Positivity of $L\left(\frac{1}{2}, \pi\right)$ for symplectic representations

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Abstract

Let $\pi$ a cuspidal generic representation of $\text{SO}(2n+1)$. We prove that $L\left(\frac{1}{2}, \pi\right) \geq 0$. To cite this article: E. Lapid, S. Rallis, C. R. Acad. Sci. Paris, Ser. I 334 (2002) 101–104. © 2002 Académie des sciences/Éditions scientifiques et médicales Elsevier SAS

1. Introduction

Let $\pi$ be a cuspidal automorphic representation of $\text{GL}_n(\mathbb{A})$ where $\mathbb{A}$ is the adèles ring of a number field $F$. Suppose that $\pi$ is self-dual. Then the standard $L$-function $L(s, \pi)$ is real for $s \in \mathbb{R}$ and positive for $s > 1$. By the generalized Riemann hypothesis we expect that $L(s, \pi) > 0$ for $s > \frac{1}{2}$ and in particular, $L\left(\frac{1}{2}, \pi\right) \geq 0$. However, the latter is unknown even in the case of quadratic Dirichlet characters. In general, if $\pi$ is self dual then $\pi$ is either symplectic or orthogonal, i.e., exactly one of the (partial) $L$-functions $L^s(s, \pi, \wedge^2)$ (exterior square) or $L^s(s, \pi, \text{sym}^2)$ (symmetric square) has a pole at $s = 1$. In the first case, $n$ is even and the central character of $\pi$ is trivial [9].

Our main result in [13] is:

THEOREM 1. – Let $\pi$ be a symplectic cuspidal representation of $\text{GL}_n(\mathbb{A})$. Then $L\left(\frac{1}{2}, \pi\right) \geq 0$.

We remark that in the formulation of Theorem 1 we could take the partial $L$-function instead of the completed one.

In the case, $n = 2$, $\pi$ is symplectic exactly when the central character of $\pi$ is trivial. In this case more precise information is known about $L\left(\frac{1}{2}, \pi\right)$, at least in special cases (cf. [11]), and the theorem was proved before using a variant of Jacquet’s relative trace formula [7]. Even for this case our proof is different.

However, the relative trace formula may yield more information (cf. [8]).

The Tannakian formalism suggests that the symplectic (resp. orthogonal) representations are precisely the functorial images from groups whose $L$-group is a symplectic (resp. orthogonal) group. In fact, it has been proved [5,2] that generic cuspidal representations of $\text{SO}(2n+1)$ are in one-to-one (functorial)
correspondence with the set of families \( \{ \pi_1, \ldots, \pi_k \} \) of distinct cuspidal symplectic representations of \( \text{GL}_{n_i}(\mathbb{A}) \) with \( n_1 + \cdots + n_k = n \). As a consequence:

**Theorem 2.** Let \( \pi \) be a cuspidal generic representation of \( \text{SO}(2n + 1)(\mathbb{A}) \). Then \( L^\times(\frac{1}{2}, \pi) \geq 0 \).

Here the (partial) \( L \)-function corresponds to the standard imbedding of the \( L \)-group \( \text{Sp}_n \) in \( \text{GL}_{2n} \). We could have also taken the completed \( L \)-function as defined by Shahidi.

As a by-product of the proof we also obtain the following result.

**Theorem 3.** Let \( \pi \) be a self-dual cuspidal representation of \( \text{GL}_n(\mathbb{A}) \). Then the root numbers \( \varepsilon(\frac{1}{2}, \pi, \text{sym}^2) \) and \( \varepsilon(\frac{1}{2}, \pi, \wedge^2) \) are equal to one.

A priori one knows that these root numbers are \( \pm 1 \). In [15] Prasad and Ramakrishnan, motivated by results of Fröhlich and Queyrut [4] and Deligne [3], conjectured that \( \varepsilon(\frac{1}{2}, \pi) = 1 \) for any orthogonal representation of \( \text{GL}_n \). Theorem 3 is compatible with this conjecture and Langlands functoriality. We also remark that it is not difficult to prove that \( \varepsilon(\frac{1}{2}, \pi \otimes \pi) = 1 \) for any cuspidal representation \( \pi \) of \( \text{GL}_n \) (cf. [1]).

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2. Reduction to a local statement

As mentioned before, Theorems 1 and 3 are proved using the theory of Eisenstein series on classical groups. Let \( G \) be a split connected classical group (symplectic or special orthogonal) of rank \( n \). We identify \( \text{GL}_n \) with the Levi subgroup \( M = MU \) of \( G \). Let \( \pi \) be a cuspidal representation of \( \text{GL}_n(\mathbb{A}) \) and identify the induced space \( I(\pi, s) \) with the space \( \mathcal{A}_F(\pi, s) \) of automorphic forms \( \varphi \) on \( \text{U}(\mathbb{A}) \mathcal{M}(F) \backslash \text{G}(\mathbb{A}) \) such that the function \( m \mapsto |\det(m)|^{-\delta_P(m)^{-1/2}}\varphi(m) \) belongs to the space of \( \pi \) for any \( k \in \mathbb{K} \), where \( \delta_P \) is the modulus function of \( P(\mathbb{A}) \). We denote by \( E(g, \varphi, s) \) (the meromorphic continuation of) the Eisenstein series for \( \varphi \in I(\pi, s) \). We will be interested in the case where \( E(\bullet, \varphi, s) \) has a pole at \( s = \frac{1}{2} \). A necessary condition is that \( \pi \) is self-dual and that \( P \) is conjugate to its opposite (i.e., \( G \neq \text{SO}(4m + 2) \)). From now on we assume that these conditions are satisfied. Let \( w \in G \) be such that the map \( m \mapsto wmw^{-1} \) induces the involution \( x^2 = w^{-1}x^{-1}w \) on \( \text{GL}_m \) where \( (w_n)_{i,j} = (-1)^{j-i-n+1} \). Let \( E_{-1}(\bullet, \varphi, s) \) be the residue of \( E(g, \varphi, s) \) at \( s = \frac{1}{2} \). Up to a positive constant depending on normalization of measures, the inner product of residues of Eisenstein series is given by

\[
\int_{G/G(\mathbb{A})} E_{-1}(g, \varphi_1)E_{-1}(g, \varphi_2) dg = \int_{\mathcal{M}(\mathbb{A})} \mathcal{M}_{-1}\varphi_1(m)\varphi_2(m) dm, \tag{1}
\]

where \( \mathcal{M}_{-1} \) is the residue of the intertwining operator \( \mathcal{M}(s) : \mathcal{A}_F(\pi, s) \to \mathcal{A}_F(\pi, -s) \) at \( s = \frac{1}{2} \).

Let \( \pi^\natural \) be the (abstract) representation of \( \mathcal{M}(\mathbb{A}) \) on \( V_\pi \) defined by \( \pi^\natural(m)v = \pi(m^2)v \). It is equivalent to the contragredient of \( \pi \). Let \( \mathcal{M}(s) = \mathcal{M}(\pi, s) : I(\pi, s) \to I(\pi^\natural, -s) \) be the “abstract” intertwining operator. Since \( \pi \) is self-dual, and multiplicity one holds for \( \text{GL}_n \), we have an intertwining operator \( \iota = \iota_\pi : \pi^\natural \to \pi \) which does not depend on the automorphic realization of \( \pi \) and which is given by \( \iota(\varphi) = \varphi^\natural \) where \( \varphi^\natural(m) = \varphi(m^2) \). We write \( \iota(s) = \iota(\pi, s) \) for the induced map \( I(\pi^\natural, s) \to I(\pi, s) \) given by \( [\iota(s)f](g) = \iota(f(g)) \). By our identifications we have \( \mathcal{M}(s) = \iota(-s) \circ \mathcal{M}(s) \).

In the local case we can define \( \pi_\pi^\natural \) and the local intertwining operators \( \mathcal{M}_\pi(s) : I(\pi_\pi, s) \to I(\pi_\pi^\natural, -s) \) in the same way. If \( \pi_\pi \) is a local self-dual irreducible generic representation of \( \text{GL}_{n_\pi} \), then fixing an additive character \( \psi_\pi \) we may define an intertwining map \( \iota_\pi = \iota_{\pi_\pi} : \pi_\pi^\natural \to \pi_\pi \) by \( \iota_{\pi_\pi}(W) = W^\natural \) on the Whittaker model. This map does not depend on the choice of Whittaker model. Suppose that \( \pi = \otimes_{\pi_\pi} \pi_\pi \) and \( \psi \) is a global additive character. Then we have \( \iota_\pi = \prod_{\pi_\pi} \iota_{\pi_\pi} \).

Shahidi has defined normalization factors \( m_{\psi}(\pi_\pi, s) = m_{\psi}(s) \) for the local intertwining operators [16]. (We suppress their dependence on \( \psi_\pi \).) Thus we may write \( m_{\psi}(\pi_\pi, s) = m_{\psi}(\pi_\pi, s) R_{\psi}(\pi_\pi, s) \) where \( R_{\psi}(s) = R_{\psi}(\pi_\pi, s) \) are the normalized intertwining operators. Let \( m(s) = m(\pi, s) = \prod_{\pi_\pi} m_{\psi}(\pi_\pi, s) \) and

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\( R(s) = \prod \nu R_\nu(\pi_\nu, s) \) so that \( M(s) = m(s) R(s) \). We have

\[
m(s) = \begin{cases} 
L(s, \pi) & G = \text{Sp}_n, \\
\varepsilon(s, \pi) L(s+1, \pi) & G = \text{SO}(2n+1), \\
\varepsilon(2s, \pi) L(2s+1, \pi) & G = \text{SO}(2n).
\end{cases}
\]

In particular, the residue \( m_{-1} \) at \( s = \frac{1}{2} \) is given by

\[
m_{-1} = \begin{cases} 
\varepsilon(\frac{1}{2}, \pi) L(\frac{1}{2}, \pi) & G = \text{Sp}_n, \\
\varepsilon(1, \pi, \text{sym}^2) L(2, \pi, \text{sym}^2) & G = \text{SO}(2n+1), \\
\varepsilon(1, \pi, \text{sym}^2) L(2, \pi, \text{sym}^2) & G = \text{SO}(2n).
\end{cases}
\]

The Eisenstein series \( E(\bullet, \varphi, s) \) has a pole at \( s = \frac{1}{2} \) if and only if \( m_{-1} \neq 0 \). Note that \( L(3/2, \pi) > 0 \) and that if \( \varepsilon(\frac{1}{2}, \pi) = -1 \) then \( L(\frac{1}{2}, \pi) = 0 \) by the functional equation. Thus, Theorem 1 would follow, if we knew that \( m_{-1} \geq 0 \) in the first and last case. Moreover, the factor \( \frac{\text{res}_{s=1} L(s, \pi, \text{sym}^2)}{L(2, \pi, \text{sym}^2)} \) is positive since \( L(s, \pi, \text{sym}^2) \) is holomorphic and non-zero for \( \text{Re}(s) > 1 \) and real for \( s \in \mathbb{R} \). Similarly for \( \frac{\text{res}_{s=1} L(s, \pi, \text{sym}^3)}{L(2, \pi, \text{sym}^3)} \). On the other hand \( \varepsilon(s, \pi, \text{sym}^2) \) are exponential functions and \( \varepsilon(\frac{1}{2}, \pi, \text{sym}^2) \cdot \varepsilon(\frac{1}{2}, \pi, \text{sym}^3) = \varepsilon(\frac{1}{2}, \pi \otimes \pi) = 1 \).

Hence Theorem 3 would follow, if we knew in addition that \( m_{-1} > 0 \) in the second case. Therefore it remains to show that \( m_{-1} \geq 0 \) in all cases. Let \( \mathcal{B}(s) = \mathcal{B}(\pi, s) \) be the operator \( i(-s) \circ R(s) : I(\pi, s) \to I(\pi, s) \) where \( i \) denotes the Hermitian dual. It is Hermitian for \( s \in \mathbb{R} \) and \( \mathcal{B}(\pi, 0) \) is an involution. Since \( 2\mathcal{B}_{-1} = m_{-1} \cdot \mathcal{B}(\frac{1}{2}) \) the relation, (1) yields that \( \mathcal{B}(\frac{1}{2}) \) is semi-definite and has the same sign as \( m_{-1} \). It remains to show that the sign is positive. In the case where \( \pi_\nu \) is everywhere unramified this follows from the fact that \( \tau_\nu \) and \( R_\nu \) act trivially on the unramified vector. In the general case one knows by [12], Proposition 6.3 that \( \mathcal{B}(\pi, 0) \) has a nontrivial \(+1\)-eigenspace. It remains to show the following:

**Proposition 4.** Suppose that \( \mathcal{B}(\pi, \frac{1}{2}) \) is semi-definite. Then \( \mathcal{B}(\pi, 0) \) is definite (i.e., a scalar, necessarily \( \pm 1 \)), and has the same sign as \( \mathcal{B}(\pi, \frac{1}{2}) \).

This global statement follows from its local counterpart (with analogous notation).

3. Local analysis

Let \( \pi \) be a self-dual generic irreducible unitarizable representation of \( \text{GL}_n(F) \) where \( F \) is now a local field. We say that \( \pi \) is of \( G \)-type if \( \mathcal{B}(\pi, 0) \) is a scalar (necessarily \( \pm 1 \)). We first prove Proposition 4 in the square-integrable case. This requires an analysis of the reducibility points of \( I(\pi, s) \) for \( \pi \) square-integrable, which involves among other things the theory of \( R \)-groups. Such an analysis was carried out by Shahidi, Tadic, Muić Jantzen, Goldberg and others. (See [17, 16, 18, 14, 10, 6].) This analysis also shows that if \( \pi \) is tempered and \( 0 < s < \frac{1}{2} \) then \( I(\pi, s) \) is irreducible.

To prove the local analogue of Proposition 4 for a general \( \pi \), the following elementary lemma from linear algebra will be useful.

**Lemma 5.** Let \( \mathcal{B}_\alpha \), \( 0 \leq \alpha \leq 1 \), be a continuous family of Hermitian operators on a finite dimensional inner product space. Suppose that \( \mathcal{B}_0 \) is positive semi-definite and that the rank of \( \mathcal{B}_\alpha \) is constant for \( 0 \leq \alpha < 1 \). Then \( \mathcal{B}_1 \) is positive semi-definite.
Let $\pi$ be as before. Since any generic irreducible representation of $\text{GL}_n(F)$ is parabolically induced from essentially square-integrable representations, we may write $\pi = \Sigma \oplus \Omega$ ($\Sigma$ stands for induction) where $\Sigma$ is induced from mutually inequivalent, self-dual, square-integrable representations which are not of $G$-type, and $\Omega$ is induced from square integrable self-dual representations of $G$-type, and representations of the form $\rho_j \oplus \rho_j'$ where $\rho_j$ is essentially square-integrable. Moreover, since $\pi$ is unitarizable the central exponents of the $\rho_j$'s are less than $\frac{1}{2}$ in absolute value.

The first step is to reduce to the case where $\pi$ is tempered. By twisting the $\rho_j$'s by unramified characters, we obtain a continuous “deformation” $\{\pi_\alpha\}$ of $\pi$ into a tempered representation. The reduction is achieved by applying Lemma 5 to both families $\mathcal{B}(\pi_\alpha, \frac{1}{2})$ and $\mathcal{B}(\pi_\alpha, 0)$.

Suppose now that $\pi$ is tempered. The main step is to show that $\Sigma = 0$. Indeed, if $\Sigma = 0$ then $\pi$ is of $G$-type and the operators $\mathcal{B}(\pi, s)$ are non-degenerate for $0 \leq s < \frac{1}{2}$ since $I(\pi, s)$ is irreducible for $0 < s < \frac{1}{2}$. We may then apply Lemma 5 once again. To prove that $\Sigma = 0$, we consider a family of intertwining operators $\mathcal{B}(\alpha)$ on $I(\Sigma, \bullet |\alpha| \oplus \Omega \bullet |\frac{1}{2}|, 0)$. As before, we may use Lemma 5 to deform $\alpha$ to 0. On the other hand, if $\mathcal{B}(0)$ were semi-definite, then the same would be true for $\mathcal{B}(\Sigma, 0)$. However, by the theory of $R$-groups, $\mathcal{B}(\Sigma, 0)$ is of order exactly two unless $\Sigma = 0$.

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References

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