Vibration of a pre-constrained elastic thin shell I: Modeling and regularity of the solutions

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Abstract   We study the vibration of an elastic thin shell which is pre-constrained by a large displacement with a small deformation. In this first Note we prove the solutions exist and we investigate both the interior regularity and the boundary regularity which is known to be important in the shape differentiation of hyperbolic equations. To cite this article: J. Cagnol, J.-P. Zolésio, C. R. Acad. Sci. Paris, Ser. I 334 (2002) 161–166. © 2002 Académie des sciences/Éditions scientifiques et médicales Elsevier SAS

Vibration d'une coque élastique mince pré-contrainte I

Résumé   On étudie la vibration d'une coque élastique pré-contrainte par grand déplacement en petites déformations. Dans cette première Note on modélise la vibration et on démontre l'existence des solutions et la régularité intérieure. On termine par une étude de la régularité sur le bord, laquelle est connue pour intervenir dans la dérivée par rapport au domaine dans les équations hyperboliques. Pour citer cet article : J. Cagnol, J.-P. Zolésio, C. R. Acad. Sci. Paris, Ser. I 334 (2002) 161–166. © 2002 Académie des sciences/Éditions scientifiques et médicales Elsevier SAS

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Soit $\Omega^0 \subset \mathbb{R}^3$ une coque mince dans sa position au repos et $\Omega \subset \mathbb{R}^3$ cette coque dans son état contraint qui est un équilibre statique. Soit T_0 une application de $\mathbb{R}^3$ dans $\mathbb{R}^3$ telle que $T_0(\Omega^0) = \Omega$. On suppose que T_0 vérifie que la matrice jacobienne E définie par $DT_0 \circ T_0^{-1}$ a ses coefficients dans $W^{1,\infty}(\mathbb{R}^3)$.

La coque Ω est soumise a une vibration. Soit τ le temps final et $t < \tau$, on note $\Omega(t)$ la coque au temps t et $T(t)$ l'application telle que $T(t)(\Omega) = \Omega(t)$. On suppose que Ω^0 et $\Omega(t)$ sont homogènes et isotropes. On suppose également que T appartient à $L^2([0, \tau], H^1(\Omega, \mathbb{R}^3)) \cap H^1([0, \tau], L^2(\Omega, \mathbb{R}^3))$ et satisfait $T_{|\Gamma_D} = I$ où Γ_D est une partie du bord. Cette application est une perturbation de l'identité, on note $T = I + u$ avec u petit dans H^1. Notons $H^1_{\Gamma_D}(\Omega; \mathbb{R}^3)$ l'ensemble des fonctions de $H^1(\Omega, \mathbb{R}^3)$ qui s'annulent sur Γ_D. Le paramètre u appartient à l'ensemble H qui est défini par (1).

On note $\mathrm{T} = T \circ T_0$. Soit $\widetilde{u} = u \circ T_0$, on a $\mathrm{T}(t) = T_0 + \widetilde{u}(t)$. On note C le tenseur élastique d'ordre 4 et on définit $\varepsilon(u)$ et $\Sigma(u)$ par (2). Soit ϕ and ψ les valeurs de u et $\partial_t u$ à $t = 0$. On suppose $\phi \in H^1(\Omega, \mathbb{R}^3)$ et $\psi \in L^2(\Omega, \mathbb{R}^3)$.

En utilisant le principe d'extrémisation de l'action on montre que l'équation satisfaite par E est (6). On note $\tau < +\infty$ le temps final et $\sigma_D = [0; \tau] \times \Gamma_D$. L'équation de la vibration autour de la forme précontrainte Ω est hyperbolique d'ordre de 2, nous établissons qu'elle est donnée par (3).

THÉORÈME 1. –
 (i) *L'équation* (3) *a une solution unique u dans* $C([0; \tau]; H^1_{\Gamma_D}(\Omega)) \cap C^1([0; \tau]; L^2(\Omega))$;
 (ii) *Pour tout $v \in C^1(\Gamma_D)$ avec $v \geqslant 0$ sur Γ_D et* supp $v \in \Gamma_D$, *on a $v|\varepsilon(u)| \in L^2(\sigma_D)$.*

L'existence des solutions et la régularité intérieure se démontrent en utilisant les propriétés de l'opérateur spatial et les résultats de semi-groupes de [1], la coercivité nécessaire s'obtenant par l'inégalité de Korn et l'estimation (7).

Pour la régularité sur le bord on considère la dérivée, par rapport à s, de :

$$\frac{1}{2} \int_{Q_s} \frac{1}{|\det(E)|} \left({}^*E\varepsilon\left(\varphi \circ T_s^{-1}\right)E \right) .. C .. \left({}^*E\varepsilon\left(\varphi \circ T_s^{-1}\right)E \right),$$

où $A..B = A_{ij}B_{ij}$ représente la double contraction des tenseurs A et B. On note $\mathcal{E}$ cette dérivée en 0. On effectue la dérivation sur le domaine mobile en utilisant [4, lemma 12], il vient (8). On dérive ensuite en se ramenant au domaine fixe grâce à un changement de variables ce qui donne (9). On montre que les expressions obtenues donnent une égalité entre intégrale de volume et intégrale de bord ce qui permet d'établir la régularité du bord, au moyen d'un argument de densité.

L'intérêt de cette régularité sur le bord réside dans sa supériorité par rapport à celle que l'on peut obtenir par les théorèmes de trace. Cette différence de régularité joue un rôle important dans la dérivabilité par rapport au domaine des équations hyperboliques (*cf.* [3,2,4]).

Cette note sera suivie d'une seconde portant sur le modèle $p(d.\infty)$ en géométrie intrinsique.

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1. Introduction

The vibration of a shell constrained to be in a specific shape and the vibration of a shell with that shape at its natural reference position are known to be different by physicists.

Let $\Omega^0 \subset \mathbb{R}^3$ be a shell in its unconstrained state and $\Omega \subset \mathbb{R}^3$ be that shell in a constrained state which is a static equilibrium. Let T_0 be the *static displacement*, that is the mapping from $\mathbb{R}^3$ onto $\mathbb{R}^3$ such that $T_0(\Omega^0) = \Omega$. We assume T_0 is such that the coefficients of the matrix $E = DT_0 \circ T_0^{-1}$ belong to $W^{1,\infty}(\mathbb{R}^3)$, which is the case, for instance, if $T_0 \in W^{2,\infty}(\Omega^0, \mathbb{R}^3)$.

We suppose Ω is under a vibration. Let τ be the final time and $t < \tau$, we note $\Omega(t)$ the shell at the time t and $T(t)$ the mapping such that $T(t)(\Omega) = \Omega(t)$. We suppose Ω^0 and $\Omega(t)$ are homogeneous. Moreover, for the sake of simplicity, we will suppose they are isotropic. We assume T belongs to $L^2([0, \tau], H^1(\Omega, \mathbb{R}^3)) \cap H^1([0, \tau], L^2(\Omega, \mathbb{R}^3))$ and satisfies the embedding condition of the shell to Γ_D, that is $T_{|\Gamma_D} = I$. The mapping T is a perturbation of I, we have $T = I + u$ where u is small in H^1. From now, u will be the parameter to be considered. We note $H^1_{\Gamma_D}(\Omega; \mathbb{R}^3)$ the set of functions of $H^1(\Omega, \mathbb{R}^3)$ that vanish on Γ_D. The parameter u belongs to:

$$H = L^2\left([0, \tau], H^1_{\Gamma_D}\left(\Omega, \mathbb{R}^3\right)\right) \cap H^1\left([0, \tau], L^2\left(\Omega, \mathbb{R}^3\right)\right). \tag{1}$$

We note $T = T \circ T_0$. Let $\tilde{u} = u \circ T_0$, we have $T(t) = T_0 + \tilde{u}(t)$. We note C the 4-order elastic tensor:

$$\varepsilon(u) = \frac{1}{2}\left({}^*Du + Du \right) \quad \text{and} \quad \Sigma(u) = \frac{2}{|\det(E)|} E \left(C .. \left({}^*E\varepsilon(u)E \right) \right) {}^*E, \tag{2}$$

where $A..B = A_{ij}B_{ij}$ and E is the matrix $(DT_0) \circ T_0^{-1}$.

Let ϕ and ψ be the value of u and $\partial_t u$ at $t = 0$. We suppose $\phi \in H^1(\Omega, \mathbb{R}^3)$ and $\psi \in L^2(\Omega, \mathbb{R}^3)$. The equation of the vibration around the natural shape of the joint Ω is given by the hyperbolic equation:

$$\begin{cases} \rho \partial_t^2 u - \operatorname{div}(\Sigma(u)) = 0 & \text{on } [0; \tau[\times \Omega, \\ \Sigma(u) \cdot n = 0 & \text{on } \sigma = [0; \tau[\times \Gamma, \\ u(0) = \phi, \ \partial_t u(0) = \psi & \text{on } \Omega. \end{cases} \tag{3}$$

We will note $P(u) = \rho \partial_t^2 u - \operatorname{div}(\Sigma(u))$. We first prove the system above is the equation of the vibration in Section 2. Then, in Section 3, we prove:

THEOREM 1. – *Let $\sigma_D = [0; \tau[\times \Gamma_D$.*
(i) *System (3) has a unique solution u in $C([0; \tau]; H^1_{\Gamma_D}(\Omega)) \cap C^1([0; \tau]; L^2(\Omega))$;*
(ii) *$\forall v \in C^1(\Gamma_D)$ with $v \geqslant 0$ on Γ_D and supp $v \in \Gamma_D$ we have $v|\varepsilon(u)| \in L^2(\sigma_D)$.*

This note will be followed by another one on the $p(d.\infty)$ model with intrinsic geometry.

2. The modeling

2.1. Equation of the vibration

Let C be the 4-order elastic tensor. It satisfies $\forall (i, j, k, l) \in \{1, 2, 3\}^4$, $C_{ijkl} = C_{jikl} = C_{klij}$ and it is constant since the shell is homogeneous. Moreover we assume $\exists \alpha > 0$, $\forall \xi$, $C_{ijkl}\xi_{ij}\xi_{kl} > \alpha \xi_{i,j}^2$ this allows the definition of norm $|\cdot|$ such that $|\Xi|^2 = \Xi..C..\Xi$. We consider:

$$\bar{\varepsilon}(u) = \frac{1}{2}(^*DT\, DT - I);$$

the elements of this matrix belong to $H^1([0, \tau], L^2(\Omega, \mathbb{R}^3)) \cap L^2([0, \tau], H^{-1}(\Omega, \mathbb{R}^3))$. When no confusion is possible we shall note $\bar{\varepsilon}$ instead of $\bar{\varepsilon}(u)$. A similar notation will be used for all subsequent functions depending on u and will not be pointed out again. The elastic energy is given by $E_p = \int_{\Omega^0} \bar{\varepsilon}..C..\bar{\varepsilon}$. At the first order $2\bar{\varepsilon} = {}^*DT_0((I + 2\varepsilon) \circ T_0)DT_0 - I$. The elastic energy of Ω is given by

$$E_p = \frac{1}{4}\int_\Omega (^*E(I + 2\varepsilon(u))E - I)..C..(^*E(I + 2\varepsilon(u))E - I)\frac{1}{|\det(E)|}.$$

Let ρ be the density and v be the speed vector field of the vibrating body. It is defined on $\Omega(t)$ by $v(t, x) = \partial_t T \circ T^{-1}$. The kinetic energy of $\Omega(t)$ is given by

$$E_k(t) = \frac{1}{2}\rho \int_\Omega (\partial_t u)^2.$$

Let $\tau < +\infty$ be the final time and the action A be defined by $A = \int_0^\tau (E_p - E_k(t))\, dt$. We have:

$$A(u) = \frac{1}{4}\int_0^\tau \int_\Omega (^*E(I + 2\varepsilon(u))E - I)..C..(^*E(I + 2\varepsilon(u))E - I)\frac{1}{|\det(E)|} - 2\rho(\partial_t u)^2.$$

Let θ be a real and $w \in \mathbb{R}^3$; we note $A'(u; w) = \frac{\partial}{\partial \theta}A(u + \theta w)|_{\theta=0}$. The physical evolutions of the structures are characterized by the functions u, defined on $[0; \tau[\times \Omega$, such that $A'(u; w) = 0$ for all test w such that $w(0) = 0$ and $w(\tau) = 0$. Using the linearity of ε we get:

$$A'(u; w) = \int_0^\tau \int_\Omega \Sigma(u)..\varepsilon(w) + (^*EE - I)..C..(^*E\varepsilon(w)E)\frac{1}{|\det(E)|} - \rho \partial_t u \partial_t w.$$

When $u = 0$ we have $T = T_0$ which gives a minimum for the energy, thus we get $A'(0; w) = 0$. Hence

$$\int_0^\tau \int_\Omega (\,^*EE - I)..C..(\,^*E\varepsilon(w)E) \frac{1}{|\det(E)|} = 0. \tag{4}$$

Using this identity in the general expression of $A'(u; w) = 0$, we get

$$\int_0^\tau \int_\Omega \Sigma(u)..\varepsilon(w) - \rho\partial_t u \partial_t w = 0 \tag{5}$$

from Green's formula $\int_0^\tau\int_\Omega - \mathrm{div}(\Sigma(u))w + \rho\partial_t^2 u w + \int_0^\tau\int_\Gamma \Sigma(u)nw - \int_\Omega \partial_t u(0)w(0) + \partial_t u(\tau)w(\tau) = 0$ that equality holds for all tests w therefore the vibration is governed by (3).

Remark 1. – The operator Σ depends on E.

2.2. Equation satisfied by E

The matrix E is $DT_0 \circ T_0^{-1}$ where T_0 is a local minimum for the elastic energy. We have obtained (4) which strong formulation leads to:

$$\mathrm{div}\left(\frac{1}{\det E} E\left((\,^*EE - I)..C \right) \,^*E \right) = 0, \tag{6}$$

that is $\mathrm{div}((\frac{1}{\det(DT_0)} DT_0(\bar\varepsilon(0)..C)\,^*DT_0) \circ T_0^{-1}) = 0$; moreover T_0 is assumed to be a small deformation hence $\|\bar\varepsilon(0)\|_{L^\infty}$ is small as compared to $\|I\|_{L^\infty}$. On the other hand from that last assertion comes the existence of a real $c > -1$ such that

$$\forall v \in \mathbb{R}^3, \quad \left\| \,^*EEv \right\|_{L^2} \geqslant (1 + c)\|v\|_{L^2}; \tag{7}$$

the same inequality holds for $E\,^*E$.

3. Existence and regularity of the solutions

3.1. Interior regularity

The existence and regularity of the solutions will be derived from the theory of semi-groups. Let Λ be the linear operator in $L^2(\Omega)$ defined by $D(\Lambda) = H^2(\Omega; \mathbb{R}^3) \cap H^1_{\Gamma_D}(\Omega; \mathbb{R}^3)$ and $\Lambda = -\frac{1}{\rho} \mathrm{div}(\Sigma)$.

LEMMA 2. – *The operator Λ is self-adjoint.*

LEMMA 3. – *The operator Λ is coercive in $H^1_{\Gamma_D}(\Omega)$.*

Proof. – We use the Hooke's law, [6] and $\lambda \geqslant 0$ to derive:

$$\int_\Omega \Lambda(u)u \geqslant \frac{2\mu}{\rho} \int_\Omega \frac{1}{|\det(E)|} \left(\,^*E\varepsilon(u)E \right)..\left(\,^*E\varepsilon(u)E \right);$$

then using (7), the Korn's inequality and $u_{|\Gamma_D} = 0$ we obtain the the existence of a nonnegative real c such that

$$\int_\Omega \Lambda(u)u \geqslant \left(\frac{2kc\mu}{\rho} \frac{1}{\max_\Omega |\det(E)|} \right) \|u\|_{H^1(\Omega)}. \qquad \square$$

We note

$$U = \begin{pmatrix} u \\ \partial_t u \end{pmatrix}, \quad U_0 = \begin{pmatrix} \phi \\ \psi \end{pmatrix} \quad \text{and} \quad A = \begin{pmatrix} 0 & 1 \\ -\Lambda & 0 \end{pmatrix}$$

then (3) is equivalent to:

$$\partial_t U = AU, \quad U(0) = U_0.$$

Following [1], Proposition 2.12, we have

PROPOSITION 4. – *The operator A is the infinitesimal generator of a strongly continuous semi-group of contraction S on* H. *Moreover $^*A = -A$.*

The first part of Theorem 1 derives from [1], Proposition 3.3 and Proposition 4.

3.2. Boundary regularity

Let φ be a solution to (3). We consider a flow mapping T_s and the associated vector field V (*cf.* [7] or [4]). Let us compute the derivative with respect to s of

$$\frac{1}{2} \int_{Q_s} \frac{1}{|\det(E)|} \left({}^*E\varepsilon \left(\varphi \circ T_s^{-1} \right) E \right) .. C .. \left({}^*E\varepsilon \left(\varphi \circ T_s^{-1} \right) E \right)$$

at $s = 0$, via two different ways. That derivative will be denoted $\mathcal{E}$ and is call the *extractor*. We want the distributed integral to be defined for $\varphi \in C([0; \tau]; H^1_{\Gamma_D}(\Omega)) \cap C^1([0; \tau]; L^2(\Omega))$ such that $P(\varphi) \in L^2(\Omega)$.

Using [4] we have:

LEMMA 5. – *One has*:

$$\mathcal{E} = \int_Q \frac{1}{|\det(E)|} \left(\operatorname{div}\left(V(0) \right) |\partial_t \varphi|^2 + \left\langle D\varphi . \partial_t V(0), \partial_t \varphi \right\rangle \right) dx \, dt$$

$$- \int_\sigma \frac{1}{|\det(E)|} \left(\left\langle D\varphi . V(0), \Sigma(\varphi)n \right\rangle + \frac{1}{2} \left({}^*E\varepsilon(\varphi)E \right) .. C .. \left({}^*E\varepsilon(\varphi)E - |\partial_t \varphi|^2 \right) \left\langle V(0), n \right\rangle \right). \quad (8)$$

From the change of variable T_s in the left-hand side and the computation of the derivatives with respect to s we derive the:

LEMMA 6. – *One has*

$$\mathcal{E} = \int_Q \frac{1}{|\det(E)|} \left(\frac{1}{2} \operatorname{tr}\left(E^{-1} . DE . V(0) \right) \left({}^*E\varepsilon(\varphi)E \right) .. C .. \left({}^*E\varepsilon(\varphi)E \right) \right.$$

$$+ \left({}^*DE . V(0)\varepsilon(\varphi) \right) .. C .. \left({}^*E\varepsilon(\varphi)E \right) + \frac{1}{2} {}^*E\varepsilon \left(D\varphi . DV(0) \right) E \right) .. C .. \left({}^*E\varepsilon(\varphi)E \right)$$

$$\left. + \left({}^*E\varepsilon(\varphi)DE . V(0) \right) .. C .. \left({}^*E\varepsilon(\varphi)E \right) + \frac{1}{2} \operatorname{div}\left(V(0) \right) \left({}^*E\varepsilon(\varphi)E \right) .. C .. \left({}^*E\varepsilon(\varphi)E \right) \right) dx \, dt. \quad (9)$$

LEMMA 7. – *On Γ_D we have*:

$$\left\langle \Sigma(\varphi).n, D\varphi . V(0) \right\rangle = \left({}^*E\varepsilon(\varphi)E \right) .. C .. \left({}^*E\varepsilon(\varphi)E \right) \left\langle V(0), n \right\rangle = \left| {}^*E(D\varphi.n) {}^*nE \right| \left\langle V(0), n \right\rangle.$$

PROPOSITION 8. – *One has*:

$$\int_{\sigma_D} \frac{1}{|\det(E)|} \left| {}^*E(D\varphi.n) {}^*nE \right|^2 \left\langle V(0), n \right\rangle = B(\varphi),$$

where B is defined by

$$B(\varphi) = \int_Q \frac{2}{|\det(E)|}\bigg(\operatorname{div}\big(V(0)\big)|\partial_t\varphi|^2 + \big\langle D\varphi.\partial_t V(0), \partial_t\varphi\big\rangle + \big\langle D\varphi.V(0), P(\varphi)\big\rangle$$

$$-\frac{1}{2}\big({}^*E\varepsilon\big(D\varphi.DV(0)\big)E\big)..C..\big({}^*E\varepsilon(\varphi)E\big) - \frac{1}{2}\operatorname{div}\big(V(0)\big)\big({}^*E\varepsilon(\varphi)E\big)..C..\big({}^*E\varepsilon(\varphi)E\big)$$

$$-\frac{1}{2}\operatorname{tr}\big(E^{-1}.DE.V(0)\big)\big({}^*E\varepsilon(\varphi)E\big)..C..\big({}^*E\varepsilon(\varphi)E\big)$$

$$-\big({}^*DE.V(0)\varepsilon(\varphi)E + {}^*E\varepsilon(\varphi)DE.V(0)\big)..C..\big({}^*E\varepsilon(\varphi)E\big)\bigg).$$

Proof. – We use Lemmas 5 and 6 to compute $\mathcal{E}$ via two different ways. We have $\langle V(0), n\rangle = 0$ on $\overline{\Gamma_N}$ moreover $\partial_t\varphi = 0$ on Γ_D. We use Lemma 7 to prove the proposition. $\quad\square$

Remark 9. – The real $B(\varphi)$ is defined when $\varphi \in C([0; \tau]; H^1_{\Gamma_D}(\Omega)) \cap C^1([0; \tau]; L^2(\Omega))$ and $P(\varphi) \in L^1([0, \tau[, L^2(\Omega))$.

Let $(\varphi^m)_m$ be a sequence of functions of $C^\infty(Q)$ that vanish on Γ_D and such that $\varphi^m \to u$ in $H^1(Q)$ and $P\varphi^m \to Pu$ in $L^2(Q)$. The proof of the density is similar to the one given in [5]. Since $B(u)$ exists, $\||\varepsilon(\varphi^m)|\sqrt{\langle V(0), n\rangle}\|_{L^2(\sigma_D)}$ is bounded, hence there exists a function ξ in $L^2(\Gamma_D)$ and a subsequence such that

$$\varepsilon(\varphi^m)\sqrt{\langle V(0), n\rangle} \rightharpoonup \xi \quad \text{weakly in } L^2(\Gamma_D) \text{ as } m_k \to +\infty.$$

On σ_D Green's theorem proves $\xi = \Sigma(u)..\varepsilon(u)\sqrt{\langle V(0), n\rangle}$.

Therefore for all $\nu \in C^1(\Gamma_D)$ with $\nu \geqslant 0$ on Γ_D and $\operatorname{supp}\nu \in \Gamma_D$ we have $\nu|\varepsilon(u)| \in L^2(\sigma_D)$. The second part of theorem 1 derives.

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