniformément elliptique dans le sens de [1]. Le domaine \( \Omega \subset \mathbb{R}^n \) est non borné et les coefficients des opérateurs \( A_i \) et \( B_i \), ainsi que la frontière \( \partial \Omega \) du domaine satisfont les conditions de
regularité :

\[ a_{ik}^\beta \in C^{1-s_i+\alpha}(\Omega), \quad b_{ik}^\beta \in C^{1-\sigma_i+\alpha}(\partial\Omega), \quad a\Omega \in C^{1+\lambda+\delta}, \]

où les entiers \( s_i, \sigma_i \) sont les mêmes que dans la condition d’ellipticité, \( C^{m+\alpha}(\Omega) \) est un espace de Hôlder. Notons \( A = (A_1, \ldots, A_p), B = (B_1, \ldots, B_r), L = (A, B), L : E_0 \to E \).

On définit les problèmes limites correspondants. Pour cela, il faut définir les domaines limites et les opérateurs limites. Pour définir les domaines limites on considère une suite \( x_n \in \mathbb{M} \) telle que \( |x_n| \to \infty \). Soit \( \chi(x) \) la fonction caractéristique de \( \Omega \). Notons \( \Omega_\ast \) le domaine avec la fonction caractéristique \( \chi(x + x_n) \). Nous disons que \( \Omega_\ast \) est un domaine limite si \( \Omega_n \) converge vers \( \Omega_\ast \) localement dans la métrique de Hausdorff : \( \Omega_n \to \Omega_\ast \) dans \( \mathcal{E}_{loc} \).

**Théorème 1.** – Si le domaine \( \Omega \) satisfait la Condition D (voir ci-dessous), alors il existe une fonction \( f(x) \) telle que :

1. \( f(x) \in C^{k+\alpha}(\mathbb{R}^n) \);
2. \( f(x) > 0 \) ssi \( x \in \Omega \);
3. \( |\nabla f(x)| \geq 1 \) pour \( x \in \partial\Omega \);
4. \( \min(d(x), 1) \leq |f(x)|, \) où \( d(x) \) est la distance de \( x \) à \( \partial\Omega \).

**Théorème 2.** – Soit \( \Omega \) un domaine non borné qui satisfait la Condition D, \( x_m \in \mathbb{M}, |x_m| \to \infty, \) et \( f(x) \) la fonction construite dans le Théorème 1. Alors il existe une sous-suite \( x_{m_i} \), et la fonction \( f_*(x) \) telle que \( f_m(x) \equiv f(x + x_m) \to f_*(x) \) dans \( C^0_{loc}(\mathbb{R}^n) \), et le domaine \( \Omega_\ast = \{ x : f_*(x) > 0 \} \) satisfait la Condition D, où \( \Omega_\ast = \mathbb{R}^n \).

En plus, \( \Omega_{m_i} \to \Omega_\ast \) dans \( \mathcal{E}_{loc} \), où \( \Omega_{m_i} = \{ x : f_{m_i}(x) > 0 \} \).

Pour définir les opérateurs limites, on considère les suites des coefficients \( a_{ik}^\beta(x + x_n), b_{ik}^\beta(x + x_n) \) et choisit les sous-suites convergentes vers certaines fonctions limites uniformement dans chaque ensemble borné. Ces fonctions limites sont les coefficients des opérateurs limites \( \hat{L} \). On a donc défini les problèmes limites.

**Condition NS.** – Pour chaque domaine limite \( \Omega_\ast \) et chaque opérateur limite \( \hat{L} \) le problème limite \( \hat{L} = 0 \) a une seule solution \( u = 0 \).

**Théorème 3.** – Si la Condition NS est satisfaite, alors l’opérateur \( L \) est résoluble normalement et son noyau a une dimension finie.

Le résultat inverse a lieu sous les conditions un peu plus fortes sur les coefficients et le domaine :

\[ a_{ik}^\beta \in C^{1-s_i+\delta}(\Omega), \quad b_{ik}^\beta \in C^{1-\sigma_i+\delta}(\partial\Omega), \quad \partial\Omega \in C^{1+\lambda+\delta}, \]

avec \( \alpha < \delta < 1 \).

**Théorème 4.** – Si l’opérateur \( L \) est normalement résoluble et son noyau a une dimension finie, alors la Condition NS est satisfaite.

Notons \( E^0 \) le sous-espace de \( E \) qui contient les fonctions convergentes vers 0 à l’infini. Soit \( E^* \) l’espace dual à \( E, \) \( \tilde{E} \) le sous-espace de fonctionnelles qui s’annulent sur \( E^0 \) et \( \tilde{E} \) le sous-espace complémentaire dans \( E^* \).

**Théorème 5.** – Soit l’opérateur \( L : E_0 \to E \) normalement résoluble et son noyau a une dimension finie. Supposons que le problème \( Lu = f \) et résoluble si et seulement si \( \psi_i(f) = 0 \), \( i = 1, \ldots, N \), où \( \psi_i \in E^* \) sont des fonctionnelles linéairement indépendantes. Alors, \( \psi_i \in \tilde{E} \).

Ce résultat nous permet de démontrer le théorème suivant.
1. Operators and spaces

Let $\beta = (\beta_1, \ldots, \beta_n)$ be a multi-index, $\beta_i$ be nonnegative integers, $|\beta| = \beta_1 + \cdots + \beta_n$, $D^\beta = D_1^{\beta_1} \cdots D_n^{\beta_n}$, $D_i = \partial \partial x_i$. We consider the following operators:

$$A_iu = \sum_{k=1}^p \sum_{|\beta| \leq \beta_k} a^\beta_{ij}(x) D^\beta u_k \quad (i = 1, \ldots, p), \quad x \in \Omega \subset \mathbb{R}^n,$$

$$B_iu = \sum_{k=1}^p \sum_{|\beta| \leq \beta_k} b^\beta_{ij}(x) D^\beta u_k \quad (i = 1, \ldots, r), \quad x \in \partial \Omega.$$

Here $\beta_{ik}$, $\gamma_{ik}$ are given numbers. As in [1] we suppose that there exist integers $s_1, \ldots, s_p, t_1, \ldots, t_p, \sigma_1, \ldots, \sigma_r$ such that

$$\beta_{ij} \leq s_i + t_j, \quad i, j = 1, \ldots, p; \quad \gamma_{ij} \leq \sigma_i + t_j, \quad i = 1, \ldots, r, \quad j = 1, \ldots, p; \quad s_i \leq 0.$$

We suppose that the problem is elliptic in the sense of [2,1], i.e., the system is elliptic and supplementary and complementing (Shapiro–Lopatinskit) conditions are satisfied. Moreover we assume that the system is uniformly elliptic.

We consider the space $E_0 = C^{l_1+\alpha} + \cdots + C^{l_p+\alpha} (\Omega)$ of vector valued functions $u(x) = (u_1(x), \ldots, u_p(x))$, $u_j \in C^{l_j+\alpha} (\Omega)$, $j = 1, \ldots, p$, where $l$ and $\alpha$ are given numbers, $l = \max(0, \sigma_i)$, $0 < \alpha < 1$. The space $C^{l_j+\alpha} (\Omega)$ is a Hölder space of functions bounded together with their derivatives up to the order $l + t_j$, and the latter satisfy the Hölder condition uniformly in $x$.

The domain $\Omega$ is supposed to be of class $C^{l+\alpha}$, where $l = \max(-s_i - \sigma_i, t_j)$, $a^\beta_{ij} \in C^{l-s_i + \alpha} (\Omega)$, $b^\beta_{ij} \in C^{l-\sigma_i + \alpha} (\partial \Omega)$. We assume that $\lambda \geq 1$.

Operator $A_i$ acts from $E_0$ into $C^{l-s_i + \alpha} (\Omega)$. Denote $L_1 = (A_1, \ldots, A_p)$. Then $L_1$ acts from $E_0$ into $E_1 = C^{l-s_1 + \alpha} (\Omega) \times \cdots \times C^{l-s_p + \alpha} (\Omega)$. Operator $B_i$ acts from $E_0$ into $C^{l-\sigma_i + \alpha} (\partial \Omega)$. Denote $L_2 = (B_1, \ldots, B_r)$. Then $L_2$ acts from $E_0$ into $E_2 = C^{l-\sigma_1 + \alpha} (\partial \Omega) \times \cdots \times C^{l-s_r + \alpha} (\partial \Omega)$. Let $E = E_1 \times E_2$. Then $L = (L_1, L_2)$ acts from $E_0$ into $E$.

2. Limiting domains

In this section we define limiting domains for unbounded domains in $\mathbb{R}^n$, show their existence and study some of their properties. We consider an unbounded domain $\Omega \subset \mathbb{R}^n$, which satisfies the following condition:

**Condition D.** – For each $x_0 \in \bar{\Omega}$ there exists a neighbourhood $U(x_0)$ such that

1. $U(x_0)$ contains a sphere with the radius $\delta$ and the center $x_0$, where $\delta$ is independent of $x_0$;
2. There exists a homeomorphism $\psi(x; x_0)$ of the neighbourhood $U(x_0)$ on the unit sphere $B = [y : |y| < 1]$ in $\mathbb{R}^n$ such that the images of $\Omega \cap U(x_0)$ and $\bar{\Omega} \cap U(x_0)$ coincide with $B_+ = [y : y_n > 0, |y| < 1]$ and $B_0 = [y : y_n = 0, |y| < 1]$ respectively;
(3) The function $\psi(x; x_0)$ and its inverse belong to the Hölder space $C^{k+\alpha}$ with $k = l + \lambda$. Their $\| \cdot \|_{k+\alpha}$-norms are bounded uniformly in $x_0$.

To define convergence of domains we use the following Hausdorff metric space. Let $A$ and $B$ denote two nonempty closed sets in $\mathbb{R}^n$. Denote $\zeta(A, B) = \sup_{a \in A} \rho(a, B)$, $\zeta(B, A) = \sup_{b \in B} \rho(b, A)$, where $\rho(a, B)$ denotes the distance from a point $a$ to a set $B$, and let

$$
\zeta(A, B) = \max \{ \zeta(A, B), \zeta(B, A) \}.
$$

We denote $\Xi$ a metric space of bounded nonempty sets in $\mathbb{R}^n$ with the distance given by (1). We say that a sequence of domains $\Omega_m$ converges to a domain $\Omega$ in $\Xi_{\text{loc}}$ if

$$
\zeta(\overline{\Omega}_m \cap \overline{B}_R, \overline{\Omega} \cap \overline{B}_R) \to 0, \quad m \to \infty,
$$

for any $R > 0$ and $B_R = \{ x : |x| < R \}$. Here bar denotes the closure of domains.

**Definition.** – Let $\Omega \subset \mathbb{R}^n$ be an unbounded domain, $x_m \in \overline{\Omega}$, $|x_m| \to \infty$ as $m \to \infty$; $\chi(x)$ be a characteristic function of $\Omega$, and $\Omega_m$ be a shifted domain defined by the characteristic function $\chi_m(x) = \chi(x + x_m)$.

We say that $\Omega_s$ is a limiting domain of the domain $\Omega$ if $\Omega_m \to \Omega_s$ in $\Xi_{\text{loc}}$ as $m \to \infty$.

**Theorem 2.1.** – If a domain $\Omega$ satisfies Condition $D$, then there exists a function $f(x)$ defined in $\mathbb{R}^n$ such that:

1. $f(x) \in C^{k+\alpha}(\mathbb{R}^n)$;
2. $f(x) > 0$ iff $x \in \Omega$;
3. $|\nabla f(x)| \geq 1$ for $x \in \partial \Omega$;
4. $\min(d(x), 1) \leq |f(x)|$, where $d(x)$ is the distance from $x$ to $\partial \Omega$.

**Theorem 2.2.** – Let $\Omega$ be an unbounded domain satisfying Condition $D$, $x_m \in \overline{\Omega}$, $|x_m| \to \infty$, and $f(x)$ be the function constructed in Theorem 2.1. Then there exists a subsequence $x_m$ and a function $f_\infty(x)$ such that $f_{m_\infty}(x) \equiv f(x + x_m) \to f_\infty(x)$ in $C^k_{\text{loc}}(\mathbb{R}^n)$, and the domain $\Omega_\infty = \{ x : f_\infty(x) > 0 \}$ satisfies Condition $D$, or $\Omega_\infty = \mathbb{R}^n$.

Moreover $\Omega_m \to \Omega_\infty$ in $\Xi_{\text{loc}}$, where $\Omega_m = \{ x : f_m(x) > 0 \}$.

3. Limiting problems

In the previous section we have introduced limiting domains. Here we will define the corresponding limiting problems. Let $\Omega$ be a domain satisfying Condition $D$ and $\chi(x)$ be its characteristic function. Consider a sequence $x_m \in \overline{\Omega}$, $|x_m| \to \infty$ and the shifted domains $\Omega_m$ defined by the shifted characteristic functions $\chi_m(x) = \chi(x + x_m)$. We suppose that the sequence of domains $\Omega_m$ converge in $\Xi_{\text{loc}}$ to some limiting domain $\Omega_s$.

**Definition 1.** – Let $u_m(x) \in C^k(\Omega_m)$, $m = 1, 2, \ldots$. We say that $u_m(x)$ converges to a limiting function $u_\infty(x) \in C^k(\Omega_\infty)$ in $C^k_{\text{loc}}(\Omega_m \to \Omega_\infty)$ if there exists an extension $v_m(x) \in C^k(\mathbb{R}^n)$ of $u_m(x)$, $m = 1, 2, \ldots$, and an extension $v_\infty(x) \in C^k(\mathbb{R}^n)$ of $u_\infty(x)$ such that

$$
v_m \to v_\infty \quad \text{in} \quad C^k_{\text{loc}}(\mathbb{R}^n).
$$

**Definition 2.** – Let $u_m(x) \in C^k(\partial \Omega_m)$, $m = 1, 2, \ldots$. We say that $u_m(x)$ converges to a limiting function $u_\infty(x) \in C^k(\partial \Omega_\infty)$ in $C^k_{\text{loc}}(\partial \Omega_m \to \partial \Omega_\infty)$ if there exists an extension $v_m(x) \in C^k(\mathbb{R}^n)$ of $u_m(x)$, $m = 1, 2, \ldots$, and an extension $v_\infty(x) \in C^k(\mathbb{R}^n)$ of $u_\infty(x)$ such that

$$
v_m \to v_\infty \quad \text{in} \quad C^k_{\text{loc}}(\mathbb{R}^n).
$$

**Remark.** – In these definitions $u_\infty(x)$ does not depend on the choice of the extensions $v_m(x)$ and $v_\infty(x)$. Indeed, in Definition 1 for any point $x \in \overline{\Omega_\infty}$ there exists a sequence $\hat{x}_m \in \overline{\Omega_m}$ such that $\hat{x}_m \to x$. Therefore

$$
u_\infty(x) = v_\infty(x) = \lim_{m \to \infty} v_m(\hat{x}_m) = \lim_{m \to \infty} u_m(\hat{x}_m).
$$
Similarly it can be checked for Definition 2.

**Theorem 3.1.** Let $u_m \in C^{k+\sigma}(\Omega_m)$, $\|u_m\|_{C^{k+\sigma}} \leq M$, where the constant $M$ is independent of $m$, $0 \leq k \leq l + \lambda$. Then there exists a function $u_s \in C^{k+\sigma}(\Omega_s)$, $\|u_s\|_{C^{k+\sigma}} \leq M$ and a subsequence $u_{m_j}$ such that $u_{m_j} \to u_s$ in $C^k(\Omega_m) \to \Omega_s$.

Let $u_m \in C^{k+\alpha}(\partial \Omega_m)$, $\|u_m\|_{C^{k+\alpha}} \leq M$. Then there exists a function $u_s \in C^{k+\alpha}(\partial \Omega_s)$, $\|u_s\|_{C^{k+\alpha}} \leq M$ and a subsequence $u_{m_j}$ such that $u_{m_j} \to u_s$ in $C^k(\partial \Omega_m) \to \partial \Omega_s$.

We recall that the operator $L$ consists of a pair of operators, $L = (L_1, L_2)$ where the operator $L_1$ acts inside the domain and $L_2$ is a boundary operator. So we can represent the boundary problem as

$$L_1 u = f_1, \quad L_2 u = f_2,$$

where $u \in E_0(\Omega)$, $f_1 \in E_1(\Omega)$, $f_2 \in E_2(\partial \Omega)$, $E = E_1 \times E_2$. The coefficients $a_{ij}(x)$ of the operator $L_1$ are defined in $\Omega$ and the coefficients $b_{ij}(x)$ of $L_2$ in $\partial \Omega$. The shifted coefficients $a_{ij}(x + x_m)$ and $b_{ij}(x + x_m)$ satisfy conditions of Theorem 3.1. Therefore we can define the limiting problem

$$\hat{L}_1 u = f, \quad \hat{L}_2 u = g,$$

where $u \in E_0(\Omega_m)$, $f \in E_1(\Omega_m)$, $g \in E_2(\partial \Omega_m)$, $\hat{L}_1$ and $\hat{L}_2$ are operators with limiting coefficients $a^\beta_{ij}(x) \in C^{\alpha + \lambda, \delta}(\Omega_m)$, $b^\beta_{ij}(x) \in C^{\beta + \alpha, \delta}(\partial \Omega_m)$. The operator $\hat{L} = (\hat{L}_1, \hat{L}_2) : E_0(\Omega_m) \to E(\Omega_m)$ will be called limiting operator.

We note that for a given problem (2) there can exist a set of limiting problems corresponding to different sequences $x_m$ and to different converging subsequences of coefficients of the operators.

**4. Normal solvability**

We introduce limiting domains and limiting operators defined above.

**Condition NS.** For any limiting domain $\Omega$ and any limiting operator $\hat{L}$ the problem $\hat{L}u = 0$, $u \in E_0(\Omega)$ has only zero solution.

**Theorem 4.1.** Let Condition NS be satisfied. Then the operator $L$ is normally solvable and its kernel is finite dimensional.

We suppose now that a little more smoothness of operators and domains is required:

$$a_{ik}^\beta \in C^{\alpha, \delta}(\Omega), \quad b_{ik}^\beta \in C^{\alpha, \delta}(\partial \Omega)$$

and the domain $\Omega$ is of class $C^{\lambda, \delta}$ with $\alpha < \delta < 1$.

**Theorem 4.2.** Let the operator $L$ be normally solvable and its kernel be finite dimensional. Then Condition NS is satisfied.

**5. Dual spaces; Invertibility of limiting operators**

We consider now the space $E = E(\Omega)$ defined above and the space $E^0$, which consists of functions $u \in E$ converging to 0 at infinity in the norm $E$, i.e., $\|u\|_{E(\overline{\Omega} \cap \{|x| \leq N\})} \to 0$ as $N \to \infty$. We say that $u_n \to u_0$ in $E_{\text{loc}}(\Omega)$ if this convergence holds in $\overline{\Omega} \cap \{|x| \leq N\}$ for any $N$.

**Lemma 5.1.** Let $\phi$ be a functional in the dual space $(E^0)^*$, $u \in E$ and $u \notin E^0$, $u_n \in E^0$, $\|u_n\|_E \leq 1$, and $u_n \to u$ in $E_{\text{loc}}$. Then there exists a limit

$$\hat{\phi} = \lim_{n \to \infty} \phi(u_n).$$

It does not depend on the sequence $u_n$. If $u_n \to 0$ in $E_{\text{loc}}$, then $\phi(u_n) \to 0$.

We can extend now the functional $\phi$ to the space $E(\Omega)$. For any $u \in E(\Omega)$ we put $\hat{\phi}(u) = \phi(u)$ if $u \in E^0(\Omega)$ and $\hat{\phi}(u) = \lim_{n \to \infty} \phi(u_n)$, where $u_n \in E^0(\Omega)$ is an arbitrary sequence converging to $u$ in $E_{\text{loc}}$. This is a linear bounded functional on $E(\Omega)$. 461
Denote all such functionals \( \hat{E} \). It is a linear subspace in \( E^* \). Suppose that \( \hat{E} \neq E^* \). We take a functional \( \psi \in E^* \), which does not belong to \( \hat{E} \). Let \( \psi_0 \) be a restriction of \( \psi \) on \( E^0 \). Then \( \psi_0 \in (E^0)^* \). As above we can define the functional \( \hat{\psi}_0 \in (E)^* \). By assumption \( \psi \neq \hat{\psi}_0 \). Denote \( \bar{\psi} = \psi - \hat{\psi}_0 \). Then \( \bar{\psi} = 0 \), \( \forall u \in E^0 \).

Thus we have the following theorem.

**Theorem 5.2.** – The dual space \( E^* \) is a direct sum of the extension \( \hat{E} \) of \((E^0)^*\) on \( E \) and of the subspace \( \hat{E} \) consisting of all functionals satisfying (3).

**Lemma 5.3.** – Suppose that the operator \( L : E_0 \to E \) is normally solvable with a finite dimensional kernel, and the problem \( Lu = f \), \( f \in E \), is solvable if and only if \( \psi_i(f) = 0 \), \( i = 1, \ldots, N \), where \( \psi_i \) are linearly independent functionals in \( E^* \). Then \( \psi_i \in \hat{E} \).

These results allow us to prove invertibility of limiting operators (cf. Theorem 4.2).

**Theorem 5.4.** – If the operator \( L \) is Fredholm, then any its limiting operator is invertible.

**Corollary 5.5.** – If an operator \( L \) coincides with its limiting operator, and it is Fredholm, then it is invertible.

The last result shows in particular that the spectrum of operators with constant, periodic or quasi-periodic coefficients in unbounded cylinders does not contain eigenvalues and consists only of points of the essential spectrum.

**References**


