

ANNALES DE L'I. H. P., SECTION C

P. DELANOË

**Classical solvability in dimension two of the
second boundary-value problem associated with
the Monge-Ampère operator**

Annales de l'I. H. P., section C, tome 8, n° 5 (1991), p. 443-457

[<http://www.numdam.org/item?id=AIHPC_1991__8_5_443_0>](http://www.numdam.org/item?id=AIHPC_1991__8_5_443_0)

© Gauthier-Villars, 1991, tous droits réservés.

L'accès aux archives de la revue « Annales de l'I. H. P., section C » (<http://www.elsevier.com/locate/anihpc>) implique l'accord avec les conditions générales d'utilisation (<http://www.numdam.org/conditions>). Toute utilisation commerciale ou impression systématique est constitutive d'une infraction pénale. Toute copie ou impression de ce fichier doit contenir la présente mention de copyright.

NUMDAM

Article numérisé dans le cadre du programme
Numérisation de documents anciens mathématiques
<http://www.numdam.org/>

Classical solvability in dimension two of the second boundary-value problem associated with the Monge-Ampère operator

by

P. DELANOË*

C.N.R.S., Université de Nice-Sophia Antipolis,
I.M.S.P., parc Valrose, 06034 Nice Cedex, France

ABSTRACT. — Given two bounded strictly convex domains of \mathbb{R}^n and a positive function on their product, all data being smooth, find a smooth strictly convex function whose gradient maps one domain onto the other with Jacobian determinant proportional to the given function. We solve this problem under the (technical) condition $n = 2$.

Key words : Strictly convex functions, prescribed gradient image, Monge-Ampère operator, continuity method, *a priori* estimates.

RÉSUMÉ. — Soit deux domaines bornés strictement convexes de \mathbb{R}^n et une fonction positive définie sur leur produit, ces données étant lisses, trouver une fonction lisse strictement convexe dont le gradient applique un domaine sur l'autre avec déterminant Jacobien proportionnel à la fonction donnée. Nous résolvons ce problème sous la condition (technique) $n = 2$.

Classification A.M.S. : 35 J 65, 35 B 45, 53 C 45.

(*) Partially supported by the CEE contract GADGET #SC1-0105-C.

I. INTRODUCTION

Let D and D^* be bounded C^∞ strictly convex domains of \mathbb{R}^n . We denote by $S(D, D^*)$ the subset of $C^\infty(\bar{D})$ consisting of strictly convex real functions ⁽¹⁾ whose gradient maps D onto D^* . Given any $u \in C^\infty(\bar{D})$, we denote by $A(u)$ the Jacobian determinant of the gradient mapping $x \rightarrow du(x)$. The nonlinear second order differential operator A is called the *Monge-Ampère* operator on D . Basic features of A restricted to $S(D, D^*)$ are listed in the preliminary

PROPOSITION 1. — A sends $S(D, D^*)$ into

$$\Sigma := \{f \in C^\infty(\bar{D}), f > 0, \langle f \rangle = |D^*|/|D|\}$$

($\langle f \rangle$ denotes the average of f over D and $|D|$, the Lebesgue measure of D). On $S(D, D^*)$, A is elliptic and its derivative is divergence-like. Given any defining function h^* of D^* , the boundary operator $u \rightarrow B(u) := h^*(du)|_{\partial D}$ is co-normal with respect to A on $S(D, D^*)$. Furthermore, given any $u \in S(D, D^*)$ and any $x \in \partial D$, the co-normal direction at x with respect to the derivative of A at u is nothing but the normal direction of ∂D^* at $du(x)$.

We postpone the proof of proposition 1 till the end of this section. The second boundary-value problem consists in showing that $A : S(D, D^*) \rightarrow \Sigma$ is onto. More generally, we wish to solve in $S(D, D^*)$ two kinds of equations namely

$$\text{Log } A(u) = f(x, du) + \langle u \rangle \quad (1)$$

$$\text{Log } A(u) = F(x, du, u) \quad (2)$$

where $f \in C^\infty(\bar{D} \times \bar{D}^*)$ and $F \in C^\infty(\bar{D} \times \bar{D}^* \times \mathbb{R})$, the latter being uniformly increasing in u . We aim at the following

THEOREM. — Equations (1) and (2) are uniquely solvable in $S(D, D^*)$ provided $n=2$.

The second boundary-value problem was first posed and solved (with $n=2$ but the methods, geometric in nature, extend to any dimension) in a generalized sense in [18] chapter V section 3 (see also [3] theorem 2, where the whole plane is taken in place of D). The elliptic Monge-Ampère operator with a quasilinear Neumann boundary condition is treated in [16], in any dimension, and it is further treated with a quasilinear oblique boundary condition in [21] provided $n=2$. A general study of nonlinear oblique boundary-value problems for nonlinear second order uniformly

⁽¹⁾ Here the meaning of "strictly convex" is restricted to having a positive-definite hessian matrix, which rules out e.g. the strictly convex function $u(x) = |x - y|^4$ near $y \in D$, as pointed out to us by Martin Zerner.

elliptic equations is performed in [15]. Quite recently, the following problem was solved [5]: existence and regularity on a given bounded domain D of \mathbb{R}^n (no convexity assumption, no restriction on n) of a diffeomorphism from \bar{D} to itself, reducing to the identity on ∂D , with prescribed positive Jacobian determinant (of average 1 on D).

Remarks. — 1. The restriction $n=2$ is unsatisfactory but we could not draw second order boundary estimates without it. In May 1988, in Granada (Spain), Neil Trudinger informed us that Kai-Sing Tso had treated the problem in *any* dimension; however, from that time on, Tso's preprint has not been available due to a serious gap in his proof, as he himself wrote us [20]. In June 1989, John Urbas visited us in Antibes and he kindly advised us to submit our own 2-dimensional result; it is a pleasure to thank him for his thorough reading of the present paper. This may be the right place to thank also the Referee for pointing out a mistake at the end of the original proof of proposition 2 below, and a few inaccuracies (particularly one in remark 6).

2. We do *not* assume the non-emptiness of $S(D, D^*)$ to prove the theorem; we thus *obtain* it (when $n=2$) as a by-product of our proof. In fact, we found no straightforward way of exhibiting any member of $S(D, D^*)$ —except, of course, if $D=D^*$ —, although we can write down explicitly a $C^\infty(\bar{D})$ convex (but not *strictly* convex) function with gradient image D^* , constructed from any suitable *support function* for D^* . Provided non-emptiness, it is possible to prove that $S(D, D^*)$ is a locally closed Fréchet submanifold of the open subset of strictly convex functions in $C^\infty(\bar{D})$, as the fiber of a submersion.

3. From the proof below, it appears that, given any $\alpha \in (0, 1)$ $C^{2,\alpha}(\bar{D})$ solutions may be derived (by approximation) from the above theorem under the sole regularity assumptions: D and D^* are $C^{2,1}$, f and F are $C^{1,1}$. We did not study further 2-dimensional global regularity refinements as done in [19], [14] for the Dirichlet problem.

4. The uniqueness for (1) shows that, in general, the equation $\text{Log } A(u) = f(x, du)$ is *not* well-posed on $S(D, D^*)$. The idea of introducing in (1) the average term goes back to [6] and it proved to be useful in various contexts ([2], [8], [9], [10]). If $u \in S(D, D^*)$ solves (1), then $v = u + \text{Const.}$ solves in $S(D, D^*)$ the equation $\text{Log } A(v) = f(x, dv) + \langle u \rangle$, while the Legendre transform v^* of v solves in $S(D^*, D)$ the “dual” equation $\text{Log } A(v^*) = -f(dv^*, x) - \langle u \rangle$. In case $f(x, x^*) = f_1(x) - f_2(x^*)$, the value of $\langle u \rangle$ is *a priori* fixed by the constraint (due to the “Jacobian” structure of A)

$$\int_{D^*} e^{f_2(x^*)} dx^* = e^{\langle u \rangle} \int_D e^{f_1(x)} dx.$$

The prescribed Gauss-curvature equation is an example of this type.

Proof of proposition 1. — By its very definition, as the *Jacobian* of the gradient mapping, A readily sends $S(D, D^*)$ into the submanifold Σ .

Let $u \in S(D, D^*)$. In euclidean co-ordinates (x^1, \dots, x^n) , $A(u)$ reads

$$A(u) = \det(u_{ij})$$

and the derivative of A at u reads

$$\delta u \in C^\infty(\bar{D}) \rightarrow dA(u)(\delta u) = A^{ij}(\delta u)_{ij}$$

where

$$A^{ij} = A(u) u^{ij}$$

(indices denote partial derivatives, Einstein's convention holds, (u^{ij}) is the matrix inverse of (u_{ij}) and (A^{ij}) , its co-matrix). Since u is strictly convex, A is indeed *elliptic* at u . Furthermore, one easily verifies the following identity: for any $\delta u \in C^\infty(\bar{D})$,

$$A^{ij}(\delta u)_{ij} \equiv [A^{ij}(\delta u)]_j.$$

So, as asserted, $dA(u)$ is *divergence-like*. The co-normal boundary operator associated with A at u is

$$\delta u \in C^\infty(\bar{D}) \rightarrow \beta(\delta u) = A^{ij} N^i(\delta u)_j \in C^\infty(\partial D),$$

N standing for the outward unit normal on ∂D . Fix a defining function h^* for D^* (i.e. $h^* \in C^\infty(\bar{D}^*)$ is strictly convex and vanishes on ∂D^*). Since $u \in S(D, D^*)$, the function $H := h^*(du) \in C^\infty(\bar{D})$ is negative inside D and vanishes on ∂D . Moreover, a straightforward computation yields in D :

$$u^{ij} H_{ij} - u^{ij} [\text{Log } A(u)]_i H_j = u_{ij} h_{ij}^* > 0.$$

Hopf's lemma [12] implies that $H_N > 0$ on ∂D . Since

$$H_i = u_{ij} h_j^*$$

the boundary operators satisfy

$$A(u) dB(u) = H_N \beta.$$

So B is indeed *co-normal* with respect to A at u .

Last, the geometric interpretation of the co-normal direction β given at the end of proposition 1, simply follows from the fact that $dB(u)(x)$ equals the derivative in the direction of $dh^*[du(x)]$ which is precisely (outward) *normal* to ∂D^* at $du(x)$. \square

II. THE CONTINUITY METHOD

Fix $(x_0, x_0^*) \in D \times D^*$ and $\lambda \in (0, 1]$ such that the gradient of

$$v_0 = \frac{\lambda}{2} |x - x_0|^2 + x_0^* \cdot x$$

maps \bar{D} into D^* ($|\cdot|$ stands for the standard euclidean norm, \cdot for the euclidean scalar product). Set $u_0 := v_0 - \langle v_0 \rangle$, $D_0 := du_0(D)$. A routine verification shows that D_0 is C^∞ strictly convex. Let $t \in [0, 1] \rightarrow D_t$ be a smooth path of bounded C^∞ strictly convex domains connecting D_0 to $D_1 = D^*$, with $D_t \subset D_{t'}$ for $t < t'$; fix $t \rightarrow h_t$ a smooth path of corresponding defining functions. For each $t \in [0, 1]$, consider in $S(D, D_t)$ the two following equations:

$$\text{Log } A(u) = tf(x, du) + (1-t)n \text{Log } \lambda + \langle u \rangle \quad (1.t)$$

$$\text{Log } A(u) = tF(x, du, u) + (1-t)(u - u_0 + n \text{Log } \lambda). \quad (2.t)$$

By construction u_0 solves both equations for $t=0$, so (for $i=1, 2$) the sets $T_i := \{t \in [0, 1], (i.t) \text{ admits a solution in } S(D, D_t)\}$ are non-empty. Hereafter, we show that they are both relatively open and closed in $[0, 1]$: if so, by connectedness, they coincide with all of $[0, 1]$. The solutions for $t=1$ are those announced in the theorem; their uniqueness is established at the end of this section.

Let us show that T_1 is relatively open in $[0, 1]$; similar, more standard (due to the monotonicity assumption of F), reasonings hold for T_2 . Fix $\alpha \in (0, 1)$ and denote by $U^{2,\alpha}$ the open subset of $C^{2,\alpha}(\bar{D})$ consisting of strictly convex functions. On $[0, 1] \times U^{2,\alpha}$, consider the smooth map (M, B) defined by

$$\begin{aligned} M(t, u) &:= \text{Log } A(u) - tf(x, du) - (1-t)n \text{Log } \lambda - \langle u \rangle, \\ B(t, u) &:= h_t(du)|_{\partial D}, \end{aligned}$$

and ranging in $C^{0,\alpha}(\bar{D}) \times C^{1,\alpha}(\partial D)$. Let $t_0 \in T$; there thus exists u_0 in $U^{2,\alpha}$ such that $(M, B)(t_0, u_0) = (0, 0)$. The proof is based on the Banach implicit function theorem applied to (M, B) at (t_0, u_0) . We want to show that the map

$$(m, b) := [M_u(t_0, u_0), B_u(t_0, u_0)]: C^{2,\alpha}(\bar{D}) \rightarrow C^{0,\alpha}(\bar{D}) \times C^{1,\alpha}(\partial D)$$

is an isomorphism. Record the following expression of (m, b) in euclidean co-ordinates:

$$\begin{aligned} m(\delta u) &= u_0^{ij}(\delta u)_{ij} - t_0 f_{u_i}(x, du_0)(\delta u)_i - \langle \delta u \rangle, \\ b(\delta u) &= (h_t)_i(du_0)(\delta u)_i. \end{aligned}$$

From proposition 1, we know that b is oblique; so Hopf's maximum principle [11] combined with Hopf's lemma [12] imply that any $\delta u \in \text{Ker}(m, b)$ is constant, hence actually $\langle \delta u \rangle = 0$ and $\delta u \equiv 0$. Therefore (m, b) is one-to-one.

Now we fix $(\delta M_0, \delta B_0) \in C^{0,\alpha}(\bar{D}) \times C^{1,\alpha}(\partial D)$ and we look for δu in $C^{2,\alpha}(\bar{D})$ solving: $(m, b)(\delta u_0) = (\delta M_0, \delta B_0)$. Consider the auxiliary map

$$(m', b') := \{A(u_0)(m + \langle \cdot \rangle), [A(u_0)/H_N]b\},$$

where $H = h_t(du_0)$. It follows from proposition 1 that, given any $(\delta M', \delta B') \in C^{0,\alpha}(\bar{D}) \times C^{1,\alpha}(\partial D)$, the function $\delta u' \in C^{2,\alpha}(\bar{D})$ solves:

$$(m', b')(\delta u') = (\delta M', \delta B'), \quad (3)$$

if and only if, for every $\delta v' \in W^{1,2}(D)$,

$$L(\delta u', \delta v') = \int_{\partial D} \delta B' \delta v \, da - \int_D \delta M' \delta v \, dx$$

(da is the measure induced on ∂D by dx), where L is the continuous bilinear form on $W^{1,2}(D)$ given by

$$L(\delta u', \delta v') := \int_D A(u_0) [u_0^{ij} (\delta u')_i (\delta v')_j + t_0 f_{u_i}(x, du_0) (\delta u')_i \delta v'] \, dx.$$

Let us argue on (m', b') as in [6]. Combining the ellipticity of m' and the obliqueness of b' (asserted by proposition 1), with Hopf's maximum principle, Schauder's estimates and Fredholm's theory of *compact* operators, we know that the kernel of the adjoint of (m', b') (formally obtained by varying the first argument of L instead of the second, and by integrating by parts) is *one-dimensional*, let $\delta w \in C^{2,\alpha}(\bar{D})$ span it, and that (3) is solvable up to an additive constant if and only if

$$\int_{\partial D} \delta B' \delta w \, da - \int_D \delta M' \delta w \, dx = 0. \quad (4)$$

Observe that

$$\int_D A(u_0) \delta w \, dx \neq 0$$

since, otherwise, one could solve (3) with $(\delta M', \delta B') = [A(u_0), 0]$ contradicting the maximum principle. We may thus normalize δw by

$$\int_D A(u_0) \delta w \, dx = 1.$$

Then we can solve (3) with right-hand side equals:

$$\left\{ A(u_0) \left[\delta M_0 + \int_{\partial D} [A(u_0)/H_N] \delta B_0 \delta w \, da - \int_D A(u_0) \delta M_0 \delta w \, dx \right], [A(u_0)/H_N] \delta B_0 \right\}$$

since the latter satisfies (4). If $\delta u'_0$ is a solution, then

$$\delta u_0 = \delta u'_0 - \langle \delta u'_0 \rangle + \int_{\partial D} [A(u_0)/H_N] \delta B_0 \delta w \, da - \int_D A(u_0) \delta M_0 \delta w \, dx$$

solves the original equation

$$(m, b)(\delta u_0) = (\delta M_0, \delta B_0).$$

So (m, b) is also *onto*. The implicit function theorem thus implies the existence of a real $\delta > 0$ and of a smooth map

$$t \in (t_0 - \delta, t_0 + \delta) \cap [0, 1] \rightarrow u_t \in U^{2, \alpha}$$

such that $(M, B)(t, u) = (0, 0)$. By proposition 1 and standard elliptic regularity [1], $u_t \in S(D, D_t)$, hence T_1 is relatively open. \square

Assuming $n=2$, we shall carry out a $C^{2, \alpha}(\bar{D})$ *a priori* bound, independent of $t \in [0, 1]$, on the solutions in $S(D, D_t)$ of equations (1. t) and (2. t). Provided such a bound exists, the closedness of T_i ($i=1, 2$) follows in a standard way from Ascoli's theorem combined with proposition 1 and elliptic regularity [1].

Last, let us prove that (1) admits *at most one* solution in $S(D, D^*)$; a similar argument holds for (2). By contradiction, let u_0 and u_1 be two distinct solutions of (1) in $S(D, D^*)$. Then, for $t \in [0, 1]$, $u_t := tu_1 + (1-t)u_0 \in S(D, D^*)$ and $u := u_1 - u_0$ solves the linear boundary-value problem:

$$\begin{aligned} \left(\int_0^1 u_t^{ij} dt \right) u_{ij} - \left[\int_0^1 f_{u_t}(x, du_t) dt \right] u_i - \langle u \rangle &= 0 \quad \text{in } D, \\ \left[\int_0^1 (h_1)_i(du_t) dt \right] u_i &= 0 \quad \text{on } \partial D \end{aligned}$$

which is elliptic inside D and oblique on ∂D by proposition 1. The maximum principle implies $u \equiv 0$, contradicting the assumption. \square

III. PRELIMINARY *A PRIORI* ESTIMATES

In this section, we do *not* need yet the condition $n=2$. For any $v \in S(D, D_t)$, $dv \in D^*$, hence $|dv|$ is bounded above by $\rho(D^*) := \max_{x^* \in D^*} |x^*|$.

Set $|f|_0 = \max_{\bar{D} \times \bar{D}^*} |f(x, x^*)|$, and let $u \in S(D, D_t)$ solve (1. t), then

$$e^{-|f|_0} A(u) \leq e^{\langle u \rangle} \leq e^{|f|_0 + n |\text{Log } \lambda|} A(u).$$

Integrating this over D yields for $\langle u \rangle$ the pinching:

$$\text{Log} |D_0| - |f|_0 \leq \langle u \rangle \leq \text{Log} |D^*| + |f|_0 + n |\text{Log } \lambda|.$$

Since $|du| \leq \rho(D^*)$, u is *a priori* bounded in $C^1(\bar{D})$ in terms of $|D^*|$, $\rho(D^*)$, $|f|_0$, $|D_0|$, λ and n .

By assumption, there exists $\mu \in (0, 1]$ such that $F_u \geq \mu$ on $\bar{D} \times \bar{D}^* \times \mathbb{R}$. The right-hand side of equation (2. t), let us denote it by

$$f(t, x, du, u),$$

thus satisfies $f_u \geq \mu$ as well, on $[0, 1] \times \bar{D} \times \bar{D}^* \times \mathbb{R}$. Let $u \in S(D, D_t)$ solve (2. t). Set

$$M := \max_{\bar{D}} (u), \quad m := \min_{\bar{D}} (u)$$

$$M_0 := \max_{[0, 1] \times \bar{D} \times \bar{D}^*} [f(t, x, x^*, 0)], \quad m_0 := \min_{[0, 1] \times \bar{D} \times \bar{D}^*} [f(t, x, x^*, 0)].$$

From the mean value theorem, we know that

$$M - m \leq \rho(D^*) \delta(D),$$

$\delta(D)$ standing for the diameter of D . If $M \geq 0$ and $m \leq 0$, it implies $|u| \leq \rho(D^*) \delta(D)$ and we are done. If not, say for instance $M < 0$, then $A(u) = \exp[f(t, x, du, u)] \leq \exp[M_0 + \mu M]$. Integrating this over D yields: $\mu M \geq [\text{Log}(|D_0|/|D|) - M_0]$, hence under our assumption $[\text{Log}(|D_0|/|D|) - M_0] < 0$ and

$$-m = \max_{\bar{D}} |u| \leq \rho(D^*) \delta(D) + [M_0 - \text{Log}(|D_0|/|D|)]/\mu.$$

Similarly, $m > 0$ yields $[\text{Log}(|D^*|/|D|) - m_0] > 0$ and

$$M = \max_{\bar{D}} |u| \leq \rho(D^*) \delta(D) + [\text{Log}(|D^*|/|D|) - m_0]/\mu.$$

In any case, we obtain a $C^1(\bar{D})$ *a priori* bound on u in terms of $|D^*|$, $\rho(D^*)$, $|D|$, $\delta(D)$, $|D_0|$, M_0 , m_0 and μ .

For simplicity, let us give a unified treatment of higher order *a priori* estimates for equations (1. t) and (2. t) by rewriting these equations into a single general form

$$\text{Log } A(u) = \Gamma(t, x, du, u, \langle u \rangle). \quad (*)$$

Let $u \in S(D, D_t)$ solve (*). In this section, a constant will be said *under control* provided it depends only on the following quantities: $|u|_1$, i. e. the $C^1(\bar{D})$ -norm of u , on the C^2 -norm of Γ on

$$K := [0, 1] \times \bar{D} \times \bar{D}^* \times I \times I,$$

where $I = [-|u|_1, |u|_1]$, on the $C^0([0, 1], C^2)$ -norm of $t \rightarrow h_t$ (the fixed path of defining functions, cf. *supra*), and on the *positive* real

$$\sigma := \min_{t \in [0, 1]} \sigma(t)$$

where $\sigma(t)$ is the smallest eigenvalue of $[(h_t)_{ij}]$ over \bar{D}_t .

Since u is convex, a $C^2(\bar{D})$ bound on u follows from a bound on

$$M_2 := \max_{(x, \theta) \in \bar{D} \times S} [u_{\theta\theta}(x)]$$

S standing for the unit sphere of \mathbb{R}^n . Set $H := h_t(du)$ and consider

$$Q: (c, \theta, x) \in (0, \infty) \times S \times \bar{D} \rightarrow Q(c, \theta, x) = \text{Log}[u_{\theta\theta}(x)] + c H(x).$$

PROPOSITION 2. — *There exists $C \in (0, \infty)$ under control such that, if $\max_{(\theta, x) \in S \times \bar{D}} [Q(C, \theta, x)]$ occurs at $(z, x_0) \in S \times D$ with x_0 interior to D , then*

M_2 is under control.

This proposition does not refer to any boundary condition and constitutes by no means an interior estimate (it is rather the type of argument suited on a compact manifold). A similar proposition (with Δu and $|du|^2$, respectively in place of $u_{\theta\theta}$ and H) is lemma 2 of [13], later (and independently) reproved in [7] (p. 694); a similar argument is used in [4] (p. 398). Here proposition 2 may serve for the higher dimensional theorem, due to the special form of Q ; so for completeness, we provide a detailed proof of it.

Proof. — Fix $(c, \theta) \in (0, \infty) \times S$ and consider Q as a function of x only. Let us record some auxiliary formulae: differentiating twice equation (*) in the θ -direction yields,

$$u^{ij} u_{\theta ij} = (\Gamma)_{\theta} \equiv \Gamma_{\theta} + \Gamma_u u_{\theta} + \Gamma_{u_i} u_{\theta i} \quad (5)$$

$$u^{ij} u_{\theta\theta ij} = (\Gamma)_{\theta\theta} + u^{ik} u^{jm} u_{\theta ij} u_{\theta km} \quad (6)$$

with

$$(\Gamma)_{\theta\theta} \equiv \Gamma_{u_i} u_{\theta\theta i} + \Gamma_{u_i u_j} u_{\theta i} u_{\theta j} + 2(\Gamma_{\theta u_i} + u_{\theta} \Gamma_{uu_i}) u_{\theta i} + \Gamma_u u_{\theta\theta} + [\Gamma_{\theta\theta} + 2u_{\theta} \Gamma_{\theta u} + \Gamma_{uu}(u_{\theta})^2].$$

Differentiating twice H yields (with the subscript t , of h , dropped),

$$H_i = h_k u_{ik} \quad (7)$$

$$H_{ij} = h_k u_{ijk} + h_{km} u_{ik} u_{jm} \quad (8)$$

and similarly for Q ,

$$Q_i = (u_{\theta\theta i}/u_{\theta\theta}) + c H_i$$

$$Q_{ij} = (u_{\theta\theta ij}/u_{\theta\theta}) - [u_{\theta\theta i} u_{\theta\theta j}/(u_{\theta\theta})^2] + c H_{ij}.$$

Combining (8) with (5) and (7), we get

$$u^{ij} H_{ij} = h_i (\Gamma_i + \Gamma_u u_i) + \Gamma_{u_i} H_i + h_{ij} u_{ij} \quad (9)$$

while from (6) we get,

$$u^{ij} Q_{ij} = [(\Gamma)_{\theta\theta}/u_{\theta\theta}] + (1/u_{\theta\theta}) [u^{ik} u^{jm} u_{\theta ij} u_{\theta km} - (1/u_{\theta\theta}) u^{ij} u_{\theta\theta i} u_{\theta\theta j}] + c u^{ij} H_{ij}.$$

Expanding the square

$$(u_{\theta\theta} u_{\theta ij} - u_{\theta i} u_{\theta\theta j}) (u_{\theta\theta} u_{\theta km} - u_{\theta k} u_{\theta\theta m}) u^{ik} u^{jm}$$

one immediately verifies the identity:

$$u^{ik} u^{jm} u_{\theta ij} u_{\theta km} \geq (1/u_{\theta\theta}) u^{ij} u_{\theta\theta i} u_{\theta\theta j}.$$

So,

$$u^{ij} Q_{ij} \geq [(\Gamma)_{\theta\theta}/u_{\theta\theta}] + c u^{ij} H_{ij}.$$

Combining the expression of $(\Gamma)_{\theta\theta}$ with that of Q_i and (9) yields,

$$u^{ij} Q_{ij} - \Gamma_{u_i} Q_i \geq ch_{ij} u_{ij} + (1/u_{\theta\theta}) \Gamma_{u_i u_j} u_{\theta i} u_{\theta j} \\ + (2/u_{\theta\theta}) (\Gamma_{\theta u_i} + u_{\theta} \Gamma_{uu_i}) u_{\theta i} + \Gamma_u + ch_i (\Gamma_i + \Gamma_u u_i) \\ + (1/u_{\theta\theta}) [\Gamma_{\theta\theta} + 2 u_{\theta} \Gamma_{\theta u} + \Gamma_{uu} (u_{\theta})^2]. \quad (10)$$

Introducing the constant σ (defined above) we get

$$(1/u_{\theta\theta}) \Gamma_{u_i u_j} u_{\theta i} u_{\theta j} + \frac{1}{3} ch_{ij} u_{ij} \\ \geq (1/u_{\theta\theta}) u_{ik} u_{jm} \left(\theta^k \theta^m \Gamma_{u_i u_j} + \frac{1}{3} c \sigma u_{\theta\theta} \delta_{ij} u^{km} \right) \\ \geq (1/u_{\theta\theta}) u_{\theta i} u_{\theta j} \left(\Gamma_{u_i u_j} + \frac{1}{3} c \sigma \delta_{ij} \right)$$

this last inequality being obtained by noting that, identically for u strictly convex, $u_{\theta\theta} u^{km} \geq \theta^k \theta^m$. Hence our first requirement on c is:

$$\left(\Gamma_{u_i u_j} + \frac{1}{3} c \sigma \delta_{ij} \right) \geq 0$$

in the sense of symmetric matrices, over K . To express our second requirement on c , we first note that the inequality between the arithmetic and the geometric means of n positive numbers applied to the eigenvalues of (u_{ij}) and combined with $(*)$, yields on D :

$$\Delta u \geq n \exp \left(\frac{1}{n} \min_K \Gamma \right) =: \gamma.$$

Then we take c such that

$$2 \min [\Gamma_{yu_y}(r) + u_y(x) \Gamma_{uu_y}(r)] + \frac{1}{3} c \sigma \gamma \geq 0$$

the minimum being taken on $(r, x, y) \in K \times \bar{D} \times S$. From now on, c has a fixed value under control, C , meeting both requirements and we take $(\theta, x) = (z, x_0)$ as defined in proposition 2. In particular, $u_{zz}(x_0)$ is now the *maximum* eigenvalue of $[u_{ij}(x_0)]$; diagonalizing the latter and using the *second* requirement on C , we obtain at x_0 :

$$\frac{1}{3} C h_{ij} u_{ij} + (2/u_{zz}) (\Gamma_{zu_i} + u_z \Gamma_{uu_i}) u_{zi} \geq \frac{1}{3} C \sigma \Delta u + 2 (\Gamma_{zu_z} + u_z \Gamma_{uu_z}) \geq 0.$$

Now (10) yields for $x \rightarrow Q = Q(C, z, x)$ at x_0 ,

$$u^{ij} Q_{ij} - \Gamma_{u_i} Q_i \geq C \left(\frac{1}{3} \sigma u_{zz} - C' \right) - C'' (1/u_{zz}), \quad (11)$$

for some positive constants under control C', C'' . Since $Q(C, z, \cdot)$ assumes its *maximum* at $x_0 \in D$, (11) implies a controlled bound from above on

$u_{zz}(x_0)$, hence also on $(\theta, x) \rightarrow Q(C, \theta, x)$ and on $(\theta, x) \rightarrow u_{\theta\theta}(x)$. Therefore M_2 is under control. \square

According to proposition 2, we may assume, without loss of generality, that the point x_0 above lies on ∂D , hence a $C^2(\bar{D})$ *a priori* bound on u follows from an *a priori* bound on $u_{zz}(x_0)$ which, in turn, coincides with

$$\max_{(\theta, x) \in S \times \partial D} [u_{\theta\theta}(x)].$$

IV. A PRIORI ESTIMATES OF SECOND DERIVATIVES ON THE BOUNDARY ($n=2$)

In this section we fix a defining function of D , denoted by k , and we include in the definition of constants *under control* the possible dependence on $|k|_3$, on $\tau := \min_{\partial D} k_N > 0$ and on the minimum over \bar{D} of the smallest eigenvalue of (k_{ij}) , denoted by $s > 0$.

We still let $u \in S(D, D_t)$ solve equation $(*)$. According to proposition 1 $H = h_t(du)$ which vanishes on ∂D , satisfies there $H_N > 0$; moreover, (7) implies on ∂D (dropping the subscript t of h):

$$h_i[du(x)] = H_N u^{ij} N^j(x). \quad (12)$$

In particular, the function on ∂D

$$\varphi(x) := N^i(x) h_i[du(x)]$$

is *positive*. Fix an arbitrary point $x_0 \in \partial D$ and a direct system of euclidean co-ordinates (O, x^1, x^2) satisfying $N(x_0) = \partial/\partial x^2$. Then (12) reads at x_0 ,

$$\left. \begin{aligned} u_{11}(x_0) &= (e^\Gamma/H_N) \varphi(x_0) \\ u_{12}(x_0) &= -(e^\Gamma/H_N)(x_0) h_1[du(x_0)] \end{aligned} \right\} \quad (13)$$

while equation $(*)$ itself provides for $u_{22}(x_0) = u_{NN}(x_0)$,

$$\varphi u_{22}(x_0) = H_N(x) + (e^\Gamma/H_N)(x_0) \{h_1[du(x_0)]\}^2. \quad (14)$$

We thus need positive lower bounds under control on $H_N(x_0)$ and $\varphi(x_0)$, as well as a controlled upper bound on $H_N(x_0)$.

Let us start with $H_N(x_0)$. Aside from (9), H also satisfies in D [still by combining (8), (5), (7)],

$$u^{ij} H_{ij} - u^{ij} (\Gamma)_i H_j = h_{ij} u_{ij}. \quad (15)$$

Set $T = u_{11} + u_{22}$, $T^* = u^{11} + u^{22}$, and note the identity: $T^* = A(u)T$. It implies the existence of positive constants under control, α, β , such that

$$\alpha T^* \leq T \leq \beta T^*, \quad (16)$$

which we simply denote by: $T \simeq T^*$. Consider the function

$$(c, x) \in (0, \infty) \times \bar{D} \rightarrow w(c, x) = H(x) - ck(x).$$

From (9) and $T \geq \gamma$ (cf. *supra*), we infer

$$u^{ij}[w(c, \cdot)]_{ij} \leq -T \left[\frac{1}{2} c(s/\beta) - (u_{ij}/T)(h_{ij} + \Gamma_{u_i} h_j) \right] - \left[\frac{1}{2} \gamma c(s/\beta) - h_i(\Gamma_i + \Gamma_u u_i) \right],$$

and there readily exists $c = C > 1$, under control, such that the latter right-hand side is non-positive. Similarly (15) (16) yield:

$$u^{ij}[w(c, \cdot)]_{ij} - u^{ij}(\Gamma)_i[w(c, \cdot)]_j \geq \frac{1}{2} \sigma T - T \max \{ 0, (c/\alpha) u^{ij}/T^* [k_{ij} - k_i(\Gamma_j + \Gamma_u u_j)] \} + \left(\frac{1}{2} \sigma \gamma + ck_i \Gamma_{u_i} \right),$$

(σ was defined at the beginning of section III) and there exists $c \in (0, 1)$ under control such that the right-hand side is nonnegative. Since w identically vanishes on $(0, \infty) \times \partial D$, Hopf's maximum principle [11] implies the following pinching under control on ∂D :

$$c\tau \leq ck_N \leq H_N \leq Ck_N \leq C|k|_1. \quad (17)$$

Combined with (13), it implies a controlled upper bound on $|u_{11}(x_0)| + |u_{12}(x_0)|$. Furthermore, combined with (14), it implies also (the notation \simeq is defined at (16))

$$u_{22}(x_0) \simeq 1/\varphi(x_0). \quad (18)$$

We now turn to a lower bound on $\varphi(x_0)$ and consider the function

$$(c, x) \in (0, \infty) \times \bar{D} \rightarrow P(c, x) = \psi - ck,$$

where

$$\psi(x) := k_i(x) h_i[du(x)].$$

A routine computation using (5) yields in D :

$$u^{ij} \psi_{ij} = k_i h_{ij} (\Gamma_j + \Gamma_u u_j + \Gamma_{u_m} u_{jm}) + 2k_{ij} h_{ij} + k_i h_{ijm} u_{jm} + u^{ij} k_{ijm} h_m.$$

It implies the existence of a constant c_1 under control such that, in D ,

$$u^{ij} P_{ij} \leq c_1 (1 + T) - c(s/\beta) T = - \left[\frac{1}{2} c \gamma (s/\beta) - c_1 \right] - T \left[\frac{1}{2} c (s/\beta) - c_1 \right];$$

let us choose $c = C_0 := 2c_1 \beta/s \min(1, \gamma)$, so that $u^{ij} [P(C_0, \cdot)]_{ij} \leq 0$ in D . By Hopf's maximum principle [11], $P(C_0, \cdot)$ necessarily assumes its *minimum* over \bar{D} at a boundary point y_0 where

$$\psi_N \leq C_0 k_N. \quad (19)$$

Pick a euclidean system of co-ordinates (O, y^1, y^2) such that $N(y_0) = \partial/\partial y^2$. Then $dk(y_0) = k_N \partial/\partial y^2$ while, using (13) (17):

$$|u_{12}(y_0)| \leq C_1 := e^{|\Gamma|_0} |h|_1 / c\tau$$

is under control, and (19) reads:

$$\begin{aligned} u_{22}(y_0) k_N(y_0) h_{22}[du(y_0)] \\ \leq C_0 k_N(y_0) - k_{2i}(y_0) h_i[du(y_0)] \\ - k_N(y_0) u_{12}(y_0) h_{12}[du(y_0)]. \end{aligned}$$

It implies

$$\sigma \gamma u_{22}(y_0) \leq C_0 |k|_1 + |h|_1 |k|_2 + C_1 |k|_1 |h|_2$$

i.e. a controlled bound from above on $u_{22}(y_0)$. Recalling (18), it means a controlled positive bound from below, λ , on $\varphi(y_0)$. Since on ∂D , $P(C_0, \cdot) \equiv k_N \varphi$, and since $P(C_0, \cdot)$ assumes its *minimum* at y_0 , we infer on ∂D :

$$\varphi(x) \geq \lambda k_N(y_0)/k_N(x) \geq \lambda \tau / |k|_1.$$

Using (18) again, we obtain a controlled upper bound on $u_{22}(x_0)$. The second derivatives of u are thus *a priori* bounded on ∂D . \square

Remarks. — 5. Proposition 1 and (12) show that the lower bound $\varphi \geq \lambda$ ensures *a priori* the *uniform obliqueness* of the boundary operator at u . Geometrically, it implies another positive lower bound on the scalar product of the outward unit normals, to ∂D at x and to ∂D_t at $du(x)$.

6. Let (T, N) and (T^*, N^*) be direct orthonormal moving frames on ∂D and on ∂D_t respectively (N^* stands for the outward unit normal on ∂D_t) and let z_0 be a critical point of: $x \in \partial D \rightarrow N(x) \cdot N^*[du(x)]$. Denote by $J du$ the Jacobian (or differential) of the gradient mapping du . With the help of Frénet's formulae, one verifies that

$$|J du[T(z_0)]| = (R_0^*/R_0), \quad (20)$$

R_0 (resp. R_0^*) standing for the curvature radius of ∂D at z_0 [resp. of ∂D_t at $du(z_0)$]. Equation (*) implies that the area of the parallelogram $[J du(T), J du(N)]$ equals $\exp(\Gamma)$, in particular, it is uniformly bounded *below* by a positive constant. What happens if we drop the *strict* convexity of ∂D at z_0 , but keep that of ∂D_t at $du(z_0)$, *i.e.* if we let R_0 go to infinity and R_0^* remain bounded? From (20), $|J du[T(z_0)]|$ goes to zero hence $|J du[N(z_0)]|$ goes to infinity. In a direct system of euclidean co-ordinates $(0, x^1, x^2)$ such that $N(z_0) = \partial/\partial x^2$, it implies that $|u_{11}(z_0)| + |u_{12}(z_0)|$ goes to zero while $|u_{22}(z_0)|$ blows up like R_0 *i.e.* the control on $u_{NN}(z_0)$ is lost.

V. HIGHER ORDER *A PRIORI* ESTIMATES

Let $u \in S(D, D_t)$ solve equation (*). Fix a generic point $x \in \bar{D}$ and choose a euclidean co-ordinates system which puts $[u_{ij}(x)]$ into a *diagonal* form.

Observe that for each $i \in \{1, \dots, n\}$,

$$u_{ii}(x) = A(u) / \prod_{j \neq i} u_{jj}(x) \geq \gamma / (|u|_2)^{n-1}. \quad (21)$$

In case $n=2$, the $C^2(\bar{D})$ *a priori* estimate drawn on u in the two preceding sections thus implies the controlled *uniform* ellipticity of $d[\text{Log } A(u)]$ on \bar{D} . Given $\alpha \in (0, 1)$, a $C^{2,\alpha}(\bar{D})$ *a priori* bound on u now follows from the general theory of [15] (section 6); however, this bound is so straightforward for $n=2$ that we include it for completeness.

First of all, given any interior subdomain D' of D and any $z \in S$, the 2-dimensional regularity theory of [17] applied to u_z , which satisfies (5) in D' , yields a $C^{1,\alpha}(\bar{D}')$ *a priori* bound under control on u_z , hence, since z is arbitrary, a controlled $C^{2,\alpha}(\bar{D}')$ *a priori* bound on u . The theory of [17] also applies to H which satisfies (9) in D and vanishes on ∂D : it yields a $C^{1,\alpha}(\bar{D})$ *a priori* bound under control on H . Solving for u_{11} , u_{12} and u_{22} the 3×3 system given by (7) and equation (*), we get (dropping the subscript t of h):

$$\begin{cases} u_{11} = [(H_1)^2 + (h_2)^2 e^\Gamma] / \Delta \\ u_{12} = (H_1 H_2 - h_1 h_2 e^\Gamma) / \Delta \\ u_{22} = [(H_2)^2 + (h_1)^2 e^\Gamma] / \Delta, \end{cases} \quad (22)$$

where

$$\Delta(x) := H_i(x) h_i [du(x)].$$

Note that (7) and (21) imply

$$\Delta(x) \geq (\gamma / |u|_2) |dh[du(x)]|^2. \quad (23)$$

Given any small enough $\delta \in (0, 1)$, let

$$D_\delta := \{x \in D, \text{dist}(x, \partial D) < \delta\}.$$

From the $C^2(\bar{D})$ *a priori* estimate precedingly drawn on u , it follows that the gradient image $du(D_\delta)$ is contained in $(D_t)_{C_\delta}$ for some positive constant C under control. If $\tau^* := \min_{t \in [0, 1]} (\min_{\partial_t D} |dh_t|)$,

then there readily exists $\delta_0 \in (0, 1)$ under control such that, for any $x \in D_{\delta_0}$, $|dh_t[du(x)]| \geq \tau^*/2$. Therefore (22) and (23) imply a $C^\alpha(\bar{D}_{\delta_0})$ *a priori* bound under control on the second derivatives of u . A $C^{2,\alpha}(\bar{D})$ *a priori* bound on u follows.

Actually, a straightforward “bootstrap” argument now provides $C^{k,\alpha}(\bar{D})$ *a priori* bounds on u for each integer $k > 2$.

REFERENCES

- [1] A. AGMON, D. DOUGLIS and L. NIRENBERG, Estimates Near the Boundary for Solutions of Elliptic Partial Differential Equations Satisfying General Boundary Conditions, I,

- Comm. Pure Appl. Math.*, Vol. **12**, 1959, pp. 623-727; II, *Ibid.*, Vol. **17**, 1964, pp. 35-92.
- [2] T. AUBIN, Réduction du cas positif de l'équation de Monge-Ampère sur les variétés Kähleriennes compactes à la démonstration d'une inégalité, *J. Funct. Anal.*, Vol. **53**, 1983, pp. 231-245.
- [3] I. BAKEL'MAN, Generalized Solutions of Monge-Ampère Equations, *Dokl. Akad. Nauk. S.S.S.R.*, Vol. **114**:6, 1957, pp. 1143-1145 (in russian).
- [4] L. CAFFARELLI, L. NIRENBERG and J. SPRUCK, The Dirichlet Problem for Nonlinear Second-Order Elliptic Equations I. Monge-Ampère equation, *Comm. Pure Appl. Math.*, Vol. **37**, 1984, pp. 369-402.
- [5] B. DACOROGNA and J. MOSER, On a Partial Differential Equation Involving the Jacobian Determinant, *Ann. Inst. Henri Poincaré Analyse non linéaire*, Vol. **7**:1, 1990, pp. 1-26.
- [6] P. DELANOË, Equations du type de Monge-Ampère sur les variétés Riemanniennes compactes II, *J. Funct. Anal.*, Vol. **41**, 1981, pp. 341-353.
- [7] P. DELANOË, Equations de Monge-Ampère en dimension deux, *C. R. Acad. Sci. Paris*, **294**, série I, 1982, pp. 693-696.
- [8] P. DELANOË, Plongements radiaux $S^n \rightarrow \mathbb{R}^{n+1}$ à courbure de Gauss positive prescrite, *Ann. Sci. Ec. Norm. Sup.*, Vol. **18**, 1985, pp. 635-649.
- [9] P. DELANOË, Remarques sur les variétés localement Hessiennes, *Osaka J. Math.*, Vol. **26**, 1989, pp. 65-69.
- [10] P. DELANOË, Viscosity Solutions of Eikonal and Lie Equations on Compact Manifolds, *Ann. Global Anal. Geom.*, Vol. **7**:2, 1989, pp. 79-83.
- [11] E. HOPF, Elementare Bemerkungen über die Lösungen partieller Differential-gleichungen zweiter Ordnung vom elliptischen Typus, *Sitz. Ber. Preuß. Akad. Wissensch. Berlin, Math.-Phys. Kl.*, Vol. **19**, 1927, pp. 147-152.
- [12] E. HOPF, A Remark on Linear Elliptic Differential Equations of Second Order, *Proc. Am. Math. Soc.*, Vol. **3**, 1952, pp. 791-793.
- [13] N. M. IVOTCHKINA, The *a priori* Estimate $\|u\|_{C(\bar{\Omega})}^2$ on Convex Solutions of the Dirichlet problem for the Monge-Ampère Equation, *Zapisk. Nauchn. Semin. LOMI*, Vol. **96**, 1980, pp. 69-79.
- [14] L. Y. LIAO and F. SCHULZ, Regularity of Solutions of Two-Dimensional Monge-Ampère Equations, *Transact. Am. Math. Soc.*, Vol. **307**:1, 1988, pp. 271-277.
- [15] G. M. LIEBERMAN and N. S. TRUDINGER, Nonlinear Oblique Boundary Value Problems for Nonlinear Elliptic Equations, *Transact. Am. Math. Soc.*, **295**:2, 1986, pp. 509-546.
- [16] P.-L. LIONS, N. S. TRUDINGER and J. I. E. URBAS, The Neumann problem for Equations of Monge-Ampère Type, *Comm. Pure Appl. Math.*, Vol. **39**, 1986, pp. 539-563.
- [17] L. NIRENBERG, On Nonlinear Elliptic Partial Differential Equations and Hölder Continuity, *Comm. Pure Appl. Math.*, Vol. **6**, 1953, pp. 103-156.
- [18] A. V. POGORELOV, Monge-Ampère Equations of Elliptic Type, Noordhoff Ltd, 1964.
- [19] F. SCHULZ, Boundary Estimates for Solutions of Monge-Ampère Equations in the Plane, *Ann. Sc. Norm. Pisa*, Vol. **11**:3, 1984, pp. 431-440.
- [20] K.-S. TSO, Personal Letters from the Chinese University of Hong-Kong sent on July 12, 1988 and on June 7, 1989.
- [21] J. I. E. URBAS, *The Oblique Derivative Problem for Equations of Monge-Ampère Type in Two Dimensions*, Preprint, Courant Institute and CMA at Canberra, 1987.

(Manuscript received November 2nd, 1989)

(Revised July 30th, 1990.)