RIGIDITY OF GENERIC SINGULARITIES OF MEAN CURVATURE FLOW

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ABSTRACT

Shrinkers are special solutions of mean curvature flow (MCF) that evolve by rescaling and model the singularities. While there are infinitely many in each dimension, Colding and Minicozzi II (Ann. Math. 175(2):755–833, 2012) showed that the only generic are round cylinders $\mathbf{S}^k \times \mathbf{R}^{n-k}$. We prove here that round cylinders are rigid in a very strong sense. Namely, any other shrinker that is sufficiently close to one of them on a large, but compact, set must itself be a round cylinder.

To our knowledge, this is the first general rigidity theorem for singularities of a nonlinear geometric flow. We expect that the techniques and ideas developed here have applications to other flows.

Our results hold in all dimensions and do not require any a priori smoothness.

0. Introduction

The mean curvature flow is an evolution equation where a hypersurface evolves over time by locally moving in the direction of steepest descent for the volume element.

A hypersurface $\Sigma \subset \mathbf{R}^{n+1}$ is said to be a *self-similar shrinker*, or just *shrinker*, if it is the $t = -1$ time-slice of a mean curvature flow ("MCF") moving by rescalings.[1] Being self-similar is easily seen to be equivalent to being stationary for the Gaussian surface area.

By a monotonicity formula of Huisken and an argument of Ilmanen and White, blow-ups of singularities of a MCF are shrinkers. The only generic shrinkers are round cylinders by [CM1].

Our main theorem is the following rigidity or uniqueness theorem for cylinders:

Theorem **0.1.** — *Given n, λ_0 and C, there exists $\mathrm{R} = \mathrm{R}(n, \lambda_0, \mathrm{C})$ so that if $\Sigma^n \subset \mathbf{R}^{n+1}$ is a shrinker with entropy $\lambda(\Sigma) \leq \lambda_0$ satisfying*

($\dagger$) *Σ is smooth in B_R with $\mathrm{H} \geq 0$ and $|\mathrm{A}| \leq \mathrm{C}$ on $\mathrm{B}_\mathrm{R} \cap \Sigma$,*

then Σ is a generalized cylinder $\mathbf{S}^k \times \mathbf{R}^{n-k}$ for some $k \leq n$.

Smooth will always mean smooth and embedded.

The entropy λ is the supremum of the Gaussian surface areas of Σ over all centers and scales. When Σ is a shrinker, there is no need to take a supremum as the entropy is always achieved by the standard Gaussian surface area (see Section 7 of [CM1]).

[1] This means that the time t slice of the MCF is given by $\sqrt{-t}\,\Sigma$. See [A], [Ch], [KKM], and [N] for examples of shrinkers.

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In particular, Theorem 0.1 implies that a shrinker that is close to a cylinder in a sufficiently large ball must be isometric to the cylinder.

We will say that a singular point is *cylindrical* if there is at least one tangent flow that is a multiplicity one generalized cylinder $\mathbf{S}^k \times \mathbf{R}^{n-k}$. We will prove the following gap theorem for singularities in a neighborhood of a cylindrical singular point:

Theorem **0.2.** — *Let* M_t *be a MCF in* $\mathbf{R}^{n+1}$. *Each cylindrical singular point has a space-time neighborhood where every non-cylindrical singular point has entropy at least* $\epsilon = \epsilon(n) > 0$ *below that of the cylinder.*

As a corollary of the theorem, we get uniqueness of type for cylindrical tangent flows:

Corollary **0.3.** — *If one tangent flow at a singular point of a mean curvature flow is a multiplicity one cylinder, then they all are.*

By definition, a tangent flow is the limit of a sequence of rescalings at the singularity, where the convergence is on compact subsets. For this reason, it is essential for applications of Theorem 0.1, like Corollary 0.3, that Theorem 0.1 only requires closeness on a fixed compact set.

These results are the first general uniqueness theorems for tangent flows to a geometric flow at a non-compact singularity. Some special cases for MCF were previously analyzed assuming either some sort of convexity or that the hypersurface is a surface of rotation; see [H1], [H2], [HS], [W1], [SS], [AAG], and Section 3.2 in the book [GGS]. In contrast, uniqueness for blowups at compact singularities is better understood; cf. [AA] and [Si].

In each dimension, the sphere has the lowest entropy among closed shrinkers, but in higher dimensions there are smooth noncompact shrinkers, very close to Simons cones, with entropy below the cylinders; see [CIMW] or [I1] for details. The results of this paper may be compared and contrasted with Brakke's regularity theorem, [B], which shows not only that the hyperplane isolated among shrinkers, but there is a gap to the entropy of all other shrinkers.

0.1. *Rigidity without smoothness.* — The rigidity theorem holds even when the shrinker is not required to be smooth outside of the ball of radius R. This is important in applications, including Theorem 0.2 and Corollary 0.3.

Smooth shrinkers are stationary for Gaussian surface area (called the F functional in [CM1]). Accordingly, weak solutions of the shrinker equation can be defined as F-stationary n-dimensional integer rectifiable varifolds. We will refer to these simply as F-stationary varifolds. Theorem 0.1 holds for F-stationary varifolds that are smooth in B_R with the given estimates.

0.2. *Outline of proof.* — The proof of Theorem 0.1 uses iteration and improvement. Roughly speaking, the theorem assumes that the shrinker is cylindrical on some large scale. The iterative step then shows that it is cylindrical on an even larger scale, but with some loss in the estimates. The improvement step then comes back and says that there was actually no loss if the scale is large enough. Applying these two steps repeatedly gives that the shrinker is roughly cylindrical on all scales, which will easily give the theorem. We will make this outline more precise in Section 2.

Corollary 0.3 shows that if one tangent flow at a singular point of a MCF is a multiplicity one cylinder, then all are. However, it leaves open the possibility that the direction of the axis (the $\mathbf{R}^k$ factor) depends on the sequence of rescalings. This has since been settled in [CM3] where it was shown that even the axis is independent of the sequence of rescalings. The proof given there, in particular, the first Lojasiewicz type inequality there, has its roots in ideas and inequalities of this paper and in fact implicitly uses that cylinders are isolated among shrinkers by Theorem 0.1. The results here and in [CM3] were also used in [CM4], where new rectifiability results for the singular set were obtained, and in [CM5].

The results of this paper were discussed in [CMP].

1. Notation and background from [CM1]

We begin by recalling the classification of smooth, embedded mean convex shrinkers in arbitrary dimension from theorem 0.17 in [CM1]:

Theorem **1.1** *[CM1].* — $\mathbf{S}^k \times \mathbf{R}^{n-k}$ *are the only smooth complete embedded shrinkers without boundary, with polynomial volume growth, and* $H \geq 0$ *in* $\mathbf{R}^{n+1}$.

The $\mathbf{S}^k$ factor in Theorem 1.1 is round, centered at 0, and has radius $\sqrt{2k}$; we allow the possibilities of a sphere ($n - k = 0$) or a hyperplane (i.e., $k = 0$), although Brakke's theorem rules out the multiplicity one hyperplane as a tangent flow at a singular point.

The classification of smooth embedded shrinkers with $H \geq 0$ began with [H1], where Huisken showed that round spheres are the closed ones. In a second paper, [H2], Huisken showed that the generalized cylinders $\mathbf{S}^k \times \mathbf{R}^{n-k} \subset \mathbf{R}^{n+1}$ are the open ones with polynomial volume growth and $|A|$ bounded. Theorem 0.17 in [CM1] completed the classification by removing the $|A|$ bound.

1.1. *Notation.* — Let $\Sigma \subset \mathbf{R}^{n+1}$ be a hypersurface, $\mathbf{n}$ its unit normal, Δ its Laplace operator, A its second fundamental form, $H = \operatorname{div}_\Sigma \mathbf{n}$ its mean curvature, x is the position vector, and let v^T denote the tangential projection of a vector field v onto Σ.

It is easy to see that a hypersurface Σ is a shrinker if it satisfies the equation

$$H = \frac{\langle x, \mathbf{n} \rangle}{2}. \tag{1.2}$$

We will use the operators $\mathcal{L}$ and L on shrinkers from [CM1] defined by

$$\mathcal{L} = \Delta - \frac{1}{2}\nabla_{x^{\mathrm{T}}}, \tag{1.3}$$

$$\mathrm{L} = \mathcal{L} + |\mathrm{A}|^2 + \frac{1}{2}. \tag{1.4}$$

With our convention on H, a one-parameter family of hypersurfaces $\Sigma_t \subset \mathbf{R}^{n+1}$ flows by mean curvature if

$$(\partial_t x)^\perp = -\mathrm{H}\mathbf{n}. \tag{1.5}$$

It is convenient to recall the family of functionals on the space of hypersurfaces given by integrating Gaussian weights with varying centers and scales. These are often referred to as Gaussian surface areas. For $t_0 > 0$ and $x_0 \in \mathbf{R}^{n+1}$, define F_{x_0,t_0} by

$$\mathrm{F}_{x_0,t_0}(\Sigma) = (4\pi t_0)^{-n/2} \int_\Sigma \mathrm{e}^{-\frac{|x-x_0|^2}{4t_0}}\, d\mu. \tag{1.6}$$

We will think of x_0 as being the point in space that we focus on and $\sqrt{t_0}$ as being the scale. Write $\mathrm{F} = \mathrm{F}_{0,1}$.

The entropy is the supremum over all Gaussians and is given by

$$\lambda(\Sigma) = \sup_{x_0,t_0} \mathrm{F}_{x_0,t_0}(\Sigma). \tag{1.7}$$

Here the supremum is over all $t_0 > 0$ and $x_0 \in \mathbf{R}^{n+1}$. The entropy is invariant under dilation and rigid motions and, as a consequence of a result of Huisken, [H1], is monotone nonincreasing under both MCF and rescaled MCF.

Note that both the F-functionals and the entropy are defined for rectifiable varifolds.

2. Key ingredients for the rigidity of the cylinder

In this section, we will prove the main theorem using the iterative step (Proposition 2.1 below) and the improvement step (Proposition 2.2 below). These propositions will be proven in Sections 3 and 4, respectively. We will use the main theorem to prove Theorem 0.2 and Corollary 0.3 in Section 2.2.

Throughout this section $\Sigma \subset \mathbf{R}^{n+1}$ is an n-dimensional F-stationary varifold. On a first reading, the reader may prefer to ignore the technicalities that arise because Σ is not assumed to be smooth. The extra difficulties dealing with non-smoothness are very minor and easy to overcome.

Smooth will always mean smooth and embedded.

Proposition **2.1.** — *Given* $\lambda_0 < 2$ *and* n*, there exist positive constants* R_0*,* δ_0*,* C_0 *and* θ *so that if* $\lambda(\Sigma) \leq \lambda_0$*,* $R \geq R_0$*, and*

- $B_R \cap \Sigma$ *is smooth with* $H \geq 1/4$ *and* $|A| \leq 2$*,*

then $B_{(1+\theta)R} \cap \Sigma$ *is smooth with* $H \geq \delta_0$ *and* $|A| \leq C_0$*.*

From now on, δ_0 and C_0 will be given by the previous proposition.

Proposition **2.2.** — *Given* n*,* $\lambda_0 > 0$*,* $\delta_0 > 0$ *and* C_0*, there exists* R_1 *so that if* $\lambda(\Sigma) \leq \lambda_0$*,* $R \geq R_1$*, and*

- $B_R \cap \Sigma$ *is smooth with* $H \geq \delta_0$ *and* $|A| \leq C_0$*,*

then $H \geq 1/4$ *and* $|A| \leq 2$ *on* $B_{R-3} \cap \Sigma$*.*

When we apply these iteratively, it will be important that Proposition 2.1 extends the scale of the "cylindrical region" by a factor greater than one, while Proposition 2.2 only forces one to come in by a constant amount to get the improvement. This makes the iteration work as long as the initial scale is large enough.

2.1. *The proof of the rigidity theorem.* — We will see that the main theorem follows from the following proposition, where we also assume a positive lower bound for H and an upper bound for the entropy that is less than two.

Proposition **2.3.** — *Suppose that* Σ *satisfies the hypotheses of Theorem 0.1. If, in addition, we have the stronger assumptions that* $\lambda_0 < 2$ *and* $H \geq \delta > 0$ *on* B_{R-1} *where* $R = R(n, \lambda_0, C, \delta)$*, then* Σ *is a generalized cylinder* $\mathbf{S}^k \times \mathbf{R}^{n-k}$ *for some* $k \leq n$*.*

Proof of Proposition 2.3 using Propositions 2.1 and 2.2. — This follows by applying first Proposition 2.1 then Proposition 2.2 and repeating this. □

We will use the following elementary lemma in the proof of the main theorem.

Lemma **2.4.** — *There exists* C *depending only on* n *such that if* $R > 0$ *and* μ *is a measure with* $\lambda(\mu) \leq \lambda_0$*, then*

$$F\big(\mu \lfloor (\mathbf{R}^{n+1} \setminus B_R)\big) \leq C e^{-R^2/8} \lambda_0. \tag{2.5}$$

In particular, if μ *is the mass measure of an* F*-stationary varifold* Σ*, then*

$$\big|\lambda(\Sigma) - F(\Sigma \cap B_R)\big| \leq C e^{-R^2/8} \lambda_0. \tag{2.6}$$

Proof. — Write $\rho_{x,t}(z) := e^{-|x-z|^2/4t}/(4\pi t)^{n/2}$. Compute

$$F_{0,1}\big(\mu\lfloor(\mathbf{R}^{n+1}\setminus B_R)\big) \le \sup_{|z|\ge R}\frac{\rho_{0,1}(z)}{\rho_{0,2}(z)}F_{0,2}(\Sigma\setminus B_R) \le 2^{n/2}e^{-R^2/8}\lambda_0.$$

When Σ is F-stationary, [CM1] gives that $\lambda(\Sigma) = F_{0,1}(\Sigma)$, yielding the second statement. □

We will also use the following compactness theorem for sequences of F-stationary varifolds:

Lemma **2.7.** — *Let $\Sigma_i \subset \mathbf{R}^{n+1}$ be a sequence of F-stationary varifolds with $\lambda(\Sigma_i) \le \lambda_0$ and*

(**2.8**) $$B_{R_i}\cap\Sigma_i \text{ is smooth with } |A| \le C,$$

where $R_i \to \infty$. Then there exists a subsequence Σ_i' that converges smoothly and with multiplicity one to a complete embedded shrinker Σ with $|A| \le C$ and

(**2.9**) $$\lim_{i\to\infty}\lambda\big(\Sigma_i'\big) = \lambda(\Sigma).$$

Proof. — All of this discussion is done in great detail in [CM2] for a similar case, so we will sketch the argument here.

Combining the a priori $|A|$ bound with elliptic estimates for the shrinker equation, the Arzela-Ascoli theorem, the strong maximum principle, and a diagonal argument, we get that a subsequence converges in $C^{2,\alpha}$ with finite multiplicity to a smooth embedded shrinker Σ with $\lambda(\Sigma) \le \lambda_0$.

We argue as in [CM2] to see why the multiplicity must be one. Namely, Proposition 3.2 in [CM2] gives that Σ is L-stable if the multiplicity is greater than one.[2] On the other hand, by [CM1] (see Theorem 0.5 in [CM2]), there are no complete L-stable shrinkers with polynomial volume growth, so the limit must have been multiplicity one.

Finally, (2.9) now follows from (2.6). □

We will now give the proof of the main rigidity theorem. Though nearly all of the work involves smooth computations, the argument actually proves the general version where Σ is F-stationary and need not be smooth everywhere. The proof uses Proposition 2.3 which relies on the two key ingredients, Propositions 2.1 and 2.2, which will be proven in the next two sections.

Proof of Theorem 0.1 using Proposition 2.3. — By the compactness of Lemma 2.7 and the classification of complete embedded mean convex shrinkers in [H1] and [H2], we can assume that Σ is smoothly close to $\mathbf{S}^k\times\mathbf{R}^{n-k}$ in B_{R_1} for some $k\in\{0,\dots,n\}$, where R_1 can be taken as large as we wish. Note that $F_{0,1}(\mathbf{S}^k\times\mathbf{R}^{n-k}) \le \sqrt{2\pi/e} \approx 1.52$ by [S].

[2] Proposition 3.2 in [CM2] is stated for surfaces (i.e., $n = 2$) but the proof applies in arbitrary dimension.

Then, using the closeness to the cylinder and (**2.6**), we can arrange that

$$\lambda(\Sigma) < 1.6. \tag{2.10}$$

We now consider two cases depending on k:

- When $k = 0$, then (**2.6**) allows us to make $\lambda(\Sigma)$ as close to 1 as we wish, so Brakke's theorem ([B, Theorem 6.11]; cf. [W2]) gives that Σ is a hyperplane.
- When $k > 1$, we get that H is approximately $\sqrt{k/2}$ on $B_{R_1} \cap \Sigma$ and the theorem follows from Proposition 2.3. □

2.2. *The proofs of Theorem 0.2 and Corollary 0.3.* — Theorem 0.2 will follow from a slightly more general form of the rigidity theorem for the cylinder and a compactness result for mean curvature flows.

To make this precise, we will define a distance d_V between Radon measures on $\mathbf{R}^{n+1}$ with the Gaussian weight. Namely, let f_n be a countable dense subset of the unit ball in the space of continuous functions with compact support and define

$$d_V(\mu_1, \mu_2) = \sum_k 2^{-k} \left| \int f_k e^{-|x|^2/4}\, d\mu_1 - \int f_k e^{-|x|^2/4}\, d\mu_2 \right|. \tag{2.11}$$

It is then easy to see that d_V is a metric on the space of Radon measures satisfying $F(\mu) < \infty$ and $\mu_j \to \mu$ in the standard weak topology if and only if $d_V(\mu_j, \mu) \to 0$.

Corollary **2.12.** — *Given n and λ_0, there exists $\kappa > 0$ so that if $\Sigma^n \subset \mathbf{R}^{n+1}$ is an F-stationary varifold with entropy $\lambda(\Sigma) \le \lambda_0$ and $d_V(\Sigma, \mathbf{S}^k \times \mathbf{R}^{n-k}) \le \kappa$, then Σ is isometric to $\mathbf{S}^k \times \mathbf{R}^{n-k}$.*

Here $\mathbf{S}^k \times \mathbf{R}^{n-k}$ has multiplicity one and the $\mathbf{S}^k$ factor is centered at 0 with radius $\sqrt{2k}$.

Proof of Corollary 2.12. — This follows immediately from Theorem 0.1 since measure-theoretic closeness to a smooth shrinker implies smooth closeness on compact sets by Allard's regularity theorem [Al]. (Note that we need the version of the theorem that assumes smoothness only in B_R.) □

The rescaled mean curvature flow is obtained from a MCF by setting $N_s = \frac{1}{\sqrt{t_0 - t}}(M_t - x_0)$, $s = -\log(t_0 - t)$, $t < t_0$, where (x_0, t_0) is some fixed point in spacetime. It satisfies the equation $(\partial_s x)^{\perp} = -H\nu + x/2$.

Proposition **2.13.** — *Given n, λ_0 and $\epsilon > 0$, there exists $\delta > 0$ so that if $N_s \subset \mathbf{R}^{n+1}$ is a rescaled MCF of integral varifolds for $s \in [0, 1]$ with $\lambda(N_0) \le \lambda_0$ and*

$$F(N_0) - F(N_1) \le \delta, \tag{2.14}$$

then there is an F-stationary varifold Σ with $\lambda(\Sigma) \le \lambda_0$ and $d_V(\Sigma, N_s) \le \epsilon$ for all $s \in [0, 1]$.

Proof. — We will argue as in the existence of tangent flows in [I1], [W3]. Suppose therefore that N^i_s is a sequence of rescaled MCFs satisfying (**2.14**) with $\delta = 1/i$, but where

$$(\mathbf{2.15}) \qquad d_V\big(\Sigma, N^i_{s_i}\big) \geq \epsilon > 0 \quad \text{for some } s_i \text{ and every such } \Sigma.$$

The Brakke compactness of Ilmanen (Lemma 7.1 in [I2]) gives a subsequence of the N^i_s's converges to a limiting Brakke flow N_s with $\lambda(N_0) \leq \lambda_0$ and $F(N_0) = F(N_1)$. In particular, N_s is a static solution of the rescaled MCF (i.e., each N_s is the same F-stationary varifold Σ). Since there is no mass loss, every sequence of time slices converges to Σ as Radon measures, contradicting (**2.15**). □

Proof of Theorem 0.2. — Let $\kappa > 0$ be given by Corollary 2.12, and let $\delta > 0$ by given by Corollary 2.12 with ϵ replaced by $\kappa/6$.

Let (x_0, t_0) be a singular point with at least one tangent flow isometric to a cylinder $\mathbf{S}^k \times \mathbf{R}^{n-k}$. Then there is a time $t < t_0$ such that $\frac{1}{\sqrt{t_0 - t}}(M_t - x_0)$ is C^2 close to a cylinder in a large ball. By (**2.5**), the Gaussian integral is predictably small outside of this ball, so there is a space-time neighborhood U of (x_0, t_0) such that for any $(x_1, t_1) \in U$,

$$d_V\left(\frac{1}{\sqrt{t_1 - t}}(M_t - x_1), O\big(\mathbf{S}^k \times \mathbf{R}^{n-k}\big)\right) \leq \frac{\kappa}{6},$$

for some $O \in SO(n+1)$, and

$$F\left(\frac{1}{\sqrt{t_1 - t}}(M_t - x_1)\right) \leq \lambda_k + \frac{\delta}{2},$$

where $\lambda_k \equiv F(\mathbf{S}^k \times \mathbf{R}^{n-k})$.

Now suppose that (x_1, t_1) has entropy at least $\lambda_k - \delta/2$. If N_s, $s \geq s_0$, is the rescaled MCF starting from $\frac{1}{\sqrt{t_1 - t}}(M_t - x_1)$, then the total variation of $F(N_s)$ is at most δ. For each integer $j \geq 0$, it follows by the choice of δ that there is an F-stationary varifold V_j so that

$$(\mathbf{2.16}) \qquad d_V(N_s, V_j) \leq \frac{\kappa}{6} \quad \text{for every } s \in [s_0 + j, s_0 + j + 1].$$

We have that V_0 is a generalized cylinder by Corollary 2.12. Furthermore, (**2.16**) and the triangle inequality implies that $d_V(V_j, V_{j+1}) \leq \kappa/3$, so every V_j is a generalized cylinder, giving the theorem. □

Proof of Corollary 0.3. — The corollary follows immediately from Theorem 0.2. □

3. Proof of Proposition 2.1

In this section, we will prove Proposition 2.1, which shows that a curvature bound on a sufficiently large ball B_R implies a slightly worse bound on a larger ball $B_{(1+\theta)R}$ for some fixed $\theta > 0$. We will do this by using Brakke's theorem to prove estimates in B_R for the mean curvature flow $\Sigma_t = \sqrt{-t}\,\Sigma$ associated to the shrinker.

3.1. *Applying Brakke to a self-shrinker.* — We will use the following consequence of Allard's theorem, [Al]:

Lemma **3.1.** — *Given n, there exists* $\epsilon_A > 0$ *so that if* $\Sigma \subset \mathbf{R}^{n+1}$ *is an* F*-stationary varifold,* $x_0 \in \Sigma$*, and there is some* $\tau > 0$ *so that*

$$F_{x_0,t_0}(\Sigma_0) \leq 1 + \epsilon_A \quad \textit{for every } t_0 \in (0, \tau), \tag{3.2}$$

then Σ *is smooth at* x_0.

Proof. — Since Σ is F-stationary, it is stationary for a conformal metric. Thus, by Allard's theorem, it suffices to show that some tangent cone to Σ at x_0 is a multiplicity one hyperplane.

Any tangent cone V must be a conical stationary integral varifold in $\mathbf{R}^{n+1}$. Since V is a cone, we can compute F(V)

$$\begin{aligned} F(V) &= (4\pi)^{-\frac{n}{2}} \int_0^\infty e^{-\frac{s^2}{4}} s^{n-1} \operatorname{Vol}(\partial B_1 \cap V) ds \\ &= \frac{\operatorname{Vol}(\partial B_1 \cap V)}{\operatorname{Vol}(\partial B_1 \cap \mathbf{R}^n)} = \frac{\operatorname{Vol}(B_1 \cap V)}{\operatorname{Vol}(B_1 \cap \mathbf{R}^n)}. \end{aligned} \tag{3.3}$$

Combining this with (**3.2**) implies that

$$\frac{\operatorname{Vol}(B_1 \cap V)}{\operatorname{Vol}(B_1 \cap \mathbf{R}^n)} = F(V) \leq 1 + \epsilon_A. \tag{3.4}$$

Finally, Allard gives $\epsilon_A > 0$ so that if V is any stationary varifold with $\frac{\operatorname{Vol}(B_1 \cap V)}{\operatorname{Vol}(B_1 \cap \mathbf{R}^n)} \leq 1 + \epsilon_A$, then $V = \mathbf{R}^n$ and has multiplicity one. □

Proposition **3.5.** — *Given* $\lambda_0 < 2$ *and* C_1*, there exist* $\theta > 0$ *and* C_2 *so that if* Σ *is* F*-stationary with*

- $\lambda(\Sigma) \leq \lambda_0$,
- $B_R \cap \Sigma$ *is smooth and has* $|A| \leq C_1$,

then $B_{(1+\theta)R-1/3} \cap \Sigma$ *is smooth. Furthermore, we get the curvature estimate*

$$\sup_{B_{(1+\theta)R-1/2} \cap \Sigma} |A| \leq C_2. \tag{3.6}$$

Proof. — We start with an almost-optimal bound on the F functionals at small scales:

($\star$) Given any $\epsilon > 0$, there exists $r_\epsilon \in (0, 1)$ so that

$$F_{x_0,t_0}(\Sigma_0) \leq 1 + \epsilon \tag{3.7}$$

for every $x_0 \in B_{R-1/4}$ and $t_0 \in (0, r_\epsilon^2)$.

Observe that the curvature bound implies that on any sufficiently small scale Σ decomposes into a collection of graphs with small gradient, but the entropy bound implies that there is only one such graph. Finally, (the scaled version of) Lemma 2.4 gives ($\star$).

The next step is to extend the entropy bound ($\star$) to a larger scale. This follows from using Huisken's monotonicity for the associated flow. We will instead use the shrinker equation and, in particular, the following result from Section 7 in [CM1] (see Equation (7.13) there):

Given $y \in \mathbf{R}^{n+1}$, $a \in \mathbf{R}$, and $s > 1$ so that $1 + as^2 > 0$, then

$$F_{sy, 1+as^2}(\Sigma) \le F_{y, 1+a}(\Sigma). \tag{3.8}$$

We apply this to each $y \in B_{R-1}$ with $a = t_0 - 1$ (for each t_0). This can be done so long

$$1 + s^2(t_0 - 1) = 1 + as^2 > 0. \tag{3.9}$$

It follows that we get some $\theta > 0$ and $\tau > 0$ so that every $x_0 \in B_{(1+\theta)(R-1/4)}$ has

$$F_{x_0, t_0}(\Sigma) \le 1 + \epsilon \quad \text{for all } t_0 \le \tau. \tag{3.10}$$

Regularity now follows from Lemma 3.1.

Since we have ($\star$), the curvature estimate now follows from applying Theorem 3.1 in [W2] to the associated MCF $\Sigma_t \equiv \sqrt{-t}\,\Sigma$. We apply [W2] on each ball $B_{r_\epsilon}(y)$ for $y \in B_{R-1/2}$ to get a curvature bound on this ball at time $r_0^2 - 1$. This yields the claimed $|A|$ bound on Σ on the larger ball. □

3.2. *Extending curvature bounds outward.* — We can now prove the first main ingredient for the rigidity.

Proof of Proposition 2.1. — Proposition 3.5 immediately extends smoothness and the $|A|$ bound to a larger scale, but it remains to extend the positivity of H to the larger scale. This will follow from getting a uniform bound on the time derivative of H for the MCF $\Sigma_t \equiv \sqrt{-t}\,\Sigma$ corresponding to Σ. This bound extends the positivity of H forward in time for Σ_t which corresponds to positivity of H on the larger scale for Σ.

We first use Proposition 3.5 to get $\delta > 0$ and C so that

$$\sup_{B_R \cap \Sigma_t} |A| \le C \quad \text{for } t \in (-1, \delta - 1). \tag{3.11}$$

Parabolic estimates of [EH] (applied on balls of unit scale) then give the uniform higher derivative bounds

$$\sup_{B_{R-1} \cap \Sigma_t} |\nabla A| + \left|\nabla^2 A\right| \le C' \quad \text{for } t \in (-1, \delta - 1). \tag{3.12}$$

Using the equation $\partial_t H = \Delta H + |A|^2 H$, these bounds on A and its derivatives give a uniform bound on $\partial_t H$ on this set and, thus, strict positivity of H propagates forward in time for some positive time interval, completing the proof. □

4. Proof of Proposition 2.2

In this section, we will prove the second key ingredient (Proposition 2.2) that gives the improvement in the cylindrical estimates. This proposition shows that if a shrinker has slightly positive mean curvature and some large curvature bound on a large ball, then we get much better bounds on H and A on a slightly smaller ball. Crucially, these uniform bounds do not depend on the initial radius, so we get a fixed improvement that can be iterated.

The proof of Proposition 2.2 uses the positivity of H to prove that the tensor $\tau \equiv \frac{\mathrm{A}}{\mathrm{H}}$ is almost parallel. This is proven over the next two subsections. We then show that the eigenvalues of τ fall into two clusters, one near zero corresponding to the translation invariant directions of the generalized cylinder and the other corresponding to the spherical part of the cylinder. This is done in the last two subsections.

Throughout this section, $\Sigma \subset \mathbf{R}^{n+1}$ will be an n-dimensional F-stationary varifold.

4.1. *Simons' equation.* — This subsection contains some calculations that will be used to show that $\tau = \mathrm{A}/\mathrm{H}$ is almost parallel. The calculations in this section are purely local and, thus, are valid at any smooth point of Σ.

Given $f > 0$, define a weighted divergence operator div_f and drift Laplacian $\mathcal{L}_f$ by

$$\operatorname{div}_f(\mathrm{V}) = \frac{1}{f} e^{|x|^2/4} \operatorname{div}_\Sigma\left(f e^{-|x|^2/4} \mathrm{V}\right), \tag{4.1}$$

$$\mathcal{L}_f u \equiv \operatorname{div}_f(\nabla u) = \mathcal{L} u + \langle \nabla \log f, \nabla u \rangle. \tag{4.2}$$

Here u may also be a tensor; in this case the divergence traces only with ∇. Note that $\mathcal{L} = \mathcal{L}_1$. We recall the quotient rule:

Lemma **4.3.** — *Given a tensor τ and a function g with $g \neq 0$, then*

$$\mathcal{L}_{g^2} \frac{\tau}{g} = \frac{g \mathcal{L}\tau - \tau \mathcal{L} g}{g^2} = \frac{g \mathrm{L}\tau - \tau \mathrm{L} g}{g^2}. \tag{4.4}$$

Proof. — We compute

$$\begin{aligned}
\mathcal{L}\frac{\tau}{g} &= e^{|x|^2/4} \operatorname{div}_\Sigma\left(e^{-|x|^2/4} \nabla \frac{\tau}{g}\right) = e^{|x|^2/4} \operatorname{div}_\Sigma\left(e^{-|x|^2/4}\left[\frac{\nabla\tau}{g} - \frac{\tau \nabla g}{g^2}\right]\right) \\
&= \frac{\mathcal{L}\tau}{g} - \frac{\tau \mathcal{L} g}{g^2} - 2\frac{\langle \nabla\tau, \nabla g\rangle}{g^2} + 2\frac{\tau |\nabla g|^2}{g^3} \\
&= \frac{\mathcal{L}\tau}{g} - \frac{\tau \mathcal{L} g}{g^2} - \left\langle \nabla \log g^2, \nabla \frac{\tau}{g} \right\rangle.
\end{aligned}$$

□

Proposition **4.5.** — *On the set where* $\mathrm{H} > 0$, *we have*

$$\mathcal{L}_{\mathrm{H}^2} \frac{\mathrm{A}}{\mathrm{H}} = 0, \tag{4.6}$$

$$\mathcal{L}_{\mathrm{H}^2} \frac{|\mathrm{A}|^2}{\mathrm{H}^2} = 2\left|\nabla \frac{\mathrm{A}}{\mathrm{H}}\right|^2. \tag{4.7}$$

Proof. — Since $\mathrm{LH} = \mathrm{H}$ and $\mathrm{LA} = \mathrm{A}$ by [CM1], the first claim follows from the quotient rule (Lemma 4.3). The second claim follows from the first since $\frac{|\mathrm{A}|^2}{\mathrm{H}^2} = \langle \frac{\mathrm{A}}{\mathrm{H}}, \frac{\mathrm{A}}{\mathrm{H}} \rangle$. □

4.2. *An effective bound for shrinkers with positive mean curvature.* — The next proposition gives exponentially decaying integral bounds for $\nabla(\mathrm{A}/\mathrm{H})$ when H is positive on a large ball. It will be important that these bounds decay rapidly.

Proposition **4.8.** — *If* $\mathrm{B_R} \cap \Sigma$ *is smooth with* $\mathrm{H} > 0$, *then for* $s \in (0, \mathrm{R})$ *we have*

$$\int_{\mathrm{B}_{\mathrm{R}-s} \cap \Sigma} \left|\nabla \frac{\mathrm{A}}{\mathrm{H}}\right|^2 |\mathrm{H}|^2 \mathrm{e}^{-|x|^2/4} \leq \frac{4}{s^2} \sup_{\mathrm{B_R} \cap \Sigma} |\mathrm{A}|^2 \operatorname{Vol}(\mathrm{B_R} \cap \Sigma) \mathrm{e}^{-\frac{(\mathrm{R}-s)^2}{4}}. \tag{4.9}$$

Proof. — Set $\tau = \mathrm{A}/\mathrm{H}$ and $u = |\tau|^2 = |\mathrm{A}|^2/\mathrm{H}^2$, so that $\mathcal{L}_{\mathrm{H}^2} u = 2|\nabla \tau|^2$ by Proposition 4.5. Fix a smooth cutoff function ϕ with support in $\mathrm{B_R}$. Using the divergence theorem, the formulas for $\mathcal{L}_{\mathrm{H}^2}$, and the absorbing inequality $4ab \leq a^2 + 4b^2$, we get

$$\begin{aligned} 0 &= \int_{\mathrm{B_R} \cap \Sigma} \operatorname{div}_{\mathrm{H}^2}\big(\phi^2 \nabla u\big) \mathrm{H}^2 \mathrm{e}^{-|x|^2/4} \\ &= \int_{\mathrm{B_R} \cap \Sigma} \big(\phi^2 \mathcal{L}_{\mathrm{H}^2} u + 4\phi \langle \nabla \phi, \nabla u \rangle\big) \mathrm{H}^2 \mathrm{e}^{-|x|^2/4} \\ &= \int_{\mathrm{B_R} \cap \Sigma} \big(2\phi^2 |\nabla \tau|^2 + 4\phi \langle \nabla \phi, \tau \cdot \nabla \tau \rangle\big) \mathrm{H}^2 \mathrm{e}^{-|x|^2/4} \\ &\geq \int_{\mathrm{B_R} \cap \Sigma} \big(\phi^2 |\nabla \tau|^2 - 4|\tau|^2 |\nabla \phi|^2\big) \mathrm{H}^2 \mathrm{e}^{-|x|^2/4}, \end{aligned} \tag{4.10}$$

from which we obtain

$$\int_{\mathrm{B_R} \cap \Sigma} \left(\phi^2 \left|\nabla \frac{\mathrm{A}}{\mathrm{H}}\right|^2 |\mathrm{H}|^2\right) \mathrm{e}^{-|x|^2/4} \leq 4 \int_{\mathrm{B_R} \cap \Sigma} |\nabla \phi|^2 |\mathrm{A}|^2 \mathrm{e}^{-|x|^2/4}. \tag{4.11}$$

The proposition follows by choosing $\phi \equiv 1$ on $\mathrm{B}_{\mathrm{R}-s}$ and going to zero linearly on $\partial \mathrm{B_R}$. □

The next corollary establishes bounds on two derivatives of the tensor $\tau = \mathrm{A}/\mathrm{H}$ by combining the integral estimates of the previous proposition with elliptic estimates.

Corollary **4.12.** — *Given $n > 0$, $\delta > 0$, $\lambda_0 > 0$ and $C_1 > 0$ there exists a constant $C_\tau > 0$ such that if $\lambda(\Sigma) \leq \lambda_0$, $R \geq 2$, and*

- $B_R \cap \Sigma$ *is smooth with* $|A| \leq C_1$ *and* $H \geq \delta > 0$,

then

$$\sup_{B_{R-2}\cap\Sigma} |\nabla\tau|^2 + R^{-2}\left|\nabla^2\tau\right|^2 \leq C_\tau R^{2n} e^{-R/4}. \tag{4.13}$$

Proof. — Throughout this proof C will be a constant that depends only on n, δ, λ_0 and C_1; C will be allowed to change from line to line.

Proposition 4.8 with $s = 1/2$ gives

$$\int_{B_{R-1/2}\cap\Sigma} |\nabla\tau|^2 e^{-|x|^2/4} \leq CR^n e^{-(R-1/2)^2/4}. \tag{4.14}$$

Since $e^{-|x|^2/4} \geq e^{-\frac{R^2-2R+1}{4}}$ on B_{R-1}, it follows that

$$\int_{B_{R-1}\cap\Sigma} |\nabla\tau|^2 \leq CR^n e^{-\frac{R}{4}}. \tag{4.15}$$

This gives the desired integral decay on $\nabla\tau$. We will combine this with elliptic theory to get the pointwise bounds. The key is that τ satisfies the elliptic equation $\mathcal{L}_{H^2}\tau = 0$. The two first order terms in the equation come from x^T in $\mathcal{L}$ and ∇H; both grow at most linearly (in the second case, we differentiate the shrinker equation and use the bound on $|A|$). Therefore, we can apply elliptic theory on balls of radius $1/R$ to get for any $p \in B_{R-2} \cap \Sigma$ that

$$\left(|\nabla\tau|^2 + R^{-2}\left|\nabla^2\tau\right|^2\right)(p) \leq CR^n \int_{B_{\frac{1}{R}}(p)\cap\Sigma} |\nabla\tau|^2. \tag{4.16}$$

Combining this with the integral bound gives the corollary. □

4.3. *Finding small eigenvalues of* A. — The next lemma shows that if we have an almost parallel symmetric 2-tensor with two distinct eigenvalues, then the plane spanned by the corresponding eigenvectors is almost flat.

The results in this subsection are valid for a general smooth hypersurface, possibly with boundary, and do not use an equation for the hypersurface.

Lemma **4.17.** — *Suppose that* B *is a symmetric two-tensor on a Riemannian manifold with* $|\nabla B|, |\nabla^2 B| \leq \epsilon \leq 1$. *If* e_1, e_2 *are unit vectors at* p *with*

$$B_p(e_1) = 0 \quad \textit{and} \quad B_p(e_2) = \kappa e_2, \quad \kappa \neq 0,$$

then the sectional curvature K *of the 2-plane spanned by* e_1 *and* e_2 *satisfies*

$$|\mathrm{K}| \leq 2\epsilon\left(\frac{1}{|\kappa|}+\frac{1}{\kappa^2}\right).$$

Proof. — Let v_1 and v_2 be smooth vector fields in a neighborhood of p with

$$v_j(p) = e_j, \qquad \nabla v_j(p) = 0, \qquad |v_j| \leq 1,$$

and set

$$w_2 \equiv \frac{\mathrm{B}(v_2)}{\kappa}$$

so $w_2(p) = e_2(p)$. Then the sectional curvature K at p is

$$\begin{aligned} \mathrm{K} &= \big\langle \mathrm{R}(e_2, e_1)e_2, e_1\big\rangle \\ &= \langle \nabla_{v_1}\nabla_{w_2} w_2, v_1\rangle - \langle \nabla_{w_2}\nabla_{v_1} w_2, v_1\rangle - \langle \nabla_{[v_1,w_2]} w_2, v_1\rangle. \end{aligned} \tag{4.18}$$

We will bound each of these terms by showing that ∇w_2 and $\nabla^2 w_2$ are orthogonal to v_1 up to small error terms.

Let x and y be vector fields with $|x(p)| = |y(p)| = 1$. We have

$$\kappa \nabla_x w_2 = (\nabla_x \mathrm{B})(v_2) + \mathrm{B}(\nabla_x v_2)$$

so at p one has

$$\begin{aligned} &\big|\nabla_x w_2(p)\big| \leq \frac{\epsilon}{|\kappa|}, \qquad \big|[v_1, w_2](p)\big| \leq \frac{\epsilon}{|\kappa|}, \\ &\big|\langle \nabla_{[v_1,w_2]} w_2, v_1\rangle(p)\big| \leq \frac{\epsilon^2}{|\kappa|^2}, \end{aligned} \tag{4.19}$$

thus estimating the third term of (**4.18**).

Differentiating again, we have

$$\begin{aligned} \kappa \nabla_y\nabla_x w_2 &= \nabla_y\big\{(\nabla_x \mathrm{B})(v_2) + \mathrm{B}(\nabla_x v_2)\big\} \\ &= \big(\nabla^2_{y,x}\mathrm{B}\big)(v_2) + (\nabla_{\nabla_y x}\mathrm{B})(v_2) + (\nabla_x \mathrm{B})(\nabla_y v_2) \\ &\quad + (\nabla_y \mathrm{B})(\nabla_x v_2) + \mathrm{B}(\nabla_y\nabla_x v_2). \end{aligned}$$

The last term on the right is in the range of B and, thus, is orthogonal to $v_1(p) = e_1$ which is in the kernel of B. We obtain at p

$$
\begin{aligned}
\big|\langle \nabla_{v_1}\nabla_{w_2} w_2, v_1\rangle(p)\big| &\le \frac{1}{|\kappa|}\Big(\big|\big\langle\big(\nabla^2_{v_1,w_2}\mathrm{B}\big)(v_2), v_1\big\rangle(p)\big| \\
&\quad + \big|\big\langle\big(\nabla_{\nabla_{v_1}w_2}\mathrm{B}\big)(v_2), v_1\big\rangle(p)\big| + 0 + 0\Big) \\
&\le \frac{1}{|\kappa|}\left(\epsilon + \frac{\epsilon^2}{|\kappa|}\right) = \frac{\epsilon}{|\kappa|} + \frac{\epsilon^2}{|\kappa|^2}.
\end{aligned} \tag{4.20}
$$

Similarly we obtain

$$
\big|\langle \nabla_{w_2}\nabla_{v_1} w_2, v_1\rangle(p)\big| \le \frac{\epsilon}{|\kappa|}. \tag{4.21}
$$

Combining (**4.18**), (**4.19**), (**4.20**) and (**4.21**) gives the desired bound. □

We will apply the above lemma when B is given by $\tau - \kappa_1 g$ where κ_1 is one of the small eigenvalues of τ at the point p.

Corollary **4.22.** — *Suppose that* $\Sigma \subset \mathbf{R}^{n+1}$ *is a hypersurface (possibly with boundary) with the following properties*

- $0 < \delta \le \mathrm{H}$ *on* Σ.
- *The tensor* $\tau \equiv \mathrm{A}/\mathrm{H}$ *satisfies* $|\nabla\tau| + |\nabla^2\tau| \le \epsilon \le 1$.
- *At the point* $p \in \Sigma$, τ_p *has at least two distinct eigenvalues* $\kappa_1 \neq \kappa_2$.

Then

$$
|\kappa_1\kappa_2| \le \frac{2\epsilon}{\delta^2}\left(\frac{1}{|\kappa_1-\kappa_2|} + \frac{1}{|\kappa_1-\kappa_2|^2}\right).
$$

Proof. — Let K_p be the sectional curvature of the plane at p spanned by an eigenvector for τ_p with eigenvalue κ_1 and one with eigenvalue κ_2. Since $\mathrm{A} = \mathrm{H}\tau$, the Gauss equation gives

$$
|\mathrm{K}_p| = \mathrm{H}^2|\kappa_1\kappa_2| \ge \delta^2|\kappa_1\kappa_2|.
$$

On the other hand, Lemma 4.17 with $\mathrm{B} = \tau - \kappa_1 g$ and $\kappa = \kappa_2 - \kappa_1$ gives

$$
|\mathrm{K}_p| \le 2\epsilon\left(\frac{1}{|\kappa_2-\kappa_1|} + \frac{1}{|\kappa_2-\kappa_1|^2}\right).
$$

The corollary follows by combining these. □

4.4. *The proof of Proposition 2.2.* — We are now ready to prove the second key proposition.

Proof of Proposition 2.2. — Fix n, $\lambda_0 > 0$, $\delta_0 > 0$, and $\mathrm{C}_0 > 0$. Let $\mathrm{R} > 0$ and assume that Σ is a shrinker in $\mathbf{R}^{n+1}$, $\lambda(\Sigma) \le \lambda_0$, $\Sigma \cap \mathrm{B}_\mathrm{R}$ is smooth and

$$
|\mathrm{A}| \le \mathrm{C}_0 \quad \text{and} \quad 0 < \delta_0 \le \mathrm{H} \quad \text{on } \Sigma \cap \mathrm{B}_\mathrm{R}.
$$

It follows by Corollary 4.12 that the tensor $\tau \equiv A/H$ satisfies

$$|\nabla \tau| + \left|\nabla^2 \tau\right| \leq \epsilon_\tau \quad \text{on } B_{R-2} \cap \Sigma,$$

where

$$\epsilon_\tau^2 := CR^{2n+2} e^{-R/4}$$

and the constant C depends only on n, C_0, δ_0 and λ_0 (and, in particular, not on R).

Now fix some small $\epsilon_0 > 0$, to be reduced as needed, but depending only on n. Combining the compactness result of Lemma 2.7 with Huisken's classification ([H1] and [H2]) of complete shrinkers with $H \geq 0$ and bounded $|A|$, there exists $R_1 > 0$ depending on λ_0, C_0, n, and ϵ_0 such that if $R \geq R_1$, then for some $k \in \{1, \dots n\}$, $B_{5\sqrt{2n}} \cap \Sigma$ is C^2 ϵ_0-close to a cylinder $\mathbf{S}^k \times \mathbf{R}^{n-k}$ where $\mathbf{S}^k$ has radius $\sqrt{2k}$. In particular, we can arrange that

$$|A|^2 \leq 3/4 \quad \text{and} \quad 1/2 \leq H \quad \text{on } B_{5\sqrt{2n}} \cap \Sigma, \tag{4.23}$$

and further, that at every p in $\Sigma \cap B_{5\sqrt{2n}}$ there are $n-k$ orthonormal eigenvectors

$$v_1(p), \dots, v_{n-k}(p),$$

of A with eigenvalues less than $1/\sqrt{100n}$, plus at least one eigenvector with eigenvalue at least $1/\sqrt{4n}$. Then we can apply Corollary 4.22 to obtain

$$\left|\kappa_j(p)\right| \leq C\epsilon_\tau, \quad j = 1, \dots, n-k,$$

where C depends only on n, C_0, δ_0 and λ_0.

Now fix p in $\Sigma \cap B_{2\sqrt{2n}}$ and define $n-k$ tangential vector fields v_i on Σ by

$$v_i = v_i(p) - \langle v_i(p), \mathbf{n} \rangle \mathbf{n}.$$

Fix a constant L (later we will take $L = 2R$) and let Ω denote the set of points in $B_{R-2} \cap \Sigma$ that can be reached from p by a path in $B_{R-2} \cap \Sigma$ of length at most L.

We will show that the v_i's have the following three properties on Ω:

$$\left|v_i - v_i(p)\right| \leq C\left(L + L^2\right)\epsilon_\tau, \tag{4.24}$$

$$\left|\tau(v_i)\right| \leq C\left(1 + L^2\right)\epsilon_\tau, \tag{4.25}$$

$$|\nabla_{v_i} A| \leq C(1+L)\left(1 + L^2\right)\epsilon_\tau. \tag{4.26}$$

To prove (**4.24**) and (**4.25**), suppose that $\gamma : [0, L] \to \Sigma$ is a curve with $\gamma(0) = p$ and $|\gamma'| \leq 1$ and that w is a parallel unit vector field along γ with $w(0) = v_i(p)$. Therefore, the bound on $\nabla\tau$ gives $|\nabla_{\gamma'} \tau(w)| \leq \epsilon_\tau$ and, thus,

$$\left|\tau(w)\right| \leq L\epsilon_\tau + \left|\tau_p\left(v_i(p)\right)\right| \leq (C+L)\epsilon_\tau.$$

In particular, we also have

$$\left|A(w)\right| = |H|\left|\tau(w)\right| \le C(C+L)\epsilon_\tau .$$

Therefore, since $\nabla^{\mathbf{R}^{n+1}}_{\gamma'} w = A(\gamma', w)$, the fundamental theorem of calculus gives

$$\left|w(t) - v_i(p)\right| = \left|w(t) - w(0)\right| \le C\big(L + L^2\big)\epsilon_\tau . \tag{4.27}$$

Since $\langle w(t), \mathbf{n}\rangle = 0$, we see that $|\langle v_i(p), \mathbf{n}\rangle| \le C(L+L^2)\epsilon_\tau$, giving (**4.24**). Adding (**4.24**) and (**4.27**) yields

$$\left|w(t) - v_i\right| \le C\big(L + L^2\big)\epsilon_\tau ,$$

so we get that

$$\left|\tau(v_i)\right| \le \left|\tau(w)\right| + \left|\tau(w - v_i)\right| \le C\big(1 + L^2\big)\epsilon_\tau$$

which is (**4.25**).

Next we will see that (**4.25**) implies (**4.26**). Namely, the Codazzi equation gives

$$\begin{aligned}
\left|(\nabla_{v_i} A)(x, y)\right| &= \left|(\nabla_x A)(v_i, y)\right| = \left|\big(\nabla_x (H\tau)\big)(v_i, y)\right| \\
&\le \left|H(\nabla_x \tau)(v_i, y)\right| + \left|(\nabla_x H)\tau(v_i, y)\right| \\
&\le C\epsilon_\tau + C(1+L)\big(1+L^2\big)\epsilon_\tau ,
\end{aligned}$$

where we have used the linear growth estimate $|\nabla H| \le 1 + R|A| \le 1 + CL$ as in the proof of Corollary 4.12. This gives (**4.26**).

The key for the bounds (**4.24**)–(**4.26**) is that ϵ_τ decays exponentially in R and, thus, these bounds are extremely small even compared to any power of R.

Finally, (**4.24**) and (**4.26**) together allow us to extend the nearly sharp bounds on A and H in $B_{5\sqrt{2n}} \cap \Sigma$ given by (**4.23**) to all of Ω. Define the linear functions

$$f_i(x) = \big\langle v_i(p), x\big\rangle, \quad i = 1, \dots, n-k .$$

From the ϵ_0 closeness to the cylinder in the ball $B_{5\sqrt{2n}}$, we know that

$$\Sigma_0 \equiv B_{5\sqrt{2n}} \cap \Sigma \cap \{f_1 = \cdots = f_{n-k} = 0\}$$

is a compact topological $\mathbf{S}^k$ of radius approximately $\sqrt{2k}$.

When $n - k = 1$ (there is only one approximate translation), then the flow of the vector field $v_1/|v_1|^2$ starting from Σ_0 preserves the level sets (in Σ) of f_1, the flow vector field nearly equals the constant vector $v_i(p)$ by (**4.24**), and the second fundamental form A is almost parallel along the flow lines, for as long as the flow remains within Ω. Note that if $R > 30n$, then the disk D of radius $\sqrt{2n}+1$ in the plane $f_1 = R - 3$ with center the point closest to the origin lies within B_{R-2}. Also, by considering the shape of the approximate cylinder, the path distance within Σ between p and a point $x(t)$ on the flow line is no more than the length of the flow line plus $(5+2\pi)\sqrt{2n}+1$. So if we take $L=2R$ and $R^3\epsilon_\tau$ small enough, then each flow line hits D before it leaves Ω. Similarly, if we flow along the negative vector field, then again each flow line hits $-D$ before it leaves Ω. Thus, the connected component of $B_{R-3}\cap\Sigma$ containing Σ_0 is contained in Ω and in the union of the flow lines. In particular, the nearly sharp bounds on A and H in $B_{5\sqrt{2n}}\cap\Sigma$ extend to this component. By the maximum principle, every component of $B_{R-3}\cap\Sigma$ meets $\overline{B}_{\sqrt{2n}}$. However, there is only one such component since $B_{5\sqrt{2n}}\cap\Sigma$ is cylindrical. Thus, the nearly sharp bounds are valid on $B_{R-3}\cap\Sigma$, and the proof is complete for this case.

We argue similarly when $n-k>1$, except that we construct a single "radial flow" rather than doing each flow successively as this seems simpler. First, define f by

$$f^2=\sum_{i=1}^{n-k}f_i^2,$$

and then let

$$v=\frac{\nabla f}{|\nabla f|^2},$$

where as before ∇ is the tangential derivative, Thus, the flow by v preserves the level sets of f. Note that

$$\nabla f=\frac{\sum f_i\nabla f_i}{f}=\sum\frac{f_i}{f}v_i.$$

Since the $v_i(p)$'s are orthonormal and each $v_i-v_i(p)$ is small by (**4.24**), it follows that $|\nabla f|$ is almost one. Furthermore, v is given at each point as a linear combination of the v_i's, so (**4.26**) implies that $|\nabla_v A|$ is small. Arguing as in the previous case completes the proof. □

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