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On the "prediction" problem

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Abstract We prove that almost every (in the Baire category sense) weight w on a circle $\mathbb{T}$ satisfies the following property: any function from $\mathrm{L}^2(w,\mathbb{T})$ can be decomposed as a series

$$\sum_{n\in\mathbb{Z}^+} c(n)\,\mathrm{e}^{int}$$

which converges in the norm.

We discuss this result in the context of the classical Szegö–Kolmogorov "prediction" theorem. ***To cite this article: A. Olveskii, C. R. Acad. Sci. Paris, Ser. I 334 (2002) 279–282.*** © 2002 Académie des sciences/Éditions scientifiques et médicales Elsevier SAS

Sur le problème de prédiction

Résumé Au sens des catégories de Baire, presque tout poids w vérifie la propriété suivante : toute fonction appartenant à $\mathrm{L}^2(w,\mathbb{T})$ est décomposable en série

$$\sum_{n\in\mathbb{Z}^+} c(n)\,\mathrm{e}^{int}$$

convergente en norme. Nous discutons la relation de ce résultat avec le théorème de « prédiction » classique de Szegö–Kolmogorov. ***Pour citer cet article : A. Olveskii, C. R. Acad. Sci. Paris, Ser. I 334 (2002) 279–282.*** © 2002 Académie des sciences/Éditions scientifiques et médicales Elsevier SAS

1. Introduction

1.1. In a recent paper [2] joint with G. Kozma, we proved that every measurable function $f:\mathbb{T}\to\mathbb{C}$ can be decomposed into a trigonometric series of analytic type, which converges in measure.

Here we are interested in the "weighted" analog of this result. By weight we mean any measurable function w, $0\leqslant w(t)\leqslant 1$. The set of all such functions endowed with an L^1 distance constitutes a complete metric space W.

Our main result is the following

THEOREM. –
(i) *There exists a weight $w>0$ a.e., such that any function $f\in\mathrm{L}^2(w,\mathbb{T})$ can be decomposed in a series*:

$$f=\sum_{n>0} c(n)\,\mathrm{e}^{int} \tag{1}$$

convergent in the norm.

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For any $p > 2$ the coefficients $\{c(n)\}$ can be chosen in l_p with an arbitrary small norm.

(ii) *The set of such weights is residual in the space W.*

1.2. The classical Szegö–Kolmogorov condition:

$$\int_{\mathbb{T}} \log w(t)\,\mathrm{d}t = -\infty \tag{2}$$

is responsible for completeness of the system $\{\mathrm{e}^{int}\}$, $n > 0$, in $\mathrm{L}^2(w, \mathbb{T})$, see, for example, [1]. In probabilistic language this means that if the spectral density $w(t)$ of a stationary stochastic process $\{X(n)\}$, $n \in \mathbb{Z}$, satisfies condition (2) (and only in this case) one can precisely predict the future from the past. So $X(0)$ might be written with an arbitrary small error ε as a finite linear combination of $X(-n)$, $n > 0$, with some coefficients $\{c(n)\}$ depending on ε.

Since "the past", $X(-n)$ is usually known with some "noise", it is quite reasonable to require coefficients to be small, or at least to satisfy the condition

$$\sup_n \bigl|c(n)\bigr| < C \text{ – a constant not depending on } \varepsilon. \tag{3}$$

We notice that condition (2) does not provide such an approximation. Moreover, it is easy to see that such a "stable forecast" cannot exist unless the spectral density w behaves extremely irregularly, having the essential infimum equal to zero at any arc. However such irregular behavior is "typical" in the Baire sense. Our theorem above shows that for a generic w (that is, for a residual set of w's) one can get not only an approximation with condition (3), but even a decomposition:

$$X(0) = \sum_{n>0} c(n) X(-n).$$

2. Lemmas

We use the following standard notation: mE – the Lebesgue measure of a set E on the circle $\mathbb{T}$, $\mathbb{1}_E$ – the indicator function of E, $\|c\|_p$ – the l_p norm of the sequence $c = \{c(n)\}$, the symbol $\hat{}$ stands for the Fourier transform.

2.1. We start with a simple

LEMMA. – *A generic w in the space W satisfies the following conditions*:

(i) $w(t) > 0$ *a.e.*

(ii) *Given a sequence of sets $V(r) \subset \mathbb{T}$, $mV(r) \to 0$, and a sequence of positive numbers $a(r)$, the inequality below holds for infinitely many r's*:

$$\int_{V(r)} w\,\mathrm{d}t < a(r). \tag{4}$$

We omit the proof.

2.2. By analytic polynomial we mean a trigonometric polynomial with a positive spectrum:

$$Q(t) = \sum_{n>0} c(n)\,\mathrm{e}^{int}.$$

We denote by $S_l(Q)$ the partial sums.

The main ingredient is the following

LEMMA. – *For any $h > 0$ and $p > 2$ one can construct an analytic polynomial Q and a set $E \subset \mathbb{T}$ with conditions*:

(i) $\|\widehat{Q}\|_p < h$;

(ii) $m(\mathbb{T}\setminus E) < h$;

(iii) $|Q(t) - 1| < h$ *on* E;

(iv) *Any partial sum $S = S_l(Q)$ can be decomposed as $S = A + B$ so that*:

$$\|A\|_{\mathrm{L}^\infty(E)} < 2, \qquad \|B\|_{\mathrm{L}^2(\mathbb{T})} < h.$$

This is Lemma 4.1 from [2] strengthened by Remark 2 on p. 383 ibid. The condition (iv) here is written in a different form, which can be seen from the proof.

3. Proof of the theorem

Given $f:\mathbb{T}\to\mathbb{C}$ and $r\in\mathbb{Z}^+$ we denote by $f_{[r]}$ the contracted function:

$$f_{[r]}(t) := f(rt), \quad t\in\mathbb{T}.$$

For $E\subset\mathbb{T}$ the set $E_{[r]}$ is defined by:

$$(\mathbb{1}_E)_{[r]} = \mathbb{1}_{E_{[r]}}.$$

Now, for $h(r)=1/r$, $p(r)=2+1/r$ we find according to Lemma 2.2 an analytic polynomial Q_r and a set $E(r)$. Put:

$$U(r) = \bigl\{E(r)\bigr\}_{[r]}, \qquad V(r) = \mathbb{T}\setminus U(r).$$

Considering the decomposition 2.2(iv):

$$S_l(Q_r) = A(l,r) + B(l,r),$$

denote

$$A(r) := \max_l \bigl\|A(l,r)\bigr\|_{\mathrm{L}^\infty(\mathbb{T})}$$

and set

$$a(r) = \frac{1}{r}\bigl(A^2(r) + \bigl\|\widehat{Q}_r\bigr\|_1^2\bigr)^{-1}. \tag{5}$$

Let W_0 be the set of all positive weights w satisfying condition (4) for infinitely many r's. According to Lemma 2.1 it is residual in W. Fix w in W_0, $f\in\mathrm{L}^2(w,\mathbb{T})$, $p>2$ and $d>0$. We will construct an expansion (1) satisfying the condition $\|c\|_p<d$.

We define the expansion by "blocks" :

$$P_n = \sum_{k\in J_n} c(k)\,\mathrm{e}^{ikt},$$

where J_n are some segments in $\mathbb{Z}^+$. It is enough to get conditions:

$$\begin{aligned}
&\text{(i)} && \min J_n > \max J_{n-1},\\
&\text{(ii)} && \|R_n\| < 1/n\|f\|, \quad R_n := f - \textstyle\sum_{j\leqslant n} P_j,\\
&\text{(iii)} && \max_l \bigl\|S_l(P_n)\bigr\| = \mathrm{O}(1/n),\\
&\text{(iv)} && \|\widehat{P}_n\|_p < d/2^n.
\end{aligned} \tag{6}$$

Here and below, by $\|\cdot\|$ with no subindex, we mean the norm in $\mathrm{L}^2(w,\mathbb{T})$.

Put $P_0=0$, fix n in $\mathbb{Z}^+$ and suppose that P_j, $j<n$, are already defined. Now approximate the remainder R_{n-1} by a trigonometric polynomial g:

$$\|R_{n-1}-g\| < \|f\|/n$$

and take r such that (4) holds. Set:

$$P_n = g\cdot\{Q_r\}_{[r]}.$$

We have to check that if r is chosen sufficiently large then all conditions (6) are fulfilled. For the first and last ones this is obvious. For the second one we write:

$$\|R_n\| = \|R_{n-1}-P_n\| < \frac{\|f\|}{n} + \|g-P_n\| < \frac{\|f\|}{n} + \bigl\|\hat{g}\bigr\|_1\biggl\{\int_{\mathbb{T}}\bigl|1-\{Q_r\}_{[r]}(t)\bigr|^2 w(t)\,\mathrm{d}t\biggr\}^{1/2}.$$

We divide the integral into two parts – over $U(r)$ and over $V(r)$. Notice that due to 2.2(iii) the integrant in the first one is less than $1/r^2$. The second integral is estimated by (4) and (5). All this gives: $\|R_n\| < \|f\|/n + \mathrm{o}(1)$ when $r \to \infty$, so we get (6)(ii) for a large r. Now we mention that if $r > 2\deg g$ than any partial sum $S_l(P_n)$, $l = sr + m$, $-r/2 < m \leqslant r/2$, can be represented as

$$S_l(P_n) = g \cdot \{S_s(Q_r)\}_{[r]} + D(l,n), \quad \|D\| \leqslant \|\hat{g}\|_1 \cdot \|\widehat{Q}_r\|_\infty$$

(compare with (10) in [2]), so using 2.2 (i) and (iv) we get:

$$\|S_l(P_n)\| \leqslant \|g\{A(s,r)\}_{[r]}\| + \|g\{B(s,r)\}_{[r]}\| + 1/r\|\hat{g}\|_1.$$

Since $w \leqslant 1$ we have:

$$\|gB_{[r]}\| \leqslant \|gB_{[r]}\|_{\mathrm{L}^2(\mathbb{T})} \leqslant \|\hat{g}\|_1 \|B\|_{\mathrm{L}^2(\mathbb{T})} < \|\hat{g}\|_1/r.$$

The estimate for A proceeds as follows:

$$\|g \cdot \{A(s,r)\}_{[r]}\|^2 = \int_{\mathbb{T}} |g(t)|^2 |A(s,r)(rt)|^2 w(t)\,\mathrm{d}t = \int_{U(r)} + \int_{V(r)}$$

$$< 4\|g\|^2 + \|g\|^2_{\mathrm{L}^\infty(\mathbb{T})} A^2(r)a(r) \leqslant 4\|g\|^2 + \mathrm{o}(1) < \frac{C\|f\|^2}{n^2} + \mathrm{o}(1) \quad (r \to \infty),$$

so, again for large r, we get (6)(iii). This completes the proof.

4. Remarks

4.1. First we clarify the remark made in 1.2. Let $\{q_j\}$ be analytic polynomials s.t.

$$\|1 - q_n\|_{\mathrm{L}^2_{(w,\mathbb{T})}} = \mathrm{o}(1), \tag{7}$$

where w is a weight, $w(t) > c > 0$ on an arc I.

Then $\|\hat{q}_n\|_\infty \to \infty$.

Indeed, if not, we get a pseudomeasure f which is the limit (in the sense of distributions) of some subsequence $1 - q_{n(j)}$, so $\hat{f}(0) = 1$, $\hat{f}(n) = 0$ for $n < 0$. The conditions (7) together with boundness of w away from zero on I implies that supp f belongs to $\mathbb{T}\backslash I$. Convolving with a smooth "hat", supported by a small neighbourhood of zero, we reach a contradiction to a classical uniqueness theorem.

4.2. Our theorem is true for some other spaces of weights. In particular, one may consider the metric space of indicator functions $\{\mathbb{1}_E\}$ of measurable sets in $\mathbb{T}$ endowed by the L^1 distance, or, equivalently, the space of $\{E\}$, with the distance $d(E,E') = m(E \blacktriangle E')$. Then the proof above works, and it gives the following result:

For a "generic" E every f in $\mathrm{L}^2(E)$ can be decomposed as a series (1) *which converges in the norm.*

Such an E can be compact with the complement of an arbitrary small measure. But lengths of the complementary intervals may not decrease too fast.

4.3. We do not know whether weights w for which the representation (1) does exist can be characterized effectively.

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