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AN ADDITIVE VERSION OF HIGHER CHOW GROUPS

BY SPENCER BLOCH AND HÉLÈNE ESNAULT

ABSTRACT. – The cosimplicial scheme

$$\Delta^\bullet = \Delta^0 \rightrightarrows \Delta^1 \overset{\rightarrow}{\rightrightarrows} \ldots; \quad \Delta^n := \operatorname{Spec}\Bigl(k[t_0,\ldots,t_n]\Big/\Bigl(\sum t_i - t\Bigr)\Bigr)$$

was used in (Bloch, S., Algebraic cycles and higher K-theory, Adv. Math. 61 (3) (1986) 267–304) to define higher Chow groups. In this note, we let t tend to 0 and replace $\Delta^\bullet$ by a degenerate version

$$Q^\bullet = Q^0 \rightrightarrows Q^1 \overset{\rightarrow}{\rightrightarrows} \ldots; \quad Q^n := \operatorname{Spec}\Bigl(k[t_0,\ldots,t_n]\Big/\Bigl(\sum t_i\Bigr)\Bigr)$$

to define an additive version of the higher Chow groups. For a field k, we show the Chow group of 0-cycles on Q^n in this theory is isomorphic to the group of absolute $(n-1)$-Kähler forms Ω_k^{n-1}.

An analogous degeneration on the level of de Rham cohomology associated to “constant modulus” degenerations of varieties in various contexts is discussed.

RÉSUMÉ. – Le schéma cosimplicial

$$\Delta^\bullet = \Delta^0 \rightrightarrows \Delta^1 \overset{\rightarrow}{\rightrightarrows} \ldots; \quad \Delta^n := \operatorname{Spec}\Bigl(k[t_0,\ldots,t_n]\Big/\Bigl(\sum t_i - t\Bigr)\Bigr)$$

a été utilisé dans (Bloch, S., Algebraic cycles and higher K-theory, Adv. Math. 61 (3) (1986) 267–304) afin de définir des groupes de Chow supérieurs. Dans cette note, nous définissons une version additive des groupes de Chow supérieurs en faisant tendre t vers 0 et en remplaçant $\Delta^\bullet$ par une version dégénérée

$$Q^\bullet = Q^0 \rightrightarrows Q^1 \overset{\rightarrow}{\rightrightarrows} \ldots; \quad Q^n := \operatorname{Spec}\Bigl(k[t_0,\ldots,t_n]\Big/\Bigl(\sum t_i\Bigr)\Bigr).$$

Nous montrons que sur un corps k, le groupe de Chow des 0-cycles dans cette théorie est isomorphe aux formes de Kähler absolues de degré $(n-1)$.

Nous discutons une dégénerescence analogue en cohomologie de de Rham apparaissant dans diverses situations pour des familles à module constant.

1. Introduction

The purpose of this note is to study a common sort of limiting phenomenon which occurs in the study of motives. Here is a simple example. Let k be a field. Let $S = \mathbb{A}^1_t = \operatorname{Spec}(k[t])$

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and let $T = \operatorname{Spec}(k[x,t]/(x(x-t)) \hookrightarrow \mathbb{A}^2_{t,x}$. Over $S[1/t]$ the Picard scheme $\operatorname{Pic}(\mathbb{A}^2_{x,t}, T)/S$ is represented by $\mathbb{G}_{m,S[1/t]}$. On the other hand, when $t = 0$ one gets $\operatorname{Pic}(\mathbb{A}^1_x, \{x^2 = 0\}) \cong \mathbb{G}_{a,k}$. In some sense, $\mathbb{G}_m$ has "jumped" to $\mathbb{G}_a$.

In higher dimension, for $t \neq 0$, the homology of the group of algebraic cycles associated to the cosimplicial scheme

$$\Delta^\bullet = \Delta^0 \rightrightarrows \Delta^1 \overset{\to}{\underset{\to}{\to}} \ldots; \quad \Delta^n := \operatorname{Spec}\Big(k[t_0, \ldots, t_n]/\Big(\sum t_i - t\Big)\Big) \tag{1.1}$$

is known to give motivic cohomology [9]. What can one say about the algebraic cycle groups (1.3) of the degenerate cosimplicial complex:

$$Q^\bullet = Q^0 \rightrightarrows Q^1 \overset{\to}{\underset{\to}{\to}} \ldots; \quad Q^n := \operatorname{Spec}\Big(k[t_0, \ldots, t_n]/\Big(\sum t_i\Big)\Big)? \tag{1.2}$$

Our main result is a calculation of the Chow groups of 0-cycles on $Q^\bullet$. Let $\mathcal{Z}^n(Q^r)$ be the free abelian group on codimension n algebraic cycles on Q^r satisfying a suitable general position condition with respect to the face maps. Let $SH^n(k,r)$ be the cohomology groups of the complex

$$\cdots \to \mathcal{Z}^n\big(Q^{r+1}\big) \to \mathcal{Z}^n\big(Q^r\big) \to \mathcal{Z}^n\big(Q^{r-1}\big) \to \cdots \tag{1.3}$$

where the boundary maps are alternating sums of pullbacks along face maps. Write $\Omega^\bullet_k$ for the absolute Kähler differentials.

THEOREM 1.1. –

$$SH^n(k,n) \cong \Omega^{n-1}_k.$$

Note for $n = 1$, this is the above $\mathbb{G}_a$.

We explain another example of this limiting phenomenon. We consider $\mathbb{P}^{n+1}$ with homogeneous coordinates $U_0, \ldots, U_{n+1}$ over a field k. Let

$$X\colon\ f(U_0, \ldots, U_{n+1}) = 0 \tag{1.4}$$

be a hypersurface X defined by a homogeneous polynomial f of degree $n+2$ (e.g. an elliptic curve in $\mathbb{P}^2$).

Write $u_i = U_i/U_0$, and define a log $(n+1)$-form on $\mathbb{P}^{n+1} \setminus X$

$$\omega_f := \frac{du_1 \wedge \cdots \wedge du_{n+1}}{f(1, u_1, \ldots, u_{n+1})}. \tag{1.5}$$

As well known, this form generates $\omega_{\mathbb{P}^{n+1}}(X) \cong \mathcal{O}_{\mathbb{P}^{n+1}}$.

Let $r_1, \ldots, r_{n+1} \geqslant 0$ be integers. We consider the action of $\mathbb{G}_m$ given by the substitutions $u_i = t^{-r_i} v_i$. Let N be minimal such that $t^N f(1, t^{-r_1}u_1, \ldots, t^{-r_{n+1}}u_{n+1})$ is integral in t. Assume $s := N - \sum r_i > 0$ and s is invertible in k. One checks easily that in the coordinates v_i one has

$$\omega_f = t^s \nu_f(t, v_1, \ldots, v_{n+1}) + s t^{s-1} dt \wedge \gamma_f(t, v_1, \ldots, v_{n+1}) \tag{1.6}$$

with

$$\nu_f(t, v_1, \dots, v_{n+1}) = \frac{dv_1 \wedge \cdots \wedge dv_{n+1}}{t^N f(1, t^{-r_1}v_1, \dots, t^{-r_{n+1}}v_{n+1})},$$
$$\gamma_f(t, v_1, \dots, v_{n+1}) = \frac{1}{s}\sum_i (-1)^i r_i v_i \frac{dv_1 \wedge \cdots \wedge \widehat{dv_i} \wedge \cdots \wedge dv_{n+1}}{t^N f(1, t^{-r_1}v_1, \dots, t^{-r_{n+1}}v_{n+1})}.$$

Thus ν_f and γ_f are forms in dv_i of degrees $n+1$ and n respectively which are integral in t. As ω_f is an $(n+1)$-form on $\mathbb{P}^{n+1}$ in the u-coordinates, it is closed, so

$$st^{s-1}dt \wedge \big(\nu_f(t, v_1, \dots, v_{n+1}) - d\gamma_f(t, v_1, \dots, v_{n+1})\big) + st^{s-1} \cdot \frac{t}{s} d\nu_f(t, v_1, \dots, v_{n+1}) = 0.$$

Dividing by t^{s-1} and restricting to $t = 0$ yields

$$\nu_f|_{t=0} = d\gamma_f|_{t=0}. \tag{1.7}$$

In other words, under the action of the 1-parameter subgroup, ν_f degenerates to the exact form $d\gamma_f$.

To see the relationship with the 0-cycles, take

$$\Delta_{n+1}\colon\ f(1, u_1, \dots, u_{n+1}) = u_1 u_2 \cdots u_{n+1}(1 - u_1 - \cdots - u_{n+1}) = 0.$$

Substitute $v_i = tu_i,\ 1 \leqslant i \leqslant n+1$. Then $s = 1$, and a calculation yields

$$\begin{aligned} \gamma_{\Delta_{n+1}}(:= \gamma_f) &= \frac{\sum_{i=1}^{n+1}(-1)^i d\log(v_1) \wedge \cdots \wedge \widehat{d\log(v_i)} \wedge \cdots \wedge d\log(v_{n+1})}{t - v_1 - \cdots - v_{n+1}}, \\ \nu_{\Delta_{n+1}}(:= \nu_f) &= \frac{dv_1 \wedge \cdots \wedge dv_{n+1}}{v_1 v_2 \cdots v_{n+1}(t - v_1 - \cdots - v_{n+1})}. \end{aligned} \tag{1.8}$$

Note the limiting configuration as $t \to 0$ is (compare (1.2))

$$v_1 v_2 \cdots v_{n+1}(v_1 + \cdots + v_{n+1}) = 0.$$

For convenience we write

$$\gamma_n := \gamma_{\Delta_{n+1}}|_{t=0}. \tag{1.9}$$

We view $\nu_{\Delta_{n+1}}|_{t=1}$ (resp. γ_n) as a map

$$\mathcal{Z}_0\big(\mathbb{A}_k^{n+1} \setminus \big\{(1 - u_1 + \cdots + u_{n+1})u_1 \cdots u_{n+1} = 0\big\}\big) \to \Omega_k^{n+1}$$

(resp.

$$\mathcal{Z}_0\big(\mathbb{A}_k^{n+1} \setminus \{v_1 v_2 \cdots v_{n+1}(v_1 + \cdots + v_{n+1}) = 0\}\big) \to \Omega_k^n).$$

Here $\mathcal{Z}_0$ denotes the free abelian group on closed points (0-cycles) and the maps are respectively

$$x \mapsto \operatorname{Tr}_{k(x)/k} \nu|_{\{x\}}; \qquad x \mapsto \operatorname{Tr}_{k(x)/k} \gamma|_{\{x\}}. \tag{1.10}$$

In the first case, the Nesterenko–Suslin–Totaro theorem [7,8] identifies the zero cycles modulo relations coming from curves in $\mathbb{A}^{n+2}$ with the Milnor K-group $K_{n+1}^M(k)$. The evaluation map

(1.10) passes to the quotient, and the resulting map $K^M_{n+1}(k)\to\Omega^{n+1}_k$ is given on symbols by the $d\log$-map

$$\{x_1,\dots,x_{n+1}\}\mapsto d\log(x_1)\wedge\cdots\wedge d\log(x_{n+1}). \tag{1.11}$$

In the second case, factoring out by the relations coming from curves on $\mathbb{A}^{n+2}$ as in (1.3) yields the Chow group of 0-cycles $SH^{n+1}(k,n+1)$, and our main result is that evaluation on γ_n gives an isomorphism

$$SH^{n+1}(k,n+1)\cong\Omega^n_k. \tag{1.12}$$

Sections 2–5 contain the proof of Theorem 1.1. Section 6 contains some brief remarks on specialization of forms as it relates to Aomoto's theory of configurations, to 0-cycles on hypersurfaces, and to Goncharov's theory of hyperbolic motives. Finally, Section 7 computes $SH^1(k,n)$, responding to a question of S. Lichtenbaum.

2. The additive Chow groups

In this section, we consider a field k, and a k-scheme X of finite type.

We will throughout use the following notations.

Notations 2.1. – We set $Q^n=\operatorname{Spec}k[t_0,\dots,t_n]/(\sum_{i=0}^n t_i)$, together with the faces

$$\partial_j:Q^{n-1}\to Q^n;\quad \partial_j^*(t_i)=\begin{cases}t_i & i<j,\\ 0 & i=j,\\ t_{i-1} & i>j.\end{cases}$$

One also has degeneracies

$$\pi_j:Q^n\to Q^{n-1};\quad \pi_j^*(t_i)=\begin{cases}t_i & i<j,\\ t_i+t_{i+1} & i=j,\\ t_{i+1} & i>j.\end{cases}$$

We denote by $\{0\}\in Q^n$ the vertex defined by $t_i=0$. We write $Q^n_X=Q^n\times_{\mathrm{Spec}(k)}X$. The above face and degeneracy maps make $Q^\bullet_X$ a cosimplicial scheme.

DEFINITION 2.2. – Let $\mathcal{SZ}_q(X,n)$ be the free abelian group on irreducible, dimension q subvarieties in Q^n_X with the property:

(i) They don't meet $\{0\}\times X$.
(ii) They meet all the faces properly, that is in dimension $\leqslant q$.

Thus the face maps induce restriction maps

$$\partial_i:\mathcal{SZ}_q(X,n)\to\mathcal{SZ}_{q-1}(X,n-1);\quad i=0,\dots,n;\ \partial=\sum_{i=0}^n(-1)^i\partial_i;$$

yielding complexes $\mathcal{SZ}_{q-\bullet}(X,\bullet)$:

$$\cdots\xrightarrow{\partial}\mathcal{SZ}_{q+1}(X,n+1)\xrightarrow{\partial}\mathcal{SZ}_q(X,n)\xrightarrow{\partial}\mathcal{SZ}_{q-1}(X,n-1)\xrightarrow{\partial}\cdots.$$

DEFINITION 2.3. – The additive higher Chow groups are given for $n \geqslant 1$ by

$$SH_q(X,n) = H_n\big(\mathcal{SZ}_{q-\bullet}(X,\bullet)\big)$$

and (for X equidimensional)

$$SH^p(X,n) = SH_{\dim \mathrm{X}-p}(X,n).$$

The groups are not defined for $n=0$.

Remarks 2.4. –

(i) The above should be compared with the higher Chow groups $CH^p(X,n)$ defined as above with $Q^\bullet$ replaced by $\Delta^\bullet$, where $\Delta^n := \operatorname{Spec}(k[t_0,\ldots,t_n]/(\sum t_i - 1))$.

(ii) The cosimplicial scheme $Q^\bullet$ admits an action of $\mathbb{G}_{m,k}$, which we define by

$$x \star (t_0,\ldots,t_n) := (t_0/x,\ldots,t_n/x).$$

(The reason for the inverse will be clear below.) By functoriality, we obtain a $k^\times$-action on the $SH^p(X,n)$.

(iii) Let $f : X' \to X$ be a proper map with $n = \dim X' - \dim X$. Then one has a push-forward map

$$f_* : \mathcal{SZ}^p(X',\bullet) \to \mathcal{SZ}^{p-n}(X,\bullet).$$

On homology this yields $SH^p(X',n) \to SH^{p-n}(X,n)$. We will be particularly interested in the case $X = \operatorname{Spec}(k)$, $X' = \operatorname{Spec}(k')$ with $[k' : k] < \infty$. We write $\operatorname{tr}_{k'/k} : SH^p(k',n) \to SH^p(k,n)$ for the resulting map. This trace map is compatible with the action of $k^\times$ from (ii) in the sense that for $x \in k^\times$ and $a' \in SH^p(k',n)$ we have $x \star \operatorname{tr}_{k'/k}(a') = \operatorname{tr}_{k'/k}(x \star a')$.

LEMMA 2.5. – *The $\star$ action of $k^\times$ on $SH^n(k,n)$ extends to an action of the multiplicative monoid k by setting $0 \star x = 0$. This action comes from a k-vector space structure on $SH^n(k,n)$.*

Proof. – We have to show that for a closed point $x = (u_0,\ldots,u_n) \in Q^n \setminus \bigcup_{i=0}^n \partial_i(Q^{n-1})$, and $a,b \in k$, one has $(a+b)\star x = a \star x + b \star x$. For either a or $b=0$, this is trivial. Thus we assume $ab \neq 0$. Let $k' = k(x)$. Then the class in $SH^n(k,n)$ of x is the trace from k' to k of a k'-rational point $x' \in SH^n(k',n)$. Using the compatibility of $\star$ and trace from Remarks 2.4(iii) above, we reduce to the case x k-rational.

Write $(t_0-u_0,\ldots,t_n-u_n)$ for the ideal of x. Define $\ell(t) = -\frac{ab}{u_0}t + a + b$. Consider the curve $W \subset Q^{n+1}$ defined parametrically by

$$W = \left\{\left(t, -t + \frac{u_0}{\ell(t)}, \frac{u_1}{\ell(t)}, \ldots, \frac{u_n}{\ell(t)}\right)\right\}.$$

To check that this parametrized locus is Zariski-closed, we consider the ideal:

$$I_W = \big((t_1+t_0)\ell(t_0) - u_0, t_2\ell(t_0) - u_1, \ldots, t_n\ell(t_0) - u_n\big).$$

If $y = (y_0,\ldots,y_{n+1})$ is a geometric point in the zero locus of I_W, then since the $u_i \neq 0$ we see that $\ell(y_0) \neq 0$. Substituting $t = y_0$, we see that y lies on the parametrized locus W.

The equation $-t + \frac{u_0}{\ell(t)} = 0$ leads to a quadratic equation in t with solutions $t = \frac{u_0}{a}$, $t = \frac{u_0}{b}$. If $a+b \neq 0$ we have

$$\begin{aligned}
&\partial_0(W) = \left(\frac{u_0}{a+b}, \dots, \frac{u_n}{a+b}\right) = (a+b) \star (u_0, \dots, u_n),\\
&\partial_1(W) = \left(\frac{u_0}{a}, \dots, \frac{u_n}{a}\right) + \left(\frac{u_0}{b}, \dots, \frac{u_n}{b}\right) = a \star (u_0, \dots, u_n) + b \star (u_0, \dots, u_n),\\
&\partial_i(W) = 0, \quad i \geqslant 2,
\end{aligned}$$

so the lemma follows in this case. If $a + b = 0$, then $\partial_0 W = 0$ as well, and again the assertion is clear. □

3. Additive Chow groups and Milnor K-theory

We consider the map (compatible with faces) $\iota : \Delta^\bullet \to Q^{\bullet+1}$ defined on k-rational points by $(u_0, \dots, u_n) \mapsto (-1, u_0, \dots, u_n)$. It induces a map of complexes $\mathcal{Z}^p(k, \bullet) \to \mathcal{SZ}^{p+1}(k, \bullet + 1)$, which in turn induces a map

$$\iota : CH^p(k, n) \to SH^{p+1}(k, n+1). \tag{3.1}$$

By [7] and [8], one has an isomorphism

$$K_n^M(k) \cong CH^n(k, n) \tag{3.2}$$

of the higher Chow groups of 0-cycles with Milnor K-theory. It is defined by:

$$(u_0, \dots, u_n) \mapsto \left\{-\frac{u_0}{u_n}, \dots, -\frac{u_{n-1}}{u_n}\right\}, \tag{3.3}$$

$$\{b_1, \dots, b_n\} \mapsto \left(\frac{b_1}{c}, \dots, \frac{b_n}{c}, -\frac{1}{c}\right); \quad c = -1 + \sum_{i=1}^n b_i. \tag{3.4}$$

Note that if $\sum_{i=1}^n b_i = 1$, then the symbol $\beta := \{b_1, \dots, b_n\}$ is trivial in Milnor K-theory, and one maps β to 0.

In this way, one obtains a map

$$\begin{aligned}
&K_{n-1}^M(k) \to SH^n(k, n);\\
&\{x_1, \dots, x_{n-1}\} \mapsto \left(-1, \frac{x_1}{-1 + \sum_{i=1}^{n-1} x_i}, \dots, \frac{x_{n-1}}{-1 + \sum_{i=1}^{n-1} x_i}, \frac{-1}{-1 + \sum_{i=1}^{n-1} x_i}\right).
\end{aligned} \tag{3.5}$$

4. Differential forms

In this section we construct a k-linear map $\Omega_k^{n-1} \to SH^n(k, n)$. (Here Ω_k^i are the absolute Kähler differential i-forms.)

The following lemma is closely related to calculations in [5].

LEMMA 4.1. – *As a k-vector space, the differential forms Ω_k^{n-1} are isomorphic to $(k \otimes_{\mathbb{Z}} \wedge^{n-1} k^\times)/\mathcal{R}$. The k-structure on $k \otimes_{\mathbb{Z}} \wedge^{n-1} k^\times$ is via multiplication on the first argument. The relations $\mathcal{R}$, for $n \geqslant 2$, are the k-subspace spanned by*

$$a \otimes (a \wedge b_1 \wedge \dots \wedge b_{n-2}) + (1-a) \otimes \big((1-a) \wedge b_1 \wedge \dots \wedge b_{n-2}\big),$$

for $b_i \in k^\times$, $a \in k$. *The map* $(k \otimes_{\mathbb{Z}} \wedge^{n-1} k^\times)/\mathcal{R} \to \Omega_k^{n-1}$ *is then defined by*

$$(a, b_1, \dots, b_{n-1}) \mapsto a d\log b_1 \wedge \cdots \wedge d\log b_{n-1}.$$

Proof. – Write $'\Omega^* := (k \otimes_{\mathbb{Z}} \wedge^* k^\times)/\mathcal{R}$. This is a quotient of the graded k-algebra $k \otimes_{\mathbb{Z}} \wedge^* k^\times$ by the graded ideal $\mathcal{R}$ and hence has a graded k-algebra structure, generated in degree 1. There is an evident surjection of graded algebras $'\Omega^* \twoheadrightarrow \Omega^*$, so, by the universal mapping property of the exterior algebra Ω^*, it suffices to check $'\Omega^1 \cong \Omega^1$.

Define $D : k \to (k \otimes_{\mathbb{Z}} k^\times)/\mathcal{R}$ by $D(a) = a \otimes a$ for $a \in k^\times$, else $D(0) = 0$. To define the required inverse, it suffices to show that D is a derivation. Clearly, $D(ab) = aD(b) + bD(a)$. Also $1 \otimes -1$ is trivial in $k \otimes k^\times$, so $D(-a) = -D(a)$. Given $a, b \in k^\times$, write $b = -ac$. We have

$$\begin{aligned} D(a+b) &= D(a-ac) = aD(1-c) + (1-c)D(a) \\ &= -aD(c) + D(a) - cD(a) = D(a) + D(-ac) = D(a) + D(b). \end{aligned} \tag{4.1}$$

Hence D is a derivation so the inverse map $\Omega^1 \to {}'\Omega^1$ is defined. □

Remark 4.2. – We will frequently use the relations in the equivalent form

$$a \otimes a \wedge (\cdots) - (1+a) \otimes (1+a) \wedge (\cdots) \sim 0.$$

PROPOSITION 4.3. – *One has a well-defined k-linear map*

$$\phi : \Omega_k^{n-1} \to SH^n(k,n),$$
$$\alpha := a d\log b_1 \wedge \cdots \wedge d\log b_{n-1} \mapsto a \star \left(-1, \frac{b_1}{\gamma}, \dots, \frac{b_{n-1}}{\gamma}, -\frac{1}{\gamma}\right),$$

where $\gamma = -1 + \sum_{i=1}^{n-1} b_i$. (*Define* $\alpha = 0$ *when* $\gamma = 0$.) *The diagram*

$$\begin{tikzcd} K_{n-1}^M(k) \arrow[r, "d\log"] \arrow[d, "\iota"', "(3.5)"] & \Omega_k^{n-1} \arrow[d, "\phi"'] \\ SH^n(k,n) \arrow[r, equal] & SH^n(k,n) \end{tikzcd}$$

is commutative.

Proof. – We write $c = -1 + a + \sum_{i=2}^{n-1} b_i$. By Lemma 4.1, we have to show

$$0 = \rho := a \star \left(-1, \frac{a}{c}, \dots, \frac{b_{n-1}}{c}, -\frac{1}{c}\right) - (a+1) \star \left(-1, \frac{a+1}{c+1}, \dots, \frac{b_{n-1}}{c+1}, -\frac{1}{c+1}\right).$$

If $a = 0$, then one has

$$\rho = -\left(-1, \frac{1}{c}, \dots, \frac{b_{n-1}}{c}, -\frac{1}{c}\right) = -\iota\{1, b_2, \dots, b_{n-1}\} = 0.$$

Similarly, $\rho = 0$ if $a = -1$. Assume now $a \neq 0, -1$. Set $b = -a \in k \setminus \{0,1\}$. One defines, for $n \geqslant 3$ and $(u_1, \dots, u_{n-1}) \in (\Delta^{n-2} \setminus \bigcup_{i=1}^{n-1} \Delta^{n-3})(k)$, the parametrized curve

$$\Gamma(b,u) := \left\{ \left(\frac{-1}{b} + t, \frac{1}{b-1}, -t, \frac{-u_1}{b(b-1)}, \dots, \frac{-u_{n-1}}{b(b-1)} \right) \right\} \subset Q^{n+1}$$

and for $n=2$

$$\Gamma(b):=\left\{\left(\frac{-1}{b}+t,\frac{1}{b-1},-t,\frac{-1}{b(b-1)}\right)\right\}\subset Q^3. \tag{4.2}$$

(See [8] for the origin of this definition.) This curve is indeed in good position, so it lies in $\mathcal{SZ}^n(Q^{n+1})$. One computes

$$\begin{aligned}\partial\Gamma(b,u)&=(1-b)\star\left(-1,1-\frac{1}{b},\frac{u_1}{b},\dots,\frac{u_{n-1}}{b}\right)\\&\quad+b\star\left(-1,\frac{b}{b-1},\frac{-u_1}{b-1},\dots,\frac{-u_{n-1}}{b-1}\right).\end{aligned} \tag{4.3}$$

(Resp. in the case $n=2$

$$\partial\Gamma(b)=(1-b)\star\left(-1,1-\frac{1}{b},\frac{1}{b}\right)+b\star\left(-1,\frac{b}{b-1},\frac{-1}{b-1}\right).)$$

Now one has

$$\begin{aligned}&\left(-1,1-\frac{1}{b},\frac{u_1}{b},\dots,\frac{u_{n-1}}{b}\right)\\&\quad=\iota\left\{\frac{1-b}{u_{n-1}},-\frac{u_1}{u_{n-1}},\dots,-\frac{u_{n-2}}{u_{n-1}}\right\}\\&\quad=\iota\left[\left\{1-b,-\frac{u_1}{u_{n-1}},\dots,-\frac{u_{n-2}}{u_{n-1}}\right\}-\left\{u_{n-1},u_1,\dots,u_{n-2}\right\}\right],\end{aligned}$$

as the rest of the multilinear expansion contains only symbols of the shape $\{\dots,u_{n-1},\dots,-u_{n-1},\dots\}$. On the other hand, since $\sum_{i=1}^{n-1}u_i=1$, one has $\{u_{n-1},u_1,\dots,u_{n-2}\}=0$. Similarly, one has

$$\left(-1,\frac{b}{b-1},\frac{-u_1}{b-1},\dots,\frac{-u_{n-1}}{b-1}\right)=\iota\left\{\frac{b}{u_{n-1}},-\frac{u_1}{u_{n-1}},\dots,-\frac{u_{n-2}}{u_{n-1}}\right\}.$$

The same argument yields that this is

$$\iota\left\{b,-\frac{u_1}{u_{n-1}},\dots,-\frac{u_{n-2}}{u_{n-1}}\right\}.$$

It follows now from (4.3) that for $n\geqslant 3$ we have the relation in $SH^n(k,n)$

$$(1-b)\star\iota\left\{1-b,-\frac{u_1}{u_{n-1}},\dots,-\frac{u_{n-2}}{u_{n-1}}\right\}+b\star\iota\left\{b,-\frac{u_1}{u_{n-1}},\dots,-\frac{u_{n-2}}{u_{n-1}}\right\}=0. \tag{4.4}$$

(The analogous relation for $n=2$ is similar.) □

PROPOSITION 4.4. – *With notation as above, the map*

$$\phi:\Omega_k^{n-1}\to SH^n(k,n)$$

is surjective. In particular, $SH^n(k,n)$ is generated by the classes of k-rational points in Q^n.

Proof. – It is easy to check that the image of ϕ coincides with the subgroup of $SH^n(k,n)$ generated by k-points. Clearly, $SH^n(k,n)$ is generated by closed points, and any closed point is the trace of a k'-rational point for some finite extension k'/k. We first reduce to the case k'/k separable. If $x \in Q^n_k$ is a closed point in good position (i.e. not lying on any face) such that $k(x)/k$ is not separable, then a simple Bertini argument shows there exists a curve C in good position on Q^{n+1} such that $\partial C = x + y$ where y is a zero cycle supported on points with separable residue field extensions over k. Indeed, let $W \subset Q^{n+1}$ be the union of the faces. View $x \in W$. Since x is in good position, it is a smooth point of W. Bertini will say that a non-empty open set in the parameter space of n-fold intersections of hypersurfaces of large degree containing x will meet W in x plus a smooth residual scheme. Since k is necessarily infinite, there will be such an n-fold intersection defined over k. Since the residual scheme is smooth, it cannot contain inseparable points. Then $x \equiv -y$ which is supported on separable points.

We assume now k'/k finite separable, and we must show that the trace of a k'-point is equivalent to a zero cycle supported on k-points. Since the image of ϕ is precisely the subgroup generated by k-rational points, it suffices to check that the diagram

(4.5)
$$\begin{tikzcd}
\Omega^{n-1}_{k'} \arrow[r, "\phi"] \arrow[d, "\mathrm{Tr}_{k'/k}"] & SH^n(k',n) \arrow[d, "\mathrm{Tr}_{k'/k}"] \\
\Omega^{n-1}_{k} \arrow[r, "\phi"] & SH^n(k,n)
\end{tikzcd}$$

commutes. Because k'/k is separable, one has $\Omega^{n-1}_k \hookrightarrow \Omega^{n-1}_{k'}$, and $\Omega^{n-1}_{k'} = k' \cdot \Omega^{n-1}_k$. One reduces to showing, for $\alpha = (-1, \alpha_1, \ldots, \alpha_n) \in Q^n(k)$ and $t \in k'$, that $\mathrm{Tr}(t \star \alpha) = (\mathrm{Tr}(t)) \star \alpha$ in $SH^n(k,n)$.

Let $P(V) = V^N + a_{N-1}V^{N-1} + \cdots + a_1 V + a_0 \in k[V]$ be the minimal polynomial of $-\frac{1}{t}$. We set $b_N = \frac{-1}{\alpha_n}$, $b_i = \frac{-a_i}{\alpha_n}$, $i = N-1, \ldots, 2$ and $b_i = a_i$, $i = 1, 0$. We define the polynomial $Q(V,u) = b_N V^{N-1}u + \cdots + b_2 V u + b_1 V + b_0 \in k[V,u]$, which by definition fulfills $Q(V, -\alpha_n V) = P(V)$. We define the ideal

$$\mathcal{I} = \big(Q(V_0,u), V_1 + \alpha_1 V_0, \ldots, V_{n-1} + \alpha_{n-1}V_0\big) \subset k[V_0, \ldots, V_{n-1}, u].$$

It defines a curve $W \subset \mathbb{A}^{n+1}$. We think of $\mathbb{A}^{n+1}$ as being Q^{n+1} with the faces $V_0 = 0$, $\ldots, V_{n-1} = 0, u = 0, u + \sum_{i=0}^{n-1} V_i = 0$. Then this curve is in general position and defines a cycle in $\mathcal{SZ}^1(k, n+1)$.

Since $b_0 \neq 0$, and $\alpha_i \neq 0$, one has

$$\partial_i W = 0, \quad i = 0, 1, \ldots, n-1. \tag{4.6}$$

One has

$$\partial_u W \text{ defined by } (a_1 V_0 + a_0, V_1 + \alpha_1 V_0, \ldots, V_{n-1} + \alpha_{n-1} V_0).$$

To compute the last face, we observe that the ideal

$$\left(u + \sum_{i=0}^{n-1} V_i, V_1 + \alpha_1 V_0, \ldots, V_{n-1} + \alpha_{n-1} V_0\right),$$

contains $u+\alpha_n V_0$. Consequently $\partial_{u+\sum_{i=0}^{n-1} V_i} W$ is defined by

$$\bigl(Q(V_0,-\alpha_n V_0), V_1+\alpha_1 V_0, \dots, V_{n-1}+\alpha_{n-1}V_0\bigr),$$

with $Q(V_0,-\alpha_n V_0)=P(V_0)$. Thus one obtains

$$0\equiv(-1)^n\partial W=\frac{a_1}{a_0}\star(-1,\alpha)-t\star(-1,\alpha).$$

Since P is the minimal polynomial of $-\frac{1}{t}$, $\frac{a_1}{a_0}$ is the trace of t. □

5. The main theorem

Recall (1.8) we have a logarithmic $(n-1)$-form γ_{n-1} on $Q^n=\operatorname{Spec}(k[v_0,\dots,v_n]/(\sum_{i=0}^n v_i))$

$$\begin{aligned}\gamma_{n-1}&=\frac{1}{v_0}\sum_1^n(-1)^i d\log(v_1)\wedge\cdots\wedge\widehat{d\log(v_i)}\wedge\cdots\wedge d\log(v_n),\\ d\gamma_{n-1}&=\nu_n=\frac{dv_1\wedge\cdots\wedge dv_n}{v_0v_1\cdots v_n}.\end{aligned}\tag{5.1}$$

Writing $v_i=V_i/V_{n+1}$, we can view γ_{n-1} as a meromorphic form on

$$\mathbb{P}^n=\operatorname{Proj}\left(k[V_0,\dots,V_{n+1}]\Big/\left(\sum_0^n V_i\right)\right).$$

Let $\mathcal{A}$: $V_0\cdots V_n=0$; ∞: $V_{n+1}=0$. The fact that $d\gamma_{n-1}$ has log poles on the divisors $V_i=0,\ 0\leqslant i\leqslant n$ implies that

$$\gamma_{n-1}\in\Gamma\bigl(\mathbb{P}^n,\Omega^{n-1}_{\mathbb{P}^n}\bigl(\log(\mathcal{A}+\infty)\bigr)(-\infty)\bigr).\tag{5.2}$$

In particular, γ_{n-1} has log poles, so we can take the residue along components of $\mathcal{A}$. (The configuration $\mathcal{A}$ does not have normal crossings. The sheaf $\Omega^{n-1}_{\mathbb{P}^n}(\log(\mathcal{A}+\infty))$ is defined to be the subsheaf of $j_*\Omega^{n-1}_{\mathbb{P}^n-\mathcal{A}-\infty}$, where $j:\mathbb{P}^n\setminus(\mathcal{A}\cup\infty)\to\mathbb{P}^n$ is the open embedding, generated by forms without poles and the evident log forms with residue 1 along one hyperplane and (-1) along another one. According to [1], the global sections of this naturally defined log sheaf compute de Rham cohomology.)

In the following, we adopt the sign convention $\operatorname{Res}_{t=0}\frac{dt}{t}\wedge\omega=\omega$. This yields

$$\operatorname{Res}_{t=0} d\left(\frac{dt}{t}\wedge\omega\right)=-d\left(\operatorname{Res}_{t=0}\frac{dt}{t}\wedge\omega\right).$$

LEMMA 5.1. – *We have the following residue formulae*

$$\operatorname{Res}_{v_i=0}\gamma_n=(-1)^i\gamma_{n-1};\quad 0\leqslant i\leqslant n+1.$$

Proof. – One can either compute directly or argue indirectly as follows:

$$\begin{aligned}-d\operatorname{Res}_{v_i=0}\gamma_n&=\operatorname{Res}_{v_i=0}d\gamma_n=\operatorname{Res}_{v_i=0}\nu_{n+1}\\ &=(-1)^{i+1}\nu_n=-d\bigl((-1)^i\gamma_{n-1}\bigr)\neq 0.\end{aligned}\tag{5.3}$$

To conclude now, it suffices to show that the space of global sections (5.2) has dimension 1. To verify the dimension 1 property, let $\mathcal{A}' \subset \mathcal{A}$ be defined by $V_1 \cdots V_n = 0$. Then $\mathcal{A}' + \infty$ consists of $n+1$ hyperplanes in general position in $\mathbb{P}^n$, so

$$\Omega^{n-1}_{\mathbb{P}^n}\big(\log(\mathcal{A}' + \infty)\big) = \wedge^{n-1}\Omega^1_{\mathbb{P}^n}\big(\log(\mathcal{A}' + \infty)\big) \cong \mathcal{O}^{\oplus n}_{\mathbb{P}^n}.$$

One looks at the evident residue

$$\Omega^{n-1}_{\mathbb{P}^n}\big(\log(\mathcal{A} + \infty)\big)(-\infty) \to \Omega^{n-2}_{\mathbb{P}^{n-1}}\big(\log(\mathcal{A} + \infty)\big)(-\infty)$$

along $V_0 = 0$. □

Remark 5.2. – The computation of the lemma shows that γ_{n-1} is the *unique* $(n-1)$-differential form in $\Gamma(\mathbb{P}^n, \Omega^{n-1}_{\mathbb{P}^n}(\log(\mathcal{A} + \infty))(-\infty))$ with $d\gamma_{n-1} = \nu_n$. The configuration $\mathcal{A}$ being given, fixing ∞ fixes γ_{n-1}.

THEOREM 5.3. – *The assignment* $x \mapsto \mathrm{Tr}_{k(x)/k}(\gamma(x))$ *gives an isomorphism for* $n \geqslant 1$

$$SH^n(k,n) \cong \Omega^{n-1}_k.$$

Proof. – Let $X \subset Q^{n+1}_k$ be a curve in good position. For a zero-cycle c in good position on Q^n we write $\mathrm{Tr}\,\gamma_{n-1}(c) \in \Omega^{n-1}_k$ (absolute differentials) for the evident linear combination of traces from residue fields of closed points. We must show $\mathrm{Tr}\,\gamma_{n-1}(\partial X) = 0$. Let $\overline{X}$ denote the closure of X in $\mathbb{P}^n$. We consider $\gamma_{n-1}|_{\overline{X}} \in \Gamma(\overline{X}, \Omega^{n-1}_{\overline{X}/\mathbb{Z}}(*D))$, where D is the pole set of $\gamma_{n-1}|_{\overline{X}}$. The form γ_{n-1} dies when restricted to ∞. Thus $D = \bigcup_{j=0}^n D_j \subset X$, with $D_j := \partial_j(Q^n) \cap \overline{X}$. We define the residue along D_j as follows. One has an exact sequence

$$0 \to \Omega^n_k \otimes \mathcal{O}_{\overline{X}} \to \Omega^n_{\overline{X}}(\log D) \to \Omega^{n-1}_k \otimes \omega_{\overline{X}}(\log D) \to 0. \tag{5.4}$$

The residue map followed by the trace

$$\omega_{\overline{X}}(\log D) \xrightarrow{\mathrm{Tr}\circ\sum \mathrm{Res}_{D_j}} k \tag{5.5}$$

yields, by the reciprocity formula, a vanishing residue map on global n-forms

$$H^0\big(\overline{X}, \Omega^n_{\overline{X}/\mathbb{Z}}(\log D)\big) \to \Omega^{n-1}_k \bigotimes_k H^0\big(\overline{X}, \omega_{\overline{X}}(\log D)\big) \xrightarrow{0} \Omega^{n-1}_k. \tag{5.6}$$

On the other hand, the residue decomposes as

$$H^0\big(\overline{X}, \Omega^n_{\overline{X}/\mathbb{Z}}(\log D)\big) \xrightarrow{\sum_j \mathrm{Res}_{D_j}} \bigoplus_{j=0}^n \Omega^{n-1}_{D_j} \xrightarrow{\sum_j \mathrm{Tr}} \Omega^{n-1}_k. \tag{5.7}$$

By Lemma 5.1, $\gamma_{n-1}(D_j) = (\mathrm{Res}_{v_j=0}\,\gamma_n)(D_j)$. The desired vanishing follows.

We now have

$$\Omega^{n-1}_k \overset{\phi}{\twoheadrightarrow} SH^n(k,n) \xrightarrow{\gamma_{n-1}} \Omega^{n-1}_k.$$

It suffices to check the composition is multiplication by $(-1)^{n+1}$. Given $b_1, \ldots, b_{n-1} \in k$ with $c := \sum b_i - 1 \neq 0$, the composition is computed to be (use (5.1) and Proposition 4.3)

$$a\cdot d\log(b_1)\wedge\cdots\wedge d\log(b_{n-1})\mapsto$$
$$-a\sum(-1)^i d\log\left(\frac{b_1}{ac}\right)\wedge\cdots\wedge\widehat{d\log\left(\frac{b_i}{ac}\right)}\wedge\cdots\wedge d\log\left(\frac{-1}{ac}\right).$$

Expanding the term on the right yields

$$-a\sum(-1)^i d\log\left(\frac{b_1}{c}\right)\wedge\cdots\wedge\widehat{d\log\left(\frac{b_i}{c}\right)}\wedge\cdots\wedge d\log\left(\frac{-1}{c}\right)-a\cdot d\log(a)\wedge(\ldots),$$

and it is easy to check that the terms involving $d\log(a)$ cancel. In this way, one reduces to the case $a=1$. Here

$$-\sum(-1)^i d\log\left(\frac{b_1}{c}\right)\wedge\cdots\wedge\widehat{d\log\left(\frac{b_i}{c}\right)}\wedge\cdots\wedge d\log\left(\frac{-1}{c}\right)$$
$$=(-1)^{n+1}\frac{db_1}{b_1}\wedge\cdots\wedge\frac{db_{n-1}}{b_{n-1}}+\frac{dc}{c}\wedge(\ldots). \tag{5.8}$$

Again the terms involving $d\log(c)$ cancel formally, completing the proof. □

Challenge 5.4. – Finally, as a challenge we remark that the Kähler differentials have operations (exterior derivative, wedge product, ...) which are not evident on the cycles $\mathcal{SZ}$. For example, one can show that the map

$$\nabla(x_0,\ldots,x_n)=\left(x_0,-\frac{x_1x_0}{1-x_0},\ldots,-\frac{x_nx_0}{1-x_0},-\frac{x_0}{1-x_0}\right) \tag{5.9}$$

satisfies

$$\gamma_n\bigl(\nabla(x)\bigr)=(-1)^n d\gamma_{n-1}(x) \tag{5.10}$$

and hence induces the exterior derivative on the 0-cycles. The map is not uniquely determined by this property, and this particular map does not preserve good position for cycles of dimension >0. Can one find a geometric correspondence on the complex $\mathcal{SZ}^\bullet$ which induces d on the 0-cycles? What about the pairings $(a,b)\mapsto a\wedge b$ or $(a,b)\mapsto a\wedge db$?

6. Specialization of forms

In this section we consider again the specialization of differential forms as in Section 1. Recall $f(U_0,\ldots,U_{n+1})$ is homogeneous of degree $n+2$, $u_i=U_i/U_0$, and $v_i=t^{r_i}u_i$. N is minimal such that $t^N f(1,t^{-r_1}v_1,\ldots,t^{-r_{n+1}}v_{n+1})$ is integral in t, and we assume $s=N-\sum r_i>0$. We have forms

$$\omega_f:=\frac{d(t^{-r_1}v_1)\wedge\cdots\wedge d(t^{-r_{n+1}}v_{n+1})}{f(1,t^{-r_1}v_1,\ldots,t^{-r_{n+1}}v_{n+1})}=t^s\nu_f+st^{s-1}dt\wedge\gamma_f \tag{6.1}$$

and $\nu_f|_{t=0}=d\gamma_f|_{t=0}$.

We have already mentioned the case

$$f=\Delta_{n+1}=\bigl(t^{-1}v_1\bigr)\cdots\bigl(t^{-1}v_{n+1}\bigr)\bigl(1-t^{-1}v_1-\cdots-t^{-1}v_{n+1}\bigr).$$

The forms $\nu_{\Delta_{n+1}}$ and $\gamma_{\Delta_{n+1}}$ are given in (1.8). As before, we write $\nu_{n+1}:=\nu_{\Delta_{n+1}}|_{t=0}$; $\gamma_n:=\gamma_{\Delta_{n+1}}|_{t=0}$. The differential form ν_{n+1} plays an interesting rôle in the computation of de

Rham cohomology of the complement of hyperplane configurations. Let $\mathcal{A}_t$ be the configuration in $\mathbb{P}^n$ of $(n+1)$ hyperplanes in general position with affine equation $u_0u_1\cdots u_n=0$, $u_0+u_1+\cdots+u_n=t\neq 0$. Then $H^n(\mathbb{P}^n\setminus\mathcal{A}_t)=H^0(\mathbb{P}^n,\Omega^n_{\mathbb{P}^n}(\log\mathcal{A}_t))$ is a pure Tate structure generated by $\nu_{\Delta n+1}$. Now make t tend to 0 and consider the degenerate configuration $\mathcal{A}_0$ with affine equation $u_0u_1\cdots u_n=0$, $u_0+u_1+\cdots+u_n=0$. Exactness of $\nu_{n+1}=d\gamma_n$ in the case (1.7) follows from Aomoto's theory [1,4]. Due to the special shape of the configurations $\mathcal{A}_t$ considered, however, more is true. We have fixed ∞, and the differential form γ_n lies in $H^0(\mathbb{P}^{n+1},\Omega^n(\log(\mathcal{A}_0+\infty))(-\infty))$. That is, it has log poles along all components of $\mathcal{A}_0$, and it vanishes as a differential form at ∞ (see Lemma 5.1). This k-vector space of differential forms is 1-dimensional. In other words, the differential form γ_n is uniquely defined by the vanishing condition at ∞ and the secondary data $d\gamma_n=\nu_{n+1}$ (see Remark 5.2). Do such primitives for more general degenerating configurations admit an interpretation identifying 0-cycles with spaces of differential forms?

Here is another example of specialization and 0-cycles. Define

$$f(1,u_1,u_2)=u_1^2-u_2^3-au_2-b;\qquad v_1=t^3u_1,\qquad v_2=t^2u_2. \tag{6.2}$$

One computes

$$\nu=\frac{dv_1\wedge dv_2}{v_1^2-v_2^3-at^4v_2-bt^6};\qquad \gamma=\frac{2v_2\,dv_1-3v_1dv_2}{v_1^2-v_2^3-at^4v_2-bt^6}. \tag{6.3}$$

One checks that $\operatorname{Res}(\gamma|_{t=0})=v_2/v_1$ and

$$d(v_2/v_1)=-dv_2/2=d\operatorname{Res}(\nu|_{t=0})\quad\text{on } v_1^2-v_2^3=0. \tag{6.4}$$

The assignment $x\mapsto \operatorname{Tr}_{k(x)/k}(v_2/v_1)(x)$ identifies the jacobian of the special fibre $v_1^2-v_2^3=0$ with $\mathbb{G}_a(k)=k$.

A final example of specialization, which we understand less well, though it was an inspiration for this article, concerns the hyperbolic motives of Goncharov [6]. The *matrix coefficients* of his theory (in the sense of [2]) are the objects $H^{2n-1}(\mathbb{P}^{2n-1}\setminus Q, M\setminus Q\cap M)$. Here $Q\subset\mathbb{P}^{2n-1}$ is a smooth quadric and M is a simplex (union of $2n$ hyperplanes in general position). The subschemes Q and M are taken in general position with respect to each other. The notation is intended to suggest a sort of abstract relative cohomology group. In de Rham cohomology, the non-trivial class in $H^{2n-1}_{DR}(\mathbb{P}^{2n-1}\setminus Q)$ is represented by

$$\omega=\frac{du_1\wedge\cdots\wedge du_{2n-1}}{(u_1^2+\cdots+u_{2n-1}^2-1)^n}. \tag{6.5}$$

Substituting $u_i=v_it$, we get

$$\omega=\frac{t^{2n-1}dv_1\wedge\cdots\wedge dv_{2n-1}+t^{2n-2}dt\wedge\sum(-1)^iv_idv_1\wedge\cdots\widehat{dv_i}\cdots\wedge dv_{2n-1}}{(t^2(v_1^2+\cdots+v_{2n-1}^2)-1)^n}. \tag{6.6}$$

We get

$$\nu|_{t=0}=\pm dv_1\wedge\cdots\wedge dv_{2n-1}$$

and

$$\gamma|_{t=0} = \frac{1}{2n-1}\sum(-1)^i v_i\, dv_1\wedge\cdots\widehat{dv_i}\cdots\wedge dv_{2n-1}.$$

Let $\Delta_M \in H_{2n-1}(\mathbb{P}^{2n-1}, M;\mathbb{Z})$ be a generator. The hyperbolic volume

$$\int\limits_{\Delta_M}\omega \tag{6.7}$$

is the *real period* [6, Section 4.1] of the Hodge structure associated to $H^{2n-1}(\mathbb{P}^{2n-1}\setminus Q)$. Goncharov remarks (op. cit., Question 6.4 and Theorem 6.5) that this volume degenerates to the euclidean volume as $t\to 0$. (More precisely, from (6.7), we see that as a relative form, $dv_1\wedge\cdots\wedge dv_{2n-1} = \lim_{t\to 0} t^{1-2n}\omega$.) He asks for an interpretation of the degenerated volume in terms of some sort of motive over $k[t]/(t^2)$.

In the Goncharov picture we can view Q as fixed and degenerate M. Suppose

$$M\colon\ L_0L_1\cdots L_{2n-1}=0$$

where

$$L_i = L_i(v_1,\ldots,v_{2n-1}) = L_i(u_1/t,\ldots,u_{2n-1}/t).$$

Assuming the L_i are general, clearing denominators and passing to the limit $t\to 0$ yields a degenerate simplex M_0 consisting of $2n$ hyperplanes meeting at the point $v=0$. This limiting configuration leads to the Chow groups $SH^*(k,2n-1)$, but we do not see how to relate $\gamma|_{t=0}$ to cycles.

7. Cycle groups of divisors

We compute $SH^1(k,n)$ for $n\geqslant 1$.

Let $\mathcal{O}_{Q^n,0}$ be the local ring at $(0,\ldots,0)\in Q^n$. Let

$$I^n \subset J^n \subset \mathcal{O}_{Q^n,0} \tag{7.1}$$

be the ideals of all (resp. all but the last) face.

LEMMA 7.1. –

$$SH^1(k,n)\cong \bigl(1+I^n\bigr)^\times/\partial_{n+1}\bigl(1+J^{n+1}\bigr)^\times.$$

Proof. – Let $D\subset Q^n$ be an effective divisor. Let

$$f\in k_n := k[t_0,\ldots,t_n]\Big/\Bigl(\sum t_i\Bigr)$$

be a defining equation for D. Then D meets faces properly in our sense if and only if $f\in\mathcal{O}^\times_{Q^n,0}$. Conversely, any $f\in\mathcal{O}^\times_{Q^n,0}$ defines a (not necessarily effective) divisor on Q^n meeting faces properly. Further

$$(f)\cdot Q_i^{n-1}=0 \quad\Leftrightarrow\quad f\in k^\times\cdot(1+t_i\mathcal{O}_{Q^n,0}). \tag{7.2}$$

By general simplicial considerations, the $SH^1(k,n)$ is given by divisors meeting all faces trivially, modulo the restriction to the last face of divisors on Q^{n+1} meeting all but the last face trivially. Scaling the f above to remove the factor $k^\times$, this is precisely the assertion of the lemma. □

PROPOSITION 7.2. – *We have* $SH^1(k,1)=k$, *and* $SH^1(k,n)=(0)$ *for* $n\geqslant 2$.

Proof. – The first assertion is part of Theorem 5.3. The exact sequence

$$0\to I^{n+1}\to J^{n+1}\xrightarrow{\partial_{n+1}} I^n\to 0 \tag{7.3}$$

yields the surjectivity $1+J^{n+1}\twoheadrightarrow 1+I^n$. We observe that when $n=1$, we have $I^1=(t_0)$ but $\partial_2(J^2)=(t_0^2)$. □

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Spencer BLOCH
Department of Mathematics,
University of Chicago,
Chicago, IL 60637, USA
E-mail: bloch@math.uchicago.edu

Hélène ESNAULT
Universität Essen,
FB6, Mathematik,
45117 Essen, Germany
E-mail: esnault@uni-essen.de