VIRTUALLY FREE PRO-p GROUPS

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ABSTRACT

We prove that in the category of pro-p groups any finitely generated group G with a free open subgroup splits either as an amalgamated free product or as an HNN-extension over a finite p-group. From this result we deduce that such a pro-p group is the pro-p completion of a fundamental group of a finite graph of finite p-groups.

1. Introduction

Let p be a prime number, and let G be a pro-p group containing an open free pro-p subgroup F. If G is torsion free, then, according to the celebrated theorem of Serre established in [17], G itself is free pro-p.

The main objective of the paper is to give a description of virtually free pro-p groups without the assumption of torsion freeness.

Theorem **1.1.** — *Let* G *be a finitely generated pro-p group with a free open subgroup* F. *Then* G *is the fundamental pro-p group of a finite graph of finite p-groups of order bounded by* $|G : F|$.

This theorem is the pro-p analogue of the description of finitely generated virtually free discrete groups proved by Karrass, Pietrovski and Solitar in [11]. In the characterization of discrete virtually free groups Stallings' theory of ends played a crucial role. In fact the proof of the theorem of Karrass, Pietrovski and Solitar uses the celebrated theorem of Stallings proved in [18] according to which every finitely generated virtually free group splits as an amalgamated free product or HNN-extension over a finite group, respectively. The theory of ends for pro-p groups has been initiated in [12]. However, it is not known whether an analogue of Stallings' Theorem holds in this context. We will prove Theorem 1.1 and such an analogue for finitely generated virtually free pro-p groups using purely combinatorial pro-p group methods combined with results on p-adic representations of finite p-groups.

Theorem **1.2.** — *Let* G *be a finitely generated virtually free pro-p group. Then* G *is either a non-trivial amalgamated free pro-p product with finite amalgamating subgroup or a non-trivial HNN-extension with finite associated subgroups.*

As a consequence of Theorem 1.1 we obtain that a finitely generated virtually free pro-p group is the pro-p completion of a virtually free discrete group. However, the discrete result is not used (and cannot be used) in the proof.

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V.A. Romankov proved in [15] that the automorphism group of a finitely generated free pro-p group $\mathrm{Aut}(\widehat{F}_n)$ of rank $n \geq 2$, is infinitely generated. Therefore, one has that, despite the fact that the automorphism group $\mathrm{Aut}(F_n)$ of a free group of rank n embeds naturally in $\mathrm{Aut}(\widehat{F}_n)$, it is by no means densely embedded there! Nevertheless, Theorem 1.1 allows us to show that, surprisingly, the number of conjugacy classes of finite p-subgroups in $\mathrm{Aut}(\widehat{F}_n)$ is not greater than the corresponding number for $\mathrm{Aut}(F_n)$.

Note that the assumption of finite generation in Theorem 1.1 is essential: there is an example of a split extension $H = F \rtimes D_4$ of a free pro-2 group F of countable rank which cannot be represented as the fundamental pro-2 group of a profinite graph of finite 2-groups (see Example 5.3).

The line of proof is as follows. In Section 3 we use a pro-p HNN-extension to embed a finitely generated virtually free pro-p group G in a split extension $E = F \rtimes K$ of a free pro-p group F and a finite p-group K with a unique conjugacy class of maximal finite subgroups. In Section 4 we prove using an inductive argument the following theorem which connects the structure of any such group $F \rtimes K$ with its action on $M := F/[F, F]$.

Theorem **1.3.** — *Let* E *be a semidirect product* $E = F \rtimes K$ *of a free pro-p group* F *of finite rank and a finite p-group* K. *Then the* K*-module* $M = F/[F, F]$ *is permutational if and only if* F *posesses a* K*-invariant basis.*

This theorem gives an HNN-extension structure on E with finite base group. In particular, E and, therefore, G acts on a pro-p tree with finite vertex stabilizers. Using this, [7, Proposition 14], and a result from [9] on pro-p groups acting on trees we prove in Section 5 Theorems 1.1 and 1.2. Finally, Section 6 deals with automorphisms of a free pro-p group.

Basic material on profinite groups can be found in [13, 19]. Throughout the paper we make the following standard assumptions. Subgroups are closed and homomorphisms are continuous. For elements x, y in a group G we will write $y^x := xyx^{-1}$ and $[x,y] := xyx^{-1}y^{-1}$. For a subset $A \subseteq G$ we denote by $(A)_G$ the normal closure of A in G, i.e., the smallest closed normal subgroup of G containing A. For profinite graphs we will use (standard) notations which can be found in [14]. The *Frattini subgroup* of G will be denoted by $\Phi(G)$, and $\mathrm{Tor}(G)$ will stand for the subset of elements of finite order in G. For a finite p-group G let $\mathrm{socle}(G) := \langle c \in Z(G) \mid c^p = 1\rangle$ denote the *socle* of G. Modules will be free $\mathbf{Z}_p$-modules of finite rank.

2. Preliminary results

2.1. *Pro-p modules.* — Modules will be left modules in the paper.

Theorem **2.1** *(Diederichsen, Heller-Reiner, [4, (2.6) Theorem]). — Let* G *be a group of order* p *and* M *a* $\mathbf{Z}_p[\mathrm{G}]$*-module, free as a* $\mathbf{Z}_p$*-module. Then*

$$\mathrm{M} = \mathrm{M}_1 \oplus \mathrm{M}_p \oplus \mathrm{M}_{p-1},$$

where M_p *is a free* G*-module,* M_1 *is a trivial* G*-module and on* M_{p-1} *the equality* $(1 + c + \cdots + c^{p-1})\mathrm{M}_{p-1} = \{0\}$ *holds for any generator* c *of* G.

Let G be a p-group. A *permutation lattice* for G (or G-*permutational* module) is a direct sum of G-modules, each of the form $\mathbf{Z}_p[\mathrm{G}/\mathrm{H}]$ for some subgroup H of G. Note that a G-module M which is a free $\mathbf{Z}_p$-module is a permutation lattice if and only if G permutes the elements of a basis of M. In particular, when $\mathrm{H} \leq \mathrm{G}$ and M is a G-permutation lattice it is an H-permutation lattice.

If $\mathrm{G} = \langle c \rangle$ is of order p, then Theorem 2.1 implies that M is a permutational lattice if and only if M_{p-1} is missing in the decomposition for M if and only if $\mathrm{M}/(c-1)\mathrm{M}$ is torsion free.

Corollary **2.2.** — *With the assumptions of Theorem 2.1 suppose that* M *admits a Heller-Reiner decomposition* $\mathrm{M} = \mathrm{M}_1 \oplus \mathrm{M}_p$*. Let* L *be a free* G*-submodule of* M *such that* M/L *is torsion free. There is a free* $\mathbf{Z}_p[\mathrm{G}]$*-submodule* M'_p *containing* L *such that* $\mathrm{M} = \mathrm{M}_1 \oplus \mathrm{M}'_p$ *is a Heller-Reiner decomposition.*

Proof. — Consider the canonical epimorphism of G-modules from M onto $\overline{\mathrm{M}} := \mathrm{M}/p\mathrm{M}$. Since M/L is torsion free, one has $p\mathrm{M} \cap \mathrm{L} = p\mathrm{L}$. From this we can deduce that $\overline{\mathrm{L}}$ is a free $\mathbf{F}_p[\mathrm{G}]$-module, and so it is injective. Therefore there is a G-invariant complement $\overline{\mathrm{N}}$ of $\overline{\mathrm{L}}$ in $\overline{\mathrm{M}}$. Since $\overline{\mathrm{M}} = \overline{\mathrm{M}}_1 \oplus \overline{\mathrm{M}}_p$ by Krull-Schmidt, $\overline{\mathrm{N}} = \bigoplus_{i\in\mathrm{I}} \overline{\mathrm{N}}_i$ is a direct sum of cyclic $\mathbf{F}_p[\mathrm{G}]$-modules N_i each of them either free or trivial.

Lift each free $\overline{\mathrm{N}}_i$ to a cyclic $\mathbf{Z}_p[\mathrm{G}]$-submodule N_i, and let $\mathrm{N}_p := \sum_{i\in\mathrm{I}} \mathrm{N}_i$. Put $\overline{\mathrm{M}}'_p := \overline{\mathrm{L}} \oplus \overline{\mathrm{N}}$, and let $\mathrm{M}'_p := \mathrm{L} + \mathrm{N}$. Since the $\mathbf{Z}_p$-rank of M'_p coincides with the $\mathbf{F}_p$-dimension of $\overline{\mathrm{M}}'_p$, it must be a free $\mathbf{Z}_p[\mathrm{G}]$-submodule of M and it contains L. Note that $\mathrm{M}/\mathrm{M}'_p \cong \mathrm{M}_1$ by Krull-Schmidt and so one has $\mathrm{M}'_p + \mathrm{M}_1 = \mathrm{M}$.

Let us show that $\mathrm{M}_1 \cap \mathrm{M}'_p = \{0\}$. There is an idempotent e with $\mathrm{M}_1 = e\mathrm{M}$ and $(1-e)\mathrm{M} = \mathrm{M}_p$. Then $e\mathrm{M}'_p = \mathrm{M}_1 \cap \mathrm{M}'_p$ and therefore $\mathrm{M}'_p = e\mathrm{M}'_p \oplus (1-e)\mathrm{M}'_p = (\mathrm{M}_1 \cap \mathrm{M}'_p) \oplus (1-e)\mathrm{M}'_p$. Since M'_p is a free $\mathbf{Z}_p[\mathrm{G}]$-module it cannot have the trivial G-module as a non-trivial direct summand. Hence $\mathrm{M}_1 \cap \mathrm{M}'_p = \{0\}$ as desired and the corollary is proved. □

2.2. *Pro-p modules and pro-p groups.* — Let $\mathrm{G} := \mathrm{F} \rtimes \mathrm{C}_p$ be a semidirect product of a finitely generated free pro-p group with a group of order p. We need to relate the Heller-Reiner decomposition of the induced C_p-module $\mathrm{F}/[\mathrm{F},\mathrm{F}]$, with a specific free product decomposition of G.

Lemma **2.3.** — *Let* G *be a split extension of a free pro-*p *group* F *of finite rank by a group of order* p*. Then*

(i) *([16])* G *has a free decomposition* $G = (\coprod_{i\in I}(C_i \times H_i)) \amalg H$*, with* $C_i \cong C_p$ *and all* H_i *and* H *free pro-*p*.*

Here I *is finite and each* C_i *is a representative of a conjugacy class of cyclic subgroups of order* p *in* G*. The subgroups* H_i *and* H *are contained in* F *and* $C_F(C_i) = H_i$*.*

(ii) *([7, Lemma 6]) Set* $M := F/[F,F]$*. Fix* $i_0 \in I$ *and a generator* c *of* C_{i_0}*. Then conjugation by* c *induces an action of* C_{i_0} *upon* M*. The* $\langle c\rangle$*-module* M *admits a Heller-Reiner decomposition* $M = M_1 \oplus M_p \oplus M_{p-1}$*.*

Moreover, the $\mathbf{Z}_p$*-ranks of the three* $\langle c\rangle$*-modules satisfy* $\operatorname{rank}(M_p) = p\operatorname{rank}(H)$, $\operatorname{rank}(M_{p-1}) = (p-1)(|I|-1)$*, and* $\operatorname{rank}(M_1) = \sum_{i\in I}\operatorname{rank}(H_i)$*.*

In particular, M *is* G/F*-permutational if and only if* $|I| = 1$*.*

We shall use also the following corollary that can be extracted from [7, Corollary 7].

Corollary **2.4.** — *If for each* $i \in I$ *a basis* B_i *of* H_i *is given and* B *is any basis of* H*, then* $\bigcup_{i\in I} B_i[F,F]/[F,F]$ *is a basis of* M_1 *and* $B[F,F]/[F,F]$ *is a basis of the* G/F*-module* M_p*. A basis of* M_{p-1} *is given by* $\{c_{i_0}^{-1}c_i \mid i \in I, i \neq i_0\}[F,F]/[F,F]$*.*

Corollary **2.5.** — *When* C_p *acts as a group of automorphisms on a finitely generated free pro-*p *group* F *and the induced action of* C_p *on* $M := F/[F,F]$ *allows an interpretation* $M = M_1 \oplus M_p$ *as a permutation module, then the image of* $C_F(C_p)$ *under the commutator quotient map intersects trivially with* M_p *and has the same* $\mathbf{Z}_p$*-rank as* M_1*.*

Proof. — Lemma 2.3 implies that $G = (C_p \times C_F(C_p)) \amalg F_0$ for a free pro-p subgroup F_0. The same lemma shows that there is a Heller-Reiner decomposition $M_1' \oplus M_p'$ with $M_1' = C_F(C_p)[F,F]/[F,F]$. Setting in Corollary 2.2 $L := M_p$ implies that $M_1' \cap M_p = \{0\}$, as claimed. The equality of $\mathbf{Z}_p$-ranks follows from Corollary 2.4, noting that $|I| = 1$. □

Lemma **2.6.** — *Suppose that* $G = F \rtimes \langle t\rangle = (\langle t\rangle \times C_F(t)) \amalg F_0$ *with* $\langle t\rangle \cong C_p$ *and* Q *is a* t*-invariant free pro-*p *factor of* F *satisfying* $C_Q(t) = \{1\}$*. Let "bar" indicate passing to the quotient modulo* $(Q)_F$*. Then* $C_F(t) \cong C_{\overline{F}}(t) = \overline{C_F(t)}$ *and* $\overline{G} \cong (C_p \times C_{\overline{F}}(t)) \amalg F_1$ *for some free pro-*p *group* F_1*.*

Proof. — Since Q is a free pro-p factor of F, we find that $Q \cap [F,F] = [Q,Q]$. Therefore by Lemma 2.3 $L := Q[F,F]/[F,F]$ is a free $\langle t\rangle$-submodule of $M := F/[F,F]$. Consider a Heller-Reiner decomposition $M = M_1 \oplus M_p$. Since M/L is a free $\mathbf{Z}_p$-module we can, using Corollary 2.2, arrange M_p such that L becomes a direct summand of M_p. Note that $\overline{F}/[\overline{F},\overline{F}] = M/L$. Since $\mathbf{Z}_p[\langle t\rangle]$ is a local ring the Krull-Schmidt theorem applies to the Heller-Reiner decomposition $\overline{F}/[\overline{F},\overline{F}] = N_1 \oplus N_p$ showing that the $\mathbf{Z}_p$-rank of M_1 coincides with the $\mathbf{Z}_p$-rank of N_1. Hence by Corollary 2.5, one has $C_{\overline{F}}(t) \cong C_F(t)$ and so certainly $\overline{C_F(t)} = C_{\overline{F}}(t)$. Moreover, Lemma 2.3 shows that $\overline{G} \cong (C_p \times C_{\overline{F}}(t)) \amalg F_1$ for some free pro-p group F_1. □

2.3. *Helpful facts on pro-p groups.*

Lemma **2.7.** — *Let* $F = (A \amalg B) \amalg C$ *be a pro-p group. Then* $(A \amalg B) \cap (A)_F = (A)_{A\amalg B}$.

Proof. — Observe that $A \amalg B/((A)_F \cap (A \amalg B)) \cong (A \amalg B)(A)_F/(A)_F \cong B$. As $(A)_{A\amalg B} \leq (A)_F \cap (A \amalg B)$ the second isomorphism theorem reads $(A \amalg B/(A)_{A\amalg B})/(((A \amalg B) \cap (A)_F)/(A)_{A\amalg B}) \cong (A \amalg B)/((A \amalg B) \cap (A)_F) \cong B$. Therefore $B \cong A \amalg B/(A)_{A\amalg B} \cong (A \amalg B)/((A \amalg B) \cap (A)_F)$ so that the canonical epimorphism from $A \amalg B/(A)_{A\amalg B}$ onto $(A \amalg B)/((A \amalg B) \cap (A)_F)$ turns out to be an isomorphism. This shows the Lemma. □

Lemma **2.8.** — *Let* $G = F \rtimes K$ *with* F *free pro-p and* K *a finite p-group. Suppose that every finite subgroup of* G *is* F*-conjugate into* K. *Then, for any* $T \leq K$,

(i) $N_G(T) = C_F(T) \rtimes N_K(T)$;
(ii) *Every finite subgroup of* $N_G(T)$ *is* $C_F(T)$*-conjugate to a subgroup of* $N_K(T)$.

Proof. — (i) observe that $g \in N_G(T)$ can be written as $g = fk$ with $f \in F$ and $k \in K$. Then $T = T^g = T^{fk}$ reads modulo F as $T = T^k$ so that $k \in N_K(T)$ and hence $f \in C_F(T)$ follows.

(ii) Let R be a finite subgroup of $N_G(T)$ and w.l.o.g. we can assume that it contains T (multiplying it by T if necessary). By the hypothesis there exists $f \in F$ with $R^f \leq K$; hence $T^f \leq K$. Therefore $TT^f \leq K$ and, since $F \lhd G$, for every element $t \in T$ one has $t^{-1}t^f \in K \cap F$. As $K \cap F = \{1\}$ it follows that $f \in C_F(T)$ as needed. □

Our proof is based on the following results from [7] and [16] frequently used in the paper.

Theorem **2.9** *[16, Theorem 1.2].* — *Let* K *be a finite p group acting on a free pro-p group* F *of finite rank. Then* $C_F(K)$ *is a free pro-p factor of* F.

Theorem **2.10** *[7, Proposition 14].* — *Let* G *be a semidirect product of a free pro-p group* F *of finite rank with a p-group* K *such that every finite subgroup is conjugate to a subgroup of* K. *Suppose that* $C_F(t) = \{1\}$ *holds for every torsion element t of* G. *Then* $G = K \amalg F_0$ *for a free pro-p factor* F_0.

3. HNN-extensions

We introduce a notion of a *pro-p HNN-extension* as a generalization of the construction described in [14, page 97].

Definition **3.1.** — *Suppose that* G *is a pro-p group, and for a finite set* I *there are given monomorphisms* $\phi_i : A_i \to G$ *for subgroups* A_i *of* G. *The HNN-extension* $\tilde{G} := \mathrm{HNN}(G, A_i, \phi_i, i \in I)$ *is defined to be the quotient of* $G \amalg F(I)$ *modulo the relations* $\phi_i(a_i) = ia_i i^{-1}$ *for all* $i \in I$. *We call* $\tilde{G}$ *an HNN-extension and* G *the base group,* I *the set of stable letters, and the subgroups* A_i *and* $B_i := \phi_i(A_i)$ *associated.*

One can see that every HNN-extension in the sense of the present definition can be obtained by successively forming *HNN-extensions*, as defined in [14], each time defining the base group to be the just constructed group and then adding a pair of associated subgroups and a new stable letter.

A pro-p HNN-extension $G = \mathrm{HNN}(H, A, f, t)$ is *proper* if the natural map from H to G is injective. Only proper pro-p HNN-extensions will be used in this paper.

A proper *HNN*-extension $\tilde{G} := \mathrm{HNN}(G, A_i, \phi_i, I)$ (viewing G as a subgroup of $\tilde{G}$) satisfies a *universal property* as follows. Given a pro-p group G, homomorphisms $f : G \to H$, $f_i : A_i \to H$ and a map $g : I \to H$ such that for all $i \in I$ and all $a_i \in A_i$ we have $f(\phi_i(a_i)) = g(i)f_i(a_i)g(i)^{-1}$, there is a unique homomorphism $\omega : \tilde{G} \to H$ which agrees with f on G, with f_i on A_i for every $i \in I$ and with g on I.

Remark **3.2.** — Every finite subgroup of $\tilde{G}$ is conjugate to a subgroup of G. This can either be seen by interpreting $\tilde{G}$ as an iterated *HNN*-extension and then using [14, Theorem 4.2(c)] or by viewing $\tilde{G}$ as the fundamental pro-p group of a graph of groups, the graph being a finite bouquet of loops using [20, Theorem 3.10].

3.1. *HNN-embedding.* — Theorem 3.4 below is an HNN-embedding result—a refined pro-p-version of the main theorem in [6]. We first prove it for semidirect products.

Proposition **3.3.** — *Let* $G = F \rtimes K$ *be a semidirect product of a free pro-p group* F *of finite rank and a finite p-group* K. *Then* G *can be embedded in a semidirect product* $\tilde{G} = E \rtimes K$ *such that every finite subgroup of* $\tilde{G}$ *is conjugate to a subgroup of* K *and* E *is free pro-p of finite rank.*

Proof. — By [16, Cor. 1.3(a)], there are only finitely many conjugacy classes of finite subgroups that are not conjugate to a subgroup in K. We proceed by induction on this number $f = f(G, K)$. For $f = 0$ there is nothing to prove. For the inductive step it suffices to show that G can be embedded into a semidirect product $\tilde{G}$ of a finitely generated free pro-p group E and (the same) K with less conjugacy classes of finite subgroups that are not conjugate to a subgroup in K. So assume that L is a finite subgroup of G not conjugate to a subgroup of K. Let $\pi : G \longrightarrow K$ be the canonical projection and $\phi = \pi_{|L}$. Put $\tilde{G} := \mathrm{HNN}(G, L, \phi)$ and observe that it is finitely generated.

For proving that G embeds in $\tilde{G}$ we need to employ [1, Theorem 1.3], according to which G embeds in $\tilde{G}$ if, and only if, the following set $\mathcal{N}$ of open normal subgroups intersects trivially: namely $\mathcal{N}$ is the set of all open normal subgroups U of G such that there is a chain of normal subgroups $U = C_0 < \cdots < C_n = G$ with $\phi(L \cap C_i) = \phi(L) \cap C_i$ and ϕ inducing the identity on each $(LC_i \cap C_{i+1})/C_i$ for all $i < n$.

Let us show that every open normal subgroup U of G properly contained in F must belong to $\mathcal{N}$. Consider the chain $C_0 := U$, $C_1 := F$ and $C_2 := G$. The conditions hold in the part below $C_1 = F$ since $L \cap F = \phi(L) \cap F = \{1\}$. It is also trivial that $\phi(L \cap C_2) = \phi(L) \cap C_2$, since $C_2 = G$. So we are left with showing that the homomorphism $\overline{\phi}$ induced by ϕ on LF/F coincides with the identity. For $g \in G$ we denote by $\overline{g}$ its image modulo F. If $\overline{x} \in LF/F$ with $x \in L$, then we have $\overline{\phi}(\overline{x}) = \overline{\phi(x)}$, and since $\phi = \pi_{|L}$, $\overline{\phi}(\overline{x}) = \overline{\pi(x)}$. By the definition of the projection π, if $x = fk$ with $f \in F$ and $k \in K$, then $\pi(x) = k$. Hence $\overline{\phi}(\overline{x}) = \overline{\pi(x)} = \overline{k} = \overline{x}$, as desired.

Note that $\pi : G \longrightarrow K$ extends to $\tilde{G} \longrightarrow K$ by the universal property of an HNN-extension, so $\tilde{G}$ is a semidirect product $E \rtimes K$ of its kernel E with K. By [6, Lemma 10], every open torsion free subgroup of $\tilde{G}$ is free pro-p. So E is free pro-p. As $\tilde{G}$ is finitely generated, E is finitely generated. Let A be any finite subgroup of $\tilde{G}$. Then, by [14, Theorem 4.2(c)], it is conjugate to a subgroup of the base group. □

Having established the HNN-embedding result for semidirect products we state and prove it for arbitrary finitely generated virtually free pro-p groups.

Theorem **3.4.** — *Let* G *be a finitely generated pro-p group possessing an open normal free pro-p subgroup* F. *Then* G *can be embedded in a semidirect product* $\tilde{G} = E \rtimes G/F$ *such that every finite subgroup of* $\tilde{G}$ *is conjugate to a subgroup of* G/F *and* E *is free pro-p. Moreover,* $\tilde{G}$ *is finitely generated.*

Proof. — Put $K := G/F$, and let $\pi : G \to K$ denote the canonical projection. Form $G_0 := G \amalg K$. By the universal property of the free pro-p product there is an epimorphism from G_0 to K which agrees with π on G and with the identity on K. As a consequence of the Kurosh subgroup theorem (see [13, Theorem 9.1.9]), its kernel, say F_0, is free pro-p and $G_0 = F_0 \rtimes K$, where K is identified with its image in G_0. One observes that G_0 is finitely generated, since G is. Now the result follows from Proposition 3.3. □

3.2. *Permutation extensions.*

Definition **3.5.** — *Given a finite p-group* K *and a finite* K*-set* X, *there is a natural extension of the action of* K *to the free pro-p group* $\tilde{F} = F(X)$. *The semidirect product* $\tilde{F} \rtimes K$ *will be called the permutational extension of* $\tilde{F}$ *by* K. *Now* K *acts on* $\tilde{F}$ *from the left by conjugation, i.e.,* $k \cdot f[\tilde{F}, \tilde{F}] := f^k[\tilde{F}, \tilde{F}]$.

Remark **3.6.** — Choosing representatives $\{A_i \mid i \in I\}$ of the conjugacy classes of all point stabilizers and letting $Z_i \subseteq X$ be a set of representatives of orbits such that $K_z = A_i$ for all $z \in Z_i$, we can rewrite the K-set X in the form $\bigcup_{i \in I} K/A_i \times Z_i$ with K acting on the cosets by left multiplication and on the second factor trivially. Then $\tilde{G} := \tilde{F} \rtimes K$ has a presentation $F(\bigcup_{i \in I} Z_i) \amalg K$ modulo the relations $[a_i, z_i]$ for all $z_i \in Z_i$ and $a_i \in A_i$, with

i running through the finite set I. The presentation shows that $\tilde{G}$ is isomorphic to an HNN-extension in the sense of Definition 3.1, with all ϕ_i the identity on the respective group A_i, and with the union $\bigcup_{i\in I} Z_i$ as the set of stable letters. We shall write $\tilde{G} = \mathrm{HNN}(K, A_i, Z_i, i \in I)$—omitting the ϕ_i from the usual notation of the *HNN*-extension.

Then $M := \tilde{F}/[\tilde{F}, \tilde{F}]$ is a K-permutation module (see the explanation after Theorem 2.1), i.e. $M = \bigoplus_{i\in I} M_i$ with $M_i := \mathbf{Z}_p[K/A_i \times Z_i]$.

Remark **3.7.** — In the presentation of $\tilde{G}$ we may, for every $i \in I$, choose $k_i \in K$ and replace every (A_i, Z_i) by $(B_i, X_i) := (A_i^{k_i}, Z_i^{k_i})$. Then $\tilde{G} = \mathrm{HNN}(K, B_i, X_i, I)$.

Lemma **3.8.** — *Let* $\tilde{F}$ *be the normal closure of* $F(\bigcup_{i\in I} Z_i)$ *in* $\tilde{G} = \mathrm{HNN}(K, A_i, Z_i, i \in I)$. *For every* $i \in I$ *choose respectively coset representative sets* R_i *of* $K/N_K(A_i)$ *and* S_i *of* $N_K(A_i)/A_i$. *Then* $C_F(A_i) = \coprod_{s\in S_i} F(Z_i)^s$ *and*

$$\tilde{F} = \coprod_{i\in I} \coprod_{r\in R_i} C_F(A_i)^r.$$

Proof. — As explained in Remark 3.6, one can view $\tilde{G}$ as the quotient of $G := F(\bigcup_{i\in I} Z_i) \amalg K$ modulo the relations $[a_i, z_i]$ for all $z_i \in Z_i$ and $a_i \in A_i$, with i running through the finite set I. By the Kurosh subgroup theorem (see [13, Theorem 9.1.9]) applied to the normal closure N of $F(\bigcup_{i\in I} Z_i)$ in G we have a free pro-p decomposition

$$N = \coprod_{i\in I} \coprod_{r\in R_i} \coprod_{s\in S_i} \coprod_{a\in A_i} F(Z_i)^{asr}.$$

The relations yield $F(Z_i)^a = F(Z_i^a) = F(Z_i)$. Since for $s \in S_i$, $a \in A_i$, $z \in Z_i$ one has $[a, z] = 1$ if, and only if, $[a^s, z] = 1$ if and only if $[a, z^{s^{-1}}] = 1$ we have

$$\tilde{F} \rtimes A_i = \Big(A_i \times \coprod_{s\in S_i} F(Z_i)^s\Big) \amalg \coprod_{r\in R_i - \{1\}} \coprod_{s\in S_i} F(Z_i)^{sr} \amalg \coprod_{j\neq i} \coprod_{k\in K} F(Z_j)^k.$$

Set $X := A_i \times \coprod_{s\in S_i} F(Z_i)^s$ and observe that $A_i \le X \cap X^g$ holds for any $g \in C_{\tilde{F}}(A_i)$. Since by Theorem [13, 9.1.12] $X \cap X^h = 1$ for every $h \notin X$, we deduce that $C_F(A_i) = X$. Thus we proved the first equality that in turn implies the second one. □

Notation **3.9.** — For a virtually free pro-p group $G = F \rtimes K$ consider the set of subgroups L of K with $C_F(L) \neq 1$ ordered by inclusion. We say that $L \le K$ is F-**c** maximal if L is maximal with respect to this ordering.

Lemma **3.10.** — *Let* $G = \mathrm{HNN}(K, A_i, Z_i, I)$ *be a permutational extension. Then for every* F-***c*** *maximal subgroup* L *of* K *there exist elements* $i \in I$ *and* $k \in K$ *such that* $L = A_i^k$.

Proof. — As in Definition 3.1, we may consider G as an iterated *HNN*-extension. By [14, Theorem 4.3(b)], in any such *HNN*-extension the group $K \cap K^x$ is contained in a conjugate of an associated subgroup for any $x \notin K$. Using this fact repeatedly for $1 \neq x \in C_F(L)$ one has that $L \leq K \cap K^x \leq A_i^g$ for a suitable element $g \in G$. Since $C_F(A_i^g) \neq \{1\}$ and L is F-**c** maximal we can conclude that $L = A_i^g$ for some $g \in G$. On the other hand, $G = F \rtimes K$ and so the canonical epimorphism $\pi : G \to K$ yields $k := \pi(g) \in K$ with $L = A_i^k$. □

The goal of the rest of this subsection is to construct a certain K-permutational free pro-p factor Q of F that will serve as a tool for the induction step in Section 4.

Proposition **3.11.** — *Let* $G = \mathrm{HNN}(K, A_i, Z_i, I)$ *be a permutational extension as described in Remark 3.6. Consider a family* $(B_j)_{j\in J}$ *of pairwise non-conjugate subgroups of* K *each being an* F-***c*** *maximal subgroup of* G. *Then* $Q := \langle C_F(B_j) \mid j \in J\rangle = \coprod_{j\in J} \coprod_{r_j \in R_j} C_F(B_j^{r_j})$ *and* Q *is a free pro-p factor of* F, *where* R_j *denotes a set of coset representatives of* $K/N_K(B_j)$.

Proof. — Lemma 3.10 and Remark 3.7 allow us to identify the family of subgroups $(B_j)_{j\in J}$ with a subfamily of $(A_i)_{i\in I}$, i.e., to assume that $J \subseteq I$ so that $B_j = A_j$ for all $j \in J$. Then Lemma 3.8 gives the result. □

In the final two lemmata of this section we do not have to assume that G is a permutational extension.

Lemma **3.12.** — *Let* $G = F \rtimes K$ *be a semidirect product with* F *free pro-p of finite rank and* K *a finite p-group. Suppose that every finite subgroup of* G *is* F*-conjugate into* K. *Then, for any* F-***c*** *maximal subgroup* L *of* K *the normalizer* $N_G(L) = \mathrm{HNN}(N_K(L), L, Z_L)$ *is a permutational extension.*

Proof. — Consider any $t \in N_K(L) \setminus L$. Then $C_{C_F(L)}(t) = \{1\}$ because otherwise there would be $f \in C_F(L)$, $f \neq 1$, fixed by $\langle L, t\rangle$ contradicting L being F-**c** maximal. Hence the induced action of $N_K(L)/L$ on $C_F(L)$ is free. Note that $C_F(L)$ is a free factor of F by Theorem 2.9 and hence is finitely generated. Since all finite subgroups of G are conjugate into K by Lemma 2.8(ii), all finite subgroups of $N_G(L)$ are conjugate into $N_K(L)$. As $L \leq K$, all finite subgroups of $N_G(L)/L$ are conjugate into $N_K(L)/L$. Therefore, Theorem 2.10 shows that $C_F(L) \rtimes (N_K(L)/L) = A \amalg F_0$ for some finite p-group A and a finitely generated free pro-p group F_0. Selecting a free pro-p base Y of F_0 we have that $N_G(L)/L \cong \mathrm{HNN}(N_K(L)/L, \{1\}, Y)$. Therefore, for $Z_L := Y$ one has $N_G(L) = \mathrm{HNN}(N_K(L), L, Z_L)$, as claimed. □

Lemma **3.13.** — *Let* $G = F \rtimes K$ *with* F *free pro-p of finite rank and* K *a finite p-group. Suppose that every finite subgroup of* G *is* F*-conjugate into* K. *Assume further that there is* $N_K(L) \leq K_0 \lhd K$ *such that* $F \rtimes K_0$ *is a permutational extension. Then*

(i) $Q := \langle C_F(L)^k \mid k \in K\rangle$ *is a* K*-invariant free pro-p factor of* F *and the subgroup* $Q \rtimes K$ *of* G *is a permutational extension.*
(ii) $\mathrm{rank}(Q) = |X_L||K : N_K(L)|$ *where* X_L *is any* $N_K(L)$*-invariant free pro-p basis of* $C_F(L)$ *on which* $N_K(L)/L$ *acts freely.*

Proof. — By Lemma 3.12, we know that $N_G(L) = \mathrm{HNN}(N_K(L), L, Z_L)$ is a permutational extension.

If $N_K(L) = K$, then $N_G(L) = Q \rtimes K$ is a permutational extension and (ii) holds.

Suppose now that $N_K(L) < K$. Fix coset representative sets T_L of $N_{K_0}(L)/L$, S of $K_0/N_{K_0}(L)$ and R_0 of K/K_0. Then, as $N_K(L) = N_{K_0}(L)$, we find that $R := R_0ST_L$ is a set of coset representatives of K/L and, as sets, $R = R_0 \times S \times T_L$. In particular, $\{L^{r_0} \mid r_0 \in R_0\}$ is a maximal set of pairwise K_0-non-conjugate K-conjugates of L. Therefore, applying Proposition 3.11 to the family $\{C_F(L^{sr_0} \mid (r_0, s) \in R_0 \times S\}$ inside the permutational extension $F \rtimes K_0$ one obtains that

$$Q_0 := \coprod_{r_0\in R_0} \coprod_{s\in S} C_F(L^{sr_0})$$

is a free pro-p factor of F. Finally, by Lemma 3.8, $X_L := \bigcup_{t\in T_L} Z_L^t$ is an $N_K(L)$-invariant free pro-p basis of $C_F(L)$. Then $\bigcup_{r\in R} Z_L^r$ is a K-invariant free pro-p basis of Q_0. Therefore Q_0 is a K-invariant free pro-p factor of F and, as $K = R_0ST_LL$, we find that $Q = Q_0$ must hold.

For showing (ii) it suffices to observe the equalities

$$\mathrm{rank}(Q) = |R_0||S||T_L||Z_L| = |X_L|\big|K : N_K(L)\big|. \qquad \square$$

4. Lifting permutational representations to $F \rtimes K$

A semidirect product $G = F \rtimes K$, where F is a finitely generated free pro-p and K is a finite p-group, will be called a *PE-group*, if every finite subgroup of G is conjugate into K.

For such a group conjugation of finite subgroups can then be achieved by elements in F. By Remark 3.2, every permutational extension is a PE-group. It is the goal of this section to show that the converse holds as well (cf. Proposition 4.8).

4.1. *Induction engine.* — Our next proposition describes properties of a "minimal" counter-example G that is a PE-group but not a permutational extension. These properties will be useful for the proof of Proposition 4.8.

Proposition **4.1.** — *Let* $G = F \rtimes K$ *be a PE-group such that any PE-group* $F' \rtimes K'$ *with either* $|K'| < |K|$ *or* $|K| = |K'|$ *and* $\mathrm{rank}(F') < \mathrm{rank}(F)$ *is a permutational extension. Suppose further that there exists a* K*-invariant free pro-p factor* Q *of* F *such that* $Q \rtimes K$ *is a permutational extension, and let* $\bar{\ } : F \to F/(Q)_F$ *denote the canonical projection. Then the following statements hold:*

(i) $\overline{F} \rtimes K$ *is a PE-group;*
(ii) *For every* $T \leq K$ *we have* $C_{\overline{F}}(T) = \overline{C_F(T)}$.

Proof. — Suppose that the proposition is false and G is a counter-example. A series of lemmata will yield a contradiction.

Lemma **4.2.** — $Z(G) = \{1\}$.

Proof. — Suppose that $Z(G) \neq \{1\}$. Then there exists $1 \neq t \in \operatorname{socle}(K)$ with $C_F(t) = F$. We claim that $G/\langle t\rangle$ satisfies (i). Indeed, when R is a finite subgroup of $G/\langle t\rangle$ then its preimage in G, say $\tilde{R}$, is F-conjugate into K. Hence R is F-conjugate into $K/\langle t\rangle$. By the minimality assumption on $|K|$ we can conclude that $\overline{F} \rtimes (K/\langle t\rangle)$ is a PE-group. Therefore (i) holds.

Let T be any subgroup of K. Then, by the minimality assumption on $|K|$, we must have $C_{\overline{F}}(T\langle t\rangle/\langle t\rangle) = \overline{C_F(T\langle t\rangle/\langle t\rangle)}$. Now (ii) follows from the equalities $C_F(T) = C_F(T\langle t\rangle) = C_F(T\langle t\rangle/\langle t\rangle)$.

Hence G is not a counter-example, a contradiction. □

Lemma **4.3.** — *Let* $\{1\} \neq t \in \operatorname{socle}(K)$. *Then either* $Q = C_Q(t)$ *or* $C_Q(t) = \{1\}$.

Proof. — Set $Q_0 := C_Q(t)$ and note that by Theorem 2.9 it is a free K-invariant factor of Q. We can assume that $Q > Q_0 > \{1\}$, else there is nothing to prove. By assumption $Q \rtimes K$ is a permutational extension and so, by Lemma 2.8(ii), $Q_0 \rtimes K = N_{Q \rtimes K}(t)$ is a PE-group. Since $\operatorname{rank}(Q_0) < \operatorname{rank}(F)$, $Q_0 \rtimes K$ is a permutational extension. If $Q = F$ then $\overline{G} = K$ and so G cannot be a counter-example to the statements of our proposition. Thus $\operatorname{rank}(Q) < \operatorname{rank}(F)$ and therefore $Q/(Q_0)_Q \rtimes K$ is a PE-group. Since $\operatorname{rank}(Q/(Q_0)_Q) < \operatorname{rank}(F)$ the quotient $Q/(Q_0)_Q \rtimes K$ is a permutational extension by our minimality assumption on G. By Theorem 2.9 there is $F_0 \leq Q$ so that $Q = Q_0 \amalg F_0$. Setting in Lemma 2.7 $A := Q_0$, $A \amalg B := Q$ implies that $(Q_0)_Q = (Q_0)_F \cap Q$ and hence $Q/(Q_0)_Q \rtimes K \cong (Q(Q_0)_F/(Q_0)_F) \rtimes K$, showing that the latter group is a permutational extension. Using that $\operatorname{rank}(Q_0) < \operatorname{rank}(F)$ and writing "tilde" for passing to the quotient modulo $(Q_0)_F$ we can deduce that statements (i) and (ii) of the proposition hold for $\tilde{G}$, i.e. $\tilde{G}$ is a PE-group and $\widetilde{C_F(t)}$ is naturally isomorphic to $C_{\tilde{F}}(\tilde{t})$. Since

$$\widetilde{(Q)}_F = (Q)_F(Q_0)_F/(Q_0)_F = (F_0)_F(Q_0)_F/(Q_0)_F = (\tilde{Q})_{\tilde{F}}$$

the second isomorphism theorem implies that $\bar{G}$ is naturally isomorphic to $(\tilde{G})/(\tilde{Q})_{\tilde{F}}$. Then observing that $\operatorname{rank}(\tilde{Q}) = \operatorname{rank}(Q(Q_0)_F/(Q_0)_F) < \operatorname{rank}(Q)$ and the pair $(\tilde{G}, \tilde{Q})$ satisfies all hypotheses of the proposition, we find that $\bar{G}$ satisfies (i) and (ii) of the proposition as well. Therefore, G cannot be a counter-example, a contradiction. □

Lemma **4.4.** — K *cannot be cyclic of order* p.

Proof. — Suppose $K \cong C_p$. Lemma 2.3(i) shows that $G = (C_F(K) \times K) \amalg F_0$ with F_0 free pro-p.

Lemma 4.3 implies that either $Q = C_Q(K)$ or $C_Q(K) = \{1\}$. In the first case $C_F(K) = Q \amalg F_Q$ and so $G/(Q)_G \cong (F_Q \times K) \amalg F_0$. Thus (i) and (ii) hold. The second case has been treated in Lemma 2.6. □

Lemma **4.5.** — *If there is* $t \in \mathrm{socle}(K)$ *with* $C_Q(t) < Q$ *then* $\overline{C_F(K)} = C_{\overline{F}}(K)$.

Proof. — Using Lemma 4.3 we find that $C_Q(t) = \{1\}$. Lemma 2.6 shows that $C_{\overline{G}}(t) = \overline{C_G(t)}$ is naturally isomorphic to $C_G(t)$. As $t \in K$ we have then $C_{\overline{F}}(K) \cong C_F(K)$ and, as $\overline{C_F(K)} \le C_{\overline{F}}(K)$, we have established the equality $\overline{C_F(K)} = C_{\overline{F}}(K)$. □

Lemma **4.6.** — *For any* $1 \neq t \in \mathrm{socle}(K)$ *such that* $Q = C_Q(t)$ *the centralizer* $C_{\overline{G}}(t)$ *is naturally isomorphic to* $C_G(t)/(Q)_{C_F(t)}$.

Proof. — Applying the Kurosh subgroup theorem (see [5, Proposition 4.1]) to the subgroup $C_F(t)$ of $F = Q \amalg F_Q$ we get that $Q = C_Q(t) = C_F(t) \cap Q$ must be a free pro-p factor of $C_F(t)$. Setting in Lemma 2.7 $A := Q$ and $A \amalg B := C_F(t)$ implies that $C_F(t) \cap (Q)_F = C_F(t) \cap (Q)_{C_F(t)}$ so that $C_{\overline{F}}(t) = C_F(t)(Q)_F/(Q)_F \cong C_F(t)/(C_F(t) \cap (Q)_F) \cong C_F(t)/(Q)_{C_F(t)}$. This equality gives $C_{\overline{G}}(t) \cong C_G(t)/(Q)_{C_F(t)}$. □

Lemma **4.7.** — *For any counter-example* G *statement (ii) holds.*

Proof. — For $\{1\} \neq T < K$ the minimality assumption on $|K|$ shows that $C_{\overline{F}}(T) = \overline{C_F(T)}$ must hold. So all we need to establish is

$$C_{\overline{F}}(K) = \overline{C_F(K)}. \tag{1}$$

Pick any $1 \neq t \in \mathrm{socle}(K)$ and note that $\langle t \rangle < K$ by Lemma 4.4. By Lemma 4.5 we may assume that $Q = C_Q(t)$.

Then by Lemma 4.6, $C_{\overline{G}}(t)$ is naturally isomorphic to $C_G(t)/(Q)_{C_F(t)}$. Therefore, as $t \in K$,

$$C_{\overline{F}}(K) = C_{C_{\overline{F}}(t)}(K) \cong C_{C_F(t)/(Q)_{C_F(t)}}(K). \tag{2}$$

By Lemma 2.8(ii), every finite subgroup of $C_G(t)$ is $C_F(t)$-conjugate into K. By Lemma 4.2, and Theorem 2.9, $\mathrm{rank}(C_F(t)) < \mathrm{rank}(F)$ and by hypothesis $Q \rtimes K$ is a permutational extension. Hence

$$\begin{aligned} C_{C_F(t)/(Q)_{C_F(t)}}(K) &= C_{C_F(t)}(K)(Q)_{C_F(t)}/(Q)_{C_F(t)} \\ &= C_F(K)(Q)_{C_F(t)}/(Q)_{C_F(t)} \\ &\cong C_F(K)/C_F(K) \cap (Q)_{C_F(t)}. \end{aligned} \tag{3}$$

Taking $C_F(K) \cap (Q)_{C_F(t)} = C_F(K) \cap (C_F(t) \cap (Q)_F) = C_F(K) \cap (Q)_F$ into account yields

$$(4) \qquad \begin{aligned} C_F(K)/C_F(K) \cap (Q)_{C_F(t)} &= C_F(K)/C_F(K) \cap (Q)_F \\ &\cong C_F(K)(Q)_F/(Q)_F \\ &= \overline{C_F(K)}. \end{aligned}$$

Combining (2), (3) and (4) yields the desired Eq. (1). □

Deriving a final contradiction. — In order to produce a final contradiction it suffices to establish (i) by Lemma 4.7.

There must be a finite subgroup R of $\overline{G}$ not $\overline{F}$-conjugate into K. If $|R| < |K|$, then taking $\bar{G}_0 = R\bar{F}$ and G_0 to be its preimage in G we see that $G_0 = F \rtimes (G_0 \cap K)$ is a PE-group and $|G_0 \cap K| < |K|$. Then by the minimality assumption on $|K|$ the group R is $\bar{F}$-conjugate into subgroup of K contradicting the hypothesis on R. Thus we must have $|R| = |K|$. Lemma 4.4 implies that $|K| > p$. Conjugating R with a suitable element in $\overline{F}$ we can achieve that $\{1\} \neq R \cap K$ is a maximal subgroup of K. Therefore, there exists $1 \neq t \in \mathrm{socle}(R) \cap \mathrm{socle}(K)$ with $R \leq C_{\overline{G}}(t)$. Lemma 4.3 implies that we can have only the following two cases:

(α) $C_Q(t) = \{1\}$.
(β) $C_Q(t) = Q$ is a free pro-p factor of $C_F(t)$.

(α) Lemma 2.6 shows that $C_F(t) \cong C_{\overline{F}}(t)$ and so $C_G(t) \cong C_{\overline{G}}(t)$. Therefore there is $R_0 \leq C_G(t)$ with $\overline{R}_0 = R$. Now R is $\overline{F}$-conjugate into K since $R_0 \cong K$ is $C_F(t)$-conjugate into K by the minimality assumption on the rank of F (remember that $\mathrm{rank}(C_F(t)) < \mathrm{rank}(F)$ by Lemma 4.2 and Theorem 2.9).

(β) An application of Lemma 4.6 gives the natural isomorphism $C_{\overline{G}}(t) \cong C_G(t)/(Q)_{C_F(t)}$. Lemma 2.8(ii) implies that $C_G(t) = C_F(t) \rtimes K$ is a PE-group. Lemma 4.2, Theorem 2.9 and the minimality assumption on the rank of F show that $C_G(t)/(Q)_{C_F(t)} = C_F(t)/(Q)_{C_F(t)} \rtimes K$ is a PE-group. Therefore, $C_{\overline{G}}(t) = C_{\overline{F}}(t) \rtimes K$ is a PE-group. In particular, R is $C_{\overline{F}}(t)$-conjugate into K, a contradiction. □

4.2. *Permutational extension criterion.*

Proposition **4.8.** — *Every PE-group* $G = F \rtimes K$ *is a permutational extension.*

Proof. — Suppose that the proposition is false. Then there is a counter-example with K of minimal order. Among all such counter-examples fix one with rank(F) minimal. If there is no finite F-**c** maximal subgroup $\{1\} \neq L \leq K$ then by Theorem 2.10 we find $G = F_0 \amalg K = \mathrm{HNN}(K, 1, Z, 1)$ where Z is a base of F_0, a contradiction. Therefore, we can fix an F-**c** maximal subgroup $\{1\} \neq L \leq K$ and set $Q := \langle C_F(L)^k \mid k \in K\rangle$. Observe that Q is K-invariant.

We claim that Q is a free pro-p factor of F and $Q \rtimes K$ is a permutational extension.

Indeed, if $L \lhd K$ then $Q = C_F(L)$ and hence by Theorem 2.9 Q is a free pro-p factor of F. Lemma 3.12 shows then that $Q \rtimes K = N_G(L) = \mathrm{HNN}(K, L, Z_L, \{L\})$ is a permutational extension. If $N_K(L) < K$ fix any maximal subgroup K_0 of K containing $N_K(L)$. By the minimality assumption on $|K|$ we can conclude that $F \rtimes K_0$ is a permutational extension and therefore the claim follows from Lemma 3.13(i).

Since $Q \rtimes K$ is a permutational extension Proposition 4.1 implies that $\overline{G} := G/(Q)_F = F/(Q)_F \rtimes K$ is a PE-group. As $\mathrm{rank}(\overline{F}) < \mathrm{rank}(F)$ the minimality assumption on $\mathrm{rank}(F)$ implies that

$$\overline{G} = \mathrm{HNN}(K, B_j, Y_j, j \in J) \tag{5}$$

is a permutational extension.

Let S_j be a set of coset representatives of $N_K(B_j)/B_j$. By Lemma 3.8, $C_{\overline{F}}(B_j) = \coprod_{s \in S_j} F(Y_j)^s$. Since $C_{\overline{F}}(B_j)$ is projective and, by virtue of Proposition 4.1(ii) $C_{\overline{F}}(B_j) = \overline{C_F(B_j)}$, we can lift Y_j to a subset Z_j of some basis of $C_F(B_j)$.

We devise a "model"-permutational extension $\tilde{G}$ that finally will turn out to be isomorphic to G.

To this end we let $\mathcal{A} = \{(B_j, Y_j) \mid j \in J\} \cup \{L, Z_L)\}$. Form $\tilde{G} := \mathrm{HNN}(K, A, Z_A, (A, Z_A) \in \mathcal{A})$ and consider a bijection ϕ which sends, for all $j \in J$ every $B_j \mapsto B_j$, $Y_j \mapsto Z_j$, $L \mapsto L$ and $Z_L \mapsto Z_L$. Using the universal property of the permutational extension $\tilde{G}$, ϕ extends to an epimorphism from $\tilde{G}$ to G.

Since $\overline{G} = G/(C_F(L)^k \mid k \in K)_F = \mathrm{HNN}(K, B_j, Y_j, j \in J)$ and the latter group is naturally isomorphic to $\tilde{G}/(Z_L)_{\tilde{G}}$, we can conclude that $\ker\phi \le (Z_L)_{\tilde{G}}$ must hold.

Set $\tilde{F} := \phi^{-1}(F)$ and note that $\tilde{G} = \tilde{F} \rtimes K$. Choose a coset representative set R_L of $K/N_K(L)$ and observe that Proposition 3.11 applied to the family $\{C_{\tilde{F}}(L^r) \mid r \in R_L\}$ yields $\tilde{Q} := \coprod_{r \in R_L} C_{\tilde{F}}(L^r)$. Now choose a coset representative set S_L of $N_K(L)/L$ then Lemma 3.8 shows that $C_{\tilde{F}}(L) = \coprod_{s \in S} F(Z_L^s)$ and so we find

$$\mathrm{rank}(\tilde{Q}) = |Z_L||K : L|. \tag{6}$$

As has been mentioned before $\tilde{F}/(\tilde{Q})_{\tilde{F}} \cong F/(Q)_F$ and so establishing

$$\mathrm{rank}(\tilde{Q}) = \mathrm{rank}(Q) \tag{7}$$

would imply $G \cong \tilde{G}$ giving the final contradiction with $\tilde{G}$ being a permutational extension.

If $N_K(L) < K$, then Lemma 3.13(ii) implies (7). Otherwise $L \lhd K$ and thus $Q = C_F(L) \cong C_{\tilde{F}}(L)$ because $N_G(L) = \mathrm{HNN}(K, L, Z_L, \{L\}) \cong N_{\tilde{G}}(L)$ (cf. Lemma 3.12). Hence (7) holds in this case as well. □

Theorem **4.9.** — *Let* G *be a semidirect product of a finitely generated free pro-p group* F *and a finite p-group* K*. The following properties are equivalent:*

(i) G *is a permutational extension.*
(ii) *Every finite subgroup of* G *is conjugate to a subgroup of* K.
(iii) $M := F/[F, F]$ *is a* K*-permutation module.*

Proof. — (i) ⇒ (ii) & (iii). If G is a permutational extension, Remark 3.2 and Remark 3.6 together imply that G is a PE-group and that $F/[F, F]$ is a permutation module.

"(ii) ⇒ (i)" has been established in Proposition 4.8.

"(iii) ⇒ (ii)". Suppose that (iii) holds but (ii) not. Then there is a counter-example G with $|K|$ minimal. Since M is a K-permutational module it is of the form

$$M := F/[F, F] = \bigoplus_{i \in I} M_i \tag{8}$$

with $M_i = \mathbf{Z}_p[(K/A_i) \times Z_i]$ for subgroups $A_i \leq K$ and some finite sets Z_i. Let R be finite subgroup of G. Note that $|R| = |RF \cap K|$ and M is also $RF \cap K$-permutational. Therefore, if $|R| < |K|$ then, by the minimality assumption on $|K|$, R is conjugate to $FR \cap K$ contradicting to the assumption. Therefore $RF = G$ so that $R \cong K$.

Fix $t \in \mathrm{socle}(R)$. Since M is a $\langle t \rangle$-permutation module, t is conjugate into K, and so we may assume $t \in \mathrm{socle}(K)$. Let $M = M_p \oplus M_1$ be the following Heller-Reiner decomposition for $\langle t \rangle$:

$$M_p := \bigoplus_{i \in I, t \notin A_i} M_i, \qquad M_1 := \bigoplus_{i \in I, t \in A_i} M_i.$$

By Lemma 2.3(i), $F = C_F(t) \amalg F_t$ for a suitable free pro-p group F_t. Corollary 2.5 implies that $C_F(t)[F, F]/[F, F]$ intersects M_p trivially and $\mathrm{rank}(C_F(t)) = \mathrm{rank}_{\mathbf{Z}_p} M_1$. The natural epimorphism from $C_F(t)$ to $C_F(t)[F, F]/[F, F]$ factors through the canonical K-module homomorphism from $C_F(t)/[C_F(t), C_F(t)]$ to $C_F(t)[F, F]/[F, F]$. Therefore, by the Krull-Schmidt theorem, $C_F(t)/[C_F(t), C_F(t)]$ and M_1 are isomorphic K-permutation modules. As a consequence, $C_G(t)/\langle t \rangle$ is a permutational extension by the minimality assumption on K and, therefore, so is $C_G(t)$. Since $R \leq C_G(t)$, we may conclude that R is conjugate into K by Remark 3.2. Since R was chosen arbitrary, we have that (ii) holds, a contradiction. □

5. Proof of the main theorems

In this section we shall use the notation and terminology of the theory of pro-p groups acting on pro-p trees from [14]. This will also be the main source of the references.

Theorem **5.1.** — *Let* G *be an infinite finitely generated virtually free pro-p group. Then* G *acts on a pro-p tree with finite vertex stabilizers.*

Proof. — By Theorem 3.4, G embeds into a group $\tilde{G} = E \rtimes G/F$ such that every finite subgroup of $\tilde{G}$ is conjugate to a subgroup of G/F and E is free pro-p.

By Theorem 4.9, $\tilde{G}$ is a permutational extension of E and so, by Remark 3.6, can be written as an HNN-extension $\mathrm{HNN}(G/F, A_i, Z_i, I)$ where the base group G/F and the associated groups in A_i are all finite. Thus $\tilde{G}$ acts on a pro-p tree T such that $T/\tilde{G}$ is a bouquet and all vertex stabilizers are finite (cf. [14, p. 89], for the situation of a single loop). □

Proof of Theorem 1.2. — By Theorem 5.1, G acts on a pro-p tree with finite vertex stabilizers. Since G is finitely generated, by [9, Theorem A], G splits as either a non-trivial amalgamated free pro-p product with finite amalgamating subgroup or a non-trivial HNN-extension with finite associated subgroups. □

Proof of Theorem 1.1. — Theorem 5.1 allows to deduce Theorem 1.1 from [9, Theorem A]. □

Combining Theorem 1.2, the main result in [9], and the main result of Hillman and Schmidt in [10] we can deduce that a pro-p group of positive deficiency having a finitely generated normal subgroup of infinite index splits into an amalgam or an HNN-extension. A pro-p group has *positive deficiency* if its minimal number of generators is greater than its number of relations, i.e. $\dim(H^1(G, \mathbf{F}_p)) - \dim(H^2(G, \mathbf{F}_p)) > 0$.

Corollary **5.2.** — *Let* G *be a finitely generated pro-p group of positive deficiency and* N *a nontrivial finitely generated normal subgroup of* G *of infinite index. Then*

(i) G *splits as an amalgamated free pro-p product or as an HNN-extension over a virtually free pro-p group.*
(ii) G *is the fundamental pro-p group of a finite graph of virtually free pro-p groups.*

Proof. — By the main result of [10] either N is procyclic and G/N is virtually free pro-p or N is virtually free pro-p and G/N is virtually procyclic. Thus (i) and (ii) follow from Theorem 1.2 and [9, Theorem A], respectively. □

We conclude this section with an example showing that the finite generation assumption on G in Theorem 1.2 is essential.

Example **5.3.** — Let A and B be groups of order 2 and $G_0 = \langle A \times B, t \mid tAt^{-1} = B\rangle$ be a pro-2 HNN extension of $A \times B$ with associated subgroups A and B. Note that G_0 admits an automorphism of order 2 that swaps A and B and inverts t. Let $G = G_0 \rtimes C$ be the holomorph. Set $H_0 = \langle \mathrm{Tor}(G_0)\rangle$ and $H = H_0 \rtimes C$. Since G_0 is virtually free pro-2, G and H are virtually free pro-2. The main result in [8] shows that H does not decompose as the fundamental pro-2 group of a profinite graph of finite 2-groups. It follows also from the proof in [8] that H does not split as a amalgamated free pro-2 product or a pro-2 HNN-extension over some finite subgroup.

6. Automorphisms

The following theorem is a consequence of Theorems 3.4 and 4.9:

Theorem **6.1.** — *Let* F_n *be a free pro-p group of finite rank n and* P *a finite p-group of automorphisms of* F. *Then there is an embedding of holomorphs* $F_n \rtimes P \longrightarrow F_m \rtimes P$ *such that* P *permutes the elements of some basis of the free pro-p group* F_m.

For a finite set X the canonical embedding of the discrete free group $\Phi(X)$ into its pro-p-completion $F(X)$ induces an embedding of $\mathrm{Aut}(\Phi(X))$ into $\mathrm{Aut}(F(X))$. This embedding is not dense [15]. The next theorem shows that nevertheless it induces a surjection (but not necessarily injection, cf. [3, Proposition 25]) on the conjugacy classes of finite groups.

Theorem **6.2.** — *Let* $F = F(X)$ *be a finitely generated free pro-p group and* $\Phi = \Phi(X)$ *be a dense abstract free subgroup of* F *on the same set of generators. Suppose that* $A \leq \mathrm{Aut}(F)$ *is a finite p-group. Then there exists an automorphism* $\beta \in \mathrm{Aut}(F)$ *such that the conjugate* A^β *is contained in* $\mathrm{Aut}(\Phi)$.

Proof. — Identifying F with its group of inner automorphisms, we may consider the holomorph $G := F \rtimes A$ as a subgroup of $\mathrm{Aut}(F)$. Since G is a finitely generated virtually free pro-p group, we may use [9, Theorem A] in order to present G as the fundamental pro-p group of a finite graph $(\mathcal{G}, \Gamma)$ of finite p-groups. By [20, Theorem 3.10], every finite subgroup of G is conjugate to a subgroup of a vertex group, so there exists $\beta_0 \in G$ with $A^{\beta_0} \in G(v)$ for some $v \in V(\Gamma)$. Let $\pi_1(\mathcal{G}, \Gamma)$ be the abstract fundamental group of the same graph of groups (cf. e.g., [2]), and set $\Phi_0 := \pi_1(\mathcal{G}, \Gamma) \cap F$. Choose a basis Y of Φ_0. Then Y is a basis of F(X), thus there exists $\alpha \in \mathrm{Aut}(F(X))$ sending X bijectively to Y. For $\beta := \beta_0\alpha^{-1}$, $A^\beta \leq \mathrm{Aut}(\Phi)$. □

Theorem **6.3.** — *Let* F *be a free pro-p group of rank n.*

(i) *The embedding* $\mathrm{Aut}(\Phi) \leq \mathrm{Aut}(F)$ *induces a surjection between the conjugacy classes of finite p-subgroups of* $\mathrm{Aut}(\Phi)$ *and* $\mathrm{Aut}(F)$.
(ii) *The* $\mathrm{Aut}(F)$*-conjugacy classes of finite subgroups of* $\mathrm{Aut}(F)$ *of order coprime to p are in one-to-one correspondence with* $\mathrm{Aut}(F/\Phi(F))$*-conjugacy classes of finite subgroups of* $\mathrm{Aut}(F/\Phi(F)) \cong GL_n(\mathbf{F}_p)$ *of order coprime to p.*

Proof. — Statement (i) is a consequence of Theorem 6.2.

We begin the proof of (ii) by defining a homomorphism $\lambda : \mathrm{Aut}(F) \to \mathrm{Aut}(F/\Phi(F))$ setting

$$\lambda(\alpha)\big(f\Phi(F)/\Phi(F)\big) := \alpha(f)\Phi(F)/\Phi(F).$$

By [13, Lemma 4.5.5], the kernel $K := \ker\lambda$ is a pro-p group. Moreover, λ is an epimorphism, since every automorphism $\alpha \in \mathrm{Aut}(F/\Phi(F))$ can be lifted to an automorphism of F (as a consequence of [13, Lemma 4.5.5]).

Let us first show that every p'-subgroup Q (i.e., coprime to p subgroup) of $\mathrm{Aut}(F/\Phi(F))$ is of the form $Q = \lambda(Q_0)$ for a suitable p'-subgroup Q_0 of Aut(F). Indeed, $\lambda^{-1}(Q)$ contains the normal p-Sylow subgroup K and, therefore, by the profinite version of the Schur-Zassenhaus theorem [13, 2.3.15], $\lambda^{-1}(Q)$ is a split extension of the pro-p group K by a p'-group Q_0, i.e., $\lambda^{-1}(Q) = K \rtimes Q_0$, and so $Q = \lambda(Q_0)$, as desired.

Next suppose that A and B are p'-subgroups of Aut(F) so that $\lambda(A)$ and $\lambda(B)$ are conjugate in $\mathrm{Aut}(F/\Phi(F))$. Then there exists $g \in F$ so that $A^gK = BK$. Now K is a closed normal p-Sylow subgroup of BK and $K \cap A^g = K \cap B = \{1\}$ shows that A^g and B are complements of K in BK. Therefore, again by [13, Theorem 2.3.15], they are conjugates in BK. Hence A and B are conjugate in G. □

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