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An asymptotic-preserving well-balanced scheme for the hyperbolic heat equations

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Abstract We propose here a well-balanced numerical scheme for the one-dimensional Goldstein–Taylor system which is endowed with all the stability properties inherent to the continuous problem and works in both rarefied and diffusive regimes. ***To cite this article: L. Gosse, G. Toscani, C. R. Acad. Sci. Paris, Ser. I 334 (2002) 337–342.*** © 2002 Académie des sciences/Éditions scientifiques et médicales Elsevier SAS

Un schema « équilibre » et « asymptotic-preserving » pour les equations de la chaleur hyperboliques

Résumé On propose un schéma numérique « équilibre » pour le système de Goldstein–Taylor monodimensionnel possédant toutes les propriétés de stabilité du problème continu et qui fonctionne dans les regimes raréfiés et diffusifs. ***Pour citer cet article : L. Gosse, G. Toscani, C. R. Acad. Sci. Paris, Ser. I 334 (2002) 337–342.*** © 2002 Académie des sciences/Éditions scientifiques et médicales Elsevier SAS

Version française abrégée

On s'interesse à l'approximation numérique du modèle linéaire de Goldstein–Taylor (1) pour l'équation de Boltzmann. Celui-ci présente plusieurs échelles caractéristiques lorsque $\varepsilon \to 0$ et dans ce dernier cas, la densité macroscopique est solution de l'équation de la chaleur (7) au sens des distributions, [14]. On considère une grille de calcul cartésienne de paramètres Δx, Δt positifs représentant respectivement les pas d'espace et de temps. Il est possible de construire explicitement un schéma de Godunov « équilibre », [8–10] pour (1) basé sur la résolution séquentielle de problèmes de Riemann pour le système nonconservatif (2). Ceux-ci ont la structure (3), (5) et on en déduit le schéma explicite (6) bien adapté aux écoulements en régime raréfiés $\varepsilon \geqslant \Delta x$ (*cf.* Lemme 1) sous la restriction CFL hyperbolique $\Delta t \leqslant \varepsilon \Delta x$. On présente ensuite une version partiellement implicite de ce schéma (9) adaptée aux régimes diffusifs pour lesquels $\varepsilon \to 0$. Celle-ci présente les mêmes propriétés de stabilité (*cf.* Lemmes 2, 3) sous une condition CFL parabolique standard $\Delta t \leqslant \Delta x^2$ independante de ε que l'on peut faire tendre vers zéro. La structure décentrée du schéma proposé correspond à une discrétisation correcte (10) de l'équation asymptotique (7) lorsque les inconnues sont proches de la distribution Maxwellienne. Le schéma est donc « asymptotic preserving » au sens de [11,12]. Il s'en déduit en particulier un résultat de convergence (*cf.* Théorème 1)

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vers la solution faible de (7) lorsque le pas d'espace Δx et ε tendent vers zéro. Ceci est illustré par une validation numérique (*voir* Fig. 1) qui résout les problèmes soulevés dans [3].

1. Introduction

We are interested in proposing an efficient numerical processing of the following multiscale linear system, also known as the Goldstein–Taylor model of the Boltzmann equation, [14]:

$$\frac{\partial u}{\partial t} + \frac{1}{\varepsilon}\frac{\partial u}{\partial x} = \frac{1}{\varepsilon^2}(v-u), \quad \frac{\partial v}{\partial t} - \frac{1}{\varepsilon}\frac{\partial v}{\partial x} = \frac{1}{\varepsilon^2}(u-v), \quad 0<\varepsilon\leqslant 1. \tag{1}$$

In the kinetic theory of rarefied gases classically described by the Boltzmann equation [6], this two-velocity model is supposed to describe the evolution of the density distribution of a fictitious gas made of two kinds of particles. Both move with equal speed parallel to the x-axis, either in the positive direction with a density u, or in the negative one with a density v. The linear collision term on the right-hand side details the collision rule: the molecules are subject to spontaneous direction reversals at the jump times of a standard Poisson process with unit rate. More sophisticated nonlinear models exist (*cf.* [18] for exhaustive references), including Carleman's or Ruijgrok–Wu's. Their hydrodynamical limits have been studied for instance in [7,14], see also [4,15].

It has been numerically shown that, despite its simple linear structure, the system (1) presents serious numerical difficulties, [3]. Indeed, setting up a numerical scheme reliable for any value of $\varepsilon \in \,]0,1]$ is quite an involved task, see, e.g., [11,12]. The limiting behaviour for (1) as $\varepsilon \to 0$ corresponds to the heat equation for the macroscopic density $\rho = u+v$. It is therefore desirable for the numerical processing to be consistent with such an asymptotic property; this is the objective of the recent *asymptotic preserving* (AP) schemes, [11].

Another approach to the numerical approximation of hyperbolic systems with source terms is the so-called *well-balanced* (WB) schemes, [8–10,16]. In this case, deriving the scheme consists essentially in localizing the production term by means of a Dirac comb on a certain lattice in order to control accurately its effects by means of generalized jump relations in a Godunov-type scheme. This technique is also useful for proving theoretical results, see for instance [1].

2. Study in the rarefied regime ($\varepsilon \simeq 1$)

Let us move a bit deeper inside our subject: we consider a cartesian computational grid determined by two positive parameters Δx and Δt standing for the space and time steps respectively. We denote $x_j = j\Delta x$ and $t^n = n\Delta t$ for $j \in \mathbb{Z}$, $n \in \mathbb{N}$, and $C_j = [x_{j-1/2}, x_{j+1/2}]$ is any computational cell belonging to the chosen partition of the real line.

Let us fix $\varepsilon = 1$. Following the ideas of [1,8,9], we modify the collision term of (1) as follows:

$$\frac{\partial u}{\partial t} + \frac{\partial u}{\partial x} = \sum_{j\in\mathbb{Z}} \Delta x(v-u)\delta(x-x_{j-1/2}), \qquad \frac{\partial v}{\partial t} - \frac{\partial v}{\partial x} = \sum_{j\in\mathbb{Z}} \Delta x(u-v)\delta(x-x_{j-1/2}). \tag{2}$$

This means that we concentrate the effects of the collisions on the borders of the cells C_j. Clearly, the Riemann problem for (2) is different from the that of (1) because of the static wave induced by the Dirac masses. Along this wave, one has a generalized jump relation created by a *nonconservative product* [13]. The structure of the Riemann problem for (2) for initial data (u_L, u_R), (v_L, v_R) having a jump in $x = x_{j-1/2}$ is therefore:

$$\begin{cases} (u_L, v_L) & \text{for } x - x_{j-1/2} < -t, \\ (u_L, \tilde{v}) & \text{for } x - x_{j-1/2} \in \,]-t, 0[, \\ (\tilde{u}, v_R) & \text{for } x - x_{j-1/2} \in \,]0, t[, \\ (u_R, v_R) & \text{for } x - x_{j-1/2} > t. \end{cases} \tag{3}$$

By analogy with previous works, we consider the steady equations in (1) along this static singularity. In the notations of [9], we get:

$$\frac{\partial \bar{u}}{\partial x} = \bar{v} - \bar{u}, \quad -\frac{\partial \bar{v}}{\partial x} = \bar{u} - \bar{v}, \quad \bar{u}(0) = u_L, \quad \bar{v}(\Delta x) = v_R, \quad x \in [0, \Delta x]. \tag{4}$$

This system can be solved explicitely and one finds the missing values in (3):

$$\tilde{u} = u_L + \frac{\Delta x}{1+\Delta x}(v_R - u_L), \qquad \tilde{v} = v_R - \frac{\Delta x}{1+\Delta x}(v_R - u_L). \tag{5}$$

We are now ready to deduce a Godunov scheme for (1) by solving elementary Riemann problems for (2) at the endpoints of each cell C_j and integrating on the rectangles $C_j \times [t^n, t^{n+1}]$. In standard notation, we obtain from (3), (5): for all $j \in \mathbb{Z}$, $n \in \mathbb{N}$,

$$\begin{cases} u_j^{n+1} = u_j^n - \dfrac{\Delta t}{\Delta x}\left(u_j^n - u_{j-1}^n\right) + \dfrac{\Delta t}{1+\Delta x}\left(v_j^n - u_{j-1}^n\right), \\ v_j^{n+1} = v_j^n - \dfrac{\Delta t}{\Delta x}\left(v_{j+1}^n - v_j^n\right) - \dfrac{\Delta t}{1+\Delta x}\left(v_{j+1}^n - u_j^n\right). \end{cases} \tag{6}$$

The upwinding of the collision term ensures that the integral curves of (4) are preserved by (6); this is the WB property. This scheme produces stable and oscillation-free numerical approximations to (1) with $\varepsilon = 1$.

In the sequel, $u^{\Delta x}$, $v^{\Delta x}$ stand for the piecewise constant functions coinciding with u_j^n, v_j^n on the appropriate cells.

LEMMA 1. – *Assume* $0 \leqslant u^0, v^0 \in \mathrm{L}^1 \cap \mathrm{L}^\infty(\mathbb{R})$ *are initial data for* (1) *with* $\varepsilon = 1$. *Then under the CFL condition* $\Delta t \leqslant \Delta x$, *there holds for all* $t > 0$ *and* $1 \leqslant p \leqslant +\infty$:

$$\left\|u^{\Delta x}(\cdot,t)\right\|_{\mathrm{L}^p(\mathbb{R})} + \left\|v^{\Delta x}(\cdot,t)\right\|_{\mathrm{L}^p(\mathbb{R})} \leqslant \left\|u^0\right\|_{\mathrm{L}^p(\mathbb{R})} + \left\|v^0\right\|_{\mathrm{L}^p(\mathbb{R})},$$

and the scheme (6) *is positivity-preserving. If moreover* $u^0, v^0 \in BV(\mathbb{R})$, *under the same conditions, one has also*:

$$\begin{aligned} &\mathsf{TV}_x\left(u^{\Delta x}(\cdot,t)\right) + \mathsf{TV}_x\left(v^{\Delta x}(\cdot,t)\right) \leqslant \mathsf{TV}_x\left(u^0\right) + \mathsf{TV}_x\left(v^0\right), \\ &\left\|u^{\Delta x}(\cdot,t+\Delta t) - u^{\Delta x}(\cdot,t)\right\|_{\mathrm{L}^1(\mathbb{R})} + \left\|v^{\Delta x}(\cdot,t+\Delta t) - v^{\Delta x}(\cdot,t)\right\|_{\mathrm{L}^1(\mathbb{R})} \\ &\quad \leqslant \Delta t\left(\mathsf{TV}_x\left(u^0\right) + \mathsf{TV}_x\left(v^0\right) + 2\left(\left\|u^0\right\|_{\mathrm{L}^1(\mathbb{R})} + \left\|v^0\right\|_{\mathrm{L}^1(\mathbb{R})}\right)\right). \end{aligned}$$

Proof. – As (6) has a linear structure, one notices that under the proposed CFL condition,

$$\begin{aligned} \left|u_j^{n+1}\right| &\leqslant \left|u_j^n\right|\left(1 - \frac{\Delta t}{\Delta x}\right) + \left|u_{j-1}^n\right|\left(\frac{\Delta t}{\Delta x} - \frac{\Delta t}{1+\Delta x}\right) + \frac{\Delta t}{1+\Delta x}\left|v_j^n\right|, \\ \left|v_j^{n+1}\right| &\leqslant \left|v_j^n\right|\left(1 - \frac{\Delta t}{\Delta x}\right) + \left|v_{j+1}^n\right|\left(\frac{\Delta t}{\Delta x} - \frac{\Delta t}{1+\Delta x}\right) + \frac{\Delta t}{1+\Delta x}\left|u_j^n\right|, \end{aligned}$$

and this allows us to conclude for Δt small enough. □

These estimates imply weak or strong convergence depending on the initial data's smoothness. However, this WB scheme is not readily well suited for computations in the diffusive scaling of (1) corresponding to ε close to zero because of the prescribed CFL condition. Lemma 1 can therefore be seen as a result valid in the range $\varepsilon \geqslant \Delta x$ as the classical restriction for parabolic equations is $\Delta t \leqslant \Delta x^2$. The complementary range of parameters will be looked at in the next section.

3. Study in the diffusive regime ($\varepsilon \to 0$)

We show here that it is possible to extend easily the scheme (6) in order to handle the diffusive limit $\varepsilon \to 0$, that is to say, to make it "asymptotic-preserving" in the sense of [11], see also [12]. The main motivation is to recover a proper discretization of the heat equation

$$\frac{\partial\rho}{\partial t}=\frac{1}{2}\frac{\partial^2\rho}{\partial x^2},\quad \rho(\cdot,t=0)=u^0+v^0,\tag{7}$$

as ε vanishes. Performing a rescaling in (4) and rearranging terms, one finds:

$$\tilde{u}=v_R+\frac{\varepsilon(u_L-v_R)}{\varepsilon+\Delta x},\qquad \tilde{v}=u_L+\frac{\varepsilon(v_R-u_L)}{\varepsilon+\Delta x}.\tag{8}$$

We treat implicitly the stiff convection terms in (6) and we find the following diagonally-dominant system for u_j^{n+1}, v_j^{n+1}:

$$\begin{cases} u_j^{n+1}+\dfrac{\Delta t}{\varepsilon\Delta x}\left(u_j^{n+1}-v_j^{n+1}\right)=u_j^n+\dfrac{\Delta t}{\Delta x(\Delta x+\varepsilon)}\left(u_{j-1}^n-v_j^n\right),\\ v_j^{n+1}-\dfrac{\Delta t}{\varepsilon\Delta x}\left(u_j^{n+1}-v_j^{n+1}\right)=v_j^n+\dfrac{\Delta t}{\Delta x(\Delta x+\varepsilon)}\left(v_{j+1}^n-u_j^n\right).\end{cases}\tag{9}$$

Once again, one notices immediately that adding both equations in (9), the upwinding of the collision term allows the derivation of the following scheme:

$$u_j^{n+1}+v_j^{n+1}=u_j^n+v_j^n+\frac{\Delta t}{\Delta x(\Delta x+\varepsilon)}\left(v_{j+1}^n-u_j^n-v_j^n+u_{j-1}^n\right).\tag{10}$$

This matches a correct discretization of (7) with a physical amount of dissipation under the assumption that $|u_j^n-v_j^n|=\mathrm{O}(\varepsilon)$, that is to say, if the unknowns are close to the local Maxwellian equilibrium of (1) as it shows with obvious notation:

$$\begin{aligned}\rho_j^{n+1}&=\rho_j^n+\frac{\Delta t}{2\Delta x^2}\left(\rho_{j+1}^n-2\rho_j^n+\rho_{j-1}^n\right)\\&\quad+\frac{\varepsilon\Delta t}{2\Delta x(\Delta x+\varepsilon)}\left\{\frac{u_{j+1}^n-v_{j+1}^n}{\varepsilon}+\frac{u_{j-1}^n-v_{j-1}^n}{\varepsilon}-\frac{\rho_{j+1}^n-2\rho_j^n+\rho_{j-1}^n}{\Delta x}\right\}.\end{aligned}\tag{11}$$

It is also expected that both (9) and (10) are stable under the standard parabolic CFL condition $\Delta t\leqslant\Delta x^2$ independently of ε.

LEMMA 2. – *Assume* $0\leqslant u^0, v^0\in\mathrm{L}^1\cap\mathrm{L}^\infty(\mathbb{R})$ *are initial data for* (1). *Then under the CFL condition* $\Delta t\leqslant\Delta x^2$ *there holds for all* $t>0$ *and* $1\leqslant p\leqslant+\infty$:

$$\left\|u^{\Delta x}(\cdot,t)\right\|_{\mathrm{L}^p(\mathbb{R})}+\left\|v^{\Delta x}(\cdot,t)\right\|_{\mathrm{L}^p(\mathbb{R})}\leqslant\left\|u^0\right\|_{\mathrm{L}^p(\mathbb{R})}+\left\|v^0\right\|_{\mathrm{L}^p(\mathbb{R})}$$

and the scheme (9) *is positivity preserving. If moreover* $u^0, v^0\in BV(\mathbb{R})$, *under the same conditions, one has also*:

$$\mathsf{TV}_x\left(u^{\Delta x}\left(\cdot,t^n\right)\right)+\mathsf{TV}_x\left(v^{\Delta x}\left(\cdot,t^n\right)\right)\leqslant\mathsf{TV}_x\left(u^0\right)+\mathsf{TV}_x\left(v^0\right).$$

Proof. – For ease of writing, we denote $a=1+\frac{\Delta t}{\varepsilon\Delta x}$, $b=\frac{\Delta t}{\varepsilon\Delta x}$, $c=\frac{\Delta t}{\Delta x(\Delta x+\varepsilon)}$. The system (9) can be explicitly solved and this is a desirable feature according to [11]:

$$\begin{cases} u_j^{n+1}=\left(\dfrac{a}{a+b}-\dfrac{bc}{a+b}\right)u_j^n+\left(\dfrac{b}{a+b}-\dfrac{ac}{a+b}\right)v_j^n+\dfrac{ac}{a+b}u_{j-1}^n+\dfrac{bc}{a+b}v_{j+1}^n,\\ v_j^{n+1}=\left(\dfrac{b}{a+b}-\dfrac{ac}{a+b}\right)u_j^n+\left(\dfrac{a}{a+b}-\dfrac{bc}{a+b}\right)v_j^n+\dfrac{bc}{a+b}u_{j-1}^n+\dfrac{ac}{a+b}v_{j+1}^n.\end{cases}\tag{12}$$

The proposed CFL restriction implies $b-ac\geqslant 0$, and it turns out that the four coefficients in each line of (12) are nonnegative with sum equal to 1. Hence we obtain control on both the L^p norms and the total variation on $\mathbb{R}$. $\square$

The space regularity can be converted in time-equicontinuity taking into account the stabilizing effect of the relaxation mechanism.

LEMMA 3. – *Assume* $0 \leqslant u^0, v^0 \in \mathrm{L}^1 \cap BV(\mathbb{R})$ *are initial data for* (1). *Then under the CFL condition* $\Delta t \leqslant \Delta x^2$, *there holds for all* $\varepsilon, t > 0$:

$$\left\|u^{\Delta x}(\cdot,t) - v^{\Delta x}(\cdot,t)\right\|_{\mathrm{L}^1(\mathbb{R})} \leqslant \varepsilon\left(\mathsf{TV}_x\left(u^0\right) + \mathsf{TV}_x\left(v^0\right)\right) + \left\|u^0 - v^0\right\|_{\mathrm{L}^1(\mathbb{R})},$$

$$\left\|u^{\Delta x}(\cdot,t+\Delta t) - u^{\Delta x}(\cdot,t)\right\|_{\mathrm{L}^1(\mathbb{R})} + \left\|v^{\Delta x}(\cdot,t+\Delta t) - v^{\Delta x}(\cdot,t)\right\|_{\mathrm{L}^1(\mathbb{R})}$$
$$\leqslant \frac{\Delta t}{\Delta x + \varepsilon}\left(\mathsf{TV}_x\left(u^0\right) + \mathsf{TV}_x\left(v^0\right) + \frac{2}{\varepsilon}\left\|u^{\Delta x}(\cdot,t) - v^{\Delta x}(\cdot,t)\right\|_{\mathrm{L}^1(\mathbb{R})}\right).$$

Proof. – To check that the numerical values stay close to the Maxwellian distribution, we get from (9)

$$(1+2b)\left(u_j^{n+1} - v_j^{n+1}\right) = (1+2c)\left(u_j^n - v_j^n\right) + c\left(u_{j-1}^n - u_j^n\right) - c\left(v_{j+1}^n - v_j^n\right),$$

and deduce the following estimate by invoking the bounds of Lemma 2 recursively for $n \in \mathbb{N}_*$:

$$\left\|u^{\Delta x}(\cdot,n\Delta t) - v^{\Delta x}(\cdot,n\Delta t)\right\|_{\mathrm{L}^1(\mathbb{R})} \leqslant \left(\frac{1+2c}{1+2b}\right)^n \left\|u^0 - v^0\right\|_{\mathrm{L}^1(\mathbb{R})} + \frac{c\Delta x}{b-c}\left(\mathsf{TV}_x\left(u^0\right) + \mathsf{TV}_x\left(v^0\right)\right).$$

The first result follows by observing that $\frac{1+2c}{1+2b} \leqslant 1$ and $\frac{c\Delta x}{b-c} = \varepsilon$. Thus from (9) one can derive furthermore the two inequalities

$$(1+b)\left|u_j^{n+1} - u_j^n\right| - b\left|v_j^{n+1} - v_j^n\right| \leqslant (b-c)\left|u_j^n - v_j^n\right| + c\left|u_{j-1}^n - u_j^n\right|,$$
$$(1+b)\left|v_j^{n+1} - v_j^n\right| - b\left|u_j^{n+1} - u_j^n\right| \leqslant (b-c)\left|u_j^n - v_j^n\right| + c\left|v_{j+1}^n - v_j^n\right|.$$

Now, the last result is obtained just adding these inequalities and summing over $j \in \mathbb{Z}$. □

From these estimates a L^1-modulus of equicontinuity as $\sqrt{\Delta t} \simeq \Delta x$ follows. In consequence of both Lemmas 2 and 3, one deduces that $u^{\Delta x}$, $v^{\Delta x}$ is uniformly bounded in $BV_{\mathrm{loc}}(\mathbb{R} \times \mathbb{R}_*^+)$.

THEOREM 1. – *Assume* $0 \leqslant u^0, v^0 \in \mathrm{L}^1 \cap BV(\mathbb{R})$ *are initial data for* (1) *and satisfy* $\|u^0 - v^0\|_{\mathrm{L}^1(\mathbb{R})} \leqslant \mathrm{O}(\varepsilon)$. *Then as* $\Delta x, \varepsilon \to 0$ *under the CFL condition* $\Delta t \leqslant \Delta x^2$, *the sequence* $u^{\Delta x}$, $v^{\Delta x}$ *is relatively compact in the strong topology of* $\mathrm{L}^1_{\mathrm{loc}}(\mathbb{R} \times \mathbb{R}_*^+)$. *In particular,* $\rho^{\Delta x} = u^{\Delta x} + v^{\Delta x}$ *converges towards the unique solution of* (7) *in the sense of distributions.*

From the bounds shown in Lemma 3, one deduces that the $\mathrm{L}^1(\mathbb{R})$ norm of the remaining terms in (11) are of the order of ε for all times. If the initial data are not "well-prepared", one has to tackle a *kinetic initial layer* as in [17] which cancels the Hölder continuity in time of the sequence. In such a case, one can only claim compactness for $t \geqslant \tau > 0$, i.e., outside this initial layer.

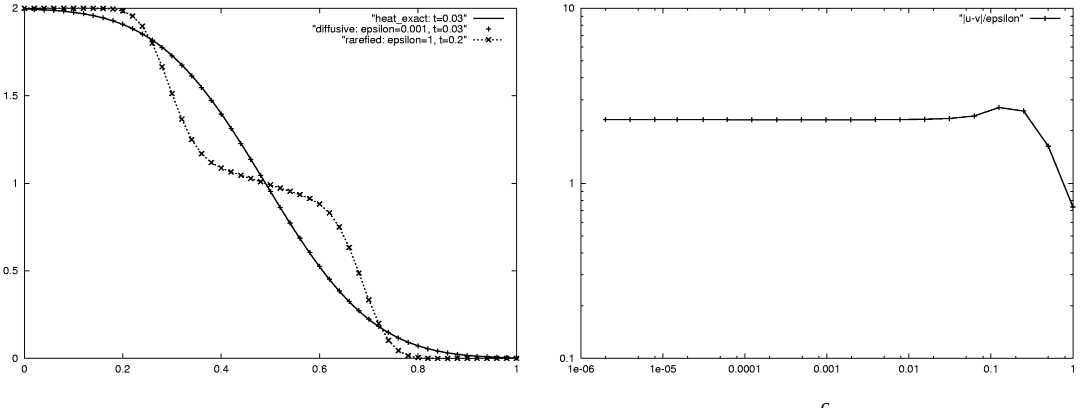

Figure 1. – Numerical results for (6) and (9) with Riemann data: $10^{-6} \leqslant \varepsilon \leqslant 1$.

4. Numerical results and conclusion

Both (6) and (9) provide an efficient and straightforward way to compute numerical approximations to the Goldstein–Taylor model (1) in both rarefied and diffusive regime. We have therefore an AP scheme with satisfying convergence properties under reasonable CFL conditions and over the full range of parameters. This answers completely the issues exposed in [3] and also improves partly the situation of [11,12] for which theoretical results are lacking, see also [2].

We close this text displaying some numerical results illustrating our statements. We choose Maxwellian initial data $u^0(x) = v^0(x) = \mathbf{1}_{x<0.5}$ to compute the solution of (1) in $x \in [0,1]$ for $\varepsilon = 1$, $t = 0.2$ and $\varepsilon = 10^{-3}$, $t = 0.03$. Of course, the exact solution of (7) is given by $\rho(x,t) = 1 - \operatorname{erf}((x-0.5)/\sqrt{2t})$. We took $\Delta x = 0.02$ and the highest value for Δt allowed by the CFL restrictions for both runs: see Fig. 1.

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