HOMOLOGICAL PROJECTIVE DUALITY

by ALEXANDER KUZNETSOV⋆

ABSTRACT

We introduce a notion of homological projective duality for smooth algebraic varieties in dual projective spaces, a homological extension of the classical projective duality. If algebraic varieties X and Y in dual projective spaces are homologically projectively dual, then we prove that the orthogonal linear sections of X and Y admit semiorthogonal decompositions with an equivalent nontrivial component. In particular, it follows that triangulated categories of singularities of these sections are equivalent. We also investigate homological projective duality for projectivizations of vector bundles.

1. Introduction

Investigation of derived categories of coherent sheaves on algebraic varieties has become one of the most important topics in the modern algebraic geometry. Among other reasons, this is because of the Homological Mirror Symmetry conjecture of Maxim Kontsevich [Ko] predicting that there is an equivalence of categories between the derived category of coherent sheaves on a Calabi–Yau variety and the derived Fukaya category of its mirror. There is an extension of Mirror Symmetry to the non Calabi–Yau case [HV]. According to this, the mirror of a manifold with non-negative first Chern class is a so-called Landau–Ginzburg model, that is an algebraic variety with a 2-form and a holomorphic function (superpotential) such that the restriction of the 2-form to smooth fibers of the superpotential is symplectic. It is expected that singular fibers of the superpotential of the mirror Landau–Ginzburg model give a decomposition of the derived category of coherent sheaves on the initial algebraic variety into semiorthogonal pieces, a semiorthogonal decomposition.

Thus from the point of view of mirror symmetry it is important to investigate when the derived category of coherent sheaves on a variety admits a semiorthogonal decomposition. The goal of the present paper is to answer the following more precise question:

(†)   *Assume that* X *is a smooth projective variety and denote by* $\mathscr{D}^b(X)$ *the bounded derived category of coherent sheaves on* X. *Supposing that we are given a semiorthogonal decomposition of* $\mathscr{D}^b(X)$, *is it possible to construct a semiorthogonal decomposition of* $\mathscr{D}^b(X_H)$, *where* X_H *is a hyperplane section of* X?

Certainly this question is closely related to the question *what does the operation of taking a hyperplane section of a projective algebraic variety mean on the side of the mirror?*

In general one cannot expect an affirmative answer to (†). However, there is an important particular case, when something can be said. Explicitly, assume that $X \subset \mathbf{P}(V)$ is a smooth projective variety, $\mathscr{O}_X(1)$ is the corresponding very ample line bundle, and assume that there is a semiorthogonal decomposition of its derived category of the following type

$$\mathscr{D}^b(X) = \langle \mathscr{A}_0, \mathscr{A}_1(1), ..., \mathscr{A}_{i-1}(i-1) \rangle,$$
$$0 \subset \mathscr{A}_{i-1} \subset \mathscr{A}_{i-2} \subset \cdots \subset \mathscr{A}_1 \subset \mathscr{A}_0,$$

where (k) stands for the twist by $\mathscr{O}_X(k)$. A decomposition of this type will be called Lefschetz decomposition because as we will see its behavior with respect to hyperplane sections is similar to that of the Lefschetz decomposition of the cohomology groups. An easy calculation shows that for any hyperplane section X_H of X with respect to $\mathscr{O}_X(1)$ the composition of the embedding and the restriction functors $\mathscr{A}_k(k) \to \mathscr{D}^b(X) \to \mathscr{D}^b(X_H)$ is fully faithful for $1 \le k \le i-1$ and $\langle \mathscr{A}_1(1), ..., \mathscr{A}_{i-1}(i-1) \rangle$ is a semiorthogonal collection in $\mathscr{D}^b(X_H)$. In other words, dropping the first (the biggest) component of the Lefschetz decomposition of $\mathscr{D}^b(X)$ we obtain a semiorthogonal collection in $\mathscr{D}^b(X_H)$. Denoting by $\mathscr{C}_H$ the orthogonal in $\mathscr{D}^b(X_H)$ to the subcategory of $\mathscr{D}^b(X_H)$ generated by this collection we consider $\{\mathscr{C}_H\}_{H \in \mathbf{P}(V^*)}$ as a family of triangulated categories over the projective space $\mathbf{P}(V^*)$. Assuming geometricity of this family, i.e. roughly speaking that there exists an algebraic variety Y with a map $Y \to \mathbf{P}(V^*)$ such that for all H we have $\mathscr{C}_H \cong \mathscr{D}^b(Y_H)$, where Y_H is the fiber of Y over $H \in \mathbf{P}(V^*)$, we prove the main result of the paper

Theorem **1.1.** — *The derived category of* Y *admits a dual Lefschetz decomposition*

$$\mathscr{D}^b(Y) = \langle \mathscr{B}_{j-1}(1-j), \mathscr{B}_{j-2}(2-j), ..., \mathscr{B}_1(-1), \mathscr{B}_0 \rangle,$$
$$0 \subset \mathscr{B}_{j-1} \subset \mathscr{B}_{j-2} \subset \cdots \subset \mathscr{B}_1 \subset \mathscr{B}_0.$$

Moreover, if $L \subset V^*$ *is a linear subspace and* $L^\perp \subset V$ *is the orthogonal subspace such that the linear sections* $X_L = X \times_{\mathbf{P}(V)} \mathbf{P}(L^\perp)$ *and* $Y_L = Y \times_{\mathbf{P}(V^*)} \mathbf{P}(L)$, *are of expected dimension* $\dim X_L = \dim X - \dim L$, *and* $\dim Y_L = \dim Y - \dim L^\perp$, *then there exists a triangulated category* $\mathscr{C}_L$ *and semiorthogonal decompositions*

$$\mathscr{D}^b(X_L) = \langle \mathscr{C}_L, \mathscr{A}_{\dim L}(1), ..., \mathscr{A}_{i-1}(i-\dim L) \rangle,$$
$$\mathscr{D}^b(Y_L) = \langle \mathscr{B}_{j-1}(\dim L^\perp - j), ..., \mathscr{B}_{\dim L^\perp}(-1), \mathscr{C}_L \rangle.$$

In other words, the derived categories of X_L and Y_L have semiorthogonal decompositions with several "trivial" components coming from the Lefschetz decompositions of the ambient varieties X and Y respectively, and with *equivalent* nontrivial components.

We would like to emphasize the similarity of derived categories and cohomology groups with respect to the hyperplane section operation. Thus, Theorem 1.1 can be considered as a homological generalization of the Lefschetz theorem about hyperplane sections.

A simple corollary of Theorem 1.1 is an equivalence of the derived categories of singularities (see [O3]) of X_L and Y_L. In particular, it easily follows that Y_L is singular if and only if X_L is singular. This means that we have an equality of the following two closed subsets of the dual projective space $\mathbf{P}(V^*)$:

$$\{H \in \mathbf{P}(V^*) \mid X_H \text{ is singular}\}$$
$$= \{\text{critical values of the projection } Y \to \mathbf{P}(V^*)\}.$$

Note that the first of these subsets is the classical projectively dual variety of X. Thus Y can be considered as a homological generalization of the projectively dual. In accordance with this we say that Y is a **homologically projectively dual** variety of X.

The simplest example of a Lefschetz decomposition is given by the standard exceptional collection $(\mathscr{O}, \mathscr{O}(1), ..., \mathscr{O}(i-1))$ on a projective space $X = \mathbf{P}^{i-1}$ (we take $\mathscr{A}_0 = \mathscr{A}_1 = \cdots = \mathscr{A}_{i-1} = \langle \mathscr{O} \rangle$). It is easy to see that the corresponding homological projectively dual variety is an empty set, and we obtain nothing interesting. However, considering a *relative* projective space we already obtain some interesting results. More precisely, consider a projectivization of a vector bundle $X = \mathbf{P}_S(E)$ over a base scheme S, embedded into the projectivization of the vector space $H^0(S, E^*)^* = H^0(X, \mathscr{O}_{X/S}(1))^*$ with the following Lefschetz decomposition

$$\mathscr{D}^b(X) = \langle \mathscr{D}^b(S), \mathscr{D}^b(S) \otimes \mathscr{O}_{X/S}(1), ..., \mathscr{D}^b(S) \otimes \mathscr{O}_{X/S}(i-1) \rangle.$$

We prove that $Y = \mathbf{P}_S(E^\perp)$, where $E^\perp = \mathsf{Ker}(H^0(S, E^*) \otimes \mathscr{O}_S \to E^*)$, is a homologically projectively dual variety of X. As a consequence we get certain semiorthogonal decompositions and equivalences between derived categories of linear sections of $\mathbf{P}_S(E)$ and $\mathbf{P}_S(E^\perp)$. For example, applying a relative version of Theorem 1.1 we can deduce that there is an equivalence of derived categories between the following two varieties related by a special birational transformation called a *flop* (it is conjectured in [BO2] that the derived categories of any pair of algebraic varieties related by a flop are equivalent). Consider a morphism of vector bundles $F \xrightarrow{\phi} E^*$ of equal ranks on S and consider

$$X_F = \{(s, e) \in \mathbf{P}_S(E) \mid \phi_s^*(e) = 0\}, \quad Y_F = \{(s, f) \in \mathbf{P}_S(F) \mid \phi_s(f) = 0\},$$
$$\text{and } Z_F = \{s \in S \mid \det \phi_s = 0\}.$$

If $\dim X_F = \dim Y_F = \dim S - 1$ then the natural projections $X_F \to Z_F$ and $Y_F \to Z_F$ are birational and the corresponding birational transformation $X_F \dashrightarrow Y_F$

is a flop. We prove an equivalence of categories $\mathscr{D}^b(X_F) \cong \mathscr{D}^b(Y_F)$ if additionally $\dim X_F \times_S Y_F = \dim S - 1$.

The next example of a Lefschetz decomposition is a decomposition of $\mathscr{D}^b(X)$ for $X = \mathbf{P}(W)$ with respect to $\mathscr{O}_X(2)$ given by $\mathscr{A}_0 = \mathscr{A}_1 = \cdots = \mathscr{A}_{i-2} = \langle \mathscr{O}_X, \mathscr{O}_X(1) \rangle$ and either $\mathscr{A}_{i-1} = \langle \mathscr{O}_X, \mathscr{O}_X(-1) \rangle$ for $\dim W = 2i$, or $\mathscr{A}_{i-1} = \langle \mathscr{O}_X \rangle$ for $\dim W = 2i - 1$. In a companion paper [K2] we show that the universal sheaf of even parts of Clifford algebras on $\mathbf{P}(S^2W^*)$ is a homologically projectively dual variety to X with respect to the double Veronese embedding $X = \mathbf{P}(W) \subset \mathbf{P}(S^2W)$. This gives immediately a proof of the theorem of Bondal and Orlov [BO2,BO3] about derived categories of intersections of quadrics.

Let us mention also that the homological projective duality for Lefschetz decompositions with $\mathscr{A}_0$ generated by exceptional pair and $\mathscr{A}_0 = \mathscr{A}_1 = \cdots = \mathscr{A}_{i-1}$ was considered in [K1]. There such decompositions were constructed for $X = \mathsf{Gr}(2,5)$, $X = \mathsf{OGr}_+(5,10)$, a connected component of the Grassmannian of 5-dimensional subspaces in k^{10} isotropic with respect to a nondegenerate quadratic form, $X = \mathsf{LGr}(3,6)$, the Lagrangian Grassmannian of 3-dimensional subspaces in k^6 with respect to a symplectic form, and $X = \mathsf{G}_2\mathsf{Gr}(2,7)$, the Grassmannian of the Lie group G_2, and it was shown that homologically projectively dual varieties for them are $Y = \mathsf{Gr}(2,5)$, $Y = \mathsf{OGr}_-(5,10)$, a quartic hypersurface in $\mathbf{P}^{13}$, and a double covering of $\mathbf{P}^{13}$ ramified in a sextic hypersurface (in the last two cases one must consider the derived category of sheaves of modules over a suitable sheaf of Azumaya algebras on Y instead of the usual derived category). Moreover, in a forthcoming paper [K4] we are going to describe the homologically projectively dual varieties to Grassmannians of lines $\mathsf{Gr}(2,W)$ (the Lefschetz decompositions for these Grassmannians were constructed in [K3]).

Finally, we would like to emphasize that aside of its purely theoretical interest homological projective duality provides a powerful tool for investigation of derived categories of linear sections of a given algebraic variety. It was already applied in [K1] for the description of derived categories of some Fano threefolds. Having in mind the role played in Mirror Symmetry by complete intersections in toric varieties it seems a good idea to investigate the homological projective duality for toric varieties. This also may shed some light on the relation of homological projective duality and Mirror Symmetry.

Now we describe the structure of the paper. In Section 2 we recall the necessary material concerning admissible subcategories, semiorthogonal decompositions, mention an important technical result, the faithful base change theorem proved in [K1], and check that the property of being fully faithful for a functor linear over a base is local over the base. In Section 3 we define *splitting functors* and give a criterion for a functor to be splitting. In Section 4 we define *Lefschetz decompositions* of triangulated categories. In Section 5 we consider derived category

of the *universal hyperplane section* of a variety admitting a Lefschetz decomposition of the derived category. In Section 6 we define *homological projective duality* and prove Theorem 1.1 and its relative versions. In Section 7 we discuss relation of the homological projective duality to the classical projective duality. In Section 8 we consider the homological projective duality for a projectivization of a vector bundle. Finally, in Section 9 we consider some explicit examples of homological projective duality.

2. Preliminaries

2.1. *Notation.* — The base field k is assumed to be algebraically closed of zero characteristic. All algebraic varieties are assumed to be embeddable (i.e. admitting a finite morphism to a smooth algebraic variety) and of finite type over k.

Given an algebraic variety X we denote by $\mathscr{D}^b(\mathrm{X})$ the bounded derived category of coherent sheaves on X. Similarly, $\mathscr{D}^-(\mathrm{X})$, $\mathscr{D}^+(\mathrm{X})$ and $\mathscr{D}(\mathrm{X})$ stand for the bounded above, the bounded below and the unbounded derived categories. Further, $\mathscr{D}^b_{qc}(\mathrm{X})$, $\mathscr{D}^-_{qc}(\mathrm{X})$, $\mathscr{D}^+_{qc}(\mathrm{X})$, and $\mathscr{D}_{qc}(\mathrm{X})$, stand for the corresponding derived categories of quasicoherent sheaves, and $\mathscr{D}^{\mathsf{perf}}(\mathrm{X})$ denotes the category of perfect complexes on X, i.e. the full subcategory of $\mathscr{D}(\mathrm{X})$ consisting of all objects locally quasiisomorphic to bounded complexes of locally free sheaves of finite rank.

Given a morphism $f : \mathrm{X} \to \mathrm{Y}$ we denote by f_* and f^* the *total* derived pushforward and the *total* derived pullback functors. The twisted pullback functor [H] is denoted by $f^!$ (it is right adjoint to f_* if f is proper). Similarly, $\otimes$ stands for the derived tensor product, and RHom, $\mathsf{R}\mathscr{H}om$ stand for the global and local RHom functors.

Given an object $\mathrm{F} \in \mathscr{D}(\mathrm{X})$ we denote by $\mathscr{H}^k(\mathrm{F})$ the k-th cohomology sheaf of F.

2.2. *Semiorthogonal decompositions.* — If $\mathscr{A}$ is a full subcategory of $\mathscr{T}$ then the right orthogonal to $\mathscr{A}$ in $\mathscr{T}$ (resp. the left orthogonal to $\mathscr{A}$ in $\mathscr{T}$) is the full subcategory $\mathscr{A}^\perp$ (resp. $^\perp\mathscr{A}$) consisting of all objects $\mathrm{T} \in \mathscr{T}$ such that $\mathsf{Hom}_{\mathscr{T}}(\mathrm{A}, \mathrm{T}) = 0$ (resp. $\mathsf{Hom}_{\mathscr{T}}(\mathrm{T}, \mathrm{A}) = 0$) for all $\mathrm{A} \in \mathscr{A}$.

Definition **2.1** ([BO1]). — *A semiorthogonal decomposition of a triangulated category $\mathscr{T}$ is a sequence of full triangulated subcategories $\mathscr{A}_1, ..., \mathscr{A}_n$ in $\mathscr{T}$ such that $\mathsf{Hom}_{\mathscr{T}}(\mathscr{A}_i, \mathscr{A}_j) = 0$*

for $i > j$ and for every object $T \in \mathcal{T}$ there exists a chain of morphisms $0 = T_n \to T_{n-1} \to \cdots \to T_1 \to T_0 = T$ such that the cone of the morphism $T_k \to T_{k-1}$ is contained in $\mathcal{A}_k$ for each $k = 1, 2, ..., n$.

In other words, every object T admits a decreasing "filtration" with factors in $\mathcal{A}_1, ..., \mathcal{A}_n$ respectively. Semiorthogonality implies that this filtration is unique and functorial.

For any sequence of subcategories $\mathcal{A}_1, ..., \mathcal{A}_n$ in $\mathcal{T}$ we denote by $\langle \mathcal{A}_1, ..., \mathcal{A}_n \rangle$ the minimal triangulated subcategory of $\mathcal{T}$ containing $\mathcal{A}_1, ..., \mathcal{A}_n$.

If $\mathcal{T} = \langle \mathcal{A}_1, ..., \mathcal{A}_n \rangle$ is a semiorthogonal decomposition then $\mathcal{A}_i = \langle \mathcal{A}_{i+1}, ..., \mathcal{A}_n \rangle^{\perp} \cap {}^{\perp}\langle \mathcal{A}_1, ..., \mathcal{A}_{i-1} \rangle$.

Definition **2.2** ([BK, B]). — *A full triangulated subcategory $\mathcal{A}$ of a triangulated category $\mathcal{T}$ is called* right admissible *if for the inclusion functor $i : \mathcal{A} \to \mathcal{T}$ there is a right adjoint $i^! : \mathcal{T} \to \mathcal{A}$, and* left admissible *if there is a left adjoint $i^* : \mathcal{T} \to \mathcal{A}$. Subcategory $\mathcal{A}$ is called* admissible *if it is both right and left admissible.*

Lemma **2.3** ([B]). — *If $\mathcal{T} = \langle \mathcal{A}, \mathcal{B} \rangle$ is a semiorthogonal decomposition then $\mathcal{A}$ is left admissible and $\mathcal{B}$ is right admissible.*

Lemma **2.4** ([B]). — *If $\mathcal{A}_1, ..., \mathcal{A}_n$ is a semiorthogonal sequence of full triangulated subcategories in a triangulated category $\mathcal{T}$ (i.e. $\mathsf{Hom}_{\mathcal{T}}(\mathcal{A}_i, \mathcal{A}_j) = 0$ for $i > j$) such that $\mathcal{A}_1, ..., \mathcal{A}_k$ are left admissible and $\mathcal{A}_{k+1}, ..., \mathcal{A}_n$ are right admissible then*

$$\langle \mathcal{A}_1, ..., \mathcal{A}_k, {}^{\perp}\langle \mathcal{A}_1, ..., \mathcal{A}_k \rangle \cap \langle \mathcal{A}_{k+1}, ..., \mathcal{A}_n \rangle^{\perp}, \mathcal{A}_{k+1}, ..., \mathcal{A}_n \rangle$$

is a semiorthogonal decomposition.

Assume that $\mathcal{A} \subset \mathcal{T}$ is an admissible subcategory. Then $\mathcal{T} = \langle \mathcal{A}, {}^{\perp}\mathcal{A} \rangle$ and $\mathcal{T} = \langle \mathcal{A}^{\perp}, \mathcal{A} \rangle$ are semiorthogonal decompositions, hence ${}^{\perp}\mathcal{A}$ is right admissible and $\mathcal{A}^{\perp}$ is left admissible. Let $i_{{}^{\perp}\mathcal{A}} : {}^{\perp}\mathcal{A} \to \mathcal{T}$ and $i_{\mathcal{A}^{\perp}} : \mathcal{A}^{\perp} \to \mathcal{T}$ be the inclusion functors.

Definition **2.5** ([B]). — *The functor $R_{\mathcal{A}} = i_{{}^{\perp}\mathcal{A}} i^!_{{}^{\perp}\mathcal{A}}$ is called the* right mutation *through $\mathcal{A}$. The functor $L_{\mathcal{A}} = i_{\mathcal{A}^{\perp}} i^*_{\mathcal{A}^{\perp}}$ is called the* left mutation *through $\mathcal{A}$.*

Lemma **2.6** ([B]). — *We have $R_{\mathcal{A}}(\mathcal{A}) = 0$ and the restriction of $R_{\mathcal{A}}$ to $\mathcal{A}^{\perp}$ is an equivalence $\mathcal{A}^{\perp} \to {}^{\perp}\mathcal{A}$. Similarly, we have $L_{\mathcal{A}}(\mathcal{A}) = 0$ and the restriction of $L_{\mathcal{A}}$ to ${}^{\perp}\mathcal{A}$ is an equivalence ${}^{\perp}\mathcal{A} \to \mathcal{A}^{\perp}$.*

Lemma **2.7** ([B]). — *If $\mathcal{A}_1, ..., \mathcal{A}_n$ is a semiorthogonal sequence of admissible subcategories in $\mathcal{T}$ then $R_{\langle \mathcal{A}_1, ..., \mathcal{A}_n \rangle} = R_{\mathcal{A}_n} \circ \cdots \circ R_{\mathcal{A}_1}$ and $L_{\langle \mathcal{A}_1, ..., \mathcal{A}_n \rangle} = L_{\mathcal{A}_1} \circ \cdots \circ L_{\mathcal{A}_n}$.*

Lemma **2.8** ([O1]). — *If* E *is a vector bundle of rank* r *on* S, $\mathbf{P}_S(E)$ *is its projectivization,* $\mathcal{O}(1)$ *is the corresponding Grothendieck ample line bundle, and* $p : \mathbf{P}_S(E) \to$ S *is the projection then the pullback* $p^* : \mathcal{D}^b(S) \to \mathcal{D}^b(\mathbf{P}_S(E))$ *is fully faithful and*

$$\mathcal{D}^b(\mathbf{P}_S(E)) = \langle p^*(\mathcal{D}^b(S)) \otimes \mathcal{O}(k), p^*(\mathcal{D}^b(S)) \otimes \mathcal{O}(k+1),$$
$$..., p^*(\mathcal{D}^b(S)) \otimes \mathcal{O}(k+r-1)\rangle$$

is a semiorthogonal decomposition for any $k \in \mathbf{Z}$.

2.3. *Saturatedness and Serre functors*

Définition **2.9** ([B]). — *A triangulated category* $\mathcal{T}$ *is called* left saturated *if every exact covariant functor* $\mathcal{T} \to \mathcal{D}^b(\mathsf{k})$ *is representable, and* right saturated *if every exact contravariant functor* $\mathcal{T} \to \mathcal{D}^b(\mathsf{k})$ *is representable. A triangulated category* $\mathcal{T}$ *is called* saturated *if it is both left and right saturated.*

Lemma **2.10** ([B]). — *A left (resp. right) admissible subcategory of a saturated category is saturated.*

Proof. — Assume that $\mathcal{A}$ is a left admissible subcategory in a saturated triangulated category $\mathcal{T}$, $i : \mathcal{A} \to \mathcal{T}$ is the inclusion functor and $i^* : \mathcal{T} \to \mathcal{A}$ is its left adjoint functor. Let $\phi : \mathcal{A} \to \mathcal{D}^b(\mathsf{k})$ be an exact covariant functor. Then $\phi \circ i^* : \mathcal{T} \to \mathcal{D}^b(\mathsf{k})$ is representable since $\mathcal{T}$ is saturated. Therefore there exists $T \in \mathcal{T}$ such that $\phi \circ i^* \cong \mathsf{Hom}_{\mathcal{T}}(T, -)$. Then

$$\phi \cong \phi \circ i^* \circ i \cong \mathsf{Hom}_{\mathcal{T}}(T, i(-)) \cong \mathsf{Hom}_{\mathcal{A}}(i^*T, -),$$

therefore i^*T represents ϕ.

Let $\psi : \mathcal{A} \to \mathcal{D}^b(\mathsf{k})$ be an exact contravariant functor. Then $\psi \circ i^* : \mathcal{T} \to \mathcal{D}^b(\mathsf{k})$ is representable since $\mathcal{T}$ is saturated. Therefore there exists $T \in \mathcal{T}$ such that $\psi \circ i^* \cong \mathsf{Hom}_{\mathcal{T}}(-, T)$. Note that $i^*(^\perp\mathcal{A}) = 0$, hence $\mathsf{Hom}_{\mathcal{T}}(^\perp\mathcal{A}, T) = 0$ which means that $T \in (^\perp\mathcal{A})^\perp = \mathcal{A}$, thus $T \cong i(A)$ with $A \in \mathcal{A}$. Finally

$$\psi \cong \psi \circ i^* \circ i \cong \mathsf{Hom}_{\mathcal{T}}(i(-), i(A)) \cong \mathsf{Hom}_{\mathcal{A}}(-, A),$$

since i is fully faithful, therefore A represents ψ.

A similar argument works for right admissible subcategories. $\square$

Lemma **2.11** ([B]). — *If* $\mathcal{A}$ *is saturated then* $\mathcal{A}$ *is admissible.*

Proof. — For any object $T \in \mathcal{T}$ consider the functor $\mathsf{Hom}_{\mathcal{T}}(T, i(-)) : \mathcal{A} \to \mathcal{D}^b(\mathsf{k})$. Since $\mathcal{A}$ is saturated there exists $A_T \in \mathcal{A}$, such that this functor is isomorphic to $\mathsf{Hom}_{\mathcal{A}}(A_T, -)$. Since $\mathsf{Hom}_{\mathcal{T}}(T, i(A_T)) \cong \mathsf{Hom}_{\mathcal{A}}(A_T, A_T)$ we have

a canonical morphism $T \to i(A_T)$ and since i is fully faithful it is easy to see that its cone is contained in $^\perp \mathscr{A}$. It follows that any morphism $T \to S$ composed with $S \to i(A_S)$ factors in a unique way as $T \to i(A_T) \to i(A_S)$. Since $\mathsf{Hom}_{\mathscr{T}}(i(A_T), i(A_S)) \cong \mathsf{Hom}_{\mathscr{A}}(A_T, A_S)$ the correspondence $T \mapsto A_T$ is a functor $\mathscr{T} \to \mathscr{A}$, left adjoint to $i : \mathscr{A} \to \mathscr{T}$. Similarly one can construct a right adjoint functor. □

Lemma **2.12** ([BV]). — *If* X *is a smooth projective variety then* $\mathscr{D}^b(\mathrm{X})$ *is saturated.*

Corollary **2.13.** — *If* X *is a smooth projective variety and* $\mathscr{A}$ *is a left* (*resp. right*) *admissible subcategory in* $\mathscr{D}^b(\mathrm{X})$ *then* $\mathscr{A}$ *is saturated.*

Définition **2.14** ([BK], [BO4]). — *Let* $\mathscr{T}$ *be a triangulated category. A covariant additive functor* $\mathsf{S} : \mathscr{T} \to \mathscr{T}$ *is a* Serre functor *if it is a category equivalence and for all objects* $\mathrm{F}, \mathrm{G} \in \mathscr{T}$ *there are given bi-functorial isomorphisms* $\mathsf{Hom}(\mathrm{F}, \mathrm{G}) \to \mathsf{Hom}(\mathrm{G}, \mathsf{S}(\mathrm{F}))^*$.

Lemma **2.15** ([BK]). — *If a Serre functor exists then it is unique up to a canonical functorial isomorphism. If* X *is a smooth projective variety then* $\mathsf{S}(\mathrm{F}) := \mathrm{F} \otimes \omega_\mathrm{X}[\dim \mathrm{X}]$ *is a Serre functor in* $\mathscr{D}^b(\mathrm{X})$.

Définition **2.16** ([BV]). — *A triangulated category* $\mathscr{T}$ *is called* Ext-*finite if for any objects* $\mathrm{F}, \mathrm{G} \in \mathscr{T}$ *the vector space* $\oplus_{n \in \mathbf{Z}} \mathsf{Hom}_{\mathscr{T}}(\mathrm{F}, \mathrm{G}[n])$ *is finite dimensional.*

Lemma **2.17** ([BK]). — *If* $\mathscr{T}$ *is an* Ext-*finite saturated category then* $\mathscr{T}$ *admits a Serre functor.*

Lemma **2.18** ([BK]). — *If* S *is a Serre functor for* $\mathscr{T}$ *and* $\mathscr{A}$ *is a subcategory of* $\mathscr{T}$ *then* $\mathsf{S}(^\perp \mathscr{A}) = \mathscr{A}^\perp$ *and* $\mathsf{S}^{-1}(\mathscr{A}^\perp) = {}^\perp \mathscr{A}$. *In particular, if* $\mathscr{T} = \langle \mathscr{A}_1, \mathscr{A}_2 \rangle$ *is a semiorthogonal decomposition then* $\mathscr{T} = \langle \mathsf{S}(\mathscr{A}_2), \mathscr{A}_1 \rangle$ *and* $\mathscr{T} = \langle \mathscr{A}_2, \mathsf{S}^{-1}(\mathscr{A}_1) \rangle$ *are semiorthogonal decompositions.*

Lemma **2.19** ([B]). — *If* $\mathscr{T}$ *admits a Serre functor* S *and* $\mathscr{A} \subset \mathscr{T}$ *is right admissible then* $\mathscr{A}$ *admits a Serre functor* $\mathsf{S}_{\mathscr{A}} = i^! \circ \mathsf{S} \circ i$, *where* $i : \mathscr{A} \to \mathscr{T}$ *is the inclusion functor.*

Proof. — If $\mathrm{A}, \mathrm{A}' \in \mathscr{A}$ then $\mathsf{Hom}_{\mathscr{A}}(\mathrm{A}, i^! \mathsf{S} i \mathrm{A}') \cong \mathsf{Hom}_{\mathscr{T}}(i\mathrm{A}, \mathsf{S} i \mathrm{A}') \cong \mathsf{Hom}_{\mathscr{T}}(i\mathrm{A}', i\mathrm{A})^* \cong \mathsf{Hom}_{\mathscr{A}}(\mathrm{A}', \mathrm{A})^*$. □

2.4. Tor *and* Ext-*amplitude.* — Let $f : \mathrm{X} \to \mathrm{Y}$ be a morphism of algebraic varieties. For any subset $\mathrm{I} \subset \mathbf{Z}$ we denote by $\mathscr{D}^\mathrm{I}(\mathrm{X})$ the full subcategory of $\mathscr{D}(\mathrm{X})$ consisting of all objects $\mathrm{F} \in \mathscr{D}(\mathrm{X})$ with $\mathscr{H}^k(\mathrm{F}) = 0$ for $k \notin \mathrm{I}$.

Definition **2.20** ([K1]). — *An object* $F \in \mathscr{D}(X)$ *has* finite Tor-amplitude *over* Y (*resp.* finite Ext-amplitude *over* Y), *if there exist integers* p, q *such that for any object* $G \in \mathscr{D}^{[s,t]}(Y)$ *we have* $F \otimes f^*G \in \mathscr{D}^{[p+s,q+t]}(X)$ (*resp.* $\mathsf{R}\mathscr{H}om(F, f^!G) \in \mathscr{D}^{[p+s,q+t]}(X)$). *Morphism* f *has* finite Tor-dimension, (*resp.* finite Ext-dimension), *if the sheaf* $\mathscr{O}_X$ *has finite* Tor-*amplitude over* Y (*resp. finite* Ext-*amplitude over* Y).

The full subcategory of $\mathscr{D}(X)$ consisting of objects of finite Tor-amplitude (resp. of finite Ext-amplitude) over Y is denoted by $\mathscr{D}_{f\mathrm{T}d/Y}(X)$ (resp. $\mathscr{D}_{f\mathrm{E}d/Y}(X)$). Both are triangulated subcategories of $\mathscr{D}^b(X)$.

The following results (Lemmas 2.21–2.25 and 2.27–2.30 below) are well known in folklore. A useful reference where all of them can be found in a compact form is [K1].

Lemma **2.21.** — *If* $i : X \to X'$ *is a finite morphism over* Y *then* $F \in \mathscr{D}_{f\mathrm{T}d/Y}(X)$ $\Leftrightarrow i_*F \in \mathscr{D}_{f\mathrm{T}d/Y}(X')$ *and* $F \in \mathscr{D}_{f\mathrm{E}d/Y}(X) \Leftrightarrow i_*F \in \mathscr{D}_{f\mathrm{E}d/Y}(X')$.

Lemma **2.22.** — *If morphism* $f : X \to Y$ *has finite* Tor-*dimension* (*resp.* Ext-*dimension*) *then any perfect complex on* X *has finite* Tor-*amplitude* (*resp.* Ext-*amplitude*) *over* Y.

Lemma **2.23.** — *If* $f : X \to Y$ *is a smooth morphism then* $\mathscr{D}_{f\mathrm{T}d/Y}(X) = \mathscr{D}^{\mathsf{perf}}(X) = \mathscr{D}_{f\mathrm{E}d/Y}(X)$.

Definition **2.24.** — *A triangulated category* $\mathscr{T}$ *is* Ext-bounded, *if for any objects* $F, G \in \mathscr{T}$ *the set* $\{n \in \mathbf{Z} \mid \mathsf{Hom}(F, G[n]) \neq 0\}$ *is finite.*

Lemma **2.25.** — *The following conditions for an algebraic variety* X *are equivalent:*

(*i*) X *is smooth;*
(*ii*) $\mathscr{D}^b(X) = \mathscr{D}^{\mathsf{perf}}(X)$;
(*iii*) *the bounded derived category* $\mathscr{D}^b(X)$ *is* Ext-*bounded.*

Lemma **2.26.** — *Assume that* $\mathscr{T} = \langle \mathscr{A}, \mathscr{B} \rangle$ *is a semiorthogonal decomposition. If both* $\mathscr{A}$ *and* $\mathscr{B}$ *are* Ext-bounded *and either* $\mathscr{A}$ *or* $\mathscr{B}$ *is admissible then* $\mathscr{T}$ *is* Ext-*bounded.*

Proof. — Let $F, G \in \mathscr{T}$. Then there exist exact triangles

$$\beta\beta^! F \to F \to \alpha\alpha^* F, \quad \beta\beta^! G \to G \to \alpha\alpha^* G.$$

Computing $\mathsf{Hom}(F, G[n])$ and using semiorthogonality of $\mathscr{A}$ and $\mathscr{B}$ we obtain a long exact sequence

$$\cdots \to \mathsf{Hom}(\alpha\alpha^* F, \beta\beta^! G[n]) \to \mathsf{Hom}(F, G[n])$$
$$\to \mathsf{Hom}(\beta^! F, \beta^! G[n]) \oplus \mathsf{Hom}(\alpha^* F, \alpha^* G[n]) \to \cdots$$

Since $\mathscr{A}$ and $\mathscr{B}$ are Ext-bounded, the third term vanishes for $|n| \gg 0$. On the other hand, if $\mathscr{A}$ is admissible then $\mathsf{Hom}(\alpha\alpha^*\mathrm{F}, \beta\beta^!\mathrm{G}[n]) \cong \mathsf{Hom}(\alpha^*\mathrm{F}, \alpha^!\beta\beta^!\mathrm{G}[n])$, hence the first term also vanishes for $|n| \gg 0$. Similarly, if $\mathscr{B}$ is admissible then $\mathsf{Hom}(\alpha\alpha^*\mathrm{F}, \beta\beta^!\mathrm{G}[n]) \cong \mathsf{Hom}(\beta^*\alpha\alpha^*\mathrm{F}, \beta^!\mathrm{G}[n])$, hence the first term also vanishes for $|n| \gg 0$. In both cases we deduce that $\mathsf{Hom}(\mathrm{F}, \mathrm{G}[n])$ vanishes for $|n| \gg 0$, hence $\mathscr{T}$ is Ext-bounded. $\qquad\square$

2.5. *Kernel functors.* — Let X_1, X_2 be algebraic varieties and let $p_i : \mathrm{X}_1 \times \mathrm{X}_2 \to \mathrm{X}_i$ denote the projections. Take any $\mathrm{K} \in \mathscr{D}_{qc}^-(\mathrm{X}_1 \times \mathrm{X}_2)$ and define functors

$$\Phi_{\mathrm{K}}(\mathrm{F}_1) := p_{2_*}(p_1^*\mathrm{F}_1 \otimes \mathrm{K}), \quad \Phi_{\mathrm{K}}^!(\mathrm{F}_2) := p_{1_*}\,\mathsf{R}\mathscr{H}\!om(\mathrm{K}, p_2^!\mathrm{F}_2).$$

Then Φ_{K} is an exact functor $\mathscr{D}_{qc}^-(\mathrm{X}_1) \to \mathscr{D}_{qc}^-(\mathrm{X}_2)$ and $\Phi_{\mathrm{K}}^!$ is an exact functor $\mathscr{D}_{qc}^+(\mathrm{X}_2) \to \mathscr{D}_{qc}^+(\mathrm{X}_1)$. We call Φ_{K} the kernel functor with kernel K, and $\Phi_{\mathrm{K}}^!$ the kernel functor of the second type with kernel K (cf. [K1]). In smooth case any kernel functor of the second type is isomorphic to a usual kernel functor: $\Phi_{\mathrm{K}}^! \cong \Phi_{\mathsf{R}\mathscr{H}\!om(\mathrm{K},\omega_{\mathrm{X}_1}[\dim \mathrm{X}_1])}$.

> *Lemma* **2.27.** — (*i*) If K *has coherent cohomologies, finite* Tor-*amplitude over* X_1 *and* $\mathsf{supp}(\mathrm{K})$ *is projective over* X_2 *then* Φ_{K} *takes* $\mathscr{D}^b(\mathrm{X}_1)$ *to* $\mathscr{D}^b(\mathrm{X}_2)$.
> (*ii*) If K *has coherent cohomologies, finite* Ext-*amplitude over* X_2 *and* $\mathsf{supp}(\mathrm{K})$ *is projective over* X_1 *then* $\Phi_{\mathrm{K}}^!$ *takes* $\mathscr{D}^b(\mathrm{X}_2)$ *to* $\mathscr{D}^b(\mathrm{X}_1)$.
> (*iii*) If both (*i*) and (*ii*) hold then $\Phi_{\mathrm{K}}^!$ is right adjoint to Φ_{K}. Moreover, Φ_{K} takes $\mathscr{D}^{\mathsf{perf}}(\mathrm{X}_1)$ to $\mathscr{D}^{\mathsf{perf}}(\mathrm{X}_2)$.

> *Lemma* **2.28.** — *If* K *is a perfect complex,* X_2 *is smooth and* $\mathsf{supp}(\mathrm{K})$ *is projective both over* X_1 *and over* X_2*, then the functor* Φ_{K} *admits a left adjoint functor* Φ_{K}^* *which is isomorphic to a kernel functor* $\Phi_{\mathrm{K}^\#}$ *with the kernel*

$$\mathrm{K}^\# := \mathsf{R}\mathscr{H}\!om(\mathrm{K}, \omega_{\mathrm{X}_2}[\dim \mathrm{X}_2]).$$

Consider kernels $\mathrm{K}_{12} \in \mathscr{D}^-(\mathrm{X}_1 \times \mathrm{X}_2)$, $\mathrm{K}_{23} \in \mathscr{D}^-(\mathrm{X}_2 \times \mathrm{X}_3)$. Denote by $p_{ij} : \mathrm{X}_1 \times \mathrm{X}_2 \times \mathrm{X}_3 \to \mathrm{X}_i \times \mathrm{X}_j$ the projections. We define the convolution of kernels as follows

$$\mathrm{K}_{23} \circ \mathrm{K}_{12} := p_{13_*}(p_{12}^*\mathrm{K}_{12} \otimes p_{23}^*\mathrm{K}_{23}).$$

> *Lemma* **2.29.** — *For* $\mathrm{K}_{12} \in \mathscr{D}^-(\mathrm{X}_1 \times \mathrm{X}_2)$, $\mathrm{K}_{23} \in \mathscr{D}^-(\mathrm{X}_2 \times \mathrm{X}_3)$ *we have* $\Phi_{\mathrm{K}_{23}} \circ \Phi_{\mathrm{K}_{12}} = \Phi_{\mathrm{K}_{23} \circ \mathrm{K}_{12}}$.

Assume that $\Phi_1, \Phi_2, \Phi_3 : \mathscr{D} \to \mathscr{D}'$ are exact functors between triangulated categories, and $\alpha : \Phi_1 \to \Phi_2$, $\beta : \Phi_2 \to \Phi_3$, $\gamma : \Phi_3 \to \Phi_1[1]$ are morphisms of

functors. We say that

$$\Phi_1 \xrightarrow{\alpha} \Phi_2 \xrightarrow{\beta} \Phi_3 \xrightarrow{\gamma} \Phi_1[1]$$

is an **exact triangle of functors**, if for any object $F \in \mathscr{D}$ the triangle

$$\Phi_1(F) \xrightarrow{\alpha(F)} \Phi_2(F) \xrightarrow{\beta(F)} \Phi_3(F) \xrightarrow{\gamma(F)} \Phi_1(F)[1]$$

is exact in $\mathscr{D}'$.

Lemma **2.30.** — *If* $K_1 \xrightarrow{\alpha} K_2 \xrightarrow{\beta} K_3 \xrightarrow{\gamma} K_1[1]$ *is an exact triangle in* $\mathscr{D}^-(X \times Y)$ *then we have the following exact triangles of functors*

$$\Phi_{K_1} \xrightarrow{\alpha_*} \Phi_{K_2} \xrightarrow{\beta_*} \Phi_{K_3} \xrightarrow{\gamma_*} \Phi_{K_1}[1]$$

$$\Phi_{K_3}^! \xrightarrow{\beta^!} \Phi_{K_2}^! \xrightarrow{\alpha^!} \Phi_{K_1}^! \xrightarrow{\gamma^!} \Phi_{K_3}^![1].$$

If additionally kernels K_1, K_2 *and* K_3 *satisfy the conditions of Lemma* 2.28 *then we have also the following exact triangle of functors*

$$\Phi_{K_3}^* \xrightarrow{\beta^*} \Phi_{K_2}^* \xrightarrow{\alpha^*} \Phi_{K_1}^* \xrightarrow{\gamma^*} \Phi_{K_3}^*[1].$$

Proof. — Evident. □

Let $\alpha : X \to Y$ be any morphism. Consider the functor $\alpha^*\alpha_* : \mathscr{D}^b(X) \to \mathscr{D}^b(X)$. Note that both the pullback and the pushforward are kernel functors (with the kernel being the structure sheaf of the graph of α). It follows that $\alpha^*\alpha_*$ is a kernel functor as well. Let $K_\alpha \in \mathscr{D}^b(X \times X)$ be its kernel, so that $\alpha^*\alpha_* \cong \Phi_{K_\alpha}$.

Lemma **2.31.** — *If* $\alpha : X \to Y$ *is a locally complete intersection embedding then we have*

$$\mathscr{H}^{-t}(K_\alpha) \cong \begin{cases} \Delta_* \Lambda^t \mathscr{N}_{X/Y}^*, & \text{if } 0 \le t \le \mathsf{codim}_Y X \\ 0, & \text{otherwise,} \end{cases}$$

where $\mathscr{N}_{X/Y}$ *is the normal bundle and* $\Delta : X \to X \times X$ *is the diagonal embedding. In particular, if* α *is a divisorial embedding then* K_α *fits into the exact triangle* $K_\alpha \to \Delta_* \mathscr{O}_X \to \Delta_* \mathscr{O}_X(-X)[2]$.

Proof. — Let $\gamma = (1 \times \alpha) \circ \Delta : X \to X \times Y$ be the graph of α. Then $\alpha_* \cong \Phi_{\gamma_* \mathscr{O}_X}$. Consider the diagram

We have $\alpha^*\alpha_*(\mathrm{F}) = \alpha^* p_{2*}(p_1^*\mathrm{F}\otimes\gamma_*\mathscr{O}_{\mathrm{X}}) \cong p_{2*}(1\times\alpha)^*(p_1^*\mathrm{F}\otimes\gamma_*\mathscr{O}_{\mathrm{X}}) \cong p_{2*}((1\times\alpha)^*p_1^*\mathrm{F}\otimes (1\times\alpha)^*\gamma_*\mathscr{O}_{\mathrm{X}})$, so it follows that $\mathrm{K}_\alpha = (1\times\alpha)^*\gamma_*\mathscr{O}_{\mathrm{X}}$. To describe K_α consider its pushforward to $\mathrm{X}\times\mathrm{Y}$:

$$(1\times\alpha)_*\mathrm{K}_\alpha = (1\times\alpha)_*(1\times\alpha)^*\gamma_*\mathscr{O}_{\mathrm{X}} \cong (1\times\alpha)_*\mathscr{O}_{\mathrm{X}\times\mathrm{X}}\otimes\gamma_*\mathscr{O}_{\mathrm{X}}$$
$$\cong \gamma_*\gamma^*(1\times\alpha)_*\mathscr{O}_{\mathrm{X}\times\mathrm{X}}.$$

Now $\gamma^*(1\times\alpha)_*\mathscr{O}_{\mathrm{X}\times\mathrm{X}} \cong \Delta^*(1\times\alpha)^*(1\times\alpha)_*\mathscr{O}_{\mathrm{X}\times\mathrm{X}}$. Since α is a locally complete intersection embedding we have $\mathscr{H}^{-t}((1\times\alpha)^*(1\times\alpha)_*\mathscr{O}_{\mathrm{X}\times\mathrm{X}}) \cong \mathscr{O}_{\mathrm{X}}\boxtimes\Lambda^t\mathscr{N}^*_{\mathrm{X}/\mathrm{Y}}$, hence $\mathscr{H}^{-t}(\Delta^*(1\times\alpha)^*(1\times\alpha)_*\mathscr{O}_{\mathrm{X}\times\mathrm{X}}) \cong \Lambda^t\mathscr{N}^*_{\mathrm{X}/\mathrm{Y}}$. Thus $\mathscr{H}^{-t}((1\times\alpha)_*\mathrm{K}_\alpha) \cong \gamma_*\Lambda^t\mathscr{N}^*_{\mathrm{X}/\mathrm{Y}} \cong (1\times\alpha)_*\Delta_*\Lambda^t\mathscr{N}^*_{\mathrm{X}/\mathrm{Y}}$ and the first part of the lemma follows since $(1\times\alpha)$ is a closed embedding. Finally, if α is divisorial then K_α has only two nontrivial cohomology, $\Delta_*\mathscr{O}_{\mathrm{X}}$ in degree 0 and $\Delta_*\mathscr{O}_{\mathrm{X}}(-\mathrm{X})$ in degree -1. Therefore it fits in the triangle as in the claim. $\square$

2.6. *Exact cartesian squares.* — Consider a cartesian square

$$
\begin{array}{ccc}
\mathrm{X}\times_{\mathrm{S}}\mathrm{Y} & \xrightarrow{\;q\;} & \mathrm{Y} \\
{\scriptstyle p}\Big\downarrow & & \Big\downarrow{\scriptstyle g} \\
\mathrm{X} & \xrightarrow{\;f\;} & \mathrm{S}.
\end{array}
$$

Consider the functors q_*p^* and $g^*f_* : \mathscr{D}^b(\mathrm{X}) \to \mathscr{D}^b(\mathrm{Y})$. It is easy to see that both are kernel functors. Explicitly, the first is given by the structure sheaf of the fiber product $\mathscr{O}_{\mathrm{X}\times_{\mathrm{S}}\mathrm{Y}}$ and the second is given by the convolution of the structure sheaves of graphs of f and g respectively. It is easy to see that the latter kernel is a complex supported on the fiber product, the top cohomology of which is isomorphic to $\mathscr{O}_{\mathrm{X}\times_{\mathrm{S}}\mathrm{Y}}$. The natural map from this complex to its top cohomology induces a morphism of functors $g^*f_* \to q_*p^*$. A cartesian square is called **exact cartesian** [K1] if this morphism of functors is an isomorphism. As explained above a square is exact cartesian if and only if the convolution of the structure sheaves of graphs of f and g is isomorphic to its top cohomology.

   Lemma **2.32** ([K1]). — *Consider a cartesian square as above.*

   (*i*) *If either f or g is flat then the square is exact cartesian.*
   (*ii*) *A square is exact cartesian, if and only if the transposed square is exact cartesian.*
   (*iii*) *If g is a closed embedding, $\mathrm{Y}\subset\mathrm{S}$ is a locally complete intersection, both S and X are Cohen–Macaulay, and $\mathrm{codim}_{\mathrm{X}}(\mathrm{X}\times_{\mathrm{S}}\mathrm{Y}) = \mathrm{codim}_{\mathrm{S}}\mathrm{Y}$, then the square is exact cartesian.*

   2.7. *Derived categories over a base.* — Consider a pair of algebraic varieties X and Y over the same smooth algebraic variety S. In other words, we have a pair of morphisms $f : \mathrm{X} \to \mathrm{S}$ and $g : \mathrm{Y} \to \mathrm{S}$.

A functor $\Phi : \mathscr{D}(X) \to \mathscr{D}(Y)$ is called S-linear [K1] if for all $F \in \mathscr{D}(X)$, $G \in \mathscr{D}^b(S)$ there are given bifunctorial isomorphisms

$$\Phi(f^*G \otimes F) \cong g^*G \otimes \Phi(F).$$

Note that since S is smooth any object $G \in \mathscr{D}^b(S)$ is a perfect complex.

Lemma **2.33** ([K1]). — *If Φ is S-linear and admits a right adjoint functor $\Phi^!$ then $\Phi^!$ is also S-linear. If $K \in \mathscr{D}^-(X \times_S Y)$ then the kernel functors Φ_{i_*K} and $\Phi^!_{i_*K}$ are S-linear.*

A strictly full subcategory $\mathscr{C} \subset \mathscr{D}(X)$ is called S-linear if for all $F \in \mathscr{C}$, $G \in \mathscr{D}^b(S)$ we have $f^*G \otimes F \in \mathscr{C}$.

Lemma **2.34** ([K1]). — *If $\mathscr{C} \subset \mathscr{D}^b(X)$ is a strictly full S-linear left (resp. right) admissible triangulated subcategory then its left (resp. right) orthogonal is also S-linear.*

2.8. *Faithful base change theorem.* — Consider morphisms $f : X \to S$ and $g : Y \to S$ with smooth S. For any base change $\phi : T \to S$ we consider the fiber products

$$X_T := X \times_S T, \quad Y_T := Y \times_S T, \quad X_T \times_T Y_T = (X \times_S Y) \times_S T$$

and denote the projections $X_T \to X$, $Y_T \to Y$, and $X_T \times_T Y_T \to X \times_S Y$ also by ϕ. For any kernel $K \in \mathscr{D}^-(X \times_S Y)$ we denote $K_T = \phi^*K \in \mathscr{D}^-(X_T \times_T Y_T)$.

Définition **2.35** ([K1]). — *A change of base $\phi : T \to S$ is called faithful with respect to a morphism $f : X \to S$ if the cartesian square*

$$\begin{array}{ccc} X_T & \longrightarrow & X \\ \downarrow & & \downarrow f \\ T & \xrightarrow{\phi} & S \end{array}$$

is exact cartesian. A change of base $\phi : T \to S$ is called faithful for a pair (X, Y) if ϕ is faithful with respect to morphisms $f : X \to S$, $g : Y \to S$, and $f \times_s g : X \times_S Y \to S$.

Using the criterions of Lemma 2.32 it is easy to deduce the following

Lemma **2.36** ([K1]). — *Let $f : X \to S$ be a morphism and $\phi : T \to S$ a base change.*

 (*i*) *If ϕ is flat then it is faithful.*
 (*ii*) *If T and X are smooth and $\dim X_T = \dim X + \dim T - \dim S$ then the base change $\phi : T \to S$ is faithful with respect to the morphism $f : X \to S$.*

Lemma **2.37** ([K1]). — *If $\phi : T \to S$ is a faithful base change for a morphism $f : X \to S$ then we have $\phi^*(\mathscr{D}_{f\mathrm{T}d/\mathrm{S}}(X)) \subset \mathscr{D}_{f\mathrm{T}d/\mathrm{T}}(X_T)$, and $\phi^*(\mathscr{D}_{f\mathrm{E}d/\mathrm{S}}(X)) \subset \mathscr{D}_{f\mathrm{E}d/\mathrm{T}}(X_T)$.*

Lemma **2.38** ([K1]). — *If* $\phi : \mathrm{T} \to \mathrm{S}$ *is a base change faithful for a pair* (X, Y), *and* f *is projective then we have* $\Phi_{\mathrm{K_T}}\phi^* = \phi^*\Phi_{\mathrm{K}}$, $\Phi_{\mathrm{K}}\phi_* = \phi_*\Phi_{\mathrm{K_T}}$, $\Phi^!_{\mathrm{K_T}}\phi^* = \phi^*\Phi^!_{\mathrm{K}}$, *and* $\Phi^!_{\mathrm{K}}\phi_* = \phi_*\Phi^!_{\mathrm{K_T}}$.

Proposition **2.39** ([K1]). — *If* ϕ *is faithful for a pair* (X, Y), *varieties* X *and* Y *are projective over* S *and smooth, and* $\mathrm{K} \in \mathscr{D}^b(\mathrm{X} \times_{\mathrm{S}} \mathrm{Y})$ *is a kernel such that* $\Phi_{\mathrm{K}} : \mathscr{D}^b(\mathrm{X}) \to \mathscr{D}^b(\mathrm{Y})$ *is fully faithful then* $\Phi_{\mathrm{K_T}} : \mathscr{D}^b(\mathrm{X_T}) \to \mathscr{D}^b(\mathrm{Y_T})$ *is fully faithful.*

Theorem **2.40** ([K1]). — *If* $\mathscr{D}^b(\mathrm{Y}) = \langle \Phi_{\mathrm{K}_1}(\mathscr{D}^b(\mathrm{X}_1)), ..., \Phi_{\mathrm{K}_n}(\mathscr{D}^b(\mathrm{X}_n)) \rangle$ *is a semiorthogonal decomposition, with* $\mathrm{K}_i \in \mathscr{D}^b(\mathrm{X}_i \times_{\mathrm{S}} \mathrm{Y})$, *the base change* ϕ *is faithful for all pairs* $(\mathrm{X}_1, \mathrm{Y}), ..., (\mathrm{X}_n, \mathrm{Y})$, *and all varieties* $\mathrm{X}_1, ..., \mathrm{X}_n, \mathrm{Y}$ *are projective over* S *and smooth then* $\mathscr{D}^b(\mathrm{Y_T}) = \langle \Phi_{\mathrm{K}_{1\mathrm{T}}}(\mathscr{D}^b(\mathrm{X}_{1\mathrm{T}})), ..., \Phi_{\mathrm{K}_{n\mathrm{T}}}(\mathscr{D}^b(\mathrm{X}_{n\mathrm{T}})) \rangle$ *is a semiorthogonal decomposition.*

Note that though $\mathrm{X}_1, ..., \mathrm{X}_n, \mathrm{Y}$ are smooth in the assumptions of the theorem, their pullbacks $\mathrm{X}_{1\mathrm{T}}, ..., \mathrm{X}_{n\mathrm{T}}, \mathrm{Y_T}$ under the base change ϕ are singular in general.

We will need also the following theorem.

Theorem **2.41.** — *If* S *and* Y *are smooth and for any point* $s \in \mathrm{S}$ *there exists an open neighborhood* $\mathrm{U} \subset \mathrm{S}$ *such that* $\mathscr{D}^b(\mathrm{Y_U}) = \langle \Phi_{\mathrm{K}_{1\mathrm{U}}}(\mathscr{D}^b(\mathrm{X}_{1\mathrm{U}})), ..., \Phi_{\mathrm{K}_{n\mathrm{U}}}(\mathscr{D}^b(\mathrm{X}_{n\mathrm{U}})) \rangle$ *is a semiorthogonal decomposition then* $\mathscr{D}^b(\mathrm{Y}) = \langle \Phi_{\mathrm{K}_1}(\mathscr{D}^b(\mathrm{X}_1)), ..., \Phi_{\mathrm{K}_n}(\mathscr{D}^b(\mathrm{X}_n)) \rangle$ *is also a semiorthogonal decomposition.*

Proof. — We must check that for every $i = 1, ..., n$ the functor $\Phi_{\mathrm{K}_i} : \mathscr{D}^b(\mathrm{X}_i) \to \mathscr{D}^b(\mathrm{Y})$ is fully faithful. Equivalently, we must show that the morphism of functors $\mathrm{id}_{\mathscr{D}^b(\mathrm{X}_i)} \to \Phi^!_{\mathrm{K}_i}\Phi_{\mathrm{K}_i}$ is an isomorphism. Note that $\mathscr{D}^b(\mathrm{X}_{i\mathrm{U}})$ being a semiorthogonal summand of an Ext-bounded category $\mathscr{D}^b(\mathrm{Y_U})$ is Ext-bounded, hence $\mathrm{X}_{i\mathrm{U}}$ is smooth, hence X_i is smooth for any i. Therefore the functors $\Phi^!_{\mathrm{K}_i}$ are kernel functors. Note also that the morphism of functors $\mathrm{id}_{\mathscr{D}^b(\mathrm{X}_i)} \to \Phi^!_{\mathrm{K}_i}\Phi_{\mathrm{K}_i}$ is induced by morphism of kernels. Moreover, restricting this morphism of kernels from S to U we obtain precisely the morphism of kernels corresponding to the canonical morphism of functors $\mathrm{id}_{\mathscr{D}^b(\mathrm{X}_{i\mathrm{U}})} \to \Phi^!_{\mathrm{K}_{i\mathrm{U}}}\Phi_{\mathrm{K}_{i\mathrm{U}}}$. Since the latter morphism is an isomorphism by assumptions for suitable U, it follows that the corresponding morphism of kernels is an isomorphism over U. Since this is true for a suitable neighborhood of every point $s \in \mathrm{S}$, we deduce that the morphism of kernels is an isomorphism over the whole S, hence Φ_{K_i} is fully faithful.

Further, we must check the semiorthogonality. Equivalently, we must show that the functor $\Phi^!_{\mathrm{K}_j}\Phi_{\mathrm{K}_i}$ is zero for all $1 \le i < j \le n$. As above we note that this functor is a kernel functor. Restricting its kernel from S to U we obtain precisely the kernel of the functor $\Phi^!_{\mathrm{K}_{j\mathrm{U}}}\Phi_{\mathrm{K}_{i\mathrm{U}}}$. Since the latter functor is zero by assumptions for suitable U, it follows that the corresponding kernel is zero over U. Since this is true for a suitable neighborhood of every point $s \in \mathrm{S}$, we deduce that the kernel is zero over the whole S, hence $\Phi^!_{\mathrm{K}_j}\Phi_{\mathrm{K}_i} = 0$.

Finally, we must check that our semiorthogonal collection generates $\mathscr{D}^b(\mathrm{Y})$. Assume that there is an object in the orthogonal to $\langle \Phi_{\mathrm{K}_1}(\mathscr{D}^b(\mathrm{X}_1)), ..., \Phi_{\mathrm{K}_n}(\mathscr{D}^b(\mathrm{X}_n)) \rangle$. Then it is easy to see that its restriction from S to U is in the orthogonal to $\langle \Phi_{\mathrm{K}_{1\mathrm{U}}}(\mathscr{D}^b(\mathrm{X}_{1\mathrm{U}})), ..., \Phi_{\mathrm{K}_{n\mathrm{U}}}(\mathscr{D}^b(\mathrm{X}_{n\mathrm{U}})) \rangle$. By assumptions we deduce that this object is zero over U. Since this is true for a suitable neighborhood of every point $s \in \mathrm{S}$, we deduce that the object is zero over the whole S.   □

3. Splitting functors

Assume that $\mathscr{A}$ and $\mathscr{B}$ are triangulated categories and $\Phi : \mathscr{B} \to \mathscr{A}$ is an exact functor. Consider the following full subcategories of $\mathscr{A}$ and $\mathscr{B}$:

$$\mathsf{Ker}\,\Phi = \{\mathrm{B} \in \mathscr{B} \mid \Phi(\mathrm{B}) = 0\} \subset \mathscr{B},$$
$$\mathsf{Im}\,\Phi = \{\mathrm{A} \cong \Phi(\mathrm{B}) \mid \mathrm{B} \in \mathscr{B}\} \subset \mathscr{A}.$$

Note that $\mathsf{Ker}\,\Phi$ is a triangulated subcategory of $\mathscr{B}$, and if Φ is fully faithful then $\mathsf{Im}\,\Phi$ is also triangulated. However, if Φ is not fully faithful, in general $\mathsf{Im}\,\Phi$ is not triangulated. If Φ admits an adjoint functor then we have

$$\mathsf{Hom}(\mathsf{Ker}\,\Phi, \mathsf{Im}\,\Phi^!) = 0, \quad \text{if } \Phi \text{ admits a right adjoint } \Phi^!,$$
$$\mathsf{Hom}(\mathsf{Im}\,\Phi^*, \mathsf{Ker}\,\Phi) = 0, \quad \text{if } \Phi \text{ admits a left adjoint } \Phi^*,$$

(evidently follows from the adjunction).

Definition **3.1.** — *An exact functor* $\Phi : \mathscr{B} \to \mathscr{A}$ *is called* right splitting *if* $\mathsf{Ker}\,\Phi$ *is a right admissible subcategory in* $\mathscr{B}$*, the restriction of* Φ *to* $(\mathsf{Ker}\,\Phi)^\perp$ *is fully faithful, and* $\mathsf{Im}\,\Phi$ *is right admissible in* $\mathscr{A}$ *(note that* $\mathsf{Im}\,\Phi = \mathsf{Im}(\Phi_{(\mathsf{Ker}\,\Phi)^\perp})$ *is a triangulated subcategory of* $\mathscr{A}$*). An exact functor* $\Phi : \mathscr{B} \to \mathscr{A}$ *is called* left splitting *if* $\mathsf{Ker}\,\Phi$ *is a left admissible subcategory in* $\mathscr{B}$*, the restriction of* Φ *to* $^\perp(\mathsf{Ker}\,\Phi)$ *is fully faithful, and* $\mathsf{Im}\,\Phi$ *is left admissible in* $\mathscr{A}$*.*

Lemma **3.2.** — *A right* (*resp. left*) *splitting functor has a right* (*resp. left*) *adjoint functor.*

Proof. — If $\mathsf{Ker}\,\Phi$ is right admissible then $(\mathsf{Ker}\,\Phi)^\perp$ is left admissible and we have a semiorthogonal decomposition $\mathscr{B} = \langle (\mathsf{Ker}\,\Phi)^\perp, \mathsf{Ker}\,\Phi \rangle$ by Lemmas 2.4 and 2.3. Since Φ vanishes on the second term and is fully faithful on the first term it follows that $\Phi \cong j \circ \phi \circ i^*$, where $i : (\mathsf{Ker}\,\Phi)^\perp \to \mathscr{B}$ and $j : \mathsf{Im}\,\Phi \to \mathscr{A}$ are the inclusion functors, i^* is a left adjoint to i, and $\phi : (\mathsf{Ker}\,\Phi)^\perp \to \mathsf{Im}\,\Phi$ is an equivalence of categories induced by Φ. Therefore $\Phi^! := i \circ \phi^{-1} \circ j^!$ is right adjoint to Φ (functor $j^!$ right adjoint to j exists because $\mathsf{Im}\,\Phi$ is right admissible).   □

Theorem **3.3.** — *Let* $\Phi : \mathscr{B} \to \mathscr{A}$ *be an exact functor. Then the following conditions are equivalent* $(1r) \Leftrightarrow (2r) \Leftrightarrow (3r) \Leftrightarrow (4r)$ *and* $(1l) \Leftrightarrow (2l) \Leftrightarrow (3l) \Leftrightarrow (4l)$*, where*

(1r) Φ *is right splitting;*

(2r) Φ *has a right adjoint functor* $\Phi^!$ *and the composition of the canonical morphism of functors* $\mathsf{id}_{\mathscr{B}} \to \Phi^!\Phi$ *with* Φ *gives an isomorphism* $\Phi \cong \Phi\Phi^!\Phi$;

(3r) Φ *has a right adjoint functor* $\Phi^!$, *there are semiorthogonal decompositions*

$$\mathscr{B} = \langle \mathsf{Im}\,\Phi^!, \mathsf{Ker}\,\Phi \rangle, \quad \mathscr{A} = \langle \mathsf{Ker}\,\Phi^!, \mathsf{Im}\,\Phi \rangle,$$

and the functors Φ *and* $\Phi^!$ *give quasiinverse equivalences* $\mathsf{Im}\,\Phi^! \cong \mathsf{Im}\,\Phi$;

(4r) *there exists a triangulated category* $\mathscr{C}$ *and fully faithful functors* $\alpha : \mathscr{C} \to \mathscr{A}$, $\beta : \mathscr{C} \to \mathscr{B}$, *such that* α *admits a right adjoint,* β *admits a left adjoint and* $\Phi \cong \alpha \circ \beta^*$.

(1l) Φ *is left splitting;*

(2l) Φ *has a left adjoint functor* Φ^* *and the composition of the canonical morphism of functors* $\Phi^*\Phi \to \mathsf{id}_{\mathscr{B}}$ *with* Φ *gives an isomorphism* $\Phi\Phi^*\Phi \cong \Phi$;

(3l) Φ *has a left adjoint functor* Φ^*, *there are semiorthogonal decompositions*

$$\mathscr{B} = \langle \mathsf{Ker}\,\Phi, \mathsf{Im}\,\Phi^* \rangle, \quad \mathscr{A} = \langle \mathsf{Im}\,\Phi, \mathsf{Ker}\,\Phi^* \rangle,$$

and the functors Φ *and* Φ^* *give quasiinverse equivalences* $\mathsf{Im}\,\Phi^* \cong \mathsf{Im}\,\Phi$;

(4l) *there exists a triangulated category* $\mathscr{C}$ *and fully faithful functors* $\alpha : \mathscr{C} \to \mathscr{A}$, $\beta : \mathscr{C} \to \mathscr{B}$, *such that* α *admits a left adjoint,* β *admits a right adjoint and* $\Phi \cong \alpha \circ \beta^!$.

Proof. — $(1r) \Rightarrow (2r)$: using the formula of Lemma 3.2 for $\Phi^!$ we deduce that $\Phi^!\Phi \cong i\phi^{-1}j^!j\phi i^* \cong ii^*$. Composing with Φ we obtain $\Phi\Phi^!\Phi \cong j\phi i^*ii^* \cong j\phi i^* \cong \Phi$.

$(2r) \Rightarrow (3r)$: for any $B \in \mathscr{B}$ let K_B be the object defined from the triangle

$$(\mathbf{1}) \qquad\qquad K_B \to B \to \Phi^!\Phi B.$$

Applying the functor Φ to this triangle and using the assumption we deduce that $\Phi(K_B) = 0$, i.e. $K_B \in \mathsf{Ker}\,\Phi$. Thus any object B can be included as the second vertex in a triangle with first vertex in $\mathsf{Ker}\,\Phi$ and the third vertex in $\mathsf{Im}\,\Phi^!$. Since these categories are semiorthogonal, we obtain the desired semiorthogonal decomposition for $\mathscr{B}$. Moreover, it follows from $(2r)$ that for $A \in \mathsf{Im}\,\Phi$ we have $A \cong \Phi\Phi^!A$, hence we have an isomorphism of functors $\mathsf{id} \cong \Phi\Phi^!$ on $\mathsf{Im}\,\Phi$. On the other hand, if $B \in \mathsf{Im}\,\Phi^!$ then $K_B = 0$ since K_B is the component of B in $\mathsf{Ker}\,\Phi$ with respect to the semiorthogonal decomposition $\mathscr{B} = \langle \mathsf{Im}\,\Phi^!, \mathsf{Ker}\,\Phi \rangle$. Therefore, $\mathsf{id} \cong \Phi^!\Phi$ on $\mathsf{Im}\,\Phi^!$. Thus Φ and $\Phi^!$ are quasiinverse equivalences between $\mathsf{Im}\,\Phi$ and $\mathsf{Im}\,\Phi^!$. Finally, we note that for any $B \in \mathsf{Im}\,\Phi^!$, $A \in \mathscr{A}$ we have

$$\mathsf{Hom}_{\mathscr{A}}(\Phi B, A) \cong \mathsf{Hom}_{\mathscr{B}}(B, \Phi^!A) \cong \mathsf{Hom}_{\mathscr{A}}(\Phi B, \Phi\Phi^!A)$$

since Φ is fully faithful on $\mathsf{Im}\,\Phi^!$, hence $\Phi\Phi^! : \mathscr{A} \to \mathsf{Im}\,\Phi$ is a right adjoint to the inclusion functor $\mathsf{Im}\,\Phi \to \mathscr{A}$, hence $\mathsf{Im}\,\Phi$ is right admissible, we have $\mathscr{A} = \langle (\mathsf{Im}\,\Phi)^{\perp}, \mathsf{Im}\,\Phi \rangle$ and it remains to note that $(\mathsf{Im}\,\Phi)^{\perp} = \mathsf{Ker}\,\Phi^!$.

$(3r) \Rightarrow (4r)$: take $\mathscr{C} = \mathsf{Im}\,\Phi$ with α being the inclusion functor $\mathsf{Im}\,\Phi \to \mathscr{A}$ and β being the composition of the equivalence $\mathsf{Im}\,\Phi \cong \mathsf{Im}\,\Phi^!$ and of the inclusion functor $\mathsf{Im}\,\Phi^! \to \mathscr{B}$. Then α admits a right adjoint because $\mathsf{Im}\,\Phi$ is right admissible in $\mathscr{A}$ and β admits a left adjoint because $\mathsf{Im}\,\Phi^!$ is left admissible in $\mathscr{B}$ and we evidently have $\Phi \cong \alpha \circ \beta^*$.

$(4r) \Rightarrow (1r)$: $\mathsf{Im}\,\Phi = \alpha(\mathscr{C})$ is right admissible because α admits a right adjoint functor; on the other hand $\mathsf{Ker}\,\Phi = \mathsf{Ker}(\beta^!) = {}^\perp\beta(\mathscr{C})$ is right admissible as the left orthogonal to $\beta(\mathscr{C})$ which is left admissible because β admits a left adjoint functor. Finally, $\Phi = \alpha \circ \beta^*$ restricted to $(\mathsf{Ker}\,\Phi)^\perp = \beta(\mathscr{C})$ is isomorphic to the composition of an equivalence $\beta(\mathscr{C}) \cong \mathscr{C}$ and of a fully faithful functor $\alpha : \mathscr{C} \to \mathscr{A}$, hence fully faithful.

The equivalences $(1l) \Leftrightarrow (2l) \Leftrightarrow (3l) \Leftrightarrow (4l)$ are proved by similar arguments. $\qquad\square$

Corollary **3.4.** — *If Φ is a right (resp. left) splitting functor and Ψ is its right (resp. left) adjoint then Ψ is a left (resp. right) splitting functor.*

Proof. — Compare $(3r)$ and $(3l)$ for Φ and Ψ. $\qquad\square$

Lemma **3.5.** — *If either $\mathscr{A}$ or $\mathscr{B}$ is saturated and $\Phi : \mathscr{B} \to \mathscr{A}$ is a right (resp. left) splitting functor then Φ is also a left (resp. right) splitting.*

Proof. — Assume that $\mathscr{B}$ is saturated and Φ is right admissible. Then $\mathsf{Ker}\,\Phi$ and $(\mathsf{Ker}\,\Phi)^\perp$ are saturated by Lemma 2.10. Moreover, $\mathsf{Im}\,\Phi \cong (\mathsf{Ker}\,\Phi)^\perp$, hence $\mathsf{Im}\,\Phi$ is also saturated. Hence by Lemma 2.11 both $\mathsf{Ker}\,\Phi$ and $\mathsf{Im}\,\Phi$ are left admissible. Moreover, it is easy to see that the restriction of Φ to ${}^\perp(\mathsf{Ker}\,\Phi)$ is isomorphic to the composition of the restriction of Φ to $(\mathsf{Ker}\,\Phi)^\perp$ with the mutation functor $\mathsf{L}_{\mathsf{Ker}\,\Phi}$

$$(\mathsf{Ker}\,\Phi)^\perp \xleftarrow{\quad \mathsf{L}_{\mathsf{Ker}\,\Phi} \quad} {}^\perp(\mathsf{Ker}\,\Phi).$$
$$\Phi \searrow \qquad \swarrow \Phi$$
$$\mathsf{Im}\,\Phi$$

But the upper arrow $\mathsf{L}_{\mathsf{Ker}\,\Phi}$ is fully faithful on ${}^\perp(\mathsf{Ker}\,\Phi)$ by Lemma 2.6, hence Φ is fully faithful on ${}^\perp(\mathsf{Ker}\,\Phi)$. $\qquad\square$

We will also need an analog of the faithful base change theorem for splitting functors.

Proposition **3.6.** — *In the notations of Proposition 2.39 if $\phi : \mathrm{T} \to \mathrm{S}$ is a faithful base change for a pair (X, Y) over a smooth base scheme S, X and Y are projective over S and smooth, and $\mathrm{K} \in \mathscr{D}^b(\mathrm{X} \times_\mathrm{S} \mathrm{Y})$ is a kernel such that the functor $\Phi_\mathrm{K} : \mathscr{D}^b(\mathrm{X}) \to \mathscr{D}^b(\mathrm{Y})$ is splitting then the functor $\Phi_{\mathrm{K}_\mathrm{T}} : \mathscr{D}^b(\mathrm{X}_\mathrm{T}) \to \mathscr{D}^b(\mathrm{Y}_\mathrm{T})$ is also splitting.*

Proof. — Analogous to the proof of Proposition 2.42 of [K1] using criterion (2r) or (2l) to check that the functors are splitting.   □

The class of splitting functors is a good generalization of the class of fully faithful functors having an adjoint. Recall that it was proved by Orlov in [O2] that any fully faithful functor having an adjoint between derived categories of smooth projective varieties is isomorphic to a kernel functor. It would be nice to prove the same result for splitting functors.

Conjecture **3.7.** — *A splitting functor between bounded derived categories of coherent sheaves on smooth projective varieties is isomorphic to a kernel functor.*

4. Lefschetz decompositions

Assume that X is an algebraic variety with a line bundle $\mathscr{O}_X(1)$ on X.

Définition **4.1.** — *A* Lefschetz decomposition *of the derived category* $\mathscr{D}^b(X)$ *is a semiorthogonal decomposition of* $\mathscr{D}^b(X)$ *of the form*

$$(2) \qquad \mathscr{D}^b(X) = \langle \mathscr{A}_0, \mathscr{A}_1(1), ..., \mathscr{A}_{i-1}(i-1) \rangle,$$
$$0 \subset \mathscr{A}_{i-1} \subset \mathscr{A}_{i-2} \subset \cdots \subset \mathscr{A}_1 \subset \mathscr{A}_0 \subset \mathscr{D}^b(X),$$

where $0 \subset \mathscr{A}_{i-1} \subset \mathscr{A}_{i-2} \subset \cdots \subset \mathscr{A}_1 \subset \mathscr{A}_0 \subset \mathscr{D}^b(X)$ *is a chain of admissible subcategories of* $\mathscr{D}^b(X)$*.* Lefschetz decomposition *is called* rectangular *if* $\mathscr{A}_{i-1} = \mathscr{A}_{i-2} = \cdots = \mathscr{A}_1 = \mathscr{A}_0$*.*

Let $\mathfrak{a}_k$ denote the right orthogonal to $\mathscr{A}_{k+1}$ in $\mathscr{A}_k$. The categories $\mathfrak{a}_0, \mathfrak{a}_1,$ ..., $\mathfrak{a}_{i-1}$ will be called primitive categories of the Lefschetz decomposition (2). By definition we have the following semiorthogonal decompositions:

$$(3) \qquad \mathscr{A}_k = \langle \mathfrak{a}_k, \mathfrak{a}_{k+1}, ..., \mathfrak{a}_{i-1} \rangle.$$

If the Lefschetz decomposition is rectangular then we have $\mathfrak{a}_0 = \mathfrak{a}_1 = \cdots = \mathfrak{a}_{i-2}$ $= 0$ and $\mathfrak{a}_{i-1} = \mathscr{A}_{i-1}$.

Assume that the bounded derived category of coherent sheaves on X, $\mathscr{D}^b(X)$ admits a Lefschetz decomposition (2) with respect to $\mathscr{O}_X(1)$. If X is smooth and projective then its derived category $\mathscr{D}^b(X)$ is saturated and admits a Serre functor. Therefore for every $0 \leq k \leq i-1$ the category $\mathscr{A}_k$ is saturated and has a Serre functor too. Moreover, for every $0 \leq k \leq i-1$ the primitive category $\mathfrak{a}_k$ is also saturated and has a Serre functor.

Let $\alpha_k : \mathscr{A}_k(k) \rightarrow \mathscr{D}^b(X)$ denote the embedding functor and let $\alpha_k^*, \alpha^! : \mathscr{D}^b(X) \rightarrow \mathscr{A}_k(k)$ be the left and the right adjoint functors. Let S_X denote a Serre functor of $\mathscr{D}^b(X)$, $\mathsf{S}_X(F) \cong F \otimes \omega_X[\dim X]$, and let S_0 denote a Serre functor of $\mathscr{A}_0$.

Consider the restriction of the functor $\alpha_0^* : \mathscr{D}^b(X) \to \mathscr{A}_0$ to the subcategory $\mathscr{A}_k(k+1) \subset \mathscr{D}^b(X)$. It follows from (2) that $\alpha_0^*(\mathscr{A}_{k+1}(k+1)) = 0$, hence it factors through the quotient $(\mathscr{A}_k/\mathscr{A}_{k+1})(k+1)$.

Lemma 4.2. — *The functor* $\alpha_0^* : (\mathscr{A}_k/\mathscr{A}_{k+1})(k+1) \to \mathscr{A}_0$ *is fully faithful.*

Proof. — It is clear that $\mathfrak{a}_k(k+1)$ is the right orthogonal to $\mathscr{A}_{k+1}(k+1)$ in $\mathscr{A}_k(k+1)$, hence we have to check that α_0^* is fully faithful on $\mathfrak{a}_k(k+1)$. For this we note that

$$(\textbf{4}) \qquad \mathfrak{a}_k(k+1) \subset \langle \mathscr{A}_{k+1}(k+1), ..., \mathscr{A}_{i-1}(i-1)\rangle^\perp = \langle \mathscr{A}_0, \mathscr{A}_1(1), ..., \mathscr{A}_k(k)\rangle,$$

since for $l > k+1$ we have $\mathsf{Hom}(\mathscr{A}_l(l), \mathfrak{a}_k(k+1)) = \mathsf{Hom}(\mathscr{A}_l(l-1), \mathfrak{a}_k(k))$ and $\mathscr{A}_l(l-1) \subset \mathscr{A}_{l-1}(l-1)$, $\mathfrak{a}_k(k) \subset \mathscr{A}_k(k)$, while $\mathsf{Hom}(\mathscr{A}_{k+1}(k+1), \mathfrak{a}_k(k+1)) = \mathsf{Hom}(\mathscr{A}_{k+1}, \mathfrak{a}_k) = 0$ by definition of $\mathfrak{a}_k$. On the other hand, we have

$$(\textbf{5}) \qquad \mathfrak{a}_k(k+1) \subset {}^\perp\langle \mathscr{A}_1(1), ..., \mathscr{A}_k(k)\rangle,$$

since for $1 \leq l \leq k$ we have $\mathsf{Hom}(\mathfrak{a}_k(k+1), \mathscr{A}_l(l)) = \mathsf{Hom}(\mathfrak{a}_k(k), \mathscr{A}_l(l-1))$ and $\mathscr{A}_l(l-1) \subset \mathscr{A}_{l-1}(l-1)$, $\mathfrak{a}_k(k) \subset \mathscr{A}_k(k)$. It follows from (4) that the functor α_0^* restricted to $\mathfrak{a}_k(k+1)$ is just the left mutation of $\mathfrak{a}_k(k+1)$ through $\langle \mathscr{A}_1(1), ..., \mathscr{A}_k(k)\rangle$. But the left mutation through an admissible subcategory induces an equivalence of its left orthogonal to its right orthogonal by Lemma 2.6, and $\mathfrak{a}_k(k+1)$ lies in the left orthogonal to $\langle \mathscr{A}_1(1), ..., \mathscr{A}_k(k)\rangle$ by (5). □

Lemma 4.3. — *We have the following semiorthogonal decomposition of* $\mathscr{A}_0$

$$\langle \alpha_0^*(\mathfrak{a}_0(1)), \alpha_0^*(\mathfrak{a}_1(2)), ..., \alpha_0^*(\mathfrak{a}_{i-1}(i))\rangle.$$

Proof. — For any $F \in \mathscr{A}_0$, $F' \in \mathfrak{a}_l$ we have

$$\mathsf{Hom}(\alpha_0^*(F'(l+1)), F) = \mathsf{Hom}(F'(l+1), F) = \mathsf{Hom}(F, S_X(F'(l+1)))^*$$
$$= \mathsf{Hom}(F, \alpha_0^! S_X(F'(l+1)))^*,$$

therefore $(\alpha_0^*(\mathfrak{a}_l(l+1)))^\perp = {}^\perp(\alpha_0^! S_X(\mathfrak{a}_l(l+1)))$. Thus for the semiorthogonality we should check that for any $k < l$ we have $\mathsf{Hom}(\alpha_0^*(\mathfrak{a}_k(k+1)), \alpha_0^! S_X(\mathfrak{a}_l(l+1))) = 0$. For this we note that the inclusion (5) (with k replaced by l) implies that $\mathfrak{a}_l(l+1) \subset \langle \mathscr{A}_{l+1}(l+1), ..., \mathscr{A}_{i-1}(i-1), S_X^{-1}\mathscr{A}_0\rangle$ by Lemma 2.18, hence

$$S_X(\mathfrak{a}_l(l+1)) \subset \langle S_X(\mathscr{A}_{l+1}(l+1)), ..., S_X(\mathscr{A}_{i-1}(i-1)), \mathscr{A}_0\rangle.$$

Comparing this with the inclusion (4) for $\mathfrak{a}_k(k+1)$ and taking into account that by Lemma 2.18 we have a semiorthogonal decomposition $\mathscr{D}^b(X) = \langle S_X(\mathscr{A}_{l+1}(l+1)), ..., S_X(\mathscr{A}_{i-1}(i-1)), \mathscr{A}_0, \mathscr{A}_1(1), ..., \mathscr{A}_l(l)\rangle$, we deduce that $\mathsf{Hom}(\alpha_0^*(\mathfrak{a}_k(k+1)), \alpha_0^! S_X(\mathfrak{a}_l(l+1))) = \mathsf{Hom}(\mathfrak{a}_k(k+1), S_X(\mathfrak{a}_l(l+1)))$ which by the Serre duality is dual

to $\mathrm{Hom}(\mathfrak{a}_l(l+1), \mathfrak{a}_k(k+1)) = \mathrm{Hom}(\mathfrak{a}_l(l), \mathfrak{a}_k(k))$ which is zero since $\mathfrak{a}_l(l) \subset \mathscr{A}_l(l)$ and $\mathfrak{a}_k(k) \subset \mathscr{A}_k(k)$.

Now assume that F lies in the right orthogonal to the collection $\langle \alpha_0^*(\mathfrak{a}_0(1)),$ $\alpha_0^*(\mathfrak{a}_1(2)), ..., \alpha_0^*(\mathfrak{a}_{i-1}(i)) \rangle$ in $\mathscr{A}_0$. By adjunction $\alpha_0(\mathrm{F})$ is in the right orthogonal to $\langle \mathfrak{a}_0(1), \mathfrak{a}_1(2), ..., \mathfrak{a}_{i-1}(i) \rangle$ in $\mathscr{D}^b(\mathrm{X})$. But $\alpha_0(\mathrm{F}) \in \mathscr{A}_0 = \langle \mathscr{A}_1(1), ..., \mathscr{A}_{i-1}(i-1) \rangle^\perp$, therefore

$$\alpha_0(\mathrm{F}) \in \langle \mathfrak{a}_0(1), \mathscr{A}_1(1), \mathfrak{a}_1(2), \mathscr{A}_2(2), ..., \mathfrak{a}_{i-2}(i-1), \mathscr{A}_{i-1}(i-1), \mathfrak{a}_{i-1}(i) \rangle^\perp.$$

It remains to note that by definition of subcategories $\mathfrak{a}_0, ..., \mathfrak{a}_{i-1}$ we have $\langle \mathfrak{a}_0(1), \mathscr{A}_1(1) \rangle = \mathscr{A}_0(1)$, $\langle \mathfrak{a}_1(2), \mathscr{A}_2(2) \rangle = \mathscr{A}_1(2), ..., \langle \mathfrak{a}_{i-2}(i-1), \mathscr{A}_{i-1}(i-1) \rangle = \mathscr{A}_{i-2}(i-1)$, and $\mathfrak{a}_{i-1}(i) = \mathscr{A}_{i-1}(i)$, so we see that $\alpha_0(\mathrm{F}) \in \langle \mathscr{A}_0(1), \mathscr{A}_1(2), ..., \mathscr{A}_{i-1}(i) \rangle^\perp$ which means that $\mathrm{F} = 0$ since $\langle \mathscr{A}_0(1), \mathscr{A}_1(2), ..., \mathscr{A}_{i-1}(i) \rangle$ is evidently a semiorthogonal decomposition of $\mathscr{D}^b(\mathrm{X})$. $\square$

Lemma **4.4.** — *We have* $\alpha_0^*(\langle \mathscr{A}_0(1), ..., \mathscr{A}_{r-1}(r) \rangle) \in \langle \alpha_0^*(\mathfrak{a}_0(1)), ..., \alpha_0^*(\mathfrak{a}_{r-1}(r)) \rangle$.

Proof. — We have $\mathscr{A}_k(k+1) = \langle \mathfrak{a}_k(k+1), \mathscr{A}_{k+1}(k+1) \rangle$ and $\alpha_0^*(\mathscr{A}_{k+1}(k+1)) = 0$ for any $0 \leq k \leq r-1$. $\square$

Lemma **4.5.** — *Triangulated subcategory of* $\mathscr{D}^b(\mathrm{X})$ *generated by* $\mathscr{A}_0, \mathscr{A}_0(1), ..., \mathscr{A}_0(r-1)$ *coincides with* $\langle \mathscr{A}_0, \mathscr{A}_1(1), ..., \mathscr{A}_{r-1}(r-1) \rangle$.

Proof. — It is clear that the latter category lies in the former. On the other hand, it is clear that $\mathscr{A}_0, \mathscr{A}_0(1), ..., \mathscr{A}_0(r-1) \subset \langle \mathscr{A}_r(r), ..., \mathscr{A}_{i-1}(i-1) \rangle^\perp = \langle \mathscr{A}_0, \mathscr{A}_1(1), ..., \mathscr{A}_{r-1}(r-1) \rangle$. $\square$

5. Universal hyperplane section

Assume that X is a smooth projective variety with an effective line bundle $\mathscr{O}_\mathrm{X}(1)$ on X and assume that we are given a Lefschetz decomposition (2) of its derived category. Let $\mathrm{V}^* \subset \Gamma(\mathrm{X}, \mathscr{O}_\mathrm{X}(1))$ be a vector space of global sections. Put $\mathrm{N} = \dim \mathrm{V}$. We assume that

(**6**) $\mathrm{N} > i$.

Consider the product $\mathrm{X} \times \mathbf{P}(\mathrm{V}^*)$. Let $\mathscr{A}_k(k) \boxtimes \mathscr{D}^b(\mathbf{P}(\mathrm{V}^*))$ denote the triangulated subcategory of $\mathscr{D}^b(\mathrm{X} \times \mathbf{P}(\mathrm{V}^*))$ generated by objects $\mathrm{F} \boxtimes \mathrm{G}$ with $\mathrm{F} \in \mathscr{A}_k(k) \subset \mathscr{D}^b(\mathrm{X})$ and $\mathrm{G} \in \mathscr{D}^b(\mathbf{P}(\mathrm{V}^*))$. Note that

$$\mathscr{A}_k(k) \boxtimes \mathscr{D}^b(\mathbf{P}(\mathrm{V}^*)) =$$
$$\langle \mathscr{A}_k(k) \boxtimes \mathscr{O}_{\mathbf{P}(\mathrm{V}^*)}, \mathscr{A}_k(k) \boxtimes \mathscr{O}_{\mathbf{P}(\mathrm{V}^*)}(1), ..., \mathscr{A}_k(k) \boxtimes \mathscr{O}_{\mathbf{P}(\mathrm{V}^*)}(\mathrm{N}-1) \rangle.$$

Indeed, the RHS is evidently contained in the LHS. On the other hand, take any $F \boxtimes G$ with $F \in \mathscr{A}_k(k)$ and $G \in \mathscr{D}^b(\mathbf{P}(V^*))$ and consider the decomposition of G with respect to the semiorthogonal decomposition $\mathscr{D}^b(\mathbf{P}(V^*)) = \langle \mathscr{O}_{\mathbf{P}(V^*)}, \mathscr{O}_{\mathbf{P}(V^*)}(1),$ ..., $\mathscr{O}_{\mathbf{P}(V^*)}(N-1) \rangle$. Tensoring by F we deduce that $F \boxtimes G$ is in the RHS.

Every category $\mathscr{A}_k(k) \boxtimes \mathscr{O}_{\mathbf{P}(V^*)}(l)$ is equivalent to $\mathscr{A}_k$, hence saturated, hence admissible, therefore $\mathscr{A}_k(k) \boxtimes \mathscr{D}^b(\mathbf{P}(V^*))$ is also admissible and saturated. Moreover, it is clear that we have the following semiorthogonal decomposition

$$\mathscr{D}^b(X \times \mathbf{P}(V^*)) =$$
$$\big\langle \mathscr{A}_0 \boxtimes \mathscr{D}^b(\mathbf{P}(V^*)), \mathscr{A}_1(1) \boxtimes \mathscr{D}^b(\mathbf{P}(V^*)), ..., \mathscr{A}_{i-1}(i-1) \boxtimes \mathscr{D}^b(\mathbf{P}(V^*)) \big\rangle.$$

Indeed, semiorthogonality in the RHS follows from the Küneth formula

$$\mathsf{RHom}_{X \times \mathbf{P}(V^*)}(F_1 \boxtimes G_1, F_2 \boxtimes G_2) \cong \mathsf{RHom}_X(F_1, F_2) \otimes \mathsf{RHom}_{\mathbf{P}(V^*)}(G_1, G_2)$$

for all $F_1, F_2 \in \mathscr{D}^b(X)$, $G_1, G_2 \in \mathscr{D}^b(\mathbf{P}(V^*))$ and admissibility of components of the RHS was verified above. Finally, taking any $F \in \mathscr{D}^b(X)$, $G \in \mathscr{D}^b(\mathbf{P}(V^*))$, considering the decomposition of F with respect to (2), and tensoring it by G we deduce that $F \boxtimes G$ is in the RHS. Since $\mathscr{D}^b(X \times \mathbf{P}(V^*))$ is generated by objects of the form $F \boxtimes G$ we see that the RHS equals to the LHS.

Consider the universal hyperplane section of X, that is the zero locus $\mathscr{X}_1 \subset X \times \mathbf{P}(V^*)$ of the canonical section of a line bundle $\mathscr{O}_X(1) \boxtimes \mathscr{O}_{\mathbf{P}(V^*)}(1)$. Let $\pi : \mathscr{X}_1 \to X$ and $f : \mathscr{X}_1 \to \mathbf{P}(V^*)$ denote the projections, and let $i : \mathscr{X}_1 \to X \times \mathbf{P}(V^*)$ denote the embedding. Note that $\mathscr{X}_1 \subset X \times \mathbf{P}(V^*)$ is a divisor of bidegree $(1, 1)$ and we have the following resolution of its structure sheaf

$$(\mathbf{7}) \qquad 0 \to \mathscr{O}_X(-1) \boxtimes \mathscr{O}_{\mathbf{P}(V^*)}(-1) \to \mathscr{O}_{X \times \mathbf{P}(V^*)} \to i_* \mathscr{O}_{\mathscr{X}_1} \to 0.$$

The following lemma is useful for calculations of Hom's between decomposable objects in $\mathscr{D}^b(\mathscr{X}_1)$.

Lemma **5.1.** — *For any* $F_1, F_2 \in \mathscr{D}^b(X)$, $G_1, G_2 \in \mathscr{D}^b(\mathbf{P}(V^*))$ *we have an exact triangle*

$$\mathsf{RHom}_X(F_1, F_2(-1)) \otimes \mathsf{RHom}_{\mathbf{P}(V^*)}(G_1, G_2(-1))$$
$$\to \mathsf{RHom}_X(F_1, F_2) \otimes \mathsf{RHom}_{\mathbf{P}(V^*)}(G_1, G_2)$$
$$\to \mathsf{RHom}_{\mathscr{X}_1}(i^*(F_1 \boxtimes G_1), i^*(F_2 \boxtimes G_2)).$$

Proof. — Tensoring resolution (7) by $(F_1^* \otimes F_2) \boxtimes (G_1^* \otimes G_2)$ and applying $\mathsf{R\Gamma}$ we obtain the following exact triangle

$$\mathsf{R\Gamma}(X \times \mathbf{P}(V^*), (F_1^* \otimes F_2(-1)) \boxtimes (G_1^* \otimes G_2(-1)))$$
$$\to \mathsf{R\Gamma}(X \times \mathbf{P}(V^*), (F_1^* \otimes F_2) \boxtimes (G_1^* \otimes G_2))$$
$$\to \mathsf{R\Gamma}(\mathscr{X}_1, i^*((F_1^* \otimes F_2) \boxtimes (G_1^* \otimes G_2))).$$

Rewriting $R\Gamma$ in terms of RHom's and applying Küneth formula we obtain the desired triangle. □

Corollary **5.2.** — *The functor* $\pi^* : \mathscr{D}^b(X) \to \mathscr{D}^b(\mathscr{X}_1)$ *is fully faithful. Moreover, for any* $F_1, F_2 \in \mathscr{D}^b(X)$ *and* $1 \leq k \leq N - 2$ *we have* $\mathsf{RHom}_{\mathscr{X}_1}(\pi^* F_1, \pi^* F_2 \otimes f^* \mathscr{O}_{\mathbf{P}(V^*)}(-k))$ $= 0$.

Proof. — Take $G_1 = \mathscr{O}_{\mathbf{P}(V^*)}$, $G_2 = \mathscr{O}_{\mathbf{P}(V^*)}(-k)$. Then we have isomorphisms $i^*(F_1 \boxtimes G_1) \cong \pi^* F_1$, and $i^*(F_2 \boxtimes G_2) \cong \pi^* F_2 \otimes f^* \mathscr{O}_{\mathbf{P}(V^*)}(-k)$. Since $\mathsf{RHom}_{\mathbf{P}(V^*)}(\mathscr{O}_{\mathbf{P}(V^*)}, \mathscr{O}_{\mathbf{P}(V^*)}(-k)) = 0$ for $1 \leq k \leq N - 1$ and $\mathsf{RHom}_{\mathbf{P}(V^*)}(\mathscr{O}_{\mathbf{P}(V^*)}, \mathscr{O}_{\mathbf{P}(V^*)})$ $= \mathsf{k}$, the first term in the triangle of the lemma vanishes for $0 \leq k \leq N - 2$ and the second term equals $\mathsf{RHom}_X(F_1, F_2)$ for $k = 0$ and vanishes for $1 \leq k \leq N - 1$ whereof we obtain the claim. □

Lemma **5.3.** — *For any* $1 \leq k \leq \mathsf{i} - 1$ *the functor* $\mathscr{A}_k(k) \boxtimes \mathscr{D}^b(\mathbf{P}(V^*)) \subset \mathscr{D}^b(X \times \mathbf{P}(V^*)) \xrightarrow{i^*} \mathscr{D}^b(\mathscr{X}_1)$ *is fully faithful, and the collection*

$$\left(\mathscr{A}_1(1) \boxtimes \mathscr{D}^b(\mathbf{P}(V^*)), ..., \mathscr{A}_{\mathsf{i}-1}(\mathsf{i} - 1) \boxtimes \mathscr{D}^b(\mathbf{P}(V^*)) \right) \subset \mathscr{D}^b(\mathscr{X}_1)$$

is semiorthogonal.

Proof. — Let $1 \leq k \leq l \leq \mathsf{i} - 1$, take $F_1 \in \mathscr{A}_l(l)$, $F_2 \in \mathscr{A}_k(k)$, $G_1, G_2 \in \mathscr{D}^b(\mathbf{P}(V^*))$ and consider the triangle of Lemma 5.1. Its first term vanishes since $F_1 \in \mathscr{A}_l(l)$ and $F_2(-1) \in \mathscr{A}_k(k-1) \subset \mathscr{A}_{k-1}(k-1)$. Therefore, in the case $k = l$ we see that the functor $i^* : \mathscr{A}_k(k) \boxtimes \mathscr{D}^b(\mathbf{P}(V^*)) \to \mathscr{D}^b(\mathscr{X}_1)$ is fully faithful. On the other hand, for $1 \leq k < l \leq \mathsf{i} - 1$ the second term vanishes as well, since $F_1 \in \mathscr{A}_l(l)$ and $F_2 \in \mathscr{A}_k(k)$. Therefore the above collection is semiorthogonal.

It remains to check that categories $\mathscr{A}_k(k) \boxtimes \mathscr{D}^b(\mathbf{P}(V^*))$ are admissible in $\mathscr{D}^b(\mathscr{X}_1)$. For this we note that they are saturated, hence admissible in $\mathscr{D}^b(\mathscr{X}_1)$. □

Let $\mathscr{C}$ denote the right orthogonal to the subcategory $\langle \mathscr{A}_1(1) \boxtimes \mathscr{D}^b(\mathbf{P}(V^*)), ..., \mathscr{A}_{\mathsf{i}-1}(\mathsf{i}-1) \boxtimes \mathscr{D}^b(\mathbf{P}(V^*)) \rangle$ in $\mathscr{D}^b(\mathscr{X}_1)$,

$$(\mathbf{8}) \qquad \mathscr{C} = \left\langle \mathscr{A}_1(1) \boxtimes \mathscr{D}^b(\mathbf{P}(V^*)), ..., \mathscr{A}_{\mathsf{i}-1}(\mathsf{i}-1) \boxtimes \mathscr{D}^b(\mathbf{P}(V^*)) \right\rangle^{\perp}_{\mathscr{D}^b(\mathscr{X}_1)}.$$

Let $\gamma : \mathscr{C} \to \mathscr{D}^b(\mathscr{X}_1)$ denote the inclusion functor. Since the subcategories $\mathscr{A}_1(1) \boxtimes \mathscr{D}^b(\mathbf{P}(V^*)), ..., \mathscr{A}_{\mathsf{i}-1}(\mathsf{i}-1) \boxtimes \mathscr{D}^b(\mathbf{P}(V^*))$ are admissible it follows that $\mathscr{C}$ is left admissible, hence the functor γ has a left adjoint functor $\gamma^* : \mathscr{D}^b(\mathscr{X}_1) \to \mathscr{C}$.

Note that the subcategory $\langle \mathscr{A}_1(1) \boxtimes \mathscr{D}^b(\mathbf{P}(V^*)), ..., \mathscr{A}_{\mathsf{i}-1}(\mathsf{i}-1) \boxtimes \mathscr{D}^b(\mathbf{P}(V^*)) \rangle$ $\subset \mathscr{D}^b(\mathscr{X}_1)$ is $\mathbf{P}(V^*)$-linear. In particular, the functor $F \mapsto F \otimes f^* \mathscr{O}_{\mathbf{P}(V^*)}(1)$, $\mathscr{D}^b(\mathscr{X}_1) \to \mathscr{D}^b(\mathscr{X}_1)$ restricts to an endofunctor of $\mathscr{C}$ which we denote simply by $F \mapsto F(1)$.

Consider the composition of functors $\pi_* \circ \gamma : \mathscr{C} \to \mathscr{D}^b(X)$.

Lemma **5.4.** — *The image of the functor* $\pi_* \circ \gamma$ *is contained in the strictly full subcategory* $\mathscr{A}_0 \subset \mathscr{D}^b(\mathrm{X})$.

Proof. — If $\mathrm{F} \in \mathscr{A}_k(k)$, $1 \leq k \leq i - 1$, and $\mathrm{F}' \in \mathscr{C}$ then we have $\mathsf{Hom}(\mathrm{F}, \pi_*(\gamma(\mathrm{F}'))) = \mathsf{Hom}(\pi^*\mathrm{F}, \gamma(\mathrm{F}')) = 0$ since $\pi^*\mathrm{F} \in \mathscr{A}_k(k) \boxtimes \mathscr{D}^b(\mathbf{P}(\mathrm{V}^*))$. Thus $\pi_*(\gamma(\mathrm{F}'))$ is contained in the right orthogonal to the subcategory $\langle \mathscr{A}_1(1), ..., \mathscr{A}_{i-1}(i-1) \rangle$, which by (2) coincides with $\mathscr{A}_0$. $\square$

Consider the functor $\gamma^* \circ \pi^* : \mathscr{D}^b(\mathrm{X}) \to \mathscr{C}$ which is left adjoint to $\pi_* \circ \gamma$. In Proposition 5.7 below we will show that the restriction of this functor to the subcategory $\mathscr{A}_0 \subset \mathscr{D}^b(\mathrm{X})$ is fully faithful. We start with two lemmas.

For any object $\mathrm{F} \in \mathscr{D}^b(\mathrm{X})$ consider the decomposition of $\pi^*\mathrm{F} \in \mathscr{D}^b(\mathscr{X}_1)$

$$(\mathbf{9}) \qquad \mathrm{F}_{\perp\mathscr{C}} \to \pi^*\mathrm{F} \to \mathrm{F}_{\mathscr{C}}$$

with $\mathrm{F}_{\mathscr{C}} \in \mathscr{C}$, $\mathrm{F}_{\perp\mathscr{C}} \in {}^{\perp}\mathscr{C} = \langle \mathscr{A}_1(1) \boxtimes \mathscr{D}^b(\mathbf{P}(\mathrm{V}^*)), ..., \mathscr{A}_{i-1}(i-1) \boxtimes \mathscr{D}^b(\mathbf{P}(\mathrm{V}^*)) \rangle$. Then it is clear that

$$(\mathbf{10}) \qquad \mathrm{F}_{\mathscr{C}} = \gamma\gamma^*\pi^*\mathrm{F}.$$

Lemma **5.5.** — *If* $\mathsf{RHom}(\mathrm{F}, \pi_*(\mathrm{F}'_{\perp\mathscr{C}}(0, k))) = 0$ *then we have a canonical isomorphism*

$$\mathsf{Hom}_{\mathscr{D}^b(\mathscr{X}_1)}(\pi^*\mathrm{F}, \pi^*\mathrm{F}'(0, k)) \cong \mathsf{Hom}_{\mathscr{C}}(\gamma^*\pi^*\mathrm{F}, (\gamma^*\pi^*\mathrm{F}')(k)).$$

Proof. — Applying the functor $\mathsf{RHom}(\pi^*\mathrm{F}, -)$ to the exact triangle $\mathrm{F}'_{\perp\mathscr{C}}(0, k) \to \pi^*\mathrm{F}'(0, k) \to \mathrm{F}'_{\mathscr{C}}(0, k)$ and taking into account the isomorphism $\mathsf{RHom}(\pi^*\mathrm{F}, \mathrm{F}'_{\perp\mathscr{C}}(0, k)) \cong \mathsf{RHom}(\mathrm{F}, \pi_*(\mathrm{F}'_{\perp\mathscr{C}}(0, k))) = 0$ we deduce $\mathsf{Hom}(\pi^*\mathrm{F}, \pi^*\mathrm{F}'(0, k)) \cong \mathsf{Hom}(\pi^*\mathrm{F}, \mathrm{F}'_{\mathscr{C}}(0, k))$. It remains to note that

$$\mathsf{Hom}_{\mathscr{D}^b(\mathscr{X}_1)}(\pi^*\mathrm{F}, \mathrm{F}'_{\mathscr{C}}(0, k)) \cong \mathsf{Hom}_{\mathscr{D}^b(\mathscr{X}_1)}(\pi^*\mathrm{F}, \gamma\gamma^*\pi^*\mathrm{F}'(0, k))$$
$$\cong \mathsf{Hom}_{\mathscr{C}}(\gamma^*\pi^*\mathrm{F}, \gamma^*\pi^*\mathrm{F}'(k)). \qquad \square$$

Recall the semiorthogonal decomposition $\mathscr{A}_0 = \langle \alpha_0^*(\mathfrak{a}_0(1)), ..., \alpha_0^*(\mathfrak{a}_{i-1}(i)) \rangle$ constructed in Lemma 4.3.

Lemma **5.6.** — *Let* $\mathrm{F} \in \langle \alpha_0^*(\mathfrak{a}_0(1)), ..., \alpha_0^*(\mathfrak{a}_k(k+1)) \rangle \subset \mathscr{A}_0 \subset \mathscr{D}^b(\mathrm{X})$. *Then*

$$\mathrm{F}_{\perp\mathscr{C}} \in \left\langle \begin{matrix} \mathscr{A}_1(1) \boxtimes \mathscr{O}_{\mathbf{P}(\mathrm{V}^*)}(-k), \mathscr{A}_1(1) \boxtimes \mathscr{O}_{\mathbf{P}(\mathrm{V}^*)}(1-k), ..., \mathscr{A}_1(1) \boxtimes \mathscr{O}_{\mathbf{P}(\mathrm{V}^*)}(-1), \\ \mathscr{A}_2(2) \boxtimes \mathscr{O}_{\mathbf{P}(\mathrm{V}^*)}(1-k), ..., \mathscr{A}_2(2) \boxtimes \mathscr{O}_{\mathbf{P}(\mathrm{V}^*)}(-1), \\ \vdots \\ \mathscr{A}_k(k) \boxtimes \mathscr{O}_{\mathbf{P}(\mathrm{V}^*)}(-1) \end{matrix} \right\rangle.$$

Proof. — Consider the decomposition of $\pi^*\mathrm{F}$ with respect to the semiorthogonal decomposition $\mathscr{D}^b(\mathscr{X}_1) = \langle \mathscr{C}, \mathscr{A}_1(1) \boxtimes \mathscr{D}^b(\mathbf{P}(\mathrm{V}^*)), ..., \mathscr{A}_{i-1}(i-1) \boxtimes \mathscr{D}^b(\mathbf{P}(\mathrm{V}^*)) \rangle$.

Then $F_{\perp\mathscr{C}}$ is glued from its components in the subcategories $\mathscr{A}_1(1)\boxtimes\mathscr{D}^b(\mathbf{P}(V^*))$, ..., $\mathscr{A}_{i-1}(i-1)\boxtimes\mathscr{D}^b(\mathbf{P}(V^*))$. First of all let us compute the component of π^*F in $\mathscr{A}_{i-1}(i-1)\boxtimes\mathscr{D}^b(\mathbf{P}(V^*))$. It is given by applying to π^*F the right adjoint functor to the inclusion functor $\mathscr{A}_{i-1}(i-1)\boxtimes\mathscr{D}^b(\mathbf{P}(V^*))\to\mathscr{D}^b(\mathscr{X}_1)$. To compute this we take $F_1\in\mathscr{A}_{i-1}(i-1)$, $G_1\in\mathscr{D}^b(\mathbf{P}(V^*))$, $F_2=F$, $G_2=\mathscr{O}_{\mathbf{P}(V^*)}$ and consider the triangle of Lemma 5.1. The second term of this triangle vanishes because $F_1\in\mathscr{A}_{i-1}(i-1)$ and $F_2\in\mathscr{A}_0$ and the first term vanishes because

$$\mathsf{RHom}_X(F_1,F_2(-1))=\mathsf{RHom}_X(F_1(1),F_2)=\mathsf{RHom}_{\mathscr{A}_0}(\alpha_0^*(F_1(1)),F_2)$$

and since $F_1\in\mathscr{A}_{i-1}(i-1)=\mathfrak{a}_{i-1}(i-1)$ we have $\alpha_0^*(F_1(1))\in\alpha_0^*(\mathfrak{a}_{i-1}(i))$ which is orthogonal to the category $\langle\alpha_0^*(\mathfrak{a}_0(1)),...,\alpha_0^*(\mathfrak{a}_k(k+1))\rangle$ by Lemma 4.3. Therefore the component of the object π^*F in the category $\mathscr{A}_{i-1}(i-1)\boxtimes\mathscr{D}^b(\mathbf{P}(V^*))$ is zero. Similar arguments show that the components of π^*F in $\mathscr{A}_{i-2}(i-2)\boxtimes\mathscr{D}^b(\mathbf{P}(V^*))$, ..., $\mathscr{A}_{k+1}(k+1)\boxtimes\mathscr{D}^b(\mathbf{P}(V^*))$ are also zero.

Now let us compute the component of π^*F in $\mathscr{A}_k(k)\boxtimes\mathscr{D}^b(\mathbf{P}(V^*))$. It is given by applying to π^*F the right adjoint functor to the inclusion functor $\mathscr{A}_k(k)\boxtimes\mathscr{D}^b(\mathbf{P}(V^*))\to\mathscr{D}^b(\mathscr{X}_1)$. To compute this we take $F_1\in\mathscr{A}_k(k)$, $G_1\in\mathscr{D}^b(\mathbf{P}(V^*))$, $F_2=F$, $G_2=\mathscr{O}_{\mathbf{P}(V^*)}$ and consider the triangle of Lemma 5.1. Note again that the second term vanishes because $F_1\in\mathscr{A}_k(k)$ and $F_2\in\mathscr{A}_0$, hence

$$\mathsf{RHom}_{\mathscr{X}_1}(F_1\boxtimes G_1,\pi^*F)\cong$$
$$\mathsf{RHom}_X(F_1,F(-1))\otimes\mathsf{RHom}_{\mathbf{P}(V^*)}(G_1,\mathscr{O}_{\mathbf{P}(V^*)}(-1))[1],$$

and since the embedding of $\mathscr{A}_k(k)\boxtimes\mathscr{D}^b(\mathbf{P}(V^*))$ into $\mathscr{D}^b(\mathscr{X}_1)$ is fully faithful by Lemma 5.3, we conclude that the component of π^*F in $\mathscr{A}_k(k)\boxtimes\mathscr{D}^b(\mathbf{P}(V^*))$ equals $\alpha_k\alpha_k^!(F(-1))\boxtimes\mathscr{O}_{\mathbf{P}(V^*)}(-1)[1]$.

Now let us compute the component of π^*F in $\mathscr{A}_{k-1}(k-1)\boxtimes\mathscr{D}^b(\mathbf{P}(V^*))$. It is given by applying to the fiber of the morphism $\alpha_k\alpha_k^!(F(-1))\boxtimes\mathscr{O}_{\mathbf{P}(V^*)}(-1)[1]\to\pi^*F$ the right adjoint functor to the inclusion functor $\mathscr{A}_{k-1}(k-1)\boxtimes\mathscr{D}^b(\mathbf{P}(V^*))\to\mathscr{D}^b(\mathscr{X}_1)$. To compute this we take $F_1\in\mathscr{A}_{k-1}(k-1)$, $G_1\in\mathscr{D}^b(\mathbf{P}(V^*))$, and either $F_2=\alpha_k\alpha_k^!(F(-1))$, $G_2=\mathscr{O}_{\mathbf{P}(V^*)}(-1)[1]$, or $F_2=\pi^*F$, $G_2=\mathscr{O}_{\mathbf{P}(V^*)}$. Repeating the above arguments we see that the projection of the target equals $\alpha_{k-1}\alpha_{k-1}^!(F(-1))\boxtimes\mathscr{O}_{\mathbf{P}(V^*)}(-1)[1]$, and the projection of source is the cone of the morphism $\alpha_k\alpha_k^!(F(-1))(-1)\boxtimes\mathscr{O}_{\mathbf{P}(V^*)}(-2)[1]\to\alpha_{k-1}\alpha_{k-1}^!\alpha_k\alpha_k^!(F(-1))\boxtimes\mathscr{O}_{\mathbf{P}(V^*)}(-1)[1]$. Both projections are contained in $\mathscr{A}_{k-1}(k-1)\boxtimes\langle\mathscr{O}_{\mathbf{P}(V^*)}(-2),\mathscr{O}_{\mathbf{P}(V^*)}(-1)\rangle$, hence the same is true for the corresponding component of $F_{\perp\mathscr{C}}$.

Proceeding in the same manner we deduce the rest of the lemma. $\square$

Now we can prove

Proposition **5.7.** — *The restriction of the functor* $\gamma^*\circ\pi^*:\mathscr{D}^b(X)\to\mathscr{C}$ *to the subcategory* $\mathscr{A}_0\subset\mathscr{D}^b(X)$ *is fully faithful.*

Proof. — Take $F, F' \in \mathscr{A}_0$. Then $F'_{\perp \mathscr{C}} \in \pi^*(\mathscr{D}^b(X)) \otimes \langle \mathscr{O}_{\mathbf{P}(V^*)}(1 - i), ..., \mathscr{O}_{\mathbf{P}(V^*)}(-1) \rangle$ by Lemma 5.6. On the other hand, by Corollary 5.2 the pushforward functor $\pi_* : \mathscr{D}^b(\mathscr{X}_1) \to \mathscr{D}^b(X)$ takes the subcategory $\pi^*(\mathscr{D}^b(X)) \otimes \langle \mathscr{O}_{\mathbf{P}(V^*)}(2 - N), ..., \mathscr{O}_{\mathbf{P}(V^*)}(-1) \rangle$ to zero. Since $N > i$ by assumption (6), we conclude that $\pi_* F'_{\perp \mathscr{C}} = 0$. Therefore, by Lemma 5.5 we have $\mathsf{Hom}(\gamma^*\pi^* F, \gamma^*\pi^* F') \cong \mathsf{Hom}(\pi^* F, \pi^* F')$ which is isomorphic to $\mathsf{Hom}(F, F')$ by Corollary 5.2. $\qquad \square$

The following corollary is not needed below, however we put it here as an illustration.

Corollary **5.8.** — *The functors* $\pi_* \gamma : \mathscr{C} \to \mathscr{D}^b(X)$ *and* $\gamma^*\pi^* : \mathscr{D}^b(X) \to \mathscr{C}$ *are splitting.*

Proof. — The first functor is right adjoint to the second one, hence it suffices to check that $\gamma^*\pi^*$ is splitting. Since $\mathsf{Im}(\pi_* \gamma) \subset \mathscr{A}_0$ by Lemma 5.4 and $\gamma^*\pi^*$ is fully faithful on $\mathscr{A}_0$ we deduce that $\mathsf{Ker}(\gamma^*\pi^*) = {}^\perp \mathscr{A}_0 = \langle \mathscr{A}_1(1), ..., \mathscr{A}_{i-1}(i - 1) \rangle$ hence is admissible. Moreover, $\mathsf{Im}(\gamma^*\pi^*)$ is equivalent to $\mathscr{A}_0$ and $\mathscr{A}_0$ is saturated, hence $\mathsf{Im}(\gamma^*\pi^*)$ is admissible. $\qquad \square$

Let

$$(\mathbf{11}) \qquad \mathrm{j} = \mathrm{N} - 1 - \max\{k \mid \mathscr{A}_k = \mathscr{A}_0\}$$

(note that $\mathrm{j} > 0$ by (6)). Then $\mathfrak{a}_k = 0$ for $k < \mathrm{N} - 1 - \mathrm{j}$. Consider the subcategories

$$(\mathbf{12}) \qquad \begin{aligned} \mathscr{C}_k &= \gamma^*(\pi^*(\langle \alpha_0^*(\mathfrak{a}_0(1)), ..., \alpha_0^*(\mathfrak{a}_{\mathrm{N}-k-2}(\mathrm{N} - k - 1)) \rangle))) \\ &\subset \mathscr{C}_0 = \gamma^*(\pi^*(\mathscr{A}_0)), \end{aligned}$$

where we put $\mathfrak{a}_l = 0$ for $l \geq i$ for convenience. Note that

$$(\mathbf{13}) \qquad \mathscr{C}_k = 0 \quad \text{for } k \geq \mathrm{j}.$$

If the initial Lefschetz decomposition is rectangular then $\mathrm{j} = \mathrm{N} - i$ and $\mathscr{C}_{\mathrm{j}-1} = \mathscr{C}_{\mathrm{j}-2} = \cdots = \mathscr{C}_1 = \mathscr{C}_0$.

Lemma **5.9.** — *The chain of subcategories*

$$0 = \mathscr{C}_{\mathrm{j}-1} \subset \mathscr{C}_{\mathrm{j}-2} \subset \cdots \subset \mathscr{C}_1 \subset \mathscr{C}_0 \subset \mathscr{C}$$

is a chain of admissible subcategories in $\mathscr{C}$.

Proof. — The category $\langle \alpha_0^*(\mathfrak{a}_0(1)), ..., \alpha_0^*(\mathfrak{a}_{\mathrm{N}-k-2}(\mathrm{N} - k - 1)) \rangle$ is generated by a semiorthogonal collection (see Lemma 4.3) of admissible subcategories of $\mathscr{A}_0$, hence admissible in $\mathscr{A}_0$, hence saturated. Therefore its image under fully faithful functor $\gamma^*\pi^* : \mathscr{A}_0 \to \mathscr{C}$ is admissible. $\qquad \square$

Almost the same arguments show the following

Proposition **5.10.** — *The collection* $\langle \mathscr{C}_{j-1}(1-j), \mathscr{C}_{j-2}(2-j), ..., \mathscr{C}_1(-1), \mathscr{C}_0 \rangle$ *is semiorthogonal in* $\mathscr{C}$.

Proof. — Take $F \in \mathscr{A}_0$, $F' \in \langle \alpha_0^*(\mathfrak{a}_0(1)), ..., \alpha_0^*(\mathfrak{a}_{N-k-2}(N-k-1)) \rangle$. Then by Lemma 5.6 we have an inclusion $F'_{\perp\mathscr{C}} \in \pi^*(\mathscr{D}^b(X)) \boxtimes \langle \mathscr{O}(0, k+2-N), ..., \mathscr{O}(0, -1) \rangle$, hence $\pi_*(F'_{\perp\mathscr{C}}(0, -k)) = 0$ by Corollary 5.2. Therefore, by Lemma 5.5 we have an isomorphism $\mathsf{Hom}(\gamma^*\pi^*F, \gamma^*\pi^*F'(0, -k)) = \mathsf{Hom}(\pi^*F, \pi^*F'(0, -k))$ which equals zero for $1 \le k \le j-1$ by Corollary 5.2 (note that $j-1 \le N-2$ by (11)). $\square$

In the following section we prove that the semiorthogonal collection of the proposition generates $\mathscr{C}$. For the proof we use some additional assumptions, though the fact must be true without them. It would be interesting to find a direct proof.

We conclude the section with a couple of lemmas that will be useful later.

Lemma **5.11.** — *We have* $\mathsf{Im}\,\pi_*\gamma = \mathscr{A}_0$, $\mathsf{Ker}\,\gamma^*\pi^* = \langle \mathscr{A}_1(1), ..., \mathscr{A}_{i-1}(i-1) \rangle$, *and similarly* $\mathsf{Im}\,\gamma^*\pi^* = \mathscr{C}_0$, $\mathsf{Ker}\,\pi_*\gamma = \mathscr{C}_0^\perp \supset \langle \mathscr{C}_{j-1}(1-j), ..., \mathscr{C}_1(-1) \rangle$.

Proof. — Since $\mathsf{Im}\,\pi_*\gamma \subset \mathscr{A}_0$ by Lemma 5.4 and the functor $\gamma^*\pi^*$ (which is left adjoint to $\pi_*\gamma$) is fully faithful on $\mathscr{A}_0$ by Lemma 5.7, we deduce that $\mathsf{Im}\,\pi_*\gamma = \mathscr{A}_0$, $\mathsf{Ker}\,\gamma^*\pi^* = {}^\perp\mathscr{A}_0 = \langle \mathscr{A}_1(1), ..., \mathscr{A}_{i-1}(i-1) \rangle$, and $\mathsf{Im}\,\gamma^*\pi^* = \gamma^*\pi^*(\mathscr{A}_0) = \mathscr{C}_0$. Therefore $\mathsf{Ker}\,\pi_*\gamma = \mathscr{C}_0^\perp \supset \langle \mathscr{C}_{j-1}(1-j), ..., \mathscr{C}_1(-1) \rangle$ by Proposition 5.10. $\square$

Lemma **5.12.** — *We have* $\gamma^*\pi^*(\langle \mathscr{A}_0(1), ..., \mathscr{A}_{r-2}(r-1) \rangle^\perp) \subset \mathscr{C}_{N-r}$.

Proof. — Since $\langle \mathscr{A}_1(1), ..., \mathscr{A}_{i-1}(i-1) \rangle = \mathsf{Ker}(\gamma^*\pi^*)$ we have $\gamma^*\pi^*(F) = \gamma^*\pi^*\alpha_0^*(F)$ for any $F \in \mathscr{D}^b(X)$. On the other hand, by Lemma 4.4 we have $\alpha_0^*(\langle \mathscr{A}_0(1), ..., \mathscr{A}_{r-2}(r-1) \rangle^\perp) \subset \langle \alpha_0^*(\mathfrak{a}_0(1)), ..., \alpha_0^*(\mathfrak{a}_{r-2}(r-1)) \rangle$ and by definition of $\mathscr{C}_{N-r}$ we have $\gamma^*\pi^*(\langle \alpha_0^*(\mathfrak{a}_0(1)), ..., \alpha_0^*(\mathfrak{a}_{r-2}(r-1)) \rangle) = \mathscr{C}_{N-r}$. $\square$

Remark **5.13.** — Consider any smooth base scheme S (not necessarily compact) and assume that $p: X \to S$ is an algebraic S-variety, projective over S with an S-linear Lefschetz decomposition. Then all results of this section can be proved by essentially the same arguments. We only should replace RHom_X by $p_* \mathsf{R}\mathscr{H}om_X$, and the Serre functor of X by the relative Serre functor of X over S (see [K1]).

6. Homological projective duality

Recall the assumptions of the previous section: we have a smooth projective variety X with an effective line bundle $\mathscr{O}_X(1)$, a Lefschetz decomposition (2) of

its derived category, a vector space of global sections $V^* \subset \Gamma(X, \mathscr{O}_X(1))$ such that (6) holds, i.e. $N = \dim V > i$ (i is the number of terms in the Lefschetz decomposition). Assume also that the space V^* generates $\mathscr{O}_X(1)$, so that we have a regular morphism $f : X \to \mathbf{P}(V)$.

Recall that we denoted by $\mathscr{X}_1 \subset X \times \mathbf{P}(V^*)$ the universal hyperplane section of X and by $\mathscr{C}$ the right orthogonal to the subcategory $\langle \mathscr{A}_1(1) \boxtimes \mathscr{D}^b(\mathbf{P}(V^*)), ..., \mathscr{A}_{i-1}(i-1) \boxtimes \mathscr{D}^b(\mathbf{P}(V^*)) \rangle$ in $\mathscr{D}^b(\mathscr{X}_1)$,

$$\mathscr{C} = \left\langle \mathscr{A}_1(1) \boxtimes \mathscr{D}^b(\mathbf{P}(V^*)), ..., \mathscr{A}_{i-1}(i-1) \boxtimes \mathscr{D}^b(\mathbf{P}(V^*)) \right\rangle^{\perp}_{\mathscr{D}^b(\mathscr{X}_1)}.$$

Note that the category $\mathscr{C}$ is a module category over the tensor category $\mathscr{D}^b(\mathbf{P}(V^*))$: if $F \in \mathscr{C}$ and $G \in \mathscr{D}^b(\mathbf{P}(V^*))$ then $F \otimes f^*G \in \mathscr{C}$.

Now we need some additional assumptions. Assume that $\mathscr{C}$ is *geometric*, meaning that $\mathscr{C}$ is equivalent to the derived category of coherent sheaves on some algebraic variety Y. Let $\Phi : \mathscr{D}^b(Y) \to \mathscr{D}^b(\mathscr{X}_1)$ denote the composition of the equivalence $\mathscr{D}^b(Y) \to \mathscr{C}$ with the inclusion functor $\gamma : \mathscr{C} \to \mathscr{D}^b(\mathscr{X}_1)$. Further assume that the module structure on $\mathscr{C}$ is *geometric*, meaning that there is an algebraic morphism $g : Y \to \mathbf{P}(V^*)$, such that there is an isomorphism of bifunctors

$$\Phi(F \otimes g^*G) \cong \Phi(F) \otimes f^*G, \quad F \in \mathscr{D}^b(Y), \; G \in \mathscr{D}^b(\mathbf{P}(V^*)).$$

In other words, the functor Φ is assumed to be $\mathbf{P}(V^*)$-linear. Note also that the functor $\Phi : \mathscr{D}^b(Y) \to \mathscr{D}^b(\mathscr{X}_1)$ is fully faithful, hence by Orlov's Theorem [O2] it can be represented by a kernel on $Y \times \mathscr{X}_1$, at least if Y and X are projective. Moreover, it is easy to see that $\mathbf{P}(V^*)$-linearity of the functor Φ implies that the kernel of Φ is supported set-theoretically on the fiber product $Y \times_{\mathbf{P}(V^*)} \mathscr{X}_1$. Actually, it is natural to conjecture that the kernel is supported even *scheme-theoretically* on the fiber product (this must be a relative version of the Orlov's Theorem). However, we don't address this question here, taking this as an additional assumption. Finally, note that

$$Y \times_{\mathbf{P}(V^*)} \mathscr{X}_1 = Q(X, Y) := (X \times Y) \times_{\mathbf{P}(V) \times \mathbf{P}(V^*)} Q,$$

where $Q = \{(v, H) \in \mathbf{P}(V) \times \mathbf{P}(V^*) \mid v \in H\}$ is the incidence quadric.

Definition **6.1.** — *An algebraic variety* Y *with a projective morphism* $g : Y \to \mathbf{P}(V^*)$ *is called* homologically projectively dual *to* $f : X \to \mathbf{P}(V)$ *with respect to a Lefschetz decomposition* (2), *if there exists an object* $\mathscr{E} \in \mathscr{D}^b(Q(X, Y))$ *such that the functor* $\Phi = \Phi_{\mathscr{E}} : \mathscr{D}^b(Y) \to \mathscr{D}^b(\mathscr{X}_1)$ *is fully faithful and gives the following semiorthogonal decomposition*

$$(\mathbf{14}) \qquad \mathscr{D}^b(\mathscr{X}_1) =$$
$$\left\langle \Phi(\mathscr{D}^b(Y)), \mathscr{A}_1(1) \boxtimes \mathscr{D}^b(\mathbf{P}(V^*)), ..., \mathscr{A}_{i-1}(i-1) \boxtimes \mathscr{D}^b(\mathbf{P}(V^*)) \right\rangle.$$

In the next section we will explain the relation of homological projective duality to the classical projective duality, and now we will state and prove the main theorem about homologically projectively dual varieties.

For every linear subspace $L \subset V^*$ we consider the corresponding linear sections of X and Y:

$$X_L = X \times_{\mathbf{P}(V)} \mathbf{P}(L^\perp), \quad Y_L = Y \times_{\mathbf{P}(V^*)} \mathbf{P}(L),$$

where $L^\perp \subset V$ is the orthogonal subspace to $L \subset V^*$.

Définition **6.2.** — *A subspace* $L \subset V^*$ *is called* admissible, *if*

(a) $\dim X_L = \dim X - \dim L$, *and*
(b) $\dim Y_L = \dim Y + \dim L - N$.

The main result of this paper is the following

Theorem **6.3.** — *If* Y *is homologically projectively dual to* X *then*

(i) Y *is smooth and* $\mathscr{D}^b(Y)$ *admits a dual Lefschetz decomposition*

$$\mathscr{D}^b(Y) = \langle \mathscr{B}_{j-1}(1-j), ..., \mathscr{B}_1(-1), \mathscr{B}_0 \rangle,$$
$$0 \subset \mathscr{B}_{j-1} \subset \cdots \subset \mathscr{B}_1 \subset \mathscr{B}_0 \subset \mathscr{D}^b(Y)$$

with the same set of primitive subcategories: $\mathscr{B}_k = \langle \mathfrak{a}_0, ..., \mathfrak{a}_{N-k-2} \rangle$;
(ii) *for any admissible linear subspace* $L \subset V^*$, $\dim L = r$, *there exist a triangulated category* $\mathscr{C}_L$ *and semiorthogonal decompositions*

$$\mathscr{D}^b(X_L) = \langle \mathscr{C}_L, \mathscr{A}_r(1), ..., \mathscr{A}_{i-1}(i-r) \rangle$$
$$\mathscr{D}^b(Y_L) = \langle \mathscr{B}_{j-1}(N-r-j), ..., \mathscr{B}_{N-r}(-1), \mathscr{C}_L \rangle.$$

This theorem can be illustrated by the following picture:
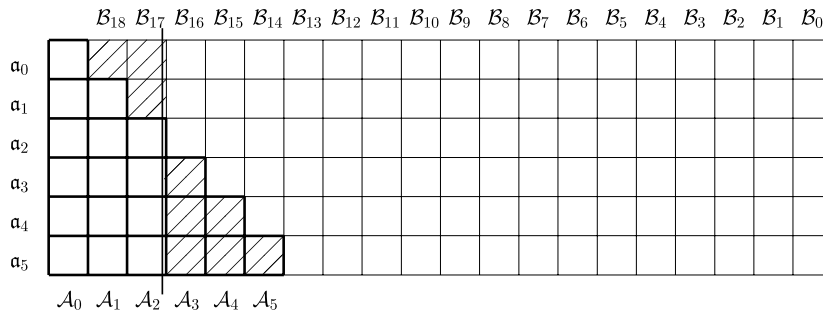
FIG. 1. — Illustration of Theorem 6.3: N = 20, i = 6, j = 19, r = 3

The bold Young diagram on the left represents the Lefschetz decomposition of $\mathscr{D}^b(X)$ and the rest of the rectangle represents the dual Lefschetz decomposition of $\mathscr{D}^b(Y)$. We divide the picture by a vertical line after the column with number $r = 3$. The hatched area on the right of this line represents the part of the Lefschetz decomposition of $\mathscr{D}^b(X)$ which is present in $\mathscr{D}^b(X_L)$, and the hatched area on the left of this line represents the part of the Lefschetz decomposition of $\mathscr{D}^b(Y)$ which is present in $\mathscr{D}^b(Y_L)$.

In fact the Lefschetz decomposition of $\mathscr{D}^b(Y)$ was constructed in the previous section. Indeed, by definition of homological projective duality the category $\mathscr{D}^b(Y)$ is equivalent to $\mathscr{C}$ and we have constructed a Lefschetz collection $\mathscr{C}_{j-1}(1-j), ..., \mathscr{C}_1(-1), \mathscr{C}_0 \subset \mathscr{C}$ in Proposition 5.10. The additional claim of the first part of the theorem is that this collection generates $\mathscr{D}^b(Y) \cong \mathscr{C}$.

The claim of the second part of the theorem can be reformulated as follows. Derived categories of orthogonal admissible linear sections of homologically projectively dual varieties admit semiorthogonal decompositions, one part of which comes from the Lefschetz decompositions of the ambient varieties, and the additional parts are equivalent. This behavior, analogous to the Lefschetz theory for cohomology of linear sections, was a motivation for our terminology.

Remark **6.4.** — Actually, to prove the claim of Theorem 6.3 we don't even need to know that Y is homologically projectively dual to X. Instead assume that Y is an algebraic variety, $g : Y \to \mathbf{P}(V^*)$ is a projective morphism, $\mathscr{E} \in \mathscr{D}^b(\mathscr{X}_1 \times_{\mathbf{P}(V^*)} Y)$ and

(1) the functor $\Phi_{\mathscr{E}} : \mathscr{D}^b(Y) \to \mathscr{D}^b(\mathscr{X}_1)$ factors as $\Phi_{\mathscr{E}} = \gamma \circ \varphi_{\mathscr{E}}$, where $\varphi_{\mathscr{E}} : \mathscr{D}^b(Y) \to \mathscr{C}$ is fully faithful;

(2) the functor $\Phi_{\mathscr{E}}^* \pi^* : \mathscr{D}^b(X) \to \mathscr{D}^b(Y)$ is fully faithful on the components $\alpha_0^*(\mathfrak{a}_k(k+1))$ of the semiorthogonal decomposition of the category $\mathscr{A}_0 \subset \mathscr{D}^b(X)$ constructed in Lemma 4.3;

(3) the subcategories $\mathscr{B}_k = \Phi_{\mathscr{E}}^* \pi^* (\langle \alpha_0^*(\mathfrak{a}_0(1)), ..., \alpha_0^*(\mathfrak{a}_{N-k-2}(N-k-1)) \rangle) = \varphi_{\mathscr{E}}^*(\mathscr{C}_k) \subset \mathscr{B}_0 = \varphi_{\mathscr{E}}^*(\mathscr{C}_0)$ form a semiorthogonal collection $\langle \mathscr{B}_{j-1}(1-j), \mathscr{B}_{j-2}(2-j), ..., \mathscr{B}_1(-1), \mathscr{B}_0 \rangle$ in $\mathscr{D}^b(Y)$.

Then we prove that $\varphi_{\mathscr{E}} : \mathscr{D}^b(Y) \to \mathscr{C}$ is an equivalence (so that a posteriori Y is homologically projectively dual to X) and that the claims of Theorem 6.3 hold true. Note also that if Y is homologically projectively dual to X then the assumptions (1)–(3) above are satisfied, so our arguments prove Theorem 6.3 as well.

From now on we take the assumptions 6.4(1)–(3).

6.1. *Universal families of linear sections.* — To prove the main theorem it is convenient to consider the universal families of linear sections of X and Y. All r-dimensional subspaces $L \subset V^*$ are parameterized by the Grassmannian $\mathsf{Gr}(r, V^*)$

which we denote for short by $\mathbf{G}_r$. Let $\mathscr{L}_r \subset V^* \otimes \mathscr{O}_{\mathbf{G}_r}$ be the tautological rank r subbundle on the Grassmannian $\mathbf{G}_r$ and let $\mathscr{L}_r^\perp := (V^* \otimes \mathscr{O}_{\mathbf{G}_r}/\mathscr{L}_r)^* \subset V \otimes \mathscr{O}_{\mathbf{G}_r}$ be the orthogonal subbundle. Then the universal families of linear sections of X and Y are

$$\mathscr{X}_r = (X \times \mathbf{G}_r) \times_{\mathbf{P}(V) \times \mathbf{G}_r} \mathbf{P}_{\mathbf{G}_r}(\mathscr{L}_r^\perp) \subset X \times \mathbf{G}_r,$$
$$\mathscr{Y}_r = (Y \times \mathbf{G}_r) \times_{\mathbf{P}(V^*) \times \mathbf{G}_r} \mathbf{P}_{\mathbf{G}_r}(\mathscr{L}_r) \subset Y \times \mathbf{G}_r.$$

It is clear that both $\mathscr{X}_r$ and $\mathscr{Y}_r$ are fibred over $\mathbf{G}_r = \mathsf{Gr}(r, V^*)$ with fibers X_L and Y_L over a point corresponding to a subspace $L \subset V^*$.

Consider the fiber product $\mathscr{X}_r \times_{\mathbf{G}_r} \mathscr{Y}_r$ and the projection $\pi_r : \mathscr{X}_r \times_{\mathbf{G}_r} \mathscr{Y}_r \to X \times Y$. Since for any vector subspace $L \subset V^*$ the product $\mathbf{P}(L^\perp) \times \mathbf{P}(L)$ is contained in the incidence quadric $Q \subset \mathbf{P}(V) \times \mathbf{P}(V^*)$ it follows that π_r factors via a map $\zeta_r : \mathscr{X}_r \times_{\mathbf{G}_r} \mathscr{Y}_r \to Q(X, Y) \subset X \times Y$.

Consider the object $\mathscr{E}_r = \zeta_r^* \mathscr{E} \in \mathscr{D}^b(\mathscr{X}_r \times_{\mathbf{G}_r} \mathscr{Y}_r)$ as a kernel on $\mathscr{X}_r \times \mathscr{Y}_r$. It gives the following kernel functors $\Phi_r = \Phi_{\mathscr{E}_r} : \mathscr{D}^b(\mathscr{Y}_r) \to \mathscr{D}^b(\mathscr{X}_r)$ and $\Phi_r^! = \Phi_{\mathscr{E}_r}^! : \mathscr{D}^b(\mathscr{X}_r) \to \mathscr{D}^b(\mathscr{Y}_r)$. We will check below that the functors Φ_r are splitting for all r and that there exist the following semiorthogonal decompositions

$$\mathscr{D}^b(\mathscr{X}_r) = \langle \mathscr{C}_r, \mathscr{A}_r(1) \boxtimes \mathscr{D}^b(\mathbf{G}_r), ..., \mathscr{A}_{i-1}(i - r) \boxtimes \mathscr{D}^b(\mathbf{G}_r) \rangle,$$
$$\mathscr{D}^b(\mathscr{Y}_r) = \langle \mathscr{B}_{j-1}(N - r - j) \boxtimes \mathscr{D}^b(\mathbf{G}_r), ..., \mathscr{B}_{N-r}(-1) \boxtimes \mathscr{D}^b(\mathbf{G}_r), \mathscr{C}_r \rangle,$$

where $\mathscr{C}_r = \mathsf{Im}\,\Phi_r$. After that we deduce from this the main Theorem 6.3 using the faithful base change Theorem 2.40.

For the proof of the above decompositions we use induction in r. Note that for $r = 1$ we have $\mathbf{G}_1 = \mathbf{P}(V^*)$, $\mathscr{Y}_1 = Y$, $\mathscr{X}_1$ is the universal hyperplane section of X and $\mathscr{E}_1 = \mathscr{E}$, hence the base of induction is given by the definition of homological projective duality ($\mathscr{C}_1 = \mathscr{C} = \mathscr{D}^b(Y)$).

To compare the universal families $\mathscr{X}_{r-1}, \mathscr{Y}_{r-1}$ and $\mathscr{X}_r, \mathscr{Y}_r$ we take for a base scheme

$$\mathbf{S}_r = \mathsf{Fl}(r - 1, r; V^*) \subset \mathsf{Gr}(r - 1, V^*) \times \mathsf{Gr}(r, V^*) = \mathbf{G}_{r-1} \times \mathbf{G}_r,$$

the partial flag variety. The scheme $\mathbf{S}_r$ parameterizes flags $L_{r-1} \subset L_r \subset V^*$ such that $\dim L_{r-1} = r - 1$, and $\dim L_r = r$. Let $\phi : \mathbf{S}_r \to \mathbf{G}_{r-1}$ and $\psi : \mathbf{S}_r \to \mathbf{G}_r$ denote the natural projections. Let $\tilde{\mathscr{L}}_{r-1} = \phi^* \mathscr{L}_{r-1}$, $\tilde{\mathscr{L}}_r = \psi^* \mathscr{L}_r$, $\tilde{\mathscr{L}}_{r-1}^\perp = \phi^* \mathscr{L}_{r-1}^\perp$, $\tilde{\mathscr{L}}_r^\perp = \psi^* \mathscr{L}_r^\perp$. Then we have the universal flags of subbundles

(**15**) $$\tilde{\mathscr{L}}_{r-1} \subset \tilde{\mathscr{L}}_r \subset V^* \otimes \mathscr{O}_{\mathbf{S}_r}, \quad \tilde{\mathscr{L}}_r^\perp \subset \tilde{\mathscr{L}}_{r-1}^\perp \subset V \otimes \mathscr{O}_{\mathbf{S}_r}.$$

Denote

$$\tilde{\mathscr{X}}_{r-1} = \mathscr{X}_{r-1} \times_{\mathbf{G}_{r-1}} \mathbf{S}_r \subset X \times \mathbf{S}_r, \quad \tilde{\mathscr{X}}_r = \mathscr{X}_r \times_{\mathbf{G}_r} \mathbf{S}_r \subset X \times \mathbf{S}_r,$$
$$\tilde{\mathscr{Y}}_{r-1} = \mathscr{Y}_{r-1} \times_{\mathbf{G}_{r-1}} \mathbf{S}_r \subset Y \times \mathbf{S}_r, \quad \tilde{\mathscr{Y}}_r = \mathscr{Y}_r \times_{\mathbf{G}_r} \mathbf{S}_r \subset Y \times \mathbf{S}_r.$$

Note that

$$\tilde{\mathscr{X}}_{r-1} = (X \times \mathbf{S}_r) \times_{\mathbf{P}(V) \times \mathbf{S}_r} \mathbf{P}_{\mathbf{S}_r}(\tilde{\mathscr{L}}_{r-1}^{\perp}), \quad \tilde{\mathscr{X}}_r = (X \times \mathbf{S}_r) \times_{\mathbf{P}(V) \times \mathbf{S}_r} \mathbf{P}_{\mathbf{S}_r}(\tilde{\mathscr{L}}_r^{\perp}),$$

$$\tilde{\mathscr{Y}}_{r-1} = (Y \times \mathbf{S}_r) \times_{\mathbf{P}(V^*) \times \mathbf{S}_r} \mathbf{P}_{\mathbf{S}_r}(\tilde{\mathscr{L}}_{r-1}), \quad \tilde{\mathscr{Y}}_r = (Y \times \mathbf{S}_r) \times_{\mathbf{P}(V^*) \times \mathbf{S}_r} \mathbf{P}_{\mathbf{S}_r}(\tilde{\mathscr{L}}_r).$$

Therefore the embeddings (15) induce embeddings $\xi : \tilde{\mathscr{X}}_r \to \tilde{\mathscr{X}}_{r-1}$ and $\eta : \tilde{\mathscr{Y}}_{r-1} \to \tilde{\mathscr{Y}}_r$. Consider the following commutative diagrams (the squares marked with $\boxed{\text{EC}}$ are exact cartesian by Lemma 2.32, because the maps ϕ and ψ are flat)

(**16**)

where f_{r-1}, f_r, g_{r-1} and g_r are the natural projections and $\hat{\phi} = \phi \circ \xi$, $\hat{\psi} = \psi \circ \eta$.

Let $\tilde{\mathscr{E}}_{r-1} \in \mathscr{D}(\tilde{\mathscr{X}}_{r-1} \times_{\mathbf{S}_r} \tilde{\mathscr{Y}}_{r-1})$ and $\tilde{\mathscr{E}}_r \in \mathscr{D}(\tilde{\mathscr{X}}_r \times_{\mathbf{S}_r} \tilde{\mathscr{Y}}_r)$ denote the pullbacks of the objects $\mathscr{E}_{r-1}$ and $\mathscr{E}_r$ via the projections $\tilde{\mathscr{X}}_{r-1} \times_{\mathbf{S}_r} \tilde{\mathscr{Y}}_{r-1} \to \mathscr{X}_{r-1} \times_{\mathbf{G}_{r-1}} \mathscr{Y}_{r-1}$, $\tilde{\mathscr{X}}_r \times_{\mathbf{S}_r} \tilde{\mathscr{Y}}_r \to \mathscr{X}_r \times_{\mathbf{G}_r} \mathscr{Y}_r$. Then we have the corresponding kernel functors $\tilde{\Phi}_{r-1}$, $\tilde{\Phi}_r$ e.t.c. between the derived categories of $\tilde{\mathscr{X}}_{r-1}$, $\tilde{\mathscr{Y}}_{r-1}$, $\tilde{\mathscr{X}}_r$ and $\tilde{\mathscr{Y}}_r$.

The induction step is based on relation of the functors Φ_{r-1}, Φ_r, $\tilde{\Phi}_{r-1}$ and $\tilde{\Phi}_r$ to the base change functors ψ^*, ψ_*, ϕ^*, ϕ_* and to the functors of the pushforward and pullback via ξ and η. The relation to ψ^*, ψ_*, ϕ^* and ϕ_* is given by Lemma 2.38. The relation to ξ and η in a sense is the key point of the proof. We prove that $\xi^*\tilde{\Phi}_{r-1} \cong \tilde{\Phi}_r\eta_*$ and that the "difference" between $\xi_*\tilde{\Phi}_r$ and $\tilde{\Phi}_{r-1}\eta^!$ is given by a very simple functor.

Other results in this section (e.g. the above semiorthogonal decompositions) are proved by similar arguments using (either ascending or descending) induction in r.

The section is organized as follows. We start with some preparations concluding with a description of the relation of the functors $\tilde{\Phi}_{r-1}$ and $\tilde{\Phi}_r$ to the pushforward and pullback via ξ and η. Then we use induction in r to prove the semiorthogonal decompositions.

6.2. *Preparations*

Lemma **6.5.** — Y *is smooth.*

Proof. — By assumption 6.4(1) the category $\mathscr{D}^b(Y)$ is a full subcategory of $\mathscr{D}^b(\mathscr{X}_1)$. On the other hand, it is easy to see that the projection $\mathscr{X}_1 \to X$ is

a projectivization of a vector bundle (the fiber of $\mathscr{X}_1$ over a point $x \in X$ consists of all hyperplanes in $\mathbf{P}(V)$ passing through $f(x) \in \mathbf{P}(V)$ which is nothing but the projectivization of the fiber of the vector bundle $f^*\Omega_{\mathbf{P}(V)}(1)$ at x). In particular, the map $\mathscr{X}_1 \to X$ is smooth, hence $\mathscr{X}_1$ is smooth as well. It follows that the category $\mathscr{D}^b(\mathscr{X}_1)$ is Ext-bounded by Lemma 2.25. Therefore $\mathscr{D}^b(Y)$ is also Ext-bounded, hence Y is smooth again by Lemma 2.25. $\qquad\square$

Recall that $\mathbf{G}_r = \mathsf{Gr}(r, V^*)$ is the Grassmannian parameterizing linear sections of X and Y, and $\mathscr{X}_r$, $\mathscr{Y}_r$ are the universal families over $\mathbf{G}_r$ of linear sections. For any variety equipped with a map to $X \times Y$ (e.g. for $Q(X, Y)$ or $\mathscr{X}_r \times_{\mathbf{G}_r} \mathscr{Y}_r$) we denote by $\mathscr{O}(k, l)$ the pullback of $\mathscr{O}_X(k) \boxtimes \mathscr{O}_Y(l)$ for brevity.

Lemma **6.6.** — *The projections* $\mathscr{X}_r \to X$, $\mathscr{Y}_r \to Y$ *and* $\zeta_r : \mathscr{X}_r \times_{\mathbf{G}_r} \mathscr{Y}_r \to Q(X, Y)$ *are smooth. In particular,* $\mathscr{X}_r$ *and* $\mathscr{Y}_r$ *are smooth and*

$$\dim \mathscr{X}_r = \dim X + \dim \mathbf{G}_r - r, \quad \dim \mathscr{Y}_r = \dim Y + \dim \mathbf{G}_r + r - N,$$
$$\dim \mathscr{X}_r \times_{\mathbf{G}_r} \mathscr{Y}_r = \dim X + \dim Y + \dim \mathbf{G}_r - N.$$

Moreover, the maps $f_r : \mathscr{X}_r \to \mathbf{G}_r$ *and* $g_r : \mathscr{Y}_r \to \mathbf{G}_r$ *are projective.*

Proof. — Note that we have the following isomorphisms

$$\mathscr{X}_r = \mathsf{Gr}_X(r, \mathscr{V}_X), \quad \mathscr{Y}_r = \mathsf{Gr}_Y(r - 1, \mathscr{V}_Y),$$
$$\mathscr{X}_r \times_{\mathbf{G}_r} \mathscr{Y}_r = \mathsf{Gr}_{Q(X,Y)}(r - 1, \mathscr{V}_Q)$$

with the relative Grassmannians, where the bundles $\mathscr{V}_X$, $\mathscr{V}_Y$ are defined from exact sequences

$$0 \to \mathscr{V}_X \to V^* \otimes \mathscr{O}_X \to \mathscr{O}_X(1) \to 0,$$
$$0 \to \mathscr{O}_Y(-1) \to V^* \otimes \mathscr{O}_Y \to \mathscr{V}_Y \to 0,$$

and $\mathscr{V}_Q$ is the middle cohomology bundle of the complex

$$\mathscr{O}_{Q(X,Y)}(0, -1) \to V^* \otimes \mathscr{O}_{Q(X,Y)} \to \mathscr{O}_{Q(X,Y)}(1, 0).$$

From this and Lemma 6.5 we easily deduce the smoothness and compute the dimensions. It is also clear that the fibers of the projections $\mathscr{X}_r \to \mathbf{G}_r$, and $\mathscr{Y}_r \to \mathbf{G}_r$ are linear sections of X and Y corresponding to subspaces $L \in \mathbf{G}_r$, so they are projective. $\qquad\square$

Lemma **6.7.** — *In parts* (i) *and* (ii) *below* k *stands either for* r *or for* $r - 1$.

(i) $\tilde{\mathscr{X}}_k$ *is the zero locus of a section of vector bundle* $\mathscr{O}_X(1) \boxtimes \tilde{\mathscr{L}}_k^*$ *on* $X \times \mathbf{S}_r$;

(ii) $\tilde{\mathscr{Y}}_k$ *is the zero locus of a section of vector bundle* $\mathscr{O}_Y(1) \boxtimes \tilde{\mathscr{L}}_k^{\perp*}$ *on* $Y \times \mathbf{S}_r$;

(*iii*) $\tilde{\mathscr{X}}_r$ *is the zero locus of a section of line bundle* $\mathscr{O}_X(1) \otimes (\tilde{\mathscr{L}}_r/\tilde{\mathscr{L}}_{r-1})^*$ *on* $\tilde{\mathscr{X}}_{r-1}$;
(*iv*) $\tilde{\mathscr{Y}}_{r-1}$ *is the zero locus of a section of line bundle* $\mathscr{O}_Y(1) \otimes (\tilde{\mathscr{L}}_{r-1}^\perp/\tilde{\mathscr{L}}_r^\perp)^* \cong$
$\mathscr{O}_Y(1) \otimes (\tilde{\mathscr{L}}_r/\tilde{\mathscr{L}}_{r-1})$ *on* $\tilde{\mathscr{Y}}_r$.

All these sections are regular.

Proof. — The parts (*i*) and (*ii*) evidently follow from the definition of $\tilde{\mathscr{X}}_k \subset$ $X \times \mathbf{S}_r$. The parts (*iii*) and (*iv*) follow from the exact sequences

$$0 \to (\tilde{\mathscr{L}}_r/\tilde{\mathscr{L}}_{r-1})^* \to \tilde{\mathscr{L}}_r^* \to \tilde{\mathscr{L}}_{r-1}^* \to 0,$$
$$0 \to (\tilde{\mathscr{L}}_{r-1}^\perp/\tilde{\mathscr{L}}_r^\perp)^* \to (\tilde{\mathscr{L}}_{r-1}^\perp)^* \to (\tilde{\mathscr{L}}_r^\perp)^* \to 0.$$

Finally, it follows from Lemma 6.6 that $\dim \tilde{\mathscr{X}}_{r-1} = \dim X + \dim \mathbf{S}_r - (r-1)$, $\dim \tilde{\mathscr{X}}_r = \dim X + \dim \mathbf{S}_r - r$, since the base changes $\mathbf{S}_r \to \mathbf{G}_{r-1}$ and $\mathbf{S}_r \to \mathbf{G}_r$ are flat. Therefore the sections in the parts (*i*) and (*iii*) are regular. The sections in the parts (*ii*) and (*iv*) are regular by similar reasons. □

Lemma **6.8.** — *For any* $r \le k \le \mathrm{i} - 1$ *the functors* $\mathscr{A}_k \boxtimes \mathscr{D}^b(\mathbf{G}_r) \subset \mathscr{D}^b(X \times \mathbf{G}_r)$ $\xrightarrow{i^*} \mathscr{D}^b(\mathscr{X}_r)$ *are fully faithful, and the collection* $(\mathscr{A}_r(1) \boxtimes \mathscr{D}^b(\mathbf{G}_r), ..., \mathscr{A}_{i-1}(\mathrm{i} - r) \boxtimes$ $\mathscr{D}^b(\mathbf{G}_r)) \subset \mathscr{D}^b(\mathscr{X}_r)$ *is semiorthogonal.*

Similarly, for any $\mathrm{N} - r \le k \le \mathrm{j} - 1$ *the functors* $\mathscr{B}_k \boxtimes \mathscr{D}^b(\mathbf{G}_r) \subset \mathscr{D}^b(Y \times \mathbf{G}_r)$ $\xrightarrow{i^*} \mathscr{D}^b(\mathscr{Y}_r)$ *are fully faithful, and the collection* $(\mathscr{B}_{j-1}(\mathrm{N} - r - \mathrm{j}) \boxtimes \mathscr{D}^b(\mathbf{G}_r), ..., \mathscr{B}_{\mathrm{N}-r}(-1)$ $\boxtimes \mathscr{D}^b(\mathbf{G}_r)) \subset \mathscr{D}^b(\mathscr{Y}_r)$ *is semiorthogonal.*

Proof. — Analogous to the proof of Lemma 5.3 using the Koszul resolutions of $\mathscr{O}_{\mathscr{X}_r}$ on $X \times \mathbf{G}_r$ and of $\mathscr{O}_{\mathscr{Y}_r}$ on $Y \times \mathbf{G}_r$ (for the second part we use assumption 6.4(3)). □

Now we describe the maps ψ and ϕ.

Lemma **6.9.** — *The maps* ϕ, ψ, $\hat{\phi}$ *and* $\hat{\psi}$ *are projectivizations of vector bundles. Explicitly,* ϕ *is the projectivization of* $V^* \otimes \mathscr{O}/\mathscr{L}_{r-1}$, ψ *is the projectivization of* $\mathscr{L}_r^*$, $\hat{\phi}$ *is the projectivization of* $\mathscr{V}_X/\mathscr{L}_{r-1}$, *and* $\hat{\psi}$ *is the projectivization of* $(\mathscr{L}_r/\mathscr{O}_{\mathbf{P}(V^*)}(-1))^*$, *where the embedding* $\mathscr{O}_{\mathbf{P}(V^*)}(-1) \to \mathscr{L}_r$ *is induced by the projection* $\mathscr{Y}_r \to \mathbf{P}_{\mathbf{G}_r}(\mathscr{L}_r) \to \mathbf{P}(V^*)$.

Proof. — By definition of $\mathbf{S}_r$ the fiber of ϕ is the set of all lines in V^*/L_{r-1} and the fiber of ψ is the set of all hyperplanes in L_r. Similarly, the fiber of $\hat{\phi}$ is the set of all lines in V^*/L_{r-1} contained in $\mathscr{V}_x$, where $\mathscr{V}_x$ is the fiber at x of the vector bundle $\mathscr{V}_X$ on X defined in the proof of Lemma 6.6, and the fiber of $\hat{\psi}$ is the set of all hyperplanes in L_r passing through a point $y \in \mathbf{P}(L_r)$. □

Applying results of [O1] we deduce the following.

Corollary **6.10.** — *The functors ϕ^*, ψ^*, $\hat{\phi}^*$ and $\hat{\psi}^*$ are fully faithful and we have*

$$\phi_* \phi^* \cong \mathsf{id}, \quad \psi_* \psi^* \cong \mathsf{id}, \quad \hat{\phi}_* \hat{\phi}^* \cong \mathsf{id}, \quad \hat{\psi}_* \hat{\psi}^* \cong \mathsf{id}.$$

Recall that we have defined the objects $\mathscr{E}_r$ on $\mathscr{X}_r \times_{\mathbf{G}_r} \mathscr{Y}_r$ as the pullbacks of $\mathscr{E} \in \mathscr{D}^b(\mathrm{Q}(\mathrm{X}, \mathrm{Y}))$ via the map $\mathscr{X}_r \times_{\mathbf{G}_r} \mathscr{Y}_r \to \mathrm{Q}(\mathrm{X}, \mathrm{Y})$, and the objects $\tilde{\mathscr{E}}_{r-1}$ and $\tilde{\mathscr{E}}_r$ as the pullbacks of $\mathscr{E}_{r-1}$ and $\mathscr{E}_r$ via the maps $\phi : \tilde{\mathscr{X}}_{r-1} \times_{\mathbf{S}_r} \tilde{\mathscr{Y}}_{r-1} \to \mathscr{X}_{r-1} \times_{\mathbf{G}_{r-1}} \mathscr{Y}_{r-1}$ and $\psi : \tilde{\mathscr{X}}_r \times_{\mathbf{S}_r} \tilde{\mathscr{Y}}_r \to \mathscr{X}_r \times_{\mathbf{G}_r} \mathscr{Y}_r$. The functors Φ_{r-1}, Φ_r, $\Phi^!_{r-1}$, $\Phi^!_r$, $\tilde{\Phi}_{r-1}$, $\tilde{\Phi}_r$, $\tilde{\Phi}^!_{r-1}$, and $\tilde{\Phi}^!_r$ are kernel functors of the first and second type corresponding to the kernels $\mathscr{E}_{r-1}$, $\mathscr{E}_r$, $\tilde{\mathscr{E}}_{r-1}$ and $\tilde{\mathscr{E}}_r$ respectively.

Lemma **6.11.** — *The functors Φ_{r-1}, Φ_r, $\tilde{\Phi}_{r-1}$ and $\tilde{\Phi}_r$ have right adjoint functors $\Phi^!_{r-1}$, $\Phi^!_r$, $\tilde{\Phi}^!_{r-1}$, and $\tilde{\Phi}^!_r$, and left adjoint functors Φ^*_{r-1}, Φ^*_r, $\tilde{\Phi}^*_{r-1}$ and $\tilde{\Phi}^*_r$. All these functors take bounded derived categories to bounded derived categories.*

Proof. — First of all note that $\mathscr{E}_r = \zeta^*_r \mathscr{E}$ is bounded because $\zeta_r : \mathscr{X}_r \times_{\mathbf{G}_r} \mathscr{Y}_r \to \mathrm{Q}(\mathrm{X}, \mathrm{Y})$ is smooth by Lemma 6.6. Since $\mathscr{X}_r$ and $\mathscr{Y}_r$ are smooth by Lemma 6.6 it follows that the pushforward of $\mathscr{E}_r$ to $\mathscr{X}_r \times \mathscr{Y}_r$ is a perfect complex. In particular it has finite Tor and Ext-amplitude over $\mathscr{X}_r$ and $\mathscr{Y}_r$. On the other hand, the projections of $\mathscr{X}_r \times_{\mathbf{G}_r} \mathscr{Y}_r$ to the factors are projective because the projections of $\mathscr{X}_r$ and $\mathscr{Y}_r$ to $\mathbf{G}_r$ are projective by Lemma 6.6. Therefore by Lemma 2.27 the functor $\Phi^!_r$ is right adjoint to Φ_r and by Lemma 2.28 there exists a left adjoint functor Φ^*_r to Φ_r. Moreover, all these functors take bounded derived categories to bounded derived categories. The same arguments prove the rest of the lemma. □

Proposition **6.12.** — *We have functorial isomorphisms*

$$\phi^* \Phi_{r-1} \cong \tilde{\Phi}_{r-1} \phi^*, \quad \psi^* \Phi_r \cong \tilde{\Phi}_r \psi^*,$$

$$\phi_* \tilde{\Phi}_{r-1} \cong \Phi_{r-1} \phi_*, \quad \psi_* \tilde{\Phi}_r \cong \Phi_r \psi_*.$$

Proof. — Note that the base changes ϕ and ψ are smooth, hence they are faithful by Lemma 2.36 and we conclude by Lemma 2.38. □

Now we go to the relation of the functors $\tilde{\Phi}_{r-1}$ and $\tilde{\Phi}_r$ to the pushforward and pullback via ξ and η.

Consider the following diagram

$$
\begin{array}{ccc}
\tilde{\mathscr{X}}_r \times_{\mathbf{S}_r} \tilde{\mathscr{Y}}_{r-1} & \xrightarrow{\eta} & \tilde{\mathscr{X}}_r \times_{\mathbf{S}_r} \tilde{\mathscr{Y}}_r \\
{\scriptstyle \xi} \downarrow & & \downarrow {\scriptstyle \xi} \\
\tilde{\mathscr{X}}_{r-1} \times_{\mathbf{S}_r} \tilde{\mathscr{Y}}_{r-1} & \xrightarrow{\eta} & \tilde{\mathscr{X}}_{r-1} \times_{\mathbf{S}_r} \tilde{\mathscr{Y}}_r \xrightarrow{\tilde{\pi}} \mathrm{X} \times \mathrm{Y}
\end{array}
$$

where $\tilde{\pi}$ is the composition $\tilde{\mathscr{X}}_{r-1} \times_{\mathbf{S}_r} \tilde{\mathscr{Y}}_r \subset \tilde{\mathscr{X}}_{r-1} \times \tilde{\mathscr{Y}}_r \subset (\mathrm{X} \times \mathbf{S}_r) \times (\mathrm{Y} \times \mathbf{S}_r) \to \mathrm{X} \times \mathrm{Y}$.

Lemma **6.13.** — *The maps ξ and η in the above diagram are divisorial embeddings and*

(**17**)
$$\xi^!(-) = \xi^*(-) \otimes \mathscr{O}(1,0) \otimes (\tilde{\mathscr{L}}_r/\tilde{\mathscr{L}}_{r-1})^*[-1],$$
$$\eta^!(-) = \eta^*(-) \otimes \mathscr{O}(0,1) \otimes (\tilde{\mathscr{L}}_r/\tilde{\mathscr{L}}_{r-1})[-1].$$

Moreover, we have the following scheme-theoretical equalities

(**18**)
$$(\tilde{\mathscr{X}}_{r-1} \times_{\mathbf{S}_r} \tilde{\mathscr{Y}}_{r-1}) \cap (\tilde{\mathscr{X}}_r \times_{\mathbf{S}_r} \tilde{\mathscr{Y}}_r) = \tilde{\mathscr{X}}_r \times_{\mathbf{S}_r} \tilde{\mathscr{Y}}_{r-1},$$
$$(\tilde{\mathscr{X}}_{r-1} \times_{\mathbf{S}_r} \tilde{\mathscr{Y}}_{r-1}) \cup (\tilde{\mathscr{X}}_r \times_{\mathbf{S}_r} \tilde{\mathscr{Y}}_r) = \tilde{\pi}^{-1}(\mathrm{Q}(\mathrm{X},\mathrm{Y}))$$

and the following square is exact carthesian

(**19**)

$$
\begin{array}{ccc}
(\tilde{\mathscr{X}}_{r-1} \times_{\mathbf{S}_r} \tilde{\mathscr{Y}}_{r-1}) \cup (\tilde{\mathscr{X}}_r \times_{\mathbf{S}_r} \tilde{\mathscr{Y}}_r) & \xrightarrow{\ i\ } & \tilde{\mathscr{X}}_{r-1} \times_{\mathbf{S}_r} \tilde{\mathscr{Y}}_r \\
\downarrow{\tilde{\xi}} & \boxed{\text{EC}} & \downarrow{\tilde{\pi}} \\
\mathrm{Q}(\mathrm{X},\mathrm{Y}) & \xrightarrow{\ i\ } & \mathrm{X} \times \mathrm{Y}.
\end{array}
$$

Proof. — Consider the projections of $\tilde{\mathscr{X}}_r \times_{\mathbf{S}_r} \tilde{\mathscr{Y}}_{r-1}$, $\tilde{\mathscr{X}}_{r-1} \times_{\mathbf{S}_r} \tilde{\mathscr{Y}}_{r-1}$, $\tilde{\mathscr{X}}_r \times_{\mathbf{S}_r} \tilde{\mathscr{Y}}_r$ and $\tilde{\mathscr{X}}_{r-1} \times_{\mathbf{S}_r} \tilde{\mathscr{Y}}_r$ to $\mathrm{X} \times \mathrm{Y}$. It is easy to check that their fibers over a point $(x,y) \in (\mathrm{X},\mathrm{Y})$ are subsets of the flag variety $\mathsf{Fl}(r-1,r;\mathrm{V}^*) = \mathbf{S}_r$ consisting of all flags $\mathrm{L}_{r-1} \subset \mathrm{L}_r$ satisfying the following incidence conditions

$$
\begin{array}{cc}
y \subset \mathrm{L}_{r-1} & y \subset \mathrm{L}_{r-1} \subset \mathscr{V}_x \\
\cap & \cap \\
\mathrm{L}_r \subset \mathscr{V}_x & \mathrm{L}_r
\end{array}
, \qquad ,
$$

$$
\begin{array}{cc}
\mathrm{L}_{r-1} & \mathrm{L}_{r-1} \subset \mathscr{V}_x \\
\cap & \cap \\
y \subset \mathrm{L}_r \subset \mathscr{V}_x & y \subset \mathrm{L}_r
\end{array}
\quad \text{and}
$$

respectively. In particular, the first three fibers are empty if $(x,y) \notin \mathrm{Q}(\mathrm{X},\mathrm{Y})$. On the other hand, over $\mathrm{Q}(\mathrm{X},\mathrm{Y})$ the first three fibers are irreducible, have dimension $(r-1)(\mathrm{N}-r)-1$, $(r-1)(\mathrm{N}-r)$ and $(r-1)(\mathrm{N}-r)$ respectively, and the first of them is the intersection of the other two. The fourth fiber is irreducible and $(r-1)(\mathrm{N}-r)$-dimensional if $(x,y) \notin \mathrm{Q}(\mathrm{X},\mathrm{Y})$ and for $(x,y) \in \mathrm{Q}(\mathrm{X},\mathrm{Y})$ it coincides with the union of the second and the third fibers (if $y \subset \mathscr{V}_x$ and $y \not\subset \mathrm{L}_{r-1}$ then $\mathrm{L}_r = \langle y, \mathrm{L}_{r-1} \rangle \subset \mathscr{V}_x$). It follows that images of ξ and η have pure codimension 1. Since they are also zero loci of line bundles by Lemma 6.7(*iii*) and (*iv*), we conclude that ξ and η are divisorial embeddings.

The above arguments also prove the first equality of (18) on the scheme-theoretical level and the second equality on the set-theoretical level. Taking into account that the RHS of the second equality is the zero locus of the line bundle $\mathscr{O}_X(1) \otimes \mathscr{O}_Y(1)$ by definition of $Q(X, Y)$, and that the LHS of the equality is the zero locus of the line bundle $(\mathscr{O}_X(1) \otimes (\tilde{\mathscr{L}}_r/\tilde{\mathscr{L}}_{r-1})^*) \otimes (\mathscr{O}_Y(1) \otimes (\tilde{\mathscr{L}}_r/\tilde{\mathscr{L}}_{r-1}))$ by Lemma 6.7(*iii*) and (*iv*), and noting that these bundles are isomorphic, we deduce that the second equality is also true on the scheme-theoretical level.

Finally, we note $X \times Y$ is Cohen–Macaulay, and that it follows from the above description of fibers of $\tilde{\pi}$ that $\tilde{\mathscr{X}}_{r-1} \times_{\mathbf{S}_r} \tilde{\mathscr{Y}}_r$ is Cohen–Macaulay as well. Therefore the square (19) is exact cartesian by Lemma 2.32(*iii*). $\square$

Consider the pullback $\widehat{\mathscr{E}} = \tilde{\zeta}^* \mathscr{E}$ of $\mathscr{E}$ from $Q(X, Y)$ to $(\tilde{\mathscr{X}}_{r-1} \times_{\mathbf{S}_r} \tilde{\mathscr{Y}}_{r-1}) \cup (\tilde{\mathscr{X}}_r \times_{\mathbf{S}_r} \tilde{\mathscr{Y}}_r)$. The following lemma gives a relation of $\tilde{\mathscr{E}}_{r-1}$ and $\tilde{\mathscr{E}}_r$.

Lemma **6.14.** — *We have the following exact triangle on* $\tilde{\mathscr{X}}_{r-1} \times_{\mathbf{S}_r} \tilde{\mathscr{Y}}_r$:

$$\textbf{(20)}\qquad \xi_* \tilde{\mathscr{E}}_r(0, -1) \otimes (\tilde{\mathscr{L}}_r/\tilde{\mathscr{L}}_{r-1})^* \to i_* \widehat{\mathscr{E}} \to \eta_* \tilde{\mathscr{E}}_{r-1}.$$

Moreover, we have an isomorphism on $\tilde{\mathscr{X}}_r \times_{\mathbf{S}_r} \tilde{\mathscr{Y}}_{r-1}$:

$$\textbf{(21)}\qquad \eta^* \tilde{\mathscr{E}}_r \cong \xi^* \tilde{\mathscr{E}}_{r-1}$$

and an isomorphism on $\tilde{\mathscr{X}}_{r-1} \times_{\mathbf{S}_r} \tilde{\mathscr{Y}}_r$

$$\textbf{(22)}\qquad i_* \widehat{\mathscr{E}} \cong \tilde{\pi}^* i_* \mathscr{E}.$$

Proof. — Since the square (19) is exact cartesian we have $i_* \widehat{\mathscr{E}} = i_* \tilde{\zeta}^* \mathscr{E} = \tilde{\pi}^* i_* \mathscr{E}$ which gives us (22). Triangle (20) can be obtained by tensoring the resolution (the twist of the left term is determined by Lemma 6.7(*iv*))

$$0 \to \mathscr{O}_{\tilde{\mathscr{X}}_r \times_{\mathbf{S}_r} \tilde{\mathscr{Y}}_r}(0, -1) \otimes (\tilde{\mathscr{L}}_r/\tilde{\mathscr{L}}_{r-1})^* \to \mathscr{O}_{(\tilde{\mathscr{X}}_{r-1} \times_{\mathbf{S}_r} \tilde{\mathscr{Y}}_{r-1}) \cup (\tilde{\mathscr{X}}_r \times_{\mathbf{S}_r} \tilde{\mathscr{Y}}_r)}$$
$$\to \mathscr{O}_{\tilde{\mathscr{X}}_{r-1} \times_{\mathbf{S}_r} \tilde{\mathscr{Y}}_{r-1}} \to 0$$

with $\widehat{\mathscr{E}}$ and applying i_*, since the pullback of $\widehat{\mathscr{E}}$ to $\tilde{\mathscr{X}}_{r-1} \times_{\mathbf{S}_r} \tilde{\mathscr{Y}}_{r-1}$ and $\tilde{\mathscr{X}}_r \times_{\mathbf{S}_r} \tilde{\mathscr{Y}}_r$ coincides with $\tilde{\mathscr{E}}_{r-1}$ and $\tilde{\mathscr{E}}_r$ respectively. Finally, (21) is evident, because both sides are isomorphic to the pullback of $\widehat{\mathscr{E}}$. $\square$

Corollary **6.15.** — *We have the following exact triangles of functors between* $\mathscr{D}^b(\tilde{\mathscr{X}}_{r-1})$ *and* $\mathscr{D}^b(\tilde{\mathscr{Y}}_r)$:

$$\textbf{(23)}\qquad \tilde{\Phi}_{r-1} \eta^! \to \xi_* \tilde{\Phi}_r \to \Phi_{i_* \widehat{\mathscr{E}}(0,1) \otimes (\tilde{\mathscr{L}}_r/\tilde{\mathscr{L}}_{r-1})},$$

$$\textbf{(24)}\qquad \Phi_{i_* \widehat{\mathscr{E}}(0,1) \otimes (\tilde{\mathscr{L}}_r/\tilde{\mathscr{L}}_{r-1})}^* \to \tilde{\Phi}_r^* \xi^* \to \eta_* \tilde{\Phi}_{r-1}^*$$

and the following canonical isomorphism of functors from $\mathscr{D}^b(\tilde{\mathscr{Y}}_{r-1})$ *to* $\mathscr{D}^b(\tilde{\mathscr{X}}_r)$:

(**25**) $$\xi^*\tilde{\Phi}_{r-1} \cong \tilde{\Phi}_r \eta_*.$$

Proof. — Twisting triangle (20) by $\mathscr{O}(0, 1) \otimes (\tilde{\mathscr{L}}_r/\tilde{\mathscr{L}}_{r-1})$, considering its terms as kernels, and taking into account the second formula of (17) we obtain triangles (23) and (24) by Lemma 2.30. Finally, isomorphism of kernels (21) gives an isomorphism of functors (25). $\square$

Lemma **6.16.** — (*i*) *The map* $\mathscr{X}_r \times_{\mathbf{G}_r} \mathscr{Y}_r \to \mathscr{Y}_r \times_Y Q(X, Y)$ *induced by the projection* $\mathscr{X}_r \times_{\mathbf{G}_r} \mathscr{Y}_r \to \mathscr{Y}_r$ *and by the map* $\zeta_r : \mathscr{X}_r \times_{\mathbf{G}_r} \mathscr{Y}_r \to Q(X, Y)$ *is a closed embedding and its image is a zero locus of a regular section of the vector bundle* $(\mathscr{L}_r/\mathscr{O}_Y(-1))^* \otimes \mathscr{O}_X(1)$ *on* $\mathscr{Y}_r \times_Y Q(X, Y)$.

(*ii*) *The map* $\mathscr{X}_r \times_{\mathbf{G}_r} \mathscr{Y}_r \to \mathscr{X}_r \times_X Q(X, Y)$ *induced by the projection* $\mathscr{X}_r \times_{\mathbf{G}_r} \mathscr{Y}_r \to \mathscr{X}_r$ *and by the map* $\zeta_r : \mathscr{X}_r \times_{\mathbf{G}_r} \mathscr{Y}_r \to Q(X, Y)$ *is a closed embedding and its image is a zero locus of a regular section of the vector bundle* $(\mathscr{L}_r^{\perp}/\mathscr{O}_X(-1))^* \otimes \mathscr{O}_Y(1)$ *on* $\mathscr{X}_r \times_X Q(X, Y)$.

Proof. — Recall the notation of the proof of Lemma 6.6. It is clear that we have

$$\mathscr{X}_r \times_{\mathbf{G}_r} \mathscr{Y}_r = \mathsf{Gr}_{Q(X,Y)}(r - 1, \mathscr{V}_Q),$$
$$\mathscr{Y}_r \times_Y Q(X, Y) = \mathsf{Gr}_{Q(X,Y)}(r - 1, \mathscr{V}_Y), \quad \text{and}$$
$$\mathscr{X}_r \times_X Q(X, Y) = \mathsf{Gr}_{Q(X,Y)}(r, \mathscr{V}_X).$$

Also, the tautological bundle on $\mathsf{Gr}_{Q(X,Y)}(r - 1, \mathscr{V}_Y)$ is isomorphic to $\mathscr{L}_r/\mathscr{O}_X(-1)$, and the tautological bundle on $\mathsf{Gr}_{Q(X,Y)}(r, \mathscr{V}_X)$ is isomorphic to $\mathscr{L}_r$. Moreover, by definition of vector bundles $\mathscr{V}_X$, $\mathscr{V}_Y$ and $\mathscr{V}_Q$ we have $\mathscr{V}_Q = \mathsf{Ker}(\mathscr{V}_Y \to \mathscr{O}_X(1)) = \mathsf{Coker}(\mathscr{O}_Y(-1) \to \mathscr{V}_X)$. Finally, it is clear that the map $\mathscr{X}_r \times_{\mathbf{G}_r} \mathscr{Y}_r \to \mathscr{Y}_r \times_Y Q(X, Y)$ is induced by the embedding $\mathscr{V}_Q \to \mathscr{V}_Y$, hence it is a zero locus of a section of the dual tautological bundle on $\mathsf{Gr}_{Q(X,Y)}(r - 1, \mathscr{V}_Y)$ tensored by $\mathscr{O}_X(1)$, which is nothing but $(\mathscr{L}_r/\mathscr{O}_Y(-1))^* \otimes \mathscr{O}_X(1)$ in this case. Similarly, the map $\mathscr{X}_r \times_{\mathbf{G}_r} \mathscr{Y}_r \to \mathscr{X}_r \times_X Q(X, Y)$ is induced by the projection $\mathscr{V}_X \to \mathscr{V}_Q$, hence it is a zero locus of a section of the tautological quotient bundle on $\mathsf{Gr}_{Q(X,Y)}(r - 1, \mathscr{V}_Y)$ tensored by $\mathscr{O}_Y(1)$, which is nothing but $(\mathscr{L}_r^{\perp}/\mathscr{O}_X(-1))^* \otimes \mathscr{O}_Y(1)$ in this case. The regularity of both sections is evident. $\square$

Lemma **6.17.** — *We have*

(*i*) $\mathsf{Im}\, \Phi_r \subset \left[\langle \mathscr{A}_r(1), ..., \mathscr{A}_{i-1}(i - r) \rangle \boxtimes \mathscr{D}^b(\mathbf{G}_r) \right]^{\perp} \subset \mathscr{D}^b(\mathscr{X}_r);$

(*ii*) $\langle \mathscr{B}_{j-1}(N - r - j), ..., \mathscr{B}_{N-r}(-1) \rangle \boxtimes \mathscr{D}^b(\mathbf{G}_r) \subset \mathsf{Ker}\, \Phi_r \subset \mathscr{D}^b(\mathscr{Y}_r).$

Proof. — (*i*) Consider the following commutative diagram

(the square marked with $\boxed{\text{EC}}$ is exact cartesian by Lemma 2.32 since the map $\pi : \mathscr{Y}_r \to Y$ is smooth by Lemma 6.6). One can check that the functor $\pi_* \circ \Phi_r : \mathscr{D}^b(\mathscr{Y}_r) \to \mathscr{D}^b(X)$ is a kernel functor with kernel $j_* \mathscr{E}_r$ on $\mathscr{Y}_r \times_Y Q(X, Y)$. Further, by definition $\mathscr{E}_r = (\pi \circ j)^* \mathscr{E} \cong j^* \pi^* \mathscr{E}$, hence $j_* \mathscr{E}_r \cong j_* j^* \pi^* \mathscr{E} \cong \pi^* \mathscr{E} \otimes j_* \mathscr{O}_{\mathscr{X}_r \times_{\mathbf{G}_r} \mathscr{Y}_r}$. On the other hand, by Lemma 6.16 we have a Koszul resolution

$$0 \to \Lambda^{r-1}(\mathscr{L}_r/\mathscr{O}_Y(-1)) \otimes \mathscr{O}_X(1 - r)$$

$$\to \cdots \to \Lambda^2(\mathscr{L}_r/\mathscr{O}_Y(-1)) \otimes \mathscr{O}_X(-2)$$

$$\to (\mathscr{L}_r/\mathscr{O}_Y(-1)) \otimes \mathscr{O}_X(-1) \to \mathscr{O}_{\mathscr{Y}_r \times_Y Q(X,Y)} \to j_* \mathscr{O}_{\mathscr{X}_r \times_{\mathbf{G}_r} \mathscr{Y}_r} \to 0.$$

Considering $\pi^* \mathscr{E} \otimes \Lambda^t(\mathscr{L}_r/\mathscr{O}_Y(-1)) \otimes \mathscr{O}_X(-t) \cong \pi^*(\mathscr{E} \otimes \mathscr{O}_X(-t)) \otimes q^* \Lambda^t(\mathscr{L}_r/\mathscr{O}_Y(-1))$ as a kernel on $\mathscr{Y}_r \times_Y Q(X, Y)$ we note that the corresponding kernel functor $\Psi_t : \mathscr{D}^b(\mathscr{Y}_r) \to \mathscr{D}^b(X)$ takes form

$$G \mapsto p_* \pi_* \big(\pi^*(\mathscr{E} \otimes \mathscr{O}_X(-t)) \otimes q^* \Lambda^t(\mathscr{L}_r/\mathscr{O}_Y(-1)) \otimes q^* G \big)$$

$$\cong p_* \big(\mathscr{E} \otimes \mathscr{O}_X(-t) \otimes \pi_* q^* \big(\Lambda^t(\mathscr{L}_r/\mathscr{O}_Y(-1)) \otimes G \big) \big)$$

$$\cong p_* \big(\mathscr{E} \otimes q^* \pi_* \big(\Lambda^t(\mathscr{L}_r/\mathscr{O}_Y(-1)) \otimes G \big) \otimes \mathscr{O}_X(-t) \big)$$

$$\cong \Phi_{\mathscr{E}} \big(\pi_* \big(\Lambda^t(\mathscr{L}_r/\mathscr{O}_Y(-1)) \otimes G \big) \big) \otimes \mathscr{O}_X(-t)$$

(in the second isomorphism we used exactness of the square marked with $\boxed{\text{EC}}$ symbol). Note that the kernel functor $\Phi_{\mathscr{E}} : \mathscr{D}^b(Y) \to \mathscr{D}^b(X)$ given by the kernel $\mathscr{E} \in \mathscr{D}^b(Q(X, Y))$ is isomorphic to the composition of the kernel functor $\Phi_{\mathscr{E}} : \mathscr{D}^b(Y) \to \mathscr{D}^b(\mathscr{X}_1)$ (which coincides with $\gamma \circ \varphi_{\mathscr{E}}$ by 6.4(1)) and of the push-forward functor $\pi_* : \mathscr{D}^b(\mathscr{X}_1) \to \mathscr{D}^b(X)$. Therefore $\text{Im}\,\Psi_t \subset (\text{Im}\,\pi_* \gamma) \otimes \mathscr{O}_X(-t)$ which coincides with $\mathscr{A}_0(-t)$ by Lemma 5.11. It follows that $\text{Im}(\pi_* \Phi_r) \subset \langle \mathscr{A}_0(1 - r), ..., \mathscr{A}_0(-1), \mathscr{A}_0 \rangle$. But the latter subcategory of $\mathscr{D}^b(X)$ coincides with $\langle \mathscr{A}_0(1 - r), ..., \mathscr{A}_{r-1} \rangle = \langle \mathscr{A}_r(1), ..., \mathscr{A}_{i-1}(i - r) \rangle^\perp$ by Lemma 4.5. Therefore we have $\text{Im}(\pi_* \Phi_r) \subset \langle \mathscr{A}_r(1), ..., \mathscr{A}_{i-1}(i - r) \rangle^\perp$. Since the functor Φ_r is $\mathbf{G}_r$-linear it follows that $\text{Im}(\Phi_r) \subset [\langle \mathscr{A}_r(1), ..., \mathscr{A}_{i-1}(i - r) \rangle \boxtimes \mathscr{D}^b(\mathbf{G}_r)]^\perp$.

(ii) Similarly, consider the following commutative diagram

$$
\begin{array}{ccccc}
\mathscr{Y}_r & \xleftarrow{\ q\ } & \mathscr{X}_r \times_{\mathbf{G}_r} \mathscr{Y}_r & \xrightarrow{\ p\ } & \mathscr{X}_r \\
& & \downarrow{\scriptstyle j} & & \| \\
\pi \downarrow & & \mathscr{X}_r \times_{\mathrm{X}} \mathrm{Q}(\mathrm{X}, \mathrm{Y}) & \xrightarrow{\ p\ } & \mathscr{X}_r \\
& & \downarrow{\scriptstyle \pi} & \boxed{\text{EC}} & \downarrow{\scriptstyle \pi} \\
\mathrm{Y} & \xleftarrow{\ q\ } & \mathrm{Q}(\mathrm{X}, \mathrm{Y}) & \xrightarrow{\ p\ } & \mathrm{X}
\end{array}
$$

(the square marked with $\boxed{\text{EC}}$ is exact cartesian by Lemma 2.32 since the map $\pi : \mathscr{X}_r \to \mathrm{X}$ is smooth by Lemma 6.6) and functor $\Phi_r \circ \pi^* : \mathscr{D}^b(\mathrm{Y}) \to \mathscr{D}^b(\mathscr{X}_r)$ which is a kernel functor with kernel $\pi^*\mathscr{E} \otimes j_*\mathscr{O}_{\mathscr{X}_r \times_{\mathbf{G}_r}\mathscr{Y}_r}$ on $\mathscr{X}_r \times_{\mathrm{X}} \mathrm{Q}(\mathrm{X}, \mathrm{Y})$. On the other hand, by Lemma 6.16 we have a Kozsul resolution

$$
0 \to \Lambda^{\mathrm{N}-r-1}(\mathscr{L}_r^\perp/\mathscr{O}_{\mathrm{X}}(-1)) \otimes \mathscr{O}_{\mathrm{Y}}(1 + r - \mathrm{N})
$$

$$
\to \cdots \to \Lambda^2(\mathscr{L}_r^\perp/\mathscr{O}_{\mathrm{X}}(-1)) \otimes \mathscr{O}_{\mathrm{Y}}(-2)
$$

$$
\to (\mathscr{L}_r^\perp/\mathscr{O}_{\mathrm{X}}(-1)) \otimes \mathscr{O}_{\mathrm{Y}}(-1) \to \mathscr{O}_{\mathscr{X}_r \times_{\mathrm{X}}\mathrm{Q}(\mathrm{X},\mathrm{Y})} \to j_*\mathscr{O}_{\mathscr{X}_r \times_{\mathbf{G}_r}\mathscr{Y}_r} \to 0.
$$

The kernel functors $\Psi_t : \mathscr{D}^b(\mathrm{Y}) \to \mathscr{D}^b(\mathscr{X}_r)$ given by the terms of this resolution tensored by $\pi^*\mathscr{E}$ take form

$$
\Psi_t : \mathrm{G} \mapsto \pi^*\Phi_\mathscr{E}(\mathrm{G}(-t)) \otimes \Lambda^t(\mathscr{L}_r^\perp/\mathscr{O}_{\mathrm{X}}(-1)), \quad t = 0, ..., \mathrm{N} - r - 1.
$$

If $\mathrm{G} \in \langle \mathscr{B}_{\mathrm{j}-1}(\mathrm{N} - r - \mathrm{j}), ..., \mathscr{B}_{\mathrm{N}-r}(-1)\rangle$ then for any $t = 0, 1, ..., \mathrm{N} - r - 1$ we have

$$
\mathrm{G}(-t) \in \langle \mathscr{B}_{\mathrm{j}-1}(1 - \mathrm{j}), ..., \mathscr{B}_1(-1)\rangle \subset \mathscr{B}_0^\perp = (\varphi_\mathscr{E}^*(\mathscr{C}_0))^\perp
$$

by assumption 6.4(3). By adjunction we have $\varphi_\mathscr{E}(\mathrm{G}(-t)) \in \mathscr{C}_0^\perp$, and by Lemma 5.11 this implies that $\Phi_\mathscr{E}(\mathrm{G}(-t)) = \pi_*\gamma\varphi_\mathscr{E}(\mathrm{G}(-t)) = 0$. In other words, we have $\Psi_t(\mathrm{G}) = 0$. Thus we obtain an inclusion $\langle \mathscr{B}_{\mathrm{j}-1}(\mathrm{N} - r - \mathrm{j}), ..., \mathscr{B}_{\mathrm{N}-r}(-1)\rangle \subset \mathsf{Ker}(\Phi_r \circ \pi^*)$, and the claim follows since Φ_r is $\mathbf{G}_r$-linear. $\qquad\square$

Lemma **6.18.** — *We have* $\Phi^*_{i_*\widehat{\mathscr{E}}(0,1)}([\langle \mathscr{A}_{r-1}(1), ..., \mathscr{A}_{\mathrm{i}-1}(\mathrm{i} - r + 1)\rangle \boxtimes \mathscr{D}^b(\mathbf{S}_r)]^\perp) \subset$ $\mathscr{B}_{\mathrm{N}-r}(-1) \boxtimes \mathscr{D}^b(\mathbf{S}_r)$.

Proof. — First of all we note that the claim is equivalent to

$$
\Phi^*_{i_*\widehat{\mathscr{E}}}\left(\left[\langle \mathscr{A}_{r-1}(1), ..., \mathscr{A}_{\mathrm{i}-1}(\mathrm{i} - r + 1)\rangle \boxtimes \mathscr{D}^b(\mathbf{S}_r)\right]^\perp\right) \subset \mathscr{B}_{\mathrm{N}-r} \boxtimes \mathscr{D}^b(\mathbf{S}_r)
$$

which we will prove. Consider the following commutative diagram

(the squares marked with $\boxed{\text{EC}}$ are exact cartesian: the first by Lemma 2.32(iii) and the second by Lemma 2.32(i)). Note also that $\pi \circ j = \tilde{\pi}$, hence $i_* \widehat{\mathscr{E}} \cong j^* \pi^* i_* \mathscr{E}$ by (22). It follows that

$$\Phi_{i_* \widehat{\mathscr{E}}}(\mathrm{G}) \cong p_*(i_* \widehat{\mathscr{E}} \otimes q^*\mathrm{G}) \cong p_* j_*(j^* \pi^* i_* \mathscr{E} \otimes q^*\mathrm{G}) \cong p_*(\pi^* i_* \mathscr{E} \otimes j_* q^*\mathrm{G})$$
$$\cong p_*(\pi^* i_* \mathscr{E} \otimes q^* j_*\mathrm{G}) = \Phi_{\pi^* i_* \mathscr{E}}(j_*\mathrm{G}),$$

that is $\Phi_{i_* \widehat{\mathscr{E}}} \cong \Phi_{\pi^* i_* \mathscr{E}} \circ j_*$, whereof by adjunction we have $\Phi^*_{i_* \widehat{\mathscr{E}}} \cong j^* \circ \Phi^*_{\pi^* i_* \mathscr{E}}$. Therefore it suffices to check that

$$\Phi^*_{\pi^* i_* \mathscr{E}}\big(\big[\langle \mathscr{A}_{r-1}(1), ..., \mathscr{A}_{i-1}(i-r+1)\rangle \boxtimes \mathscr{D}^b(\mathbf{S}_r)\big]^\perp\big) \subset \mathscr{B}_{\mathrm{N}-r} \boxtimes \mathscr{D}^b(\mathbf{S}_r)$$
$$\text{in } \mathscr{D}^b(\mathrm{Y} \times \mathbf{S}_r).$$

The LHS is evidently $\mathbf{S}_r$-linear, hence this is equivalent to

$$\pi_* \Phi^*_{\pi^* i_* \mathscr{E}}\big(\big[\langle \mathscr{A}_{r-1}(1), ..., \mathscr{A}_{i-1}(i-r+1)\rangle \boxtimes \mathscr{D}^b(\mathbf{S}_r)\big]^\perp\big) \subset \mathscr{B}_{\mathrm{N}-r}$$
$$\text{in } \mathscr{D}^b(\mathrm{Y}).$$

But

$$\Phi_{\pi^* i_* \mathscr{E}}(\pi^*\mathrm{G}) = p_*(\pi^* i_* \mathscr{E} \otimes q^* \pi^*\mathrm{G}) \cong p_*(\pi^* i_* \mathscr{E} \otimes \pi^* q^*\mathrm{G})$$
$$\cong p_* \pi^*(i_* \mathscr{E} \otimes q^*\mathrm{G}) \cong \pi^* p_*(i_* \mathscr{E} \otimes q^*\mathrm{G}) = \pi^* \Phi_{i_* \mathscr{E}}(\mathrm{G}),$$

that is $\Phi_{\pi^* i_* \mathscr{E}} \circ \pi^* \cong \pi^* \circ \Phi_{i_* \mathscr{E}}$ whereof by adjunction we deduce $\pi_\# \circ \Phi^*_{\pi^* i_* \mathscr{E}} \cong \Phi^*_{i_* \mathscr{E}} \circ \pi_\#$, where $\pi_\#$ is the left adjoint functor to π^*. Now note that for the projection $\pi : \mathrm{Y} \times \mathbf{S}_r \to \mathrm{Y}$ we have $\pi^*(\mathrm{F}) \cong \pi^!(\mathrm{F}) \otimes \omega^{-1}_{\mathbf{S}_r}[-\dim \mathbf{S}_r]$, hence $\pi_\#(\mathrm{G}) = \pi_*(\mathrm{G} \otimes \omega_{\mathbf{S}_r}[\dim \mathbf{S}_r])$. On the other hand, the projection $\pi : \tilde{\mathscr{X}}_{r-1} \to \mathrm{X}$ factors as $\tilde{\mathscr{X}}_{r-1} \to \mathrm{X} \times \mathbf{S}_r \to \mathrm{X}$, hence by Lemma 6.7($i$) we have

$$\pi_\#(\mathrm{F}) = \pi_*(\mathrm{F} \otimes \mathscr{O}_{\mathrm{X}}(r-1) \otimes \omega_{\mathbf{S}_r} \otimes \det \tilde{\mathscr{L}}^*_{r-1}[\dim \mathbf{S}_r - r + 1]).$$

Since line bundles $\omega_{\mathbf{S}_r}$ and $\det \tilde{\mathscr{L}}^*_{r-1}$ are pullbacks from $\mathbf{S}_r$, it remains to check that

$$\Phi^*_{i_*\mathscr{E}}(\langle \mathscr{A}_{r-1}(r), ..., \mathscr{A}_{i-1}(i)\rangle^\perp) \subset \mathscr{B}_{N-r} \qquad \text{in } \mathscr{D}^b(Y).$$

But it is clear that $\Phi^*_{i_*\mathscr{E}} = \varphi^*_\mathscr{E} \gamma^* \pi^*$, hence we have $\Phi^*_{i_*\mathscr{E}}(\langle \mathscr{A}_{r-1}(r), ..., \mathscr{A}_{i-1}(i)\rangle^\perp) = \varphi^*_\mathscr{E} \gamma^* \pi^*(\langle \mathscr{A}_{r-1}(r), ..., \mathscr{A}_{i-1}(i)\rangle^\perp) \subset \varphi^*_\mathscr{E}(\mathscr{C}_{N-r})$ by Lemma 5.12, and $\varphi^*_\mathscr{E}(\mathscr{C}_{N-r}) = \mathscr{B}_{N-r}$ by 6.4(3). $\qquad\square$

Lemma **6.19.** — *For all r we have*

$$\begin{aligned} \Phi^*_{i_*\widehat{\mathscr{E}}(0,1)\otimes(\tilde{\mathscr{L}}_r/\tilde{\mathscr{L}}_{r-1})} \circ \tilde{\Phi}_{r-1} &= 0, & \textit{if } r \leq N-j \\ \tilde{\Phi}_r \circ \Phi^*_{i_*\widehat{\mathscr{E}}(0,1)\otimes(\tilde{\mathscr{L}}_r/\tilde{\mathscr{L}}_{r-1})} &= 0, & \textit{if } r \geq i+1 \\ \tilde{\Phi}_r \circ \Phi^*_{i_*\widehat{\mathscr{E}}(0,1)\otimes(\tilde{\mathscr{L}}_r/\tilde{\mathscr{L}}_{r-1})} \circ \tilde{\Phi}_{r-1} &= 0, & \textit{in other cases.} \end{aligned}$$

Proof. — Note that by Lemma 6.17(i), we have, by base change to $\mathbf{S}_r$, that $\mathsf{Im}\, \tilde{\Phi}_{r-1} \subset [\langle \mathscr{A}_{r-1}(1), ..., \mathscr{A}_{i-1}(i-r+1)\rangle \boxtimes \mathscr{D}^b(\mathbf{S}_r)]^\perp$ and if $r \geq i+1$ then the RHS equals $\mathscr{D}^b(\tilde{\mathscr{X}}_{r-1})$. On the other hand, by Lemma 6.17(ii) we have $\mathscr{B}_{N-r}(-1) \otimes \mathscr{D}^b(\mathbf{S}_r) \subset \mathsf{Ker}\, \tilde{\Phi}_r$ and if $r \leq N-j$ then $N-r \geq j$ hence $\mathscr{B}_{N-r} = 0$ by (13). Thus it suffices to check that for all r we have

$$\Phi^*_{i_*\widehat{\mathscr{E}}(0,1)\otimes(\tilde{\mathscr{L}}_r/\tilde{\mathscr{L}}_{r-1})}\big([\langle \mathscr{A}_{r-1}(1), ..., \mathscr{A}_{i-1}(i-r+1)\rangle \boxtimes \mathscr{D}^b(\mathbf{S}_r)]^\perp\big)$$
$$\subset \mathscr{B}_{N-r}(-1) \boxtimes \mathscr{D}^b(\mathbf{S}_r).$$

This easily follows from Lemma 6.18 (the functor $\Phi^*_{i_*\widehat{\mathscr{E}}(0,1)\otimes(\tilde{\mathscr{L}}_r/\tilde{\mathscr{L}}_{r-1})}$ differs from $\Phi^*_{i_*\widehat{\mathscr{E}}(0,1)}$ by the $(\tilde{\mathscr{L}}_r/\tilde{\mathscr{L}}_{r-1})$-twist, and the line bundle $(\tilde{\mathscr{L}}_r/\tilde{\mathscr{L}}_{r-1})$ is a pullback from $\mathbf{S}_r$). $\qquad\square$

6.3. *The induction arguments*

Proposition **6.20.** — *The functors Φ_r are left splitting functors for all r.*

Proof. — We use induction in r. The functor Φ_1 is fully faithful and its image is left admissible by 6.4(1) (we have $\mathscr{Y}_1 = Y$ and $\Phi_1 = \Phi_\mathscr{E} : \mathscr{D}^b(Y) \to \mathscr{C} \subset \mathscr{D}^b(\mathscr{X}_1)$), hence Φ_1 is left splitting. Assume that Φ_{r-1} is left splitting. Then the functor $\tilde{\Phi}_{r-1}$ is left splitting by Proposition 3.6. Now consider the functor $\tilde{\Phi}_r \tilde{\Phi}^*_r \tilde{\Phi}_r \eta_*$. Composing the isomorphism of functors (25) with $\tilde{\Phi}_r \tilde{\Phi}^*_r$ we obtain an isomorphism

$$\tilde{\Phi}_r \tilde{\Phi}^*_r \tilde{\Phi}_r \eta_* \cong \tilde{\Phi}_r \tilde{\Phi}^*_r \xi^* \tilde{\Phi}_{r-1}.$$

Composing the exact triangle of functors (24) with $\tilde{\Phi}_r$ (on the left) and $\tilde{\Phi}_{r-1}$ (on the right) we obtain an exact triangle of functors

$$\tilde{\Phi}_r \Phi^*_{i_* \widehat{\mathscr{E}}(0,1) \otimes (\tilde{\mathscr{L}}_r / \tilde{\mathscr{L}}_{r-1})} \tilde{\Phi}_{r-1} \to \tilde{\Phi}_r \tilde{\Phi}^*_r \xi^* \tilde{\Phi}_{r-1} \to \tilde{\Phi}_r \eta_* \tilde{\Phi}^*_{r-1} \tilde{\Phi}_{r-1}.$$

Composing (25) with $\tilde{\Phi}^*_{r-1} \tilde{\Phi}_{r-1}$ we obtain an isomorphism

$$\tilde{\Phi}_r \eta_* \tilde{\Phi}^*_{r-1} \tilde{\Phi}_{r-1} \cong \xi^* \tilde{\Phi}_{r-1} \tilde{\Phi}^*_{r-1} \tilde{\Phi}_{r-1}.$$

Using the induction assumption, criterion 3.3(2l) and isomorphism (25) we deduce

$$\xi^* \tilde{\Phi}_{r-1} \tilde{\Phi}^*_{r-1} \tilde{\Phi}_{r-1} \cong \xi^* \tilde{\Phi}_{r-1} \cong \tilde{\Phi}_r \eta_*.$$

On the other hand, $\tilde{\Phi}_r \Phi^*_{i_* \widehat{\mathscr{E}}(0,1) \otimes (\tilde{\mathscr{L}}_r / \tilde{\mathscr{L}}_{r-1})} \tilde{\Phi}_{r-1} = 0$ by Lemma 6.19. Summarizing, we deduce that

$$\tilde{\Phi}_r \tilde{\Phi}^*_r \tilde{\Phi}_r \eta_* \cong \tilde{\Phi}_r \eta_*.$$

Finally since $\psi_* \eta_* \hat{\psi}^* = \hat{\psi}_* \hat{\psi}^* = \mathsf{id}$ by Corollary 6.10, we have

$$\Phi_r \Phi^*_r \Phi_r \cong \Phi_r \Phi^*_r \Phi_r \psi_* \eta_* \hat{\psi}^* \cong \psi_* \tilde{\Phi}_r \tilde{\Phi}^*_r \tilde{\Phi}_r \eta_* \hat{\psi}^*$$
$$\cong \psi_* \tilde{\Phi}_r \eta_* \hat{\psi}^* \cong \Phi_r \psi_* \eta_* \hat{\psi}^* \cong \Phi_r$$

by Proposition 6.12. Therefore Φ_r is left splitting by Theorem 3.3. $\square$

> *Corollary* **6.21.** — *We have the following semiorthogonal collections in $\mathscr{D}^b(\mathscr{X}_r)$ and $\mathscr{D}^b(\mathscr{Y}_r)$:*
>
> $$\left\langle \mathsf{Im}\,\Phi_r, \mathscr{A}_r(1) \boxtimes \mathscr{D}^b(\mathbf{G}_r), ..., \mathscr{A}_{i-1}(i-r) \boxtimes \mathscr{D}^b(\mathbf{G}_r) \right\rangle \qquad \subset \mathscr{D}^b(\mathscr{X}_r)$$
> $$\left\langle \mathscr{B}_{j-1}(N-r-j) \boxtimes \mathscr{D}^b(\mathbf{G}_r), ..., \mathscr{B}_{N-r}(-1) \boxtimes \mathscr{D}^b(\mathbf{G}_r), \mathsf{Im}\,\Phi^*_r \right\rangle \subset \mathscr{D}^b(\mathscr{Y}_r).$$
>
> *Moreover, the functors Φ_r and Φ^*_r induce an equivalence $\mathsf{Im}\,\Phi_r \cong \mathsf{Im}\,\Phi^*_r$.*

> *Proof.* — Combine Theorem 3.3 with Proposition 6.17 and Lemma 6.8. $\square$

It remains to check that these semiorthogonal collections are full, i.e. that they generate the derived categories of $\mathscr{X}_r$ and $\mathscr{Y}_r$. We begin with the case of $\mathscr{Y}_r$.

> *Proposition* **6.22.** — *For all r we have*
>
> $$\mathscr{D}^b(\mathscr{Y}_r) = \left\langle \mathscr{B}_{j-1}(N-r-j) \boxtimes \mathscr{D}^b(\mathbf{G}_r), ..., \mathscr{B}_{N-r}(-1) \boxtimes \mathscr{D}^b(\mathbf{G}_r), \mathsf{Im}\,\Phi^*_r \right\rangle.$$

Proof. — Note that for $s \geq \mathsf{j}$ we have $\mathscr{B}_s = 0$ by (13). Hence for $r \leq \mathrm{N} - \mathsf{j}$ the RHS coincides with $\mathsf{Im}\,\Phi_r^*$, and since we already know that Φ_r is splitting, for $r \leq \mathrm{N} - \mathsf{j}$ it suffices to check that $\mathsf{Ker}\,\Phi_r = 0$, that is that Φ_r is fully faithful. The arguments are the same as in the proof of Proposition 6.20, the only difference is that we use the first equality of Lemma 6.19 instead of the third.

We use induction in r. In the case $r = 1$ we have $\mathscr{Y}_1 = \mathrm{Y}$ and Φ_1 is fully faithful by 6.4(1). Now let $1 < r \leq \mathrm{N} - \mathsf{j}$ and assume that Φ_{r-1} is fully faithful. Then the functor $\tilde{\Phi}_{r-1}$ is fully faithful by Proposition 2.39. Now consider the functor $\tilde{\Phi}_r^* \tilde{\Phi}_r \eta_*$. Composing the isomorphism of functors (25) with $\tilde{\Phi}_r^*$ we obtain an isomorphism

$$\tilde{\Phi}_r^* \tilde{\Phi}_r \eta_* \cong \tilde{\Phi}_r^* \xi^* \tilde{\Phi}_{r-1}.$$

Composing the exact triangle of functors (24) with $\tilde{\Phi}_{r-1}$ we obtain an exact triangle of functors

$$\Phi_{i_* \widehat{\mathscr{E}}(0,1) \otimes (\tilde{\mathscr{L}}_r / \tilde{\mathscr{L}}_{r-1})}^* \tilde{\Phi}_{r-1} \to \tilde{\Phi}_r^* \xi^* \tilde{\Phi}_{r-1} \to \eta_* \tilde{\Phi}_{r-1}^* \tilde{\Phi}_{r-1}.$$

Since the functor $\tilde{\Phi}_{r-1}$ is fully faithful we have

$$\eta_* \tilde{\Phi}_{r-1}^* \tilde{\Phi}_{r-1} \cong \eta_*.$$

On the other hand, $\Phi_{i_* \widehat{\mathscr{E}}(0,1) \otimes (\tilde{\mathscr{L}}_r / \tilde{\mathscr{L}}_{r-1})}^* \tilde{\Phi}_{r-1} = 0$ by Lemma 6.19 since $r \leq \mathrm{N} - \mathsf{j}$. Summarizing, we deduce that

$$\tilde{\Phi}_r^* \tilde{\Phi}_r \eta_* \cong \eta_*.$$

Finally since $\psi_* \eta_* \hat{\psi}^* = \hat{\psi}_* \hat{\psi}^* = \mathsf{id}$ by Corollary 6.10, we have

$$\Phi_r^* \Phi_r \cong \Phi_r^* \Phi_r \psi_* \eta_* \hat{\psi}^* \cong \psi_* \tilde{\Phi}_r^* \tilde{\Phi}_r \eta_* \hat{\psi}^* \cong \psi_* \eta_* \hat{\psi}^* \cong \mathsf{id}_{\mathscr{Y}_r}$$

by Proposition 6.12. Therefore Φ_r is fully faithful.

For $r \geq \mathrm{N} - \mathsf{j}$ we also use induction in r. However the arguments are slightly different in this case. Assume that $r \geq \mathrm{N} - \mathsf{j} + 1$ and the claim for $r - 1$ is true. Assume that G is in the right orthogonal to the category $\langle \mathscr{B}_{\mathsf{j}-1}(\mathrm{N} - r - \mathsf{j}) \boxtimes \mathscr{D}^b(\mathbf{G}_r), ..., \mathscr{B}_{\mathrm{N}-r}(-1) \boxtimes \mathscr{D}^b(\mathbf{G}_r), \mathsf{Im}\,\Phi_r^* \rangle$. By Proposition 6.12 we have $\tilde{\Phi}_r(\psi^* \mathrm{G}) = \psi^* \Phi_r(\mathrm{G})$. But $\Phi_r(\mathrm{G}) = 0$ since $\mathrm{G} \in (\mathsf{Im}\,\Phi_r^*)^{\perp}$, hence $\tilde{\Phi}_r(\psi^* \mathrm{G}) = 0$, thus $\psi^* \mathrm{G} \in (\mathsf{Im}\,\tilde{\Phi}_r^*)^{\perp}$. On the other hand, for any $\mathrm{F} \in \mathscr{B}_{\mathrm{N}-r+t}(-1-t)$, $\mathrm{H} \in \mathscr{D}^b(\mathbf{S}_r)$ we have

$$\psi_* \mathsf{R}\mathscr{H}om(\psi^* \mathrm{F} \otimes \mathrm{H}, \psi^* \mathrm{G}) \cong \psi_*(\mathsf{R}\mathscr{H}om(\psi^* \mathrm{F}, \psi^* \mathrm{G}) \otimes \mathrm{H}^*)$$
$$\cong \psi_*(\psi^* \mathsf{R}\mathscr{H}om(\mathrm{F}, \mathrm{G}) \otimes \mathrm{H}^*)$$
$$\cong \mathsf{R}\mathscr{H}om(\mathrm{F}, \mathrm{G}) \otimes \psi_*(\mathrm{H}^*)$$
$$\cong \mathsf{R}\mathscr{H}om(\mathrm{F} \otimes (\psi_*(\mathrm{H}^*))^*, \mathrm{G})$$

whereof we deduce that $\psi^* G \in (\mathscr{B}_{N-r+t}(-1-t) \otimes \mathscr{D}^b(\mathbf{S}_r))^\perp$. Combining these two inclusions we see that $\psi^* G \in \langle \mathscr{B}_{j-1}(N-r-j) \boxtimes \mathscr{D}^b(\mathbf{S}_r), ..., \mathscr{B}_{N-r}(-1) \boxtimes \mathscr{D}^b(\mathbf{S}_r), \mathsf{Im}\, \tilde{\Phi}_r^* \rangle^\perp$. Recalling Lemma 6.18 we note that $\psi^* G$ is in the right orthogonal to the image of the first two terms of the triangle (24) applied to the subcategory $[\langle \mathscr{A}_{r-1}(1), ..., \mathscr{A}_{i-1}(i-r+1) \rangle \boxtimes \mathscr{D}^b(\mathbf{S}_r)]^\perp$. Therefore $\psi^* G$ is in the right orthogonal to the image of $\eta_* \tilde{\Phi}_{r-1}^*$ applied to the subcategory $[\langle \mathscr{A}_{r-1}(1), ..., \mathscr{A}_{i-1}(i-r+1) \rangle \boxtimes \mathscr{D}^b(\mathbf{S}_r)]^\perp$. But from Lemma 6.17 it follows by adjunction that $[\langle \mathscr{A}_{r-1}(1), ..., \mathscr{A}_{i-1}(i-r+1) \rangle \boxtimes \mathscr{D}^b(\mathbf{S}_r)] \subset \mathsf{Ker}\, \tilde{\Phi}_{r-1}^*$. Therefore $\psi^* G$ is in the right orthogonal to the image of the functor $\eta_* \tilde{\Phi}_{r-1}^*$. By adjunction we deduce $\eta^! \psi^* G \in (\mathsf{Im}\, \tilde{\Phi}_{r-1}^*)^\perp$.

On the other hand, by Lemma 6.7(*iv*) we have the following resolution

$$0 \to \mathscr{O}_{\tilde{\mathcal{Y}}_r}(-1) \otimes (\tilde{\mathscr{L}}_r/\tilde{\mathscr{L}}_{r-1})^* \to \mathscr{O}_{\tilde{\mathcal{Y}}_r} \to \eta_* \mathscr{O}_{\tilde{\mathcal{Y}}_{r-1}} \to 0$$

which implies that $\eta_*(\mathscr{B}_{N-r+t}(-1-t) \boxtimes \mathscr{D}^b(\mathbf{S}_r)) \subset \langle \mathscr{B}_{N-r+t}(-2-t), \mathscr{B}_{N-r+t}(-1-t) \rangle \boxtimes \mathscr{D}^b(\mathbf{S}_r)$ for any t, and since we have $\mathscr{B}_{N-r+t} \subset \mathscr{B}_{N-r+t-1}$ we conclude that

$$\eta_* \langle \mathscr{B}_{j-1}(N-r-j+1), ..., \mathscr{B}_{N-r+1}(-1) \rangle \boxtimes \mathscr{D}^b(\mathbf{S}_r)$$
$$\subset \langle \mathscr{B}_{j-1}(N-r-j), ..., \mathscr{B}_{N-r}(-1) \rangle \boxtimes \mathscr{D}^b(\mathbf{S}_r).$$

Therefore $\eta^! \psi^* G \in [\langle \mathscr{B}_{j-1}(N-r-j+1), ..., \mathscr{B}_{N-r+1}(-1) \rangle \boxtimes \mathscr{D}^b(\mathbf{S}_r)]^\perp$. Summarizing we deduce that $\eta^! \psi^* G = 0$ by the induction hypothesis. But $\eta^! \psi^* G = \eta^* \psi^* G \otimes \mathscr{O}_Y(1) \otimes (\tilde{\mathscr{L}}_r/\tilde{\mathscr{L}}_{r-1})[-1]$ by Lemma 6.7(*iv*), and $\eta^* \psi^* = \hat{\psi}^*$ is a fully faithful functor by Corollary 6.10, hence $G = 0$. $\qquad\square$

Corollary **6.23.** — *We have* $\mathscr{D}^b(Y) = \langle \mathscr{B}_{j-1}(1-j), ..., \mathscr{B}_1(-1), \mathscr{B}_0 \rangle$.

In particular, we deduce that $\varphi_{\mathscr{E}} : \mathscr{D}^b(Y) \to \mathscr{C}$ is an equivalence, hence Y is homologically projectively dual to X.

Proof. — Note that $\mathbf{G}_N$ is a point, $\mathscr{Y}_N = Y$, $\mathscr{X}_N = \emptyset$, and apply Proposition 6.22 for $r = N$. $\qquad\square$

Fullness of the semiorthogonal decomposition of $\mathscr{D}^b(\mathscr{X}_r)$ will be proved by a decreasing induction. The base of induction is given by the following

Lemma **6.24.** — *We have* $\mathsf{Im}\, \Phi_{N-1} = \mathscr{D}^b(\mathscr{X}_{N-1})$.

Proof. — Note that $\mathscr{X}_{N-1} = X$ and the projection $\pi : \mathscr{X}_{N-1} \to X$ is the identity. Therefore, the functor $\pi_* \circ \Phi_{N-1}$ considered in the proof of Lemma 6.17(*i*) coincides with Φ_{N-1}. Further, note that by Lemma 6.6 we have $\mathscr{Y}_{N-1} \cong \mathsf{Gr}_Y(N-2, \mathscr{V}_Y) \cong \mathbf{P}_Y(\mathscr{V}_Y^*)$, the projection to Y coincides with π and, moreover,

$\mathcal{O}_{\mathscr{Y}_{N-1}/Y}(-1) \cong \mathscr{L}_{N-1}^{\perp}$, where $\mathcal{O}_{\mathscr{Y}_{N-1}/Y}(-1)$ is the Grothendieck line bundle. It easily follows that

$$\Omega^1_{\mathscr{Y}_{N-1}/Y}(1) \cong \mathsf{Ker}(\mathscr{V}_Y \to \mathscr{L}_{N-1}^{\perp *}) \cong \mathscr{L}_{N-1}/\mathcal{O}_Y(-1).$$

Therefore, we have $\pi_*(\Lambda^t(\mathscr{L}_{N-1}/\mathcal{O}_Y(-1)) \otimes S^u(\mathscr{L}_{N-1}^{\perp})) \cong \pi_*(\Omega^t_{\mathscr{Y}_{N-1}/Y}(t-u))$ for any t, u. It is easy to check that for $0 \le t, u \le N-2$ we have

$$\pi_*\big(\Omega^t_{\mathscr{Y}_{N-1}/Y}(t-u)\big) \cong \begin{cases} \mathcal{O}_Y[-t], & \text{if } t = u \\ 0, & \text{otherwise.} \end{cases}$$

Hence in the notation of the proof of Lemma 6.17(*i*) we have

$$\Psi_t\big(\pi^*G \otimes S^u(\mathscr{L}_{N-1}^{\perp})\big) \cong \begin{cases} \Phi_{\mathscr{E}}(G) \otimes \mathcal{O}_X(-t)[-t], & \text{if } t = u \\ 0, & \text{otherwise.} \end{cases}$$

It follows that

$$\Phi_{N-1}\big(\pi^*G \otimes S^s(\mathscr{L}_{N-1}^{\perp})\big) \cong \Phi_{\mathscr{E}}(G) \otimes \mathcal{O}_X(-s).$$

But $\mathsf{Im}\,\Phi_{\mathscr{E}} = \mathscr{A}_0$ by Lemma 5.11 and Corollary 6.23, hence $\mathsf{Im}\,\Phi_{N-1}$ contains $\mathscr{A}_0(-u)$ for $0 \le u \le N-2$. It remains to note that $\mathsf{Im}\,\Phi_{N-1}$ is triangulated subcategory of $\mathscr{D}^b(X)$ since Φ_{N-1} is a splitting functor by Proposition 6.20, and on the other hand by Lemma 4.5 we have $\langle \mathscr{A}_0(2-N), ..., \mathscr{A}_0 \rangle = \langle \mathscr{A}_0(2-N), ..., \mathscr{A}_{i-1}(i+1-N)\rangle$ (note that $N > i$ by assumption (6)), and the latter category evidently coincides with $\mathscr{D}^b(X) = \mathscr{D}^b(\mathscr{X}_{N-1})$. $\qquad\square$

Proposition **6.25.** — *For all r we have*

$$\mathscr{D}^b(\mathscr{X}_r) = \big\langle \mathsf{Im}\,\Phi_r, \mathscr{A}_r(1) \boxtimes \mathscr{D}^b(\mathbf{G}_r), ..., \mathscr{A}_{i-1}(i-r) \boxtimes \mathscr{D}^b(\mathbf{G}_r)\big\rangle.$$

Proof. — The arguments are analogous to those used in the proof of Proposition 6.22.

Note that for $s \ge i$ we have $\mathscr{A}_s = 0$. Hence for $r \ge i$ the RHS coincides with $\mathsf{Im}\,\Phi_r$, and since we already know that Φ_r is splitting, for $r \ge i$ it suffices to check that $\mathsf{Ker}\,\Phi_r^* = 0$, that is that Φ_r^* is fully faithful. For this we use descending induction in r.

In the case $r = N-1$ we know that $\mathsf{Im}\,\Phi_{N-1} = \mathscr{D}^b(\mathscr{X}_{N-1})$ by Lemma 6.24. Now let $i < r \le N-1$ and assume that Φ_r is fully faithful. Then the functor $\tilde{\Phi}_r$ is fully faithful by Proposition 2.39. Now consider the functor $\xi^*\tilde{\Phi}_{r-1}\tilde{\Phi}_{r-1}^*$. Composing exact triangle of functors (24) with $\tilde{\Phi}_r$ we obtain an exact triangle of functors

$$\tilde{\Phi}_r\Phi^*_{i_*\widehat{\mathscr{E}}(0,1)\otimes(\tilde{\mathscr{L}}_r/\tilde{\mathscr{L}}_{r-1})} \to \tilde{\Phi}_r\tilde{\Phi}_r^*\xi^* \to \tilde{\Phi}_r\eta_*\tilde{\Phi}_{r-1}^*.$$

Composing isomorphism of functors (25) with $\tilde{\Phi}_{r-1}^*$ we obtain an isomorphism

$$\tilde{\Phi}_r \eta_* \tilde{\Phi}_{r-1}^* \cong \xi^* \tilde{\Phi}_{r-1} \tilde{\Phi}_{r-1}^*.$$

Since the functor $\tilde{\Phi}_r$ is fully faithful we have

$$\tilde{\Phi}_r \tilde{\Phi}_r^* \xi^* \cong \xi^*.$$

On the other hand, $\tilde{\Phi}_r \Phi_{i_* \widehat{\mathcal{E}}(0,1) \otimes (\tilde{\mathcal{L}}_r / \tilde{\mathcal{L}}_{r-1})}^* = 0$ by Lemma 6.19 since $r \geq i + 1$. Summarizing, we deduce that

$$\xi^* \tilde{\Phi}_{r-1} \tilde{\Phi}_{r-1}^* \cong \xi^*.$$

Finally, since $\hat{\phi}_* \xi^* \phi^* = \hat{\phi}_* \hat{\phi}^* = \mathsf{id}$ by Corollary 6.10, we have

$$\Phi_{r-1} \Phi_{r-1}^* \cong \hat{\phi}_* \xi^* \phi^* \Phi_{r-1} \Phi_{r-1}^* \cong \hat{\phi}_* \xi^* \tilde{\Phi}_{r-1} \tilde{\Phi}_{r-1}^* \phi^* \cong \hat{\phi}_* \xi^* \phi^* \cong \mathsf{id}_{\mathcal{X}_{r-1}}$$

by Proposition 6.12. Therefore Φ_{r-1}^* is fully faithful.

For $r < i$ we also use induction in r. However the arguments are slightly different in this case. Assume that $r \leq i$ and the claim for r is true. Assume that F is in the left orthogonal to $\mathsf{Im}\,\Phi_{r-1}$ and in the right orthogonal to the category $\langle \mathscr{A}_{r-1}(1) \boxtimes \mathscr{D}^b(\mathbf{G}_{r-1}), ..., \mathscr{A}_{i-1}(i-r+1) \boxtimes \mathscr{D}^b(\mathbf{G}_{r-1}) \rangle$. Then by the same arguments as in the proof of Proposition 6.22 we check that ϕ^*F is in the left orthogonal to $\mathsf{Im}\,\tilde{\Phi}_{r-1}$ and in the right orthogonal to the category $\langle \mathscr{A}_{r-1}(1) \boxtimes \mathscr{D}^b(\mathbf{S}_r), ..., \mathscr{A}_{i-1}(i-r+1) \boxtimes \mathscr{D}^b(\mathbf{S}_r) \rangle$. Note that by adjunction it follows from Lemma 6.18 that

$$\Phi_{i_* \widehat{\mathcal{E}}(0,1) \otimes (\tilde{\mathcal{L}}_r / \tilde{\mathcal{L}}_{r-1})} \left(\left[\mathscr{B}_{N-r}(-1) \boxtimes \mathscr{D}^b(\mathbf{S}_r) \right]^\perp \right)$$
$$\subset \left\langle \mathscr{A}_{r-1}(1) \boxtimes \mathscr{D}^b(\mathbf{S}_{r-1}), ..., \mathscr{A}_{i-1}(i-r+1) \boxtimes \mathscr{D}^b(\mathbf{S}_{r-1}) \right\rangle^{\perp\perp}.$$

We deduce that ϕ^*F is in the left orthogonal to the image of the first and the third terms of the triangle (23) applied to the subcategory $[\mathscr{B}_{N-r}(-1) \boxtimes \mathscr{D}^b(\mathbf{S}_r)]^\perp$. Therefore ϕ^*F is in the left orthogonal to the image of $\xi_* \tilde{\Phi}_r$ applied to the subcategory $[\mathscr{B}_{N-r}(-1) \boxtimes \mathscr{D}^b(\mathbf{S}_r)]^\perp$. But $[\mathscr{B}_{N-r}(-1) \boxtimes \mathscr{D}^b(\mathbf{S}_r)] \subset \mathsf{Ker}\,\tilde{\Phi}_r$ by Lemma 6.17, hence ϕ^*F is in the left orthogonal to the image of the functor $\xi_* \tilde{\Phi}_r$ and by adjunction we deduce that $\xi^* \phi^*$F $\in {}^\perp(\mathsf{Im}\,\tilde{\Phi}_r)$.

On the other hand, by Lemma 6.7(iii) we have the following resolution

$$0 \to \mathcal{O}_{\tilde{\mathcal{X}}_{r-1}}(-1) \otimes (\tilde{\mathcal{L}}_r / \tilde{\mathcal{L}}_{r-1}) \to \mathcal{O}_{\tilde{\mathcal{X}}_{r-1}} \to \xi_* \mathcal{O}_{\tilde{\mathcal{X}}_r} \to 0$$

which implies that $\xi_*(\mathscr{A}_{r+t}(1+t) \boxtimes \mathscr{D}^b(\mathbf{S}_r)) \subset \langle \mathscr{A}_{r+t}(t), \mathscr{A}_{r+t}(1+t) \rangle \boxtimes \mathscr{D}^b(\mathbf{S}_r)$ for any t, and since we have $\mathscr{A}_{r+t} \subset \mathscr{A}_{r+t-1}$ we conclude that

$$\xi_* \left\langle \mathscr{A}_r(1) \boxtimes \mathscr{D}^b(\mathbf{S}_r), ..., \mathscr{A}_{i-1}(i-r) \boxtimes \mathscr{D}^b(\mathbf{S}_r) \right\rangle$$
$$\subset \left\langle \mathscr{A}_{r-1} \boxtimes \mathscr{D}^b(\mathbf{S}_{r-1}), ..., \mathscr{A}_{i-1}(i-r) \boxtimes \mathscr{D}^b(\mathbf{S}_{r-1}) \right\rangle.$$

Therefore $\xi^! \phi^*$F$(-1) \in \langle \mathscr{A}_r(1) \boxtimes \mathscr{D}^b(\mathbf{S}_r), ..., \mathscr{A}_{i-1}(i-r) \boxtimes \mathscr{D}^b(\mathbf{S}_r) \rangle^\perp$. Finally, we have $\xi^! \phi^*$F$(-1) \cong \xi^* \phi^*$F $\otimes (\tilde{\mathcal{L}}_r / \tilde{\mathcal{L}}_{r-1})^*[-1]$ by Lemma 6.7(iii). Summarizing we deduce

that $\xi^*\phi^*F = 0$ by the induction hypothesis. But $\xi^*\phi^* = \hat{\phi}^*$ is fully faithful by Corollary 6.10, hence $F = 0$. □

6.4. *Proof of the main theorem.* — In this subsection we prove Theorem 6.3 and describe some of its generalizations.

First of all, the first claim of the theorem is proved in Lemma 6.5 and Corollary 6.23. For the second claim, let $L \subset V^*$ be an admissible subspace, $\dim L = r$. Then the map $\lambda : \mathsf{Spec}\,k \to \mathbf{G}_r$ induced by L is a faithful base change for the pair $(\mathscr{X}_r, \mathscr{Y}_r)$ by Lemmas 2.32(*iii*), 6.7, 6.6 and the definition of admissible subspace. Therefore we can apply the faithful base change Theorem 2.40. Then Theorem 6.3 follows from Proposition 6.25 and Proposition 6.22. □

Remark **6.26.** — An interesting question is what can we say about derived categories of X_L and Y_L when L is not admissible. Certainly, in this case the base change λ is no longer faithful, so we cannot apply Theorem 2.40. Looking into the proof of this theorem one can see that the problem consists in nonexactness of some cartesian squares. A natural way to fix this problem is by defining "derived fiber products", which are DG-schemes with underlying topological space being the usual fiber product carrying appropriate sheaf of DG-algebras, and by replacing all fiber products by the derived fiber products. The definition of the derived fiber product should imply exactness of all derived cartesian squares, so it would give an analogue of Theorem 2.40 for any base change. Applying it to the base change λ we would obtain semiorthogonal decomposition of Theorem 6.3 for the "derived linear sections" of X and Y (i.e. for the derived fiber products $X \overset{L}{\times}_{\mathbf{P}(V)} \mathbf{P}(L^\perp)$ and $Y \overset{L}{\times}_{\mathbf{P}(V^*)} \mathbf{P}(L)$). However, up to our knowledge, the theory of derived fiber products is not yet worked out.

The same argument with $\mathsf{Spec}\,k$ replaced by any base scheme T proves a relative version of Theorem 6.3:

Theorem **6.27.** — *Assume that* T *is a smooth algebraic variety and* $\mathscr{L} \subset V^* \otimes \mathscr{O}_T$ *is a vector subbundle,* $\mathsf{rank}\,\mathscr{L} = r$, *such that the corresponding families of linear sections of* X *and* Y

$$\mathscr{X}_{\mathscr{L}} = (X \times T) \times_{\mathbf{P}(V) \times T} \mathbf{P}_T(\mathscr{L}^\perp) \subset X \times T,$$

$$\mathscr{Y}_{\mathscr{L}} = (Y \times T) \times_{\mathbf{P}(V^*) \times T} \mathbf{P}_T(\mathscr{L}) \subset Y \times T,$$

and their fiber product $\mathscr{X}_{\mathscr{L}} \times_T \mathscr{Y}_{\mathscr{L}}$ *have expected dimension*

$$\dim \mathscr{X}_{\mathscr{L}} = \dim X + \dim T - r, \quad \dim \mathscr{Y}_{\mathscr{L}} = \dim Y + \dim T + r - N,$$

$$\dim \mathscr{X}_{\mathscr{L}} \times_T \mathscr{Y}_{\mathscr{L}} = \dim X + \dim Y + \dim T - N.$$

Then there exist a triangulated category $\mathscr{C}_{\mathscr{L}}$ and semiorthogonal decompositions

$$\mathscr{D}^b(\mathscr{X}_{\mathscr{L}}) = \langle \mathscr{C}_{\mathscr{L}}, \mathscr{A}_r(1) \boxtimes \mathscr{D}^b(\mathrm{T}), ..., \mathscr{A}_{i-1}(i-r) \boxtimes \mathscr{D}^b(\mathrm{T}) \rangle$$

$$\mathscr{D}^b(\mathscr{Y}_{\mathscr{L}}) = \langle \mathscr{B}_{j-1}(\mathrm{N} - r - \mathrm{j}) \boxtimes \mathscr{D}^b(\mathrm{T}), ..., \mathscr{B}_{\mathrm{N}-r}(-1) \boxtimes \mathscr{D}^b(\mathrm{T}), \mathscr{C}_{\mathscr{L}} \rangle.$$

Remark **6.28.** — Another relative version of Theorem 6.3 can be obtained as follows. Consider a base scheme S (not necessarily compact), assume that X and Y are algebraic varieties over S, replace the assumptions of projectivity of the maps $f : \mathrm{X} \to \mathbf{P}(\mathrm{V})$ and $g : \mathrm{Y} \to \mathbf{P}(\mathrm{V}^*)$ in the definition of homological projective duality by projectivity of the maps $\mathrm{X} \to \mathrm{S} \times \mathbf{P}(\mathrm{V})$ and $\mathrm{Y} \to \mathrm{S} \times \mathbf{P}(\mathrm{V}^*)$, and assume that we are given a Lefschetz decomposition of $\mathscr{D}^b(\mathrm{X})$ which is S-linear. We will say that Y is homologically projectively dual to X relatively over S if there exists an object $\mathscr{E} \in \mathscr{D}^b(\mathrm{Q}(\mathrm{X}, \mathrm{Y}) \times_{\mathrm{S}\times\mathrm{S}} \mathrm{S})$ (the fiber product is taken with respect to the canonical map $\mathrm{Q}(\mathrm{X}, \mathrm{Y}) \subset \mathrm{X} \times \mathrm{Y} \to \mathrm{S} \times \mathrm{S}$ and the diagonal embedding $\mathrm{S} \xrightarrow{\Delta} \mathrm{S} \times \mathrm{S}$) such that the functor $\Phi_{\mathscr{E}}$ is fully faithful and gives semiorthogonal decomposition (14). One can prove by the same arguments that Theorem 6.3 and Theorem 6.27 are true in this case as well.

7. Properties of homological projective duality

We believe that phenomenon of homological projective duality deserves to be thoroughly investigated. In this section we will discuss some basic properties of homological projective duality.

The first natural question is when a homologically projectively dual variety for a given algebraic variety X exists. From the definition of the homological projective duality it follows that it always exists *on a categorical level*, we always know the derived category of the homologically projectively dual variety. On the other hand, the question of existence of a homologically projectively dual variety *on a geometric level* seems to be of a philosophical nature. Indeed, in some sense every sufficiently good triangulated category can be considered as the derived category of coherent sheaves on a *noncommutative algebraic variety*. In fact, this is one of the ways to understand what a noncommutative algebraic variety is. From this point of view the question of existence of a homologically projectively dual variety as a usual commutative variety does not seem to be very natural and it is hard to expect a nice answer (especially if we remember that the notion of homological projective duality depends on a choice of a line bundle and a Lefschetz decomposition).

The next question is whether a homologically projectively dual variety is unique. Certainly this is true if it is a Fano variety by the Reconstruction Theorem of Bondal and Orlov [BO4]. However, in general it doesn't need to be Fano, so there are examples of several different homologically projectively dual varieties.

Another important question is how one should *construct* a homological projectively dual variety for a given variety X. A natural approach is to consider a moduli space of objects in $\mathscr{D}^b(X)$ with a given class in $K_0(X)$ and *supported on hyperplane sections* of X. However, there are two problems with this approach. The first one is of a technical nature – we don't have a good theory of moduli spaces of objects in triangulated categories yet (moduli spaces in a "good theory" should depend on a choice of stability conditions and should be "noncommutative" in general). The second problem is more complicated – how to choose a correct class in $K_0(X)$. There is a trivial restriction on this class – it should be orthogonal to subcategories $\mathscr{A}_1(1), ..., \mathscr{A}_{i-1}(i-1)$ of the Lefschetz decomposition of X. Sometimes, these restrictions determine unique class in $K_0(X)$ up to a multiplicity. However, examples considered in [K1] show that the choice of correct multiplicity turns out to be quite mysterious.

Now we turn to more specific questions.

7.1. *Disjoint unions and products.* — If algebraic variety X is a disjoint union, $X = X' \sqcup X''$ then its derived category is a completely orthogonal direct sum, $\mathscr{D}^b(X) = \mathscr{D}^b(X') \oplus \mathscr{D}^b(X'')$. If we are given Lefschetz decompositions $\mathscr{D}^b(X') = \langle \mathscr{A}'_0, \mathscr{A}'_1(1), ..., \mathscr{A}'_{i'-1}(i'-1) \rangle$, $\mathscr{D}^b(X'') = \langle \mathscr{A}''_0, \mathscr{A}''_1(1), ..., \mathscr{A}''_{i''-1}(i''-1) \rangle$, then we have a Lefschetz decomposition

$$\mathscr{D}^b(X) = \langle \mathscr{A}'_0 \oplus \mathscr{A}''_0, (\mathscr{A}'_1 \oplus \mathscr{A}''_1)(1), ..., (\mathscr{A}'_{i-1} \oplus \mathscr{A}''_{i-1})(i-1) \rangle,$$
$$i = \max\{i', i''\}.$$

If we know homologically projectively dual varieties to X' and X'', it is natural to ask what the homologically projectively dual to $X' \sqcup X''$ will be? The answer is quite simple.

Proposition **7.1.** — *If* Y' *and* Y'' *are homologically projectively dual to* X' *and* X'' *respectively then* $Y = Y' \sqcup Y''$ *is homologically projectively dual to* $X = X' \sqcup X''$.

This immediately follows from the fact that the derived category of a disjoint union has a completely orthogonal decomposition into the derived categories of the disjoint components. In other word, homological projective duality *commutes with disjoint unions.*

Now assume that $X = X' \times F$ and take $\mathscr{O}_X(1) = p^*\mathscr{O}_{X'}(1)$, where $p : X \to X'$ is the projection along F. Then we have a Lefschetz decomposition

$$\mathscr{D}^b(X) = \left\langle \mathscr{A}'_0 \boxtimes \mathscr{D}^b(F), \left(\mathscr{A}'_1 \boxtimes \mathscr{D}^b(F) \right)(1), ..., \left(\mathscr{A}'_{i'-1} \boxtimes \mathscr{D}^b(F) \right)(i'-1) \right\rangle.$$

Proposition **7.2.** — *If* Y' *is homologically projectively dual to* X' *then* $Y = Y' \times F$ *is homologically projectively dual to* $X = X' \times F$.

Proof. — Note that the universal hyperplane section of X can be represented as $\mathscr{X}_1 = \mathscr{X}_1' \times F$. Tensoring semiorthogonal decomposition $\mathscr{D}^b(\mathscr{X}_1') = \langle \mathscr{D}^b(Y'), \mathscr{A}_1'(1) \boxtimes \mathscr{D}^b(\mathbf{P}(V^*)), ..., \mathscr{A}_{i'-1}'(i'-1) \boxtimes \mathscr{D}^b(\mathbf{P}(V^*)) \rangle$ with $\mathscr{D}^b(F)$ we get

$$\mathscr{D}^b(\mathscr{X}_1) = \big\langle \mathscr{D}^b(Y') \boxtimes \mathscr{D}^b(F), \big(\mathscr{A}_1' \boxtimes \mathscr{D}^b(F)\big)(1) \boxtimes \mathscr{D}^b(\mathbf{P}(V^*)),$$
$$..., \big(\mathscr{A}_{i'-1}' \boxtimes \mathscr{D}^b(F)\big)(i'-1) \boxtimes \mathscr{D}^b(\mathbf{P}(V^*))\big\rangle$$

which shows that $Y = Y' \times F$ is homologically projectively dual to $X = X' \times F$. $\square$

7.2. *Duality.* — In this subsection we are going to check that the relation of homological projective duality is a duality indeed.

Theorem 7.3. — *If* $g : Y \to \mathbf{P}(V^*)$ *is homologically projectively dual to* $f : X \to \mathbf{P}(V)$ *then* $f : X \to \mathbf{P}(V)$ *is homologically projectively dual to* $g : Y \to \mathbf{P}(V^*)$.

Proof. — Indeed, let $\mathscr{D}^b(X) = \langle \mathscr{A}_0, \mathscr{A}_1(1), ..., \mathscr{A}_{i-1}(i-1) \rangle$ be the Lefschetz decomposition of $\mathscr{D}^b(X)$ and let $\mathscr{D}^b(Y) = \langle \mathscr{B}_{j-1}(1-j), ..., \mathscr{B}_1(-1), \mathscr{B}_0 \rangle$ be the dual Lefschetz decomposition of $\mathscr{D}^b(Y)$ given by Theorem 6.3. Dualizing, we obtain a Lefschetz decomposition $\mathscr{D}^b(Y) = \langle \mathscr{B}_0^*, \mathscr{B}_1^*(1), ..., \mathscr{B}_{j-1}^*(j-1) \rangle$. Let us show that X is homologically projectively dual to Y with respect to this Lefschetz decomposition.

Indeed, consider $\mathscr{X}_{N-1}$ and $\mathscr{Y}_{N-1}$. Note that $\mathbf{G}_{N-1} = \mathsf{Gr}(N-1, V^*) \cong \mathbf{P}(V)$, $\mathscr{X}_{N-1} \cong X$ (its embedding into $X \times \mathsf{Gr}(N-1, V^*) = X \times \mathbf{P}(V)$ is given by the graph of f) and $\mathscr{Y}_{N-1} \subset Y \times \mathbf{P}(V)$ is the universal hyperplane section of Y. Dualizing the decomposition of Proposition 6.22 with $r = N-1$ we obtain a semiorthogonal decomposition

$$\mathscr{D}^b(\mathscr{Y}_{N-1}) =$$
$$\big\langle \mathscr{D}^b(\mathscr{X}_{N-1})^*, \mathscr{B}_1^*(1) \boxtimes \mathscr{D}^b(\mathbf{P}(V)), ..., \mathscr{B}_{j-1}^*(j-1) \boxtimes \mathscr{D}^b(\mathbf{P}(V)) \big\rangle.$$

Moreover, the embedding functor $\mathscr{D}^b(\mathscr{X}_{N-1}) \to \mathscr{D}^b(\mathscr{Y}_{N-1})$ is obtained by conjugation with the duality functor of the functor Φ_{N-1}^*. Note that $\mathscr{Y}_{N-1} \times_{\mathbf{P}(V)} \mathscr{X}_{N-1} = Q(X, Y)$ and it is easy to check that the embedding functor $\mathscr{D}^b(\mathscr{X}_{N-1}) \to \mathscr{D}^b(\mathscr{Y}_{N-1})$ is a kernel functor with kernel scheme-theoretically supported on $Q(X, Y)$. Thus X is homologically projectively dual to Y. $\square$

7.3. *Dimension of the dual variety.* — It is natural to ask, what can we say about the dimension of the homologically projectively dual variety to a given variety X. This question can be answered precisely in the special case of *rectangular* Lefschetz decomposition of X (i.e. when $\mathscr{A}_0 = \mathscr{A}_1 = ... = \mathscr{A}_{i-1}$).

Proposition 7.4. — *If* $g : Y \to \mathbf{P}(V^*)$ *is homologically projectively dual to* $f : X \to \mathbf{P}(V)$ *with respect to a rectangular Lefschetz decomposition with* i *terms, then the number of*

terms j *in the dual Lefschetz decomposition of* $\mathscr{D}^b(Y)$ *and the dimension of* Y *equal*

$$j = N - i, \quad \dim Y = \dim X + N - 2i,$$

where $N = \dim V$.

Proof. — The formula for j follows immediately from (11). To get the formula for the dimension we note that in the case of rectangular Lefschetz decompositions for any i-dimensional admissible subspace $L \subset V^*$ we have by Theorem 6.3 an equivalence of categories $\mathscr{D}^b(X_L) = \mathscr{D}^b(Y_L)$. It follows that $\dim X_L = \dim Y_L$. But $\dim X_L = \dim X - i$ and $\dim Y_L = \dim Y - (N - i)$. $\qquad\square$

Remark **7.5.** — In general however it seems that it is impossible to give an explicit formula for j and $\dim Y$ in terms of i and $\dim X$. The only thing we can get is an inequality

$$j \geq N - i,$$

which follows immediately from (11). It also seems very plausible that we have in general an inequality of dimensions

$$\dim Y \geq \dim X + N - 2i,$$

where $N = \dim V$. The proof could go as follows. Consider a generic i-dimensional admissible subspace $L \subset V^*$ and note that by Theorem 6.3 we have a fully faithful functor $\mathscr{D}^b(X_L) \to \mathscr{D}^b(Y_L)$. Since $\dim X_L = \dim X - i$ and $\dim Y_L = \dim Y - (N - i)$ it would be sufficient to check that $\dim X_L \leq \dim Y_L$. However, it seems that there is yet no proof for the inequality of dimensions of varieties whose derived categories admit a fully faithful functor between them.

Another question is the relation of the number of terms of a Lefschetz decomposition to the dimension of X.

Proposition **7.6.** — *Assume that* X *is connected,* $\mathscr{D}^b(X) = \langle \mathscr{A}_0, \mathscr{A}_1(1), ..., \mathscr{A}_{i-1}(i-1) \rangle$ *is a Lefschetz decomposition and* $\mathscr{A}_{i-1} \neq 0$. *Then* $\dim X \geq i - 1$ *and equality is possible only if* $f : X \to \mathbf{P}(V)$ *is birational onto a linear subspace* $\mathbf{P}^{i-1} \subset \mathbf{P}(V)$ *and* $\mathscr{A}_{i-2} = \mathscr{A}_{i-1}$. *If moreover* $\mathscr{A}_0 = \mathscr{A}_{i-1}$ *then* $X = \mathbf{P}^{i-1}$.

Proof. — Consider generic subspace $L \subset V^*$ of dimension $i - 1$. Then X_L is a complete intersection of $i - 1$ hyperplanes in X. By Theorem 6.3 the restriction functor $\mathscr{A}_{i-1} \subset \mathscr{D}^b(X) \to \mathscr{D}^b(X_L)$ is fully faithful. Therefore $\mathscr{D}^b(X_L) \neq 0$, so X_L is not empty. Since this is true for generic L we conclude that $\dim X \geq i - 1$.

Assume that $\dim X = i - 1$ and consider generic subspace $L \subset V^*$ of dimension $i - 2$. Then X_L is a complete intersection of $i - 2$ hyperplanes in X, so X_L is a smooth connected curve for generic L. By Theorem 6.3 the restriction functors $\mathscr{A}_{i-2} \subset \mathscr{D}^b(X) \to \mathscr{D}^b(X_L)$ and $\mathscr{A}_{i-1} \subset \mathscr{D}^b(X) \to \mathscr{D}^b(X_L)$ are fully faithful and $\langle \mathscr{A}_{i-2}, \mathscr{A}_{i-1}(1) \rangle$ is a semiorthogonal collection in $\mathscr{D}^b(X_L)$. But by Lemma 9.2 the only smooth connected curve admitting a nontrivial Lefschetz decomposition is $\mathbf{P}^1$ and the decomposition necessarily takes the form $\mathscr{D}^b(\mathbf{P}^1) = \langle \mathscr{O}_{\mathbf{P}^1}(k), \mathscr{O}_{\mathbf{P}^1}(k+1) \rangle$. Therefore $X_L \cong \mathbf{P}^1$ and $\mathscr{O}_{\mathbf{P}(V)}(1)_{|X_L} \cong \mathscr{O}_{X_L}(1)$, hence f restricted to X_L is an isomorphism onto a line $\mathbf{P}^1 \subset \mathbf{P}(V)$. Since this is true for generic L, it follows that the degree of f is 1, hence $f(X) = \mathbf{P}^{i-1} \subset \mathbf{P}(V)$, and the map $f : X \to \mathbf{P}^{i-1}$ is birational. Moreover, taking any indecomposable object $E \in \mathscr{A}_{i-1}$ we deduce that its restriction to $D^b(X_L)$ is isomorphic (up to a shift) to $\mathscr{O}_{\mathbf{P}^1}(k)$ (the only indecomposable object in $\langle \mathscr{O}_{\mathbf{P}^1}(k) \rangle$). Hence E is exceptional and $(E, E(1), ..., E(i-1))$ is an exceptional collection on X.

Finally, if $\mathscr{A}_0 = \mathscr{A}_{i-1}$ then $\mathscr{D}^b(X) = \langle E, E(1), ..., E(i-1) \rangle$, hence the Grothendieck group $K_0(X)$ is a free abelian group of rank i. On the other hand, it is easy to see that if the birational map $X \to \mathbf{P}^{i-1}$ is not trivial then the rank of $K_0(X)$ is strictly greater then i. Hence $X = \mathbf{P}^{i-1}$. $\square$

Remark **7.7.** — If $X = \mathbf{P}(W) \subset \mathbf{P}(V)$, a linear subspace, $\dim W = i$, considered with the Lefschetz decomposition $\mathscr{D}^b(X) = \langle f^* \mathscr{O}_{\mathbf{P}(V)}(k), ..., f^* \mathscr{O}_{\mathbf{P}(V)}(k+i-1) \rangle$, then it follows from Corollary 8.3 below that $Y = \mathbf{P}(W^\perp) \subset \mathbf{P}(V^*)$ is homologically projectively dual to X, where $W^\perp \subset V^*$ is the orthogonal subspace (just take $S = \mathsf{Spec}\, k$ and $E = W$).

Corollary **7.8.** — *If* $g : Y \to \mathbf{P}(V^*)$ *is homologically projectively dual to* $f : X \to \mathbf{P}(V)$ *then either* $\dim X + \dim Y = \dim V - 2$, *or* $\dim X + \dim Y \geq \dim V$. *Moreover, the first case is possible only when* $X = \mathbf{P}(W) \subset \mathbf{P}(V)$ *and* $Y = \mathbf{P}(W^\perp) \subset \mathbf{P}(V^*)$ *where* $W \subset V$ *is a vector subspace.*

Proof. — Note that $i + j \geq \dim V$, hence if $\dim X \geq i$ and $\dim Y \geq j$ then $\dim X + \dim Y \geq \dim V$. Assume that $\dim X = i - 1$. If $\mathscr{A}_0 \neq \mathscr{A}_{i-1} = \mathscr{A}_{i-2}$ then $i + j \geq \dim V + 2$, hence $\dim X + \dim Y \geq (i-1) + (j-1) = i + j - 2 \geq \dim V$. Finally, if $\mathscr{A}_0 = \mathscr{A}_{i-1}$ then $X = \mathbf{P}(W)$ and $Y = \mathbf{P}(W^\perp)$ by Proposition 7.6 and Remark 7.7. $\square$

7.4. *Homological projective duality and classical projective duality.* — Given a projective morphism $f : X \to \mathbf{P}(V)$ we denote by $X^\vee \subset \mathbf{P}(V^*)$ the set of all points $H \in \mathbf{P}(V^*)$ such that the corresponding hyperplane section X_H of X is singular. It is clear that $X^\vee$ is a Zariski closed subset in $\mathbf{P}(V^*)$. Note that if $f : X \to \mathbf{P}(V)$ is an embedding then $X^\vee$ is the classical projectively dual variety to X.

The main result of this subsection is the following

Theorem **7.9.** — *Assume that* $g : Y \to \mathbf{P}(V^*)$ *is homologically projectively dual to* $f : X \to \mathbf{P}(V)$. *Then the set* $\mathsf{sing}(g) := \{$*critical values of* $g\}$ *coincides with* $X^\vee$, *the classical projectively dual variety of* X.

Proof. — Consider the universal hyperplane section $\mathscr{X}_1$ of X and the maps $f_1 : \mathscr{X}_1 \to \mathbf{P}(V^*)$ and $g : Y \to \mathbf{P}(V^*)$. Note that by definition of homological projective duality we have a semiorthogonal decomposition (14). Note also that $X^\vee = \mathsf{sing}(f_1)$ is the set of critical values of the map f_1. Thus we have to check that $\mathsf{sing}(f_1) = \mathsf{sing}(g)$.

First of all, assume that $\mathsf{sing}(g) \not\subset \mathsf{sing}(f_1)$. Let $H \in \mathbf{P}(V^*)$ be a point in $\mathsf{sing}(g)$ such that $H \notin \mathsf{sing}(f_1)$. Then it is clear that there exists a smooth hypersurface $D \subset \mathbf{P}(V^*)$ such that $H \in D$, $Y_D := Y \times_{\mathbf{P}(V^*)} D$ has a singularity over H, and $\dim Y_D = \dim Y - 1$. Let $T = D \setminus \mathsf{sing}(f_1)$. Then $H \in T$, $Y_T := Y \times_{\mathbf{P}(V^*)} T$ has a singularity over H, and $\dim Y_T = \dim Y - 1$. On the other hand, f_1 is smooth over T, hence $\mathscr{X}_{1T} = \mathscr{X}_1 \times_{\mathbf{P}(V^*)} T$ is smooth and both $\mathscr{X}_{1T}$ and $\mathscr{X}_{1T} \times_T Y_T$ have expected dimension. Therefore the base change $T \to \mathbf{P}(V^*)$ is faithful for the pair $(\mathscr{X}_1, Y)$ and we obtain by the faithful base change Theorem 2.40 a semiorthogonal decomposition

$$(\mathbf{26}) \qquad \mathscr{D}^b(\mathscr{X}_{1T}) = \big\langle \mathscr{D}^b(Y_T), \mathscr{A}_1(1) \boxtimes \mathscr{D}^b(T), ..., \mathscr{A}_{i-1}(i-1) \boxtimes \mathscr{D}^b(T)\big\rangle.$$

But category $\mathscr{D}^b(\mathscr{X}_{1T})$ is Ext-bounded since $\mathscr{X}_{1T}$ is smooth, while category $\mathscr{D}^b(Y_T)$ is not Ext-bounded since Y_T is singular (see Lemma 2.25). This is a contradiction, which shows that we must have an embedding $\mathsf{sing}(g) \subset \mathsf{sing}(f_1)$.

Similarly, assume that $\mathsf{sing}(f_1) \not\subset \mathsf{sing}(g)$. Let $H \in \mathbf{P}(V^*)$ be a point in $\mathsf{sing}(f_1)$ such that $H \notin \mathsf{sing}(g)$. Then it is clear that there exists a smooth hypersurface $D \subset \mathbf{P}(V^*)$ such that $H \in D$, $\mathscr{X}_{1D} := \mathscr{X}_1 \times_{\mathbf{P}(V^*)} D$ has a singularity over H, and $\dim \mathscr{X}_{1D} = \dim \mathscr{X}_1 - 1$. Let $T = D \setminus \mathsf{sing}(g)$. Then $H \in T$, $\mathscr{X}_{1T} := \mathscr{X}_1 \times_{\mathbf{P}(V^*)} T$ has a singularity over H, and $\dim \mathscr{X}_{1T} = \dim \mathscr{X}_1 - 1$. On the other hand, g is smooth over T, hence $Y_T = Y \times_{\mathbf{P}(V^*)} T$ is smooth and both Y_T and $\mathscr{X}_{1T} \times_T Y_T$ have expected dimension. Therefore the base change $T \to \mathbf{P}(V^*)$ is faithful for the pair $(\mathscr{X}_1, Y)$ and we again obtain by the faithful base change Theorem 2.40 a semiorthogonal decomposition (26). Now we note that category $\mathscr{D}^b(Y_T)$ is Ext-bounded since Y_T is smooth, and categories $\mathscr{A}_1(1) \boxtimes \mathscr{D}^b(T), ..., \mathscr{A}_{i-1}(i-1) \boxtimes \mathscr{D}^b(T)$ are Ext-bounded because T is smooth. Therefore category $\mathscr{D}^b(\mathscr{X}_{1T})$ is Ext-bounded by Lemma 2.26. But this is a contradiction with the fact that $\mathscr{X}_{1T}$ is singular. □

7.5. *Homological projective duality and triangulated categories of singularities.* — Recall that in [O3] to every algebraic variety X there was associated a triangulated

category $\mathscr{D}_{\mathsf{sg}}(X) := \mathscr{D}^b(X)/\mathscr{D}^{\mathsf{perf}}(X)$, the quotient category of the bounded derived category of coherent sheaves by the subcategory of perfect complexes, which was called the triangulated category of singularities of X. This definition easily generalizes to any triangulated category.

Definition **7.10** ([O4]). — *Let $\mathscr{D}$ be a triangulated category. An object* $F \in \mathscr{D}$ *is* homologically finite *if for any* $G \in \mathscr{D}$ *the set* $\{n \in \mathbf{Z} \mid \mathsf{Hom}(F, G[n]) \neq 0\}$ *is finite.*

The full subcategory of $\mathscr{D}$ consisting of homologically finite objects is denoted by $\mathscr{D}_{\mathsf{hf}}$. It is a triangulated subcategory. The quotient category $\mathscr{D}_{\mathsf{sg}} := \mathscr{D}/\mathscr{D}_{\mathsf{hf}}$ is called the triangulated category of singularities of $\mathscr{D}$.

Lemma **7.11** ([O4]). — *If $\mathscr{D} = \langle \mathscr{D}_1, \mathscr{D}_2, ..., \mathscr{D}_m \rangle$ is a semiorthogonal decomposition of $\mathscr{D}$ then its triangulated category of singularities has the following semiorthogonal decomposition*

$$\mathscr{D}_{\mathsf{sg}} = \langle \mathscr{D}_{1\mathsf{sg}}, \mathscr{D}_{2\mathsf{sg}}, ..., \mathscr{D}_{m\mathsf{sg}} \rangle.$$

Theorem **7.12.** — *If $g : Y \to \mathbf{P}(V^*)$ is homologically projectively dual to $f : X \to \mathbf{P}(V)$ and $L \subset V^*$ is an admissible subspace then $\mathscr{D}_{\mathsf{sg}}(X_L) \cong \mathscr{D}_{\mathsf{sg}}(Y_L)$.*

Proof. — Since X is smooth by assumptions and Y is smooth by Theorem 6.3 it follows from Lemma 7.11 that

$$\mathscr{A}_{0\mathsf{sg}} = \mathscr{A}_{1\mathsf{sg}} = \cdots = \mathscr{A}_{(i-1)\mathsf{sg}} = 0 \quad \text{and}$$
$$\mathscr{B}_{0\mathsf{sg}} = \mathscr{B}_{1\mathsf{sg}} = \cdots = \mathscr{B}_{(j-1)\mathsf{sg}} = 0,$$

where $\mathscr{D}^b(X) = \langle \mathscr{A}_0, \mathscr{A}_1(1), ..., \mathscr{A}_{i-1}(i-1) \rangle$ and $\mathscr{D}^b(Y) = \langle \mathscr{B}_{j-1}(1-j), ..., \mathscr{B}_1(-1), \mathscr{B}_0 \rangle$ are the Lefschetz decompositions of X and Y respectively. Using again Theorem 6.3 and Lemma 7.11 we deduce that $\mathscr{D}_{\mathsf{sg}}(X_L) \cong (\mathscr{C}_L)_{\mathsf{sg}} \cong \mathscr{D}_{\mathsf{sg}}(Y_L)$.
$\square$

8. Projective bundles

Let S be a smooth (not necessarily compact) base scheme with a vector bundle E of rank i. Let $X = \mathbf{P}_S(E)$ be a projectivization of this vector bundle with the projection $p : X \to S$, and let $\mathscr{O}_X(1)$ be the Grothendieck line bundle on $\mathbf{P}_S(E)$ over S (such that $p_* \mathscr{O}_X(1) \cong E^*$). Let $V^* \subset \Gamma(S, E^*) = \Gamma(X, \mathscr{O}_X(1))$ be a space of global sections generating E^*, and let $f : X \to \mathbf{P}(V)$ be the corresponding morphism. Let $\mathscr{A}_0 = p^*(\mathscr{D}^b(S)) \subset \mathscr{D}^b(X)$. Then by the result of Orlov [O1] we have a semiorthogonal decomposition

$$(\mathbf{27}) \qquad \mathscr{D}^b(X) = \langle \mathscr{A}_0, \mathscr{A}_1(1), ..., \mathscr{A}_{i-1}(i-1) \rangle$$

$$\text{with} \quad \mathscr{A}_0 = \mathscr{A}_1 = \cdots = \mathscr{A}_{i-1} = p^*(\mathscr{D}^b(S)).$$

Let $\mathscr{X}_1 \subset X \times \mathbf{P}(V^*)$ be the universal hyperplane section of X. Since $X \times \mathbf{P}(V^*)$ is the projectivization of the pullback of E to $S \times \mathbf{P}(V^*)$, it follows that the fiber of $\mathscr{X}_1$ over the generic point of $S \times \mathbf{P}(V^*)$ is a hyperplane in the projectivization of the fiber of E over the corresponding point of S, and over a certain closed subset of $S \times \mathbf{P}(V^*)$ the fiber of $\mathscr{X}_1$ coincides with the whole projectivization of the fiber of E. This closed subset $Y \subset S \times \mathbf{P}(V^*)$ is the zero locus of the section of the vector bundle $E^* \boxtimes \mathscr{O}_{\mathbf{P}(V^*)}(1)$, corresponding to the identity in $\Gamma(S \times \mathbf{P}(V^*), E^* \boxtimes \mathscr{O}_{\mathbf{P}(V^*)}(1)) = \Gamma(S, E^*) \otimes \Gamma(\mathbf{P}(V^*), \mathscr{O}_{\mathbf{P}(V^*)}(1)) \supset V^* \otimes V$. Note that

Lemma **8.1.** — *We have* $Y \cong \mathbf{P}_S(E^\perp)$, *where* $E^\perp = \mathsf{Ker}(V^* \otimes \mathscr{O}_S \to E^*)$. *In particular,* Y *is smooth and*

(**28**) $\mathsf{codim}_{S \times \mathbf{P}(V^*)} Y = i.$

Proof. — The fiber of Y over a point $s \in S$ consists of all sections $H \in V^* \subset H^0(S, E^*)$, that vanish at s, i.e. which are contained in the fiber of $E^\perp$ over S. □

Let $f : \mathscr{X}_1 \to S \times \mathbf{P}(V^*)$ be the canonical projection. Let $g : Y \to S \times \mathbf{P}(V^*)$ be the embedding. Let $Z = Y \times_{(S \times \mathbf{P}(V^*))} \mathscr{X}_1$ be the fiber product, and denote by $\phi : Z \to Y$ and $i : Z \to \mathscr{X}_1$ the projections. So we have the following cartesian square

$$
\begin{array}{ccc}
\mathbf{P}_Y(E) \cong Z & \xrightarrow{\ i\ } & \mathscr{X}_1 \\
{\scriptstyle \phi} \downarrow & & \downarrow {\scriptstyle f} \\
Y & \xrightarrow{\ g\ } & S \times \mathbf{P}(V^*).
\end{array}
$$

Note that by definition of Y we have $Z \cong \mathbf{P}_Y(E) \cong X \times_S Y$. We consider the kernel $\mathscr{E} = (\phi \times i)_* \mathscr{O}_Z \in \mathscr{D}^b(Y \times \mathscr{X}_1)$ and the corresponding kernel functor $\Phi_{\mathscr{E}} : \mathscr{D}^b(Y) \to \mathscr{D}^b(\mathscr{X}_1)$. Note that the functor $\Phi_{\mathscr{E}}$ is $(S \times \mathbf{P}(V^*))$-linear.

The main result of this section is the following

Theorem **8.2.** — *In the above notation and assumptions we have a semiorthogonal decomposition*

$$
\mathscr{D}^b(\mathscr{X}_1) =
$$
$$
\big\langle \Phi_{\mathscr{E}}(\mathscr{D}^b(Y)), \mathscr{A}_1(1) \boxtimes \mathscr{D}^b(\mathbf{P}(V^*)), ..., \mathscr{A}_{i-1}(i-1) \boxtimes \mathscr{D}^b(\mathbf{P}(V^*)) \big\rangle.
$$

Comparing this theorem with the definition of relative homological projective duality (see Remark 6.28) we obtain the following

Corollary **8.3.** — *If* E *is generated by global sections then* $Y = \mathbf{P}_S(E^\perp)$ *is homologically projectively dual to* $X = \mathbf{P}_S(E)$ *relatively over* S.

We start the proof with some preparations. Denote by $\pi : \mathscr{X}_1 \to X$ the projection.

Lemma 8.4. — *The subscheme* $Z \subset \mathscr{X}_1$ *is a zero locus of a section of the vector bundle* $\pi^*\Omega_{X/S}(1) \otimes f^*\mathscr{O}_{\mathbf{P}(V^*)}(1)$ *on* $\mathscr{X}_1$, *where* $\Omega_{X/S}$ *is the sheaf of relative differentials.*

Proof. — Note that it follows from the definitions that $\mathscr{X}_1$ is a zero locus of a section of the line bundle $\mathscr{O}_X(1) \boxtimes \mathscr{O}_{\mathbf{P}(V^*)}(1)$ on $X \times \mathbf{P}(V^*)$ and Z is a zero locus of a section of the vector bundle $p^*E^* \boxtimes \mathscr{O}_{\mathbf{P}(V^*)}(1)$ on $X \times \mathbf{P}(V^*)$. Moreover, it is clear that the canonical epimorphism $p^*E^* \boxtimes \mathscr{O}_{\mathbf{P}(V^*)}(1) \to \mathscr{O}_X(1) \boxtimes \mathscr{O}_{\mathbf{P}(V^*)}(1)$ takes the latter section to the former. Therefore, the latter section restricted to the zero locus of the former section is contained in the kernel of the epimorphism which is isomorphic to $(\Omega_{X/S}(1) \boxtimes \mathscr{O}_{\mathbf{P}(V^*)}(1))_{|\mathscr{X}_1} \cong \pi^*\Omega_{X/S}(1) \otimes f^*\mathscr{O}_{\mathbf{P}(V^*)}(1)$, and its zero locus coincides with Z.   □

Corollary 8.5. — *The sheaf* $i_*\mathscr{O}_Z$ *is quasiisomorphic to the Koszul complex,*

$$i_*\mathscr{O}_Z \cong \mathsf{Kosz}(\pi^*T_{X/S}(-1) \otimes f^*\mathscr{O}_{\mathbf{P}(V^*)}(-1))$$
$$:= \Lambda^\bullet(\pi^*T_{X/S}(-1) \otimes f^*\mathscr{O}_{\mathbf{P}(V^*)}(-1)).$$

In particular, $i_*\mathscr{O}_Z$ *is a perfect complex on* $\mathscr{X}_1$ *and* $i : Z \to \mathscr{X}_1$ *has finite* Tor*-dimension.*

Proof. — Since $\mathscr{X}_1$ has pure codimension 1 in $X \times \mathbf{P}(V^*)$ it follows from (28) that Z has codimension $\mathsf{i} - 1$ in $\mathscr{X}_1$, therefore the corresponding section of the vector bundle $\pi^*\Omega_{X/S}(1) \otimes f^*\mathscr{O}_{\mathbf{P}(V^*)}(1)$ is regular and $i_*\mathscr{O}_Z$ is quasiisomorphic to the Koszul complex. In particular it is a perfect complex. Finally, it follows from the projection formula that $i_*i^*F \cong F \otimes i_*\mathscr{O}_Z$ for any $F \in \mathscr{D}^b(\mathscr{X}_1)$, and since i_* is exact and conservative it follows that the Tor-dimension of i is finite.   □

Proposition 8.6. — *The functor* $\Phi_{\mathscr{E}}$ *is fully faithful.*

Proof. — Note that $\mathscr{E} = (\phi \times i)_*\mathscr{O}_Z$ is perfect on Z, Z is projective over Y and over $\mathscr{X}_1$, and $\mathscr{E}$ has finite Ext-amplitude over $\mathscr{X}_1$ by Corollary 8.5. Therefore by Lemma 2.28 the kernel functor $\Phi_{\mathscr{E}^\#} : \mathscr{D}^b(\mathscr{X}_1) \to \mathscr{D}^b(Y)$ with the kernel

$$\mathscr{E}^\# = \mathsf{R}\mathscr{H}om((\phi \times i)_*\mathscr{O}_Z, \omega_{\mathscr{X}_1 \times Y/Y}[\dim \mathscr{X}_1])$$
$$\cong (\phi \times i)_* \mathsf{R}\mathscr{H}om(\mathscr{O}_Z, (\phi \times i)^!(\omega_{\mathscr{X}_1 \times Y/Y}[\dim \mathscr{X}_1]))$$
$$\cong (\phi \times i)_*\omega_{Z/Y}[\dim Z/Y]$$

is left adjoint to $\Phi_{\mathscr{E}}$. Further, the composition $\Phi_{\mathscr{E}^\#} \circ \Phi_{\mathscr{E}} : \mathscr{D}^b(Y) \to D^b(Y)$ is given by

$$(\mathbf{29}) \qquad F \mapsto \phi_*(i^*i_*\phi^*F \otimes \omega_{Z/Y}[\dim Z/Y]) \cong \phi_*(\Phi_K(\phi^*F) \otimes \omega_{Z/Y}[\dim Z/Y])$$

where $K \in \mathscr{D}^b(Z \times Z)$ is the kernel of the functor $i^*i_* : \mathscr{D}^b(Z) \to \mathscr{D}^b(Z)$. By Lemma 2.31 we have $\mathscr{H}^{-t}(K) \cong \Delta_* \Lambda^t \mathscr{N}^*_{Z/\mathscr{X}_1}$, where $\Delta : Z \to Z \times Z$ is the diagonal embedding. On the other hand, by Lemma 8.4 we have $\mathscr{N}^*_{Z/\mathscr{X}_1} \cong (\pi^* T_{X/S}(-1) \otimes f^* \mathscr{O}_{\mathbf{P}(V^*)}(-1))_{|Z}$ and since $Z \cong \mathbf{P}_Y(E) \cong Y \times_S X$ we have $(\pi^* T_{X/S})_{|Z} \cong T_{Z/Y}$, so we conclude that

$$\mathscr{H}^{-t}(K) \cong \Delta_*\big(\Lambda^t T_{Z/Y}(-t) \otimes \phi^*g^* \mathscr{O}_{\mathbf{P}(V^*)}(-t)\big).$$

Consider the functor $\mathscr{D}^b(Y) \to \mathscr{D}^b(Y)$ defined by the RHS of the formula (29) with K replaced by $\mathscr{H}^{-t}(K)$. This functor takes $F \in \mathscr{D}^b(Y)$ to

$$(\mathbf{30}) \qquad \phi_*(\Phi_{\mathscr{H}^{-t}(K)}(\phi^*F) \otimes \omega_{Z/Y}[\dim Z/Y])$$
$$\cong \phi_*\big(\phi^*F \otimes \Lambda^t T_{Z/Y}(-t) \otimes \phi^*g^* \mathscr{O}_{\mathbf{P}(V^*)}(-t) \otimes \omega_{Z/Y}[\dim Z/Y]\big)$$
$$\cong F \otimes \phi_*\big(\Lambda^t T_{Z/Y}(-t) \otimes \phi^*g^* \mathscr{O}_{\mathbf{P}(V^*)}(-t) \otimes \omega_{Z/Y}[\dim Z/Y]\big).$$

Note that

$$\phi_*\big(\Lambda^t T_{Z/Y}(-t) \otimes \phi^*g^* \mathscr{O}_{\mathbf{P}(V^*)}(-t) \otimes \omega_{Z/Y}[\dim Z/Y]\big)$$
$$\cong \phi_* \, \mathsf{R}\mathscr{H}om\big(\Omega^t_{Z/Y}(t), \phi^!(g^* \mathscr{O}_{\mathbf{P}(V^*)}(-t))\big)$$
$$\cong \mathsf{R}\mathscr{H}om\big(\phi_*\big(\Omega^t_{Z/Y}(t)\big), g^* \mathscr{O}_{\mathbf{P}(V^*)}(-t)\big).$$

But for $t \neq 0$ we have $\phi_*(\Omega^t_{Z/Y}(t)) = 0$, while for $t = 0$ we have $\phi_*(\Omega^t_{Z/Y}(t)) = \mathscr{O}_Y$. It follows that the functor (30) is zero for $t \neq 0$ and is identity for $t = 0$. Using a devissage argument we deduce that the functor (29) is the identity. But as we have seen above this is the composition of $\Phi_{\mathscr{E}}$ with its left adjoint. Therefore $\Phi_{\mathscr{E}}$ is fully faithful. $\qquad \square$

Denote by $\alpha : \mathscr{X}_1 \to X \times \mathbf{P}(V^*)$ the embedding.

Lemma **8.7.** — *The functor* $f^*g_* : \mathscr{D}^b(Y) \to \mathscr{D}^b(\mathscr{X}_1)$ *is a kernel functor,* $f^*g_* \cong \Phi_{K(f,g)}$, *and its kernel* $K(f,g) \in \mathscr{D}^b(Y \times \mathscr{X}_1)$ *fits into exact triangle*

$$K(f,g) \to (\phi \times i)_* \mathscr{O}_Z \to (\phi \times i)_*(i^*\pi^* \mathscr{O}_X(-1) \otimes \phi^*g^* \mathscr{O}_{\mathbf{P}(V^*)}(-1))[2].$$

In particular, for any $F \in \mathscr{D}^b(Y)$ *we have an exact triangle in* $\mathscr{D}^b(\mathscr{X}_1)$

$$f^*g_*F \to i_*\phi^*F \to i_*\phi^*(F \otimes g^* \mathscr{O}_{\mathbf{P}(V^*)}(-1)) \otimes \pi^* \mathscr{O}_X(-1)[2].$$

Proof. — Consider the following commutative diagram

Note that $Z = Y \times_{(S \times \mathbf{P}(V^*))} (X \times \mathbf{P}(V^*))$ and the square in the diagram is exact cartesian, because p is flat. It follows that

$$f^* g_* = \alpha^* p^* g_* = \alpha^* (\alpha \circ i)_* \phi^* = \alpha^* \alpha_* i_* \phi^*.$$

On the other hand, it is clear that α is a divisorial embedding and $\mathscr{X}_1$ is the zero locus of the line bundle $\mathscr{O}_X(1) \boxtimes \mathscr{O}_{\mathbf{P}(V^*)}(1)$, therefore $\alpha^* \alpha_*$ is a kernel functor and by Lemma 2.31 its kernel $K_\alpha \in \mathscr{D}^b(\mathscr{X}_1 \times \mathscr{X}_1)$ fits into the exact triangle

$$K_\alpha \to \Delta_* \mathscr{O}_{\mathscr{X}_1} \to \Delta_* \alpha^* (\mathscr{O}_X(-1) \boxtimes \mathscr{O}_{\mathbf{P}(V^*)}(-1))[2],$$

where $\Delta : \mathscr{X}_1 \to \mathscr{X}_1 \times \mathscr{X}_1$ is the diagonal embedding. Computing the convolution of this triangle with the kernel $(\phi \times i)_* \mathscr{O}_Z$ of the functor $i_* \phi^*$ we obtain the desired triangle for the kernel $K(f,g)$ since $i^* \alpha^*(\mathscr{O}_X(-1) \boxtimes \mathscr{O}_{\mathbf{P}(V^*)}(-1)) \cong i^* \pi^* \mathscr{O}_X(-1) \otimes i^* f^* \mathscr{O}_{\mathbf{P}(V^*)}(-1)$ and $i^* f^* \mathscr{O}_{\mathbf{P}(V^*)}(-1) \cong \phi^* g^* \mathscr{O}_{\mathbf{P}(V^*)}(-1)$. $\qquad\square$

Proof of Theorem 8.2. — Note that by (27) we have $\mathscr{A}_k(k) \boxtimes \mathscr{D}^b(\mathbf{P}(V^*)) = f^* \mathscr{D}^b(S \times \mathbf{P}(V^*)) \otimes \pi^* \mathscr{O}_X(k)$, and that $\Phi_{\mathscr{E}}(\mathscr{D}^b(Y)) = i_* \phi^*(\mathscr{D}^b(Y))$, so we need to prove that we have the following semiorthogonal decomposition

$$(\mathbf{31}) \qquad \mathscr{D}^b(\mathscr{X}_1) = \big\langle i_* \phi^*(\mathscr{D}^b(Y)), f^*(\mathscr{D}^b(S \times \mathbf{P}(V^*))) \otimes \pi^* \mathscr{O}_X(1),$$
$$..., f^*(\mathscr{D}^b(S \times \mathbf{P}(V^*))) \otimes \pi^* \mathscr{O}_X(\mathsf{i}-1) \big\rangle.$$

The semiorthogonality in question is verified quite easily. Using Lemma 5.3 and Proposition 8.6, the question reduces to checking that $\mathsf{Hom}(f^* F \otimes \pi^* \mathscr{O}_X(k), i_* \phi^* G) = 0$ for all $F \in \mathscr{D}^b(S \times \mathbf{P}(V^*))$, $G \in \mathscr{D}^b(Y)$ and $1 \le k \le \mathsf{i}-1$. But

$$\begin{aligned}
\mathsf{Hom}(f^* F \otimes \pi^* \mathscr{O}_X(k), i_* \phi^* G) &= \mathsf{Hom}(f^* F, (i_* \phi^* G) \otimes \pi^* \mathscr{O}_X(-k)) \\
&= \mathsf{Hom}(f^* F, i_*(\phi^* G \otimes \mathscr{O}_{Z/Y}(-k))) \\
&= \mathsf{Hom}(F, f_* i_*(\phi^* G \otimes \mathscr{O}_{Z/Y}(-k))) \\
&= \mathsf{Hom}(F, g_* \phi_*(\phi^* G \otimes \mathscr{O}_{Z/Y}(-k))) \\
&= \mathsf{Hom}(F, g_*(G \otimes \phi_* \mathscr{O}_{Z/Y}(-k)))
\end{aligned}$$

and $\phi_* \mathscr{O}_{Z/Y}(-k) = 0$ for $1 \le k \le \mathsf{i}-1$ since ϕ is a projectivization of a rank i vector bundle.

It remains to check that the RHS of (31) generates $\mathscr{D}^b(\mathscr{X}_1)$. First of all we will show that it contains $i_* \phi^*(\mathscr{D}^b(Y)) \otimes \pi^* \mathscr{O}_X(k)$ for $0 \le k \le \mathsf{i}-1$. Indeed, for any $F \in \mathscr{D}^b(Y)$ the triangle of Lemma 8.7 twisted by $\pi^* \mathscr{O}_X(1)$ takes the form

$$f^* g_* F \otimes \pi^* \mathscr{O}_X(1) \to i_* \phi^* F \otimes \pi^* \mathscr{O}_X(1) \to i_* \phi^*(F \otimes g^* \mathscr{O}_{\mathbf{P}(V^*)}(-1))[2].$$

Note that its first term is contained in $f^*(\mathscr{D}^b(S \times \mathbf{P}(V^*))) \otimes \pi^* \mathscr{O}_X(1)$ and the last term is contained in $i_* \phi^*(\mathscr{D}^b(Y))$, therefore the middle term is contained in the

RHS of (31). Twisting this triangle by $\pi^*\mathcal{O}_X(1)$ we get

$$f^*g_*F \otimes \pi^*\mathcal{O}_X(2) \to i_*\phi^*F \otimes \pi^*\mathcal{O}_X(2)$$
$$\to i_*\phi^*(F \otimes g^*\mathcal{O}_{\mathbf{P}(V^*)}(-1)) \otimes \pi^*\mathcal{O}_X(1)[2].$$

Note that its first term is contained in $f^*(\mathcal{D}^b(S \times \mathbf{P}(V^*))) \otimes \pi^*\mathcal{O}_X(2)$ and the last term is contained in $i_*\phi^*(\mathcal{D}^b(Y)) \otimes \pi^*\mathcal{O}_X(1)$, therefore the middle term is contained in the RHS of (31). Continuing in this way we deduce the claim by induction.

Now assume that G is contained in the left orthogonal to the RHS of (31). Then it follows from above that for any $0 \leq k \leq i - 1$ and any $F \in \mathcal{D}^b(Y)$ we have $\mathsf{Hom}(G, i_*\phi^*F \otimes \pi^*\mathcal{O}_X(k)) = 0$. But by adjunction it equals to $\mathsf{Hom}(i^*G, \phi^*F \otimes \mathcal{O}_{Z/Y}(k))$ and since by [O1] the subcategories $\phi^*(\mathcal{D}^b(Y)) \otimes \mathcal{O}_{Z/Y}(k)$ with $0 \leq k \leq i-1$ generate $\mathcal{D}^b(Z)$ it follows that $i^*G = 0$. This means that G is supported on $\mathcal{X}_1 \setminus Z$. But $\mathcal{X}_1 \setminus Z$ is a $\mathbf{P}^{i-2}$-bundle over $(S \times \mathbf{P}(V^*)) \setminus Y$, hence the orthogonality of G to the subcategory $\langle f^*(\mathcal{D}^b(S \times \mathbf{P}(V^*))) \otimes \pi^*\mathcal{O}_X(1), ..., f^*(\mathcal{D}^b(S \times \mathbf{P}(V^*))) \otimes \pi^*\mathcal{O}_X(i-1) \rangle$ implies that $G = 0$. □

Now we are going to apply Theorems 6.3 and 6.27 for this special case of homological projective duality.

Let F be another vector bundle on S, $\mathsf{rank}(F) = r$, and let $\phi : F \to E^*$ be a morphism of vector bundles. Consider the projectivizations $\mathbf{P}_S(E)$ and $\mathbf{P}_S(F)$. Let $p : \mathbf{P}_S(E) \to S$ and $q : \mathbf{P}_S(F) \to S$ be the projections and let $\mathcal{O}_{\mathbf{P}_S(E)/S}(1)$ and $\mathcal{O}_{\mathbf{P}_S(F)/S}(1)$ denote the Grothendieck ample line bundles. Note that ϕ induces a section of the vector bundle $p^*F^* \boxtimes \mathcal{O}_{\mathbf{P}_S(E)/S}(1)$ on $\mathbf{P}_S(E)$, and a section of the vector bundle $q^*E^* \boxtimes \mathcal{O}_{\mathbf{P}_S(F)/S}(1)$ on $\mathbf{P}_S(F)$. Let $X_F \subset \mathbf{P}_S(E)$ and $Y_F \subset \mathbf{P}_S(F)$ denote their zero loci.

Theorem **8.8.** — *Assume that* $\mathsf{codim}_{\mathbf{P}_S(E)} X_F = \mathsf{rank}\, F$, $\mathsf{codim}_{\mathbf{P}_S(F)} Y_F = \mathsf{rank}\, E$ *and* $\dim X_F \times_S Y_F = \dim S - 1$. *Then there exist semiorthogonal decompositions*

$$\mathcal{D}^b(X_F) =$$
(**32**)
$$\langle \mathcal{D}^b(Y_F), p^*\mathcal{D}^b(S) \otimes \mathcal{O}_{\mathbf{P}_S(E)/S}(1), ..., p^*\mathcal{D}^b(S) \otimes \mathcal{O}_{\mathbf{P}_S(E)/S}(i - r) \rangle, \quad \text{if } r < i,$$
$$\mathcal{D}^b(Y_F) =$$
$$\langle q^*\mathcal{D}^b(S) \otimes \mathcal{O}_{\mathbf{P}_S(F)/S}(i - r), ..., q^*\mathcal{D}^b(S) \otimes \mathcal{O}_{\mathbf{P}_S(F)/S}(-1), \mathcal{D}^b(X_F) \rangle, \text{ if } r > i,$$

and an equivalence $\mathcal{D}^b(X_F) \cong \mathcal{D}^b(Y_F)$ *if* $r = i$.

Proof. — First of all, consider the case when the morphism $\phi : F \to E^*$ can be represented as a composition $F \to V^* \otimes \mathcal{O}_S \to E^*$ of a monomorphism of vector bundles to a trivial vector bundle followed by an epimorphism of vector bundles. In this case the claim of the theorem follows from Theorem 6.27.

Indeed, take $r = \mathsf{rank}\, F$, $T = \mathsf{Gr}_S(r, V^* \otimes \mathcal{O}_S)$, the relative Grassmannian, and let $\mathcal{L} \subset V^* \otimes \mathcal{O}_T$ be the tautological subbundle of rank r. Then $\mathcal{X}_{\mathcal{L}}$ and $\mathcal{Y}_{\mathcal{L}}$

in the notations of Theorem 6.27 are the universal families of linear sections of $X = \mathbf{P}_S(E)$ and $Y = \mathbf{P}_S(E^\perp)$ respectively. It is easy to see that the dimension assumptions of Theorem 6.27 are satisfied, hence we have semiorthogonal decompositions

$$\mathscr{D}^b(\mathscr{X}_{\mathscr{L}}) =$$
$$\left\langle \mathscr{D}^b(\mathscr{Y}_{\mathscr{L}}), p^*\mathscr{D}^b(\mathrm{T}) \otimes \mathscr{O}_{\mathbf{P}_S(E)/S}(1), ..., p^*\mathscr{D}^b(\mathrm{T}) \otimes \mathscr{O}_{\mathbf{P}_S(E)/S}(\mathsf{i} - r) \right\rangle, \text{ if } r < \mathsf{i},$$
$$\mathscr{D}^b(\mathscr{Y}_{\mathscr{L}}) =$$
$$\left\langle q^*\mathscr{D}^b(\mathrm{T}) \otimes \mathscr{O}_{\mathbf{P}_S(F)/S}(\mathsf{i} - r), ..., q^*\mathscr{D}^b(\mathrm{T}) \otimes \mathscr{O}_{\mathbf{P}_S(F)/S}(-1), \mathscr{D}^b(\mathscr{X}_{\mathscr{L}}) \right\rangle,$$
$$\text{if } r > \mathsf{i},$$

and an equivalence $\mathscr{D}^b(\mathscr{X}_{\mathscr{L}}) \cong \mathscr{D}^b(\mathscr{Y}_{\mathscr{L}})$, if $r = \mathsf{i}$.

The embedding $F \to V^* \otimes \mathscr{O}_S$ gives a section $\sigma : S \to T = \mathsf{Gr}_S(V^* \otimes \mathscr{O}_S)$ such that $F \cong \sigma^*\mathscr{L}$. Consider σ as a base change. Note that $\mathscr{X}_{\mathscr{L}} \times_T S = X_F$. On the other hand, $\mathscr{Y}_{\mathscr{L}} \times_T S$ is the zero locus of a section of the vector bundle $\mathscr{O}_{\mathbf{P}_S(E^\perp)/S}(1) \otimes F^{\perp*}$ on $\mathbf{P}_S(E^\perp)$. But looking at the commutative diagram

it is easy to deduce that it also can be represented as the zero locus of a section of the vector bundle $E^* \otimes \mathscr{O}_{\mathbf{P}_S(F)/S}(1)$ on $\mathbf{P}_S(F)$, i.e. that $\mathscr{Y}_{\mathscr{L}} \times_T S = Y_F$.

Finally note that the dimension assumptions of the theorem and Lemma 2.36 imply that this base change is faithful for a pair $(\mathscr{X}_{\mathscr{L}}, \mathscr{Y}_{\mathscr{L}})$. Applying the faithful base change Theorem 2.40 we deduce the claim.

The general case follows from the above case by Theorem 2.41. Indeed, all inclusion functors in the desired decompositions are S-linear and for every point $s \in S$ there exists an open neighborhood $U \subset S$ over which the morphism $\phi_{|U} : F_{|U} \to E^*_{|U}$ can be represented as a composition $F_{|U} \to V^* \otimes \mathscr{O}_U \to E^*_{|U}$ of a monomorphism of vector bundles followed by an epimorphism of vector bundles. □

Consider the case $r = \mathsf{i}$. Then the dimension assumptions of Theorem 8.8 can be rewritten as

$$\dim X_F = \dim Y_F = \dim X_F \times_S Y_F = \dim S - 1.$$

Note that $D := p(X_F) = q(Y_F) \subset S$ is the degeneration locus of ϕ, i.e. the zero locus of $\det\phi : \det F \to \det E^*$. Since $\dim D = \dim S - 1$ it follows that $p : X_F \to D$ and $q : Y_F \to D$ are birational. Therefore $q^{-1} \circ p$ is a birational transformation $X_F \dashrightarrow Y_F$. It is easy to check that $\omega_{X_F} \cong \omega_{Y_F} \cong \omega_D \cong \omega_S \otimes \det E^* \otimes \det F^*$, hence this transformation is a flop.

Corollary **8.9.** — *If* $\dim X_F = \dim Y_F = \dim X_F \times_S Y_F = \dim S - 1$ *then the kernel functor with kernel given by the structure sheaf of the fiber product* $X_F \times_D Y_F = X_F \times_S Y_F$ *is an equivalence of categories* $\mathscr{D}^b(X_F) \cong \mathscr{D}^b(Y_F)$.

For example, let $S = \mathbf{P}^4$, $E = \mathscr{O}_S(-1) \oplus \mathscr{O}_S(-1)$, $F = \mathscr{O}_S \oplus \mathscr{O}_S$ and $\phi : F \to E^*$ given by the matrix $\phi = \left(\begin{smallmatrix} x & y \\ z & u \end{smallmatrix}\right)$, where $(x : y : z : u : v)$ are the homogeneous coordinates on S. Then $D \subset S$ is the cone over $\mathbf{P}^1 \times \mathbf{P}^1$ (given by equation $xu - yz = 0$), X_F and Y_F are small resolutions of D and $X_F \dashrightarrow Y_F$ is the standard flop.

9. Examples

In this section we will give several examples of homologically projectively dual varieties.

9.1. *A stupid example.* — Let X be any smooth algebraic variety with a projective map $f : X \to \mathbf{P}(V)$. Then taking $\mathscr{A}_0 = \mathscr{D}^b(X)$ we get a Lefschetz decomposition with only one term. We will call it the stupid Lefschetz decomposition.

Proposition **9.1.** — *The universal hyperplane section* $\mathscr{X}_1 \subset X \times \mathbf{P}(V^*)$ *with the projection* $\mathscr{X}_1 \to \mathbf{P}(V^*)$ *is homologically projectively dual to* $X \to \mathbf{P}(V)$ *with respect to the stupid Lefschetz decomposition.*

Proof. — Just take $\mathscr{E}$ to be the structure sheaf of the diagonal in $\mathscr{X}_1 \times \mathscr{X}_1$. □

Alternatively, one can consider the stupid Lefschetz decomposition as a particular case of the decomposition (27) since any algebraic variety can be considered as a projectivization of a line bundle over itself.

Let us describe the claim of Theorem 6.3 in this case. Let $Y = \mathscr{X}_1$ be the universal hyperplane section of X. Let $L \subset V^*$ be a vector subspace. Then Y_L is the family of hyperplane sections of X parameterized by $\mathbf{P}(L)$. It is fibered over X with fiber equal to $\mathbf{P}(L)$ over X_L, and a hyperplane in $\mathbf{P}(L)$ over $X \setminus X_L$. Theorem 6.3 implies that we have a semiorthogonal decomposition

$$\mathscr{D}^b(Y_L) = \left\langle \mathscr{D}^b(X) \otimes \mathscr{O}_{\mathbf{P}(L)}(1 - \dim L), ..., \mathscr{D}^b(X) \otimes \mathscr{O}_{\mathbf{P}(L)}(-1), \mathscr{D}^b(X_L) \right\rangle.$$

E.g. for $\dim L = 1$ we have $Y_L = X_L$ and for $\dim L = 2$ we have Y_L is the blowup of X_L in X. In the latter case the obtained semiorthogonal decomposition coincides with the standard decomposition of the blowup.

9.2. *Curves.* — Now, assume that X is a smooth projective curve.

Lemma **9.2.** — *The derived category of a smooth connected projective curve* X *admits a nontrivial Lefschetz decomposition with respect to an effective line bundle* $\mathscr{O}_X(1)$, *only if* $X \cong \mathbf{P}^1$. *In this case* $\mathscr{O}_X(1)$ *is the positive generator of* $\mathsf{Pic}\, X$, *and the decomposition takes form* $\mathscr{D}^b(X) = \langle \mathscr{O}_{\mathbf{P}^1}(k), \mathscr{O}_{\mathbf{P}^1}(k+1) \rangle$ *for some* k.

Proof. — Assume that we have a nontrivial Lefschetz decomposition of $\mathscr{D}^b(X)$, so that $\mathscr{A}_1 \neq 0$. Let F be a nontrivial object in $\mathscr{A}_1$. By definition of a Lefschetz decomposition we have

$$\mathsf{RHom}(F(1), F) = 0.$$

Since X is a curve, every object in $\mathscr{D}^b(X)$ is a direct sum of its cohomology sheaves, and every sheaf on X is a direct sum of a torsion sheaf and of a locally free sheaf. If G is a nontrivial torsion sheaf some shift of which is a direct summand of F, then $G(1) \cong G$, hence $\mathsf{Hom}(G(1), G) \neq 0$, hence $\mathsf{Hom}(F(1), F) \neq 0$. Therefore F is a direct sum of shifts of locally free sheaves.

Since $\mathscr{A}_1$ is closed under direct summands and triangulated, there exists a locally free sheaf $F \in \mathscr{A}_1$. Then

$$\mathsf{RHom}(F(1), F) = \mathsf{R}\Gamma(X, F \otimes F^*(-1)).$$

But $F \otimes F^*$ has $\mathscr{O}_X$ as a direct summand, hence the condition $\mathsf{RHom}(F(1), F) = 0$ implies that the line bundle $\mathscr{O}_X(-1)$ on X has no cohomology. By Riemann–Roch this is possible only if $\deg \mathscr{O}_X(1) = 1 - g$, where g is the genus of X. So, if $g \geq 1$ then $\mathscr{O}_X(1)$ cannot be effective. Therefore for $g \geq 1$ we cannot have a nontrivial Lefschetz decomposition.

Now assume that $g = 0$, so $X \cong \mathbf{P}^1$. Then the above arguments show that $\mathscr{O}_X(1)$ is the positive generator of the $\mathsf{Pic}\, X$. Moreover, since any locally free sheaf on $\mathbf{P}^1$ is a direct sum of line bundles, it follows that $\mathscr{A}_1 = \langle \mathscr{O}_{\mathbf{P}^1}(k) \rangle$ for some $k \in \mathbf{Z}$. Then

$$\langle \mathscr{O}_{\mathbf{P}^1}(k) \rangle = \mathscr{A}_1 \subset \mathscr{A}_0 \subset \mathscr{A}_1(1)^\perp = \langle \mathscr{O}_{\mathbf{P}^1}(k+1) \rangle^\perp = \langle \mathscr{O}_{\mathbf{P}^1}(k) \rangle,$$

and we are done. $\square$

The above lemma shows that the only way to get a homological projective duality for a curve of positive genus is to consider the stupid Lefschetz decomposition. Then as we have shown in Proposition 9.1 the homologically projectively

dual variety is the universal hyperplane section. Note that in this case the map $\mathscr{X}_1 \to \mathbf{P}(V^*)$ is a finite covering (of degree equal to the degree of X in $\mathbf{P}(V)$) ramified over the classical projectively dual hypersurface $X^\vee \subset \mathbf{P}(V^*)$.

The case of $X = \mathbf{P}^1$ with the Lefschetz decomposition $\mathscr{D}^b(X) = \langle \mathscr{O}_{\mathbf{P}(V)}(k), \mathscr{O}_{\mathbf{P}(V)}(k+1) \rangle$ was considered in Remark 7.7.

9.3. *Hirzebruch surfaces.* — Let $S = \mathbf{P}^1$ and $E = \mathscr{O}_S \oplus \mathscr{O}_S(-d)$, so that $X = \mathbf{P}_S(E)$ is the Hirzebruch surface F_d. Take $V^* = H^0(S, E^*) \cong k \oplus k^{d+1}$. Then $f : X \to \mathbf{P}(V) = \mathbf{P}^{d+1}$ maps X onto a cone over a Veronese rational curve of degree d (the exceptional section of X is contracted to the vertex of the cone).

In this case $E^\perp = \mathsf{Ker}(V^* \otimes \mathscr{O}_S \to E^*) \cong \mathscr{O}_S(-1)^d$, hence $Y = \mathbf{P}_S(E^\perp) \cong \mathbf{P}^1 \times \mathbf{P}^{d-1}$. The map $g : Y \to \mathbf{P}(V^*)$ is a d-fold covering onto the hyperplane in $\mathbf{P}^d \subset \mathbf{P}(V^*)$, corresponding to the vertex of the cone.

9.4. *Two-dimensional quadric.* — Let $S = \mathbf{P}^1$ and $E = \mathscr{O}_S(-1) \oplus \mathscr{O}_S(-1)$, so that $X = \mathbf{P}_S(E) \cong \mathbf{P}^1 \times \mathbf{P}^1$ is the two dimensional quadric. Take $V^* = H^0(S, E^*) \cong k^4$. Then $f : X \to \mathbf{P}(V) = \mathbf{P}^3$ is the standard embedding.

In this case $E^\perp = \mathsf{Ker}(V^* \to E^*) \cong \mathscr{O}_S(-1) \oplus \mathscr{O}_S(-1)$, hence Y is also isomorphic to $\mathbf{P}^1 \times \mathbf{P}^1$ and the map $g : Y \to \mathbf{P}(V^*) = \mathbf{P}^3$ identifies it with the projectively dual quadric to X.

In a forthcoming paper [K2] we will describe homological projective duality for all quadrics.

9.5. *Springer–Grothendieck resolution.* — Let G be a semisimple algebraic group, $S = G/B$ the flag variety of G (the set of all Borel subgroups in G), $\mathfrak{g}$ the Lie algebra of G, and $\mathfrak{b} \subset \mathfrak{g} \otimes \mathscr{O}_S$, (resp. $\mathfrak{n} \subset \mathfrak{g} \otimes \mathscr{O}_S$) the vector subbundle with fiber over a point of G/B given by the corresponding Borel subalgebra (resp. nilpotent subalgebra) of $\mathfrak{g}$. Take $E = \mathfrak{n}$, which is, in fact, isomorphic to the cotangent bundle of S, so that $X = \mathbf{P}_S(E) = \mathbf{P}_{G/B}(\mathfrak{n}) \cong \mathbf{P}_{G/B}(T^*_{G/B})$, and $V^* = \mathfrak{g}^* \cong H^0(G/B, \mathfrak{n}^*)$. Then $f : X \to \mathbf{P}(\mathfrak{g})$ maps X onto the projectivization of the nilpotent cone in $\mathfrak{g}$ and is well known as the (projectivized) Springer resolution of the nilpotent cone.

In this case $E^\perp = \mathsf{Ker}(\mathfrak{g}^* \to \mathfrak{n}^*) \cong \mathsf{Ker}(\mathfrak{g} \to \mathfrak{g}/\mathfrak{b}) = \mathfrak{b}$ (we identify $\mathfrak{g}$ with $\mathfrak{g}^*$ by the Killing form), hence Y is isomorphic to $\mathbf{P}_{G/B}(\mathfrak{b})$ and the map $g : Y \to \mathbf{P}(\mathfrak{g})$ is known as the (projectivized) simultaneous Springer–Grothendieck resolution. Its generic fiber consists of $|W|$ points where W is the Weyl group of G.

REFERENCES

[B]   A. BONDAL, Representations of associative algebras and coherent sheaves (Russian), *Izv. Akad. Nauk SSSR, Ser. Mat.*, **53** (1989), 25–44; translation in *Math. USSR-Izv.*, **34** (1990), 23–42.

[BK]   A. BONDAL and M. KAPRANOV, Representable functors, Serre functors, and reconstructions (Russian), *Izv. Akad. Nauk SSSR, Ser. Mat.*, **53** (1989), 1183–1205, 1337; translation in *Math. USSR-Izv.*, **35** (1990), 519–541.

[BO1]   A. BONDAL and D. ORLOV, Semiorthogonal decomposition for algebraic varieties, preprint math.AG/9506012.

[BO2]   A. BONDAL and D. ORLOV, Derived categories of coherent sheaves, *Proceedings of the International Congress of Mathematicians*, Vol. II (Beijing, 2002), pp. 47–56, Higher Ed. Press, Beijing, 2002.

[BO3]   A. BONDAL and D. ORLOV, private communication.

[BO4]   A. BONDAL and D. ORLOV, Reconstruction of a variety from the derived category and groups of autoequivalences, *Compos. Math.*, **125** (2001), 327–344.

[BV]   A. BONDAL and M. VAN DEN BERGH, Generators and representability of functors in commutative and noncommutative geometry, *Mosc. Math. J.*, **3** (2003), 1–36, 258.

[H]   R. HARTSHORN, *Residues and Duality*, Lecture notes of a seminar on the work of A. Grothendieck, given at Harvard 1963/64. With an appendix by P. Deligne. Lect. Notes Math., **20**, Springer, Berlin, New York, 1966.

[HV]   K. HORI and C. VAFA, *Mirror Symmetry*, arXiv:hep-th/0404196.

[Ko]   M. KONTSEVICH, Homological algebra of mirror symmetry, *Proceedings of the International Congress of Mathematicians*, Vol. 1, 2 (Zürich, 1994), pp. 120–139, Birkhäuser, Basel, 1995.

[K1]   A. KUZNETSOV, Hyperplane sections and derived categories (Russian), *Izv. Ross. Akad. Nauk, Ser. Mat.*, **70** (2006), 23–128.

[K2]   A. KUZNETSOV, Derived categories of quadric fibrations and intersections of quadrics, preprint math.AG/0510670.

[K3]   A. KUZNETSOV, Exceptional collections for Grassmannians of isotropic lines, preprint math.AG/0512013.

[K4]   A. KUZNETSOV, Homological projective duality for Grassmannians of lines, preprint math.AG/0610957.

[O1]   D. ORLOV, Projective bundles, monoidal transformations, and derived categories of coherent sheaves (Russian), *Izv. Ross. Akad. Nauk, Ser. Mat.*, **56** (1992), 852–862; translation in *Russ. Acad. Sci., Izv., Math.*, **41** (1993), 133–141.

[O2]   D. ORLOV, Equivalences of derived categories and K3 surfaces, algebraic geometry, 7, *J. Math. Sci., New York*, **84** (1997), 1361–1381.

[O3]   D. ORLOV, Triangulated categories of singularities and D-branes in Landau–Ginzburg models (Russian), *Tr. Mat. Inst. Steklova*, **246** (2004), 240–262; translation in *Proc. Steklov Inst. Math.*, **246** (2004), 227–248.

[O4]   D. ORLOV, Triangulated categories of singularities and equivalences between Landau–Ginzburg models, preprint math.AG/0503630.

A. K.
Algebra Section,
Steklov Mathematical Institute, 8 Gubkin str.,
Moscow 119991, Russia
akuznet@mi.ras.ru