C. R. Acad. Sci. Paris, Ser. I 334 (2002) 209–212

Probabilités/*Probability Theory*

Integration by parts on Bessel Bridges and related stochastic partial differential equations

Lorenzo Zambotti

Scuola Normale Superiore, Piazza dei Cavalieri 7, 56126 Pisa, Italy

Received 20 December 2001; accepted 27 December 2001

Note presented by Paul Malliavin.

Abstract We prove integration by parts formulae with respect to the law of Bessel Bridges of dimension $\delta \geqslant 3$. For $\delta = 3$ we have an infinite-dimensional boundary measure, and for $\delta > 3$ a singular logarithmic derivative. We give applications to SPDEs with additive space-time white noise and singular drifts, whose solutions are non-negative. ***To cite this article: L. Zambotti, C. R. Acad. Sci. Paris, Ser. I 334 (2002) 209–212.*** © 2002 Académie des sciences/Éditions scientifiques et médicales Elsevier SAS

Integration par parties sur Ponts de Bessel et EDPS correspondantes

Résumé Nous prouvons des formules d'intégration par parties par rapport à la loi des Ponts de Bessel de dimension $\delta \geqslant 3$. Remarquons que dans le cas $\delta = 3$ nous obtenons une mesure de bord infini-dimensionelle, et pour $\delta > 3$ une dérivée logarithmique singulière. Nous donnerons aussi des applications à des EDPS avec bruit blanc en espace-temps et termes de dérive singuliers, dont les solutions sont non-négatives. ***Pour citer cet article : L. Zambotti, C. R. Acad. Sci. Paris, Ser. I 334 (2002) 209–212.*** © 2002 Académie des sciences/Éditions scientifiques et médicales Elsevier SAS

1. Introduction

Consider the Stochastic Partial Differential Equation (SPDE):

$$\begin{cases} \dfrac{\partial u}{\partial t} = \dfrac{1}{2}\dfrac{\partial^2 u}{\partial \theta^2} + f(u) + \dfrac{\partial^2 W}{\partial t \partial \theta}, \\ u(0,\theta) = u_0(\theta), \quad u(t,0) = u(t,1) = 0, \quad t \geqslant 0,\ \theta \in [0,1], \end{cases} \tag{1}$$

where $\{W(t,\theta)\}$ is a Brownian Sheet. If $f : \mathbb{R} \mapsto \mathbb{R}$ is Lipschitz-continuous, then it is well known that $P(u(t,\theta) \geqslant 0,\ \forall \theta \in [0,1]) = 0$ for all $t > 0$.

In this Note we consider equations of the form (1), where the drift term $f(u)$ is well-defined only in the class of σ-finite measures on space-time, and whose solutions are a.s. non-negative and continuous.

For the equations we consider, we identify the unique invariant measure of the solution with the law π_δ of a Bessel Bridge over $[0,1]$ of dimension $\delta \geqslant 3$. Moreover, we write an infinite-dimensional integration by parts formula on π_δ and deduce properties of the solutions. Since the law of a Bessel Bridge is naturally

E-mail address: zambotti@sns.it (L. Zambotti).

S1631-073X(02)02254-9/FLA

supported by the convex set K of non-negative continuous functions on $[0,1]$, the behaviour at the boundary of K has to be taken into account. In particular, for the solution of the SPDE with reflection introduced by Nualart and Pardoux in [7], we can prove support and decomposition theorems for the reflecting term.

2. Integration by parts formulae

Let $(x_\delta(t))_{t\geqslant[0,1]}$ be the Bessel Bridge of dimension $\delta\geqslant 3$ from 0 to 0 over $[0,1]$. For $\delta=3$, denote also by $(x_{3,r}(t))_{t\geqslant[0,1]}$ the Bessel Bridge of dimension 3 from 0 to 0 over $[0,r]$, $r\in]0,1]$: see [8]. The law π_δ of x_δ, $\delta\geqslant 3$, is concentrated on the convex set K of all continuous $x:[0,1]\mapsto\mathbb{R}_+:=[0,\infty)$ such that $x(0)=x(1)=0$.

Denote by $\mathrm{C}^1_b(\mathrm{L}^2(0,1))$ the set of all $\varphi:\mathrm{L}^2(0,1)\mapsto\mathbb{R}$ bounded and with bounded continuous Fréchet gradient $\nabla\varphi:\mathrm{L}^2(0,1)\mapsto\mathrm{L}^2(0,1)$. For all $\varphi\in\mathrm{C}^1_b(\mathrm{L}^2(0,1))$ and $h\in\mathrm{L}^2(0,1)$, let $\partial_h\varphi$ denote the directional derivative of φ along h. For all $h\in\mathrm{H}^2(0,1)$ let $h''\in\mathrm{L}^2(0,1)$ denote the second derivative of h. Finally, let $\langle\cdot,\cdot\rangle$ denote the canonical scalar product in $\mathrm{L}^2(0,1)$.

THEOREM 2.1. – *Let $\varphi\in\mathrm{C}^1_b(\mathrm{L}^2(0,1))$ and $h\in\mathrm{H}^2\cap\mathrm{H}^1_0(0,1)$. Then*:

$$\mathbb{E}\big[\partial_h\varphi(x_3)\big]=-\mathbb{E}\big[\varphi(x_3)\big\langle h'',x_3\big\rangle\big]-\int_0^1\frac{h(r)}{\sqrt{2\pi r^3(1-r)^3}}\,\mathbb{E}\big[\varphi(x_{3,r}\oplus\hat{x}_{3,1-r})\big]\,\mathrm{d}r \tag{2}$$

$$\mathbb{E}\big[\partial_h\varphi(x_\delta)\big]=-\mathbb{E}\left[\varphi(x_\delta)\left(\big\langle h'',x_\delta\big\rangle+\frac{(\delta-1)(\delta-3)}{4}\left\langle h,\frac{1}{(x_\delta)^3}\right\rangle\right)\right],\quad \delta>3, \tag{3}$$

where $x_{3,r}$ and $\hat{x}_{3,r}$ are i.i.d. and $x_{3,r}\oplus\hat{x}_{3,1-r}(\tau):=x_{3,r}(\tau)1_{[0,r]}(\tau)+\hat{x}_{3,1-r}(\tau-r)1_{]r,1]}(\tau)$, $\tau\in[0,1]$.

Remark 1. – Compare (2) and (3) with the Divergence theorem in a regular domain $O\Subset\mathbb{R}^d$:

$$\int_O(\partial_h\varphi)\rho\,\mathrm{d}x=-\int_O\varphi(\partial_h\log\rho)\rho\,\mathrm{d}x-\int_{\partial O}\varphi\langle n,h\rangle\rho\,\mathrm{d}\mathcal{H}^{d-1} \tag{4}$$

where $h\in\mathbb{R}^d$, $\varphi,\rho\in\mathrm{C}^1(\overline{O})$, $0<\inf_O\rho\leqslant\sup_O\rho<\infty$, n is the inward-pointing normal vector to the boundary ∂O and $\mathcal{H}^{d-1}$ is the $(d-1)$-dimensional Hausdorff measure. Then we have the following interpretations:

- The law π_3 of x_3 admits the field $x\mapsto x''$ as logarithmic derivative.
- Since $P(x_{3,r}(\theta)>0,\ \forall\theta\in]0,r[)=1$, $r\in]0,1]$, the second term in the right-hand side of (2) is a boundary term: indeed, it is supported by the set ∂^*K of all $x\in K$ vanishing at only one $r\in(0,1)$. Notice that, by the Dirichlet boundary condition, the boundary ∂K is K itself in the sup-norm topology.
- The definition of ∂^*K is reminiscent of the following finite-dimensional situation: the topological boundary of $\mathbb{R}^d_+:=\{(x_1,\dots,x_d):x_i\geqslant 0,\ i=1,\dots,d\}$, $\delta\geqslant 2$, is $\partial\mathbb{R}^d_+=\{x\in\mathbb{R}^d_+:\min_{i=1,\dots,d}x_i=0\}$; however, if we set $\partial^*\mathbb{R}^d_+:=\bigcup_{i=1}^d\{x_i=0,\ x_j>0,\ \forall j\neq i\}$, then $\partial^*\mathbb{R}^d_+$ is the relevant boundary for (4): indeed, $\partial\mathbb{R}^d_+\backslash\partial^*\mathbb{R}^d_+$ has Hausdorff-dimension $d-2$ and in particular $\mathcal{H}^{d-1}(\partial\mathbb{R}^d_+\backslash\partial^*\mathbb{R}^d_+)=0$.
- For all $x\in\partial^*K$ with $x(r)=0$, $r\in]0,1[$, we have that $h(r)=\langle\delta_r,h\rangle$ corresponds to $\langle n,h\rangle$, i.e., the Dirac mass δ_r at r gives the inward-pointing normal vector $n(x)$. Notice that $n\notin\mathrm{L}^2(0,1)$, which is related with the fact that K is not a C^1 domain in $\mathrm{L}^2(0,1)$.
- The logarithmic derivative of the law π_δ of x_δ, $\delta>3$, is the map $x\mapsto x''+(\delta-1)(\delta-3)/(4x^3)$. The singular term $x\mapsto 1/x^3$ gives a repulsion from 0 and substitutes the boundary term of (2). This phenomenon is reminiscent of the following finite-dimensional situation: let $m_\delta(\mathrm{d}x):=x^{\delta-1}\,\mathrm{d}x$ on $[0,\infty)$, $\delta\geqslant 1$; then, for all regular ψ with compact support:

$$\int_0^\infty\psi'\,\mathrm{d}m_1=-\psi(0),\qquad\int_0^\infty\psi'\,\mathrm{d}m_\delta=-\int_0^\infty\psi(x)\frac{\delta-1}{x}\,m_\delta(\mathrm{d}x),\quad\delta>1.$$

In this case, for $\delta = 1$ a boundary term appears, which for $\delta > 1$ is substituted by the repulsive logarithmic derivative $x \mapsto (\delta - 1)/x$.

For a general theory of integration by parts formulae in infinite dimension see [4].

3. Related stochastic partial differential equations

Let $\{W(t,\theta) : t \geqslant 0,\ \theta \in [0,1]\}$ be a Brownian sheet independent of x_δ for all $\delta \geqslant 3$. In [7] it is proved that for all $u_0 \in K$, there exists a unique pair (u, η), where $u : Q := [0,\infty) \times [0,1] \mapsto \mathbb{R}$ is continuous adapted and η is a locally finite positive measure on $[0,\infty)\times\,]0,1[$, satisfying the stochastic partial differential equation with reflection:

$$\begin{cases} \dfrac{\partial u}{\partial t} = \dfrac{1}{2}\dfrac{\partial^2 u}{\partial \theta^2} + \dfrac{\partial^2 W}{\partial t \partial \theta} + \eta(t,\theta), \\ u(0,\theta) = u_0(\theta), \quad u(t,0) = u(t,1) = 0, \\ u \geqslant 0, \quad d\eta \geqslant 0, \quad \int_Q u\, d\eta = 0. \end{cases} \tag{5}$$

Set $u_3(\cdot,\cdot,u_0) := u : Q \mapsto \mathbb{R}_+$, $u_0 \in K$.

THEOREM 3.1. –

(1) *The law π_3 of x_3 is the unique invariant probability measure of the process $(u_3(t,\cdot,u_0))_{t\geqslant 0,\ u_0\in K}$.*

(2) *For all Borel set $I \Subset (0,1)$, the process $t \mapsto \eta([0,t]\times I)$ is an additive functional of u_3, with Revuz-measure*:

$$\mathbb{E}\left[\int_0^1 \varphi\big(u_3(t,\cdot,x_3)\big)\eta(dt, I)\right] = \frac{1}{2}\int_I \frac{1}{\sqrt{2\pi r^3(1-r)^3}}\,\mathbb{E}\big[\varphi(x_{3,r} \oplus \hat{x}_{3,1-r})\big]\, dr, \tag{6}$$

for $\varphi : L^2(0,1) \mapsto \mathbb{R}$ Borel and bounded.

(3) *For all $u_0 \in K$, there exist a random Borel set $S \subset \mathbb{R}_+$ and a map $r : S \mapsto (0,1)$, such that a.s.*

$$\eta\big([\mathbb{R}_+ \times (0,1)] \setminus \{(s, r(s)) : s \in S\}\big) = 0,$$
$$\forall s \in S: \quad u\big(s, r(s)\big) = 0, \quad u(s,\theta) > 0, \quad \forall \theta \in (0,1)\setminus\{r(s)\}.$$

(4) *Let δ_r denote the Dirac mass at $r \in (0,1)$. For all $u_0 \in K$, we have a.s. on $[0,\infty)\times(0,1)$:*

$$\eta(ds, d\theta) = \delta_{r(s)}(d\theta)\,\eta\big(ds, (0,1)\big). \tag{7}$$

(5) *The process $(u_3(t,\cdot,u_0))_{t\geqslant 0,\ u_0\in K}$ is the Markov process properly associated with the symmetric Dirichlet form $(\mathcal{E}^3, D(\mathcal{E}^3))$ in $L^2(\pi_3)$, closure of the bilinear form*:

$$C_b^1\big(L^2(0,1)\big) \ni \varphi, \psi \mapsto \frac{1}{2}\int_K \langle \nabla\varphi, \nabla\psi\rangle\, d\pi_3.$$

Point 1 in Theorem 3.1 was proved independently in [2] and in [9]. For basic definitions in the theories of Dirichlet forms and additive functionals we refer to [3] and [1].

Remark 2. – By Remark 1 and by (7), we can interpret Eq. (5) as a Skorokhod problem in the infinite dimensional convex set K, writing:

$$du = \frac{1}{2}\frac{\partial^2 u}{\partial\theta^2}\, dt + dW + \frac{1}{2} n(u)\cdot dL,$$

where $\mathrm{L}_t := 2\eta([0,t] \times (0,1))$, uniquely determined by its Revuz measure (6), increases only when $u(t,\cdot) \in \partial^* K$ and $n(u(t,\cdot))$ is the Dirac mass at $r(t)$.

Let now $\delta > 3$ and consider the equation (see also [5] and [6]):

$$\begin{cases} \dfrac{\partial u}{\partial t} = \dfrac{1}{2}\dfrac{\partial^2 u}{\partial \theta^2} + \dfrac{(\delta-3)(\delta-1)}{8u^3} + \dfrac{\partial^2 W}{\partial t \partial \theta}, \\ u(0,\cdot) = u_0, \quad u(t,0) = u(t,1) = 0, \\ u \geqslant 0, \quad u^{-3} \in \mathrm{L}^1_{\mathrm{loc}}\big(\mathbb{R}_+ \times (0,1)\big). \end{cases} \tag{8}$$

THEOREM 3.2. –
(1) *For all $u_0 \in K$, there exists a unique adapted continuous solution $u_\delta(\cdot,\cdot,u_0) : Q \mapsto \mathbb{R}_+$ of* (8).
(2) *The law π_δ of x_δ is the unique invariant probability measure of the process $(u_\delta(t,\cdot,u_0))_{t \geqslant 0,\, u_0 \in K}$.*
(3) *The process $(u_\delta(t,\cdot,u_0))_{t \geqslant 0,\, u_0 \in K}$ is the Markov process properly associated with the symmetric Dirichlet Form $(\mathcal{E}^\delta, D(\mathcal{E}^\delta))$ in $\mathrm{L}^2(\pi_\delta)$, closure of the bilinear form*:

$$\mathrm{C}^1_b\big(\mathrm{L}^2(0,1)\big) \ni \varphi, \psi \mapsto \frac{1}{2}\int_K \langle \nabla\varphi, \nabla\psi \rangle \, \mathrm{d}\pi_\delta.$$

Remark 3. – The drift term $\kappa(\delta)/u^3$ in (8) has a repulsive effect from 0, which is strong enough to keep the solution u_δ non-negative without the need of the reflecting term η: this is related with the absence of boundary terms in (3).

Remark 4. – The construction of solutions of (5) and (8) uses pathwise methods: as a result u_δ, $\delta \geqslant 3$, is a strong solution, i.e., adapted to the filtration of the driving noise. The identification of $(u_\delta(t,\cdot,u_0))_{t \geqslant 0,\, u_0 \in K}$ as the Markov process associated with the Dirichlet form $(\mathcal{E}^\delta, D(\mathcal{E}^\delta))$, $\delta \geqslant 3$, is obtained a posteriori. On the other hand, the theory of Dirichlet forms is a powerful tool, which enables, for instance, to deduce the properties of u_3 listed in points (3), (4) of Theorem 3.1 from (6).

References

[1] M. Fukushima, Y. Oshima, M. Takeda, Dirichlet Forms and Symmetric Markov Processes, Walter de Gruyter, Berlin–New York, 1994.
[2] T. Funaki, S. Olla, Fluctuations for $\nabla\phi$ interface model on a wall, Stoch. Processes Appl. 94 (2001) 1–27.
[3] Z.M. Ma, M. Röckner, Introduction to the Theory of (Nonsymmetric) Dirichlet Forms, Springer-Verlag, Berlin, 1992.
[4] P. Malliavin, Stochastic Analysis, Springer, Berlin, 1997.
[5] C. Mueller, Long-time existence for signed solutions of the heat equation with a noise term, Probab. Theory Related Fields 110 (1998) 51–68.
[6] C. Mueller, E. Pardoux, The critical exponent for a stochastic PDE to hit zero, in: Stochastic Analysis, Control, Optimization and Applications, Systems Control Found. Appl., Birkhäuser Boston, 1999, pp. 325–338.
[7] D. Nualart, E. Pardoux, White noise driven quasilinear SPDEs with reflection, Probab. Theory Related Fields 93 (1992) 77–89.
[8] D. Revuz, M. Yor, Continuous Martingales and Brownian Motion, Springer-Verlag, 1991.
[9] L. Zambotti, A reflected stochastic heat equation as symmetric dynamics with respect to the 3-d Bessel Bridge, J. Funct. Anal. 180 (2001) 195–209.