

COCHAINS AND HOMOTOPY TYPE

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ABSTRACT

Finite type nilpotent spaces are weakly equivalent if and only if their singular cochains are quasi-isomorphic as E_∞ algebras. The cochain functor from the homotopy category of finite type nilpotent spaces to the homotopy category of E_∞ algebras is faithful but not full.

Introduction

Motivating questions in algebraic topology often take the form of finding algebraic invariants for some class of topological spaces that classify those spaces up to some useful equivalence relation, like homeomorphism, homotopy equivalence, or even something weaker. The fundamental work of Quillen [13] and Sullivan [18] describes a form of algebraic data that classifies simply connected spaces up to rational equivalence, the equivalence relation generated by maps of spaces that are rational homology isomorphisms. In particular, the latter work associates to each space a rational commutative differential graded algebra (CDGA) called the polynomial De Rham complex, which is closely related to the algebra of differential forms when the space is a smooth manifold and closely related to the singular cochain complex in general. Simply connected spaces of finite type (homology finitely generated in each degree) are rationally equivalent if and only if their associated polynomial De Rham complexes are quasi-isomorphic CDGAs.

More recently, in [9] the author proved an analogous theorem for p -equivalence, the equivalence relation generated by maps that are $\mathbf{Z}/p\mathbf{Z}$ homology isomorphisms. Here the relevant sort of algebra is an E_∞ algebra, an up-to-homotopy generalization of a commutative differential graded algebra. The singular cochain complex of a space with coefficients in a commutative ring \mathbf{R} has the natural structure of an E_∞ \mathbf{R} -algebra. Simply connected spaces of finite type are p -equivalent if and only if their cochains with coefficients in $\mathbf{F}_p (= \mathbf{Z}/p\mathbf{Z})$ are quasi-isomorphic E_∞ \mathbf{F}_p -algebras. As discussed below, coefficients in the algebraic closure $\bar{\mathbf{F}}_p$ lead to even stronger results.

The purpose of this paper is to prove that the singular cochains with integer coefficients, viewed as E_∞ algebras, classify finite type simply connected spaces up to weak equivalence. We prove the following theorem.

Main Theorem. — *Finite type nilpotent spaces X and Y are weakly equivalent if and only if the E_∞ algebras $C^*(X)$ and $C^*(Y)$ are quasi-isomorphic.*

Weak equivalence is the equivalence relation generated by the maps of spaces that induce bijections of components and isomorphisms of all homotopy groups for all base points; such maps are themselves called weak equivalences. By a theorem of Whitehead, spaces homotopy equivalent to CW complexes (which include most spaces of geometric interest) are weakly equivalent if and only if they are homotopy equivalent. Nilpotent is a generalization of simply connected: It means that the fundamental group at each point is nilpotent and acts nilpotently on the higher homotopy groups.

The invariants discussed in [9,13,18] do more than distinguish equivalence classes of spaces, they distinguish equivalence classes of maps. We can consider the category obtained from the category of spaces by formally inverting the rational equivalences; this category is called the rational homotopy category. Likewise, we can form the homotopy category of CDGAs by formally inverting the quasi-isomorphisms. Then [18] proves that the full subcategory of the rational homotopy category of finite type nilpotent spaces embeds as a full subcategory of the homotopy category of CDGAs. Similarly [9] proves that the full subcategory of the p -adic homotopy category (p -equivalences formally inverted) of finite type nilpotent spaces embeds as a full subcategory of the homotopy category of $E_\infty \bar{\mathbf{F}}_p$ -algebras. For the homotopy category of \mathbf{F}_p -algebras, the functor is faithful but not full.

In the integral situation, the category obtained from spaces by formally inverting the weak equivalences is called simply the homotopy category; it is equivalent to the category of CW complexes and homotopy classes of maps. We show that the singular cochain functor with integer coefficients, viewed as a functor from the homotopy category to the homotopy category of E_∞ algebras is faithful but is not full, just as in the case for coefficients in \mathbf{F}_p . First, for faithfulness, we prove the following refinement of the Main Theorem. In it, $[X, Y]$ denotes the set of maps from the space X to the space Y in the homotopy category, and $[A, B]_{\mathfrak{e}}$ denotes the set of maps from the E_∞ algebra A to the E_∞ algebra B in the homotopy category of E_∞ algebras.

Theorem 0.1. — *There is a function $\epsilon: [C^*Y, C^*X]_{\mathfrak{e}} \rightarrow [X, Y]$, natural in spaces X and finite type nilpotent spaces Y , such that the composite*

$$[X, Y] \xrightarrow{C^*} [C^*Y, C^*X]_{\mathfrak{e}} \xrightarrow{\epsilon} [X, Y]$$

*is the identity. If in addition X is finite type, then for any f in $[C^*Y, C^*X]_{\mathfrak{e}}$, the maps f and $C^*(\epsilon(f))$ induce the same map on cohomology.*

We suspect that last the statement is true for arbitrary X , but our techniques do not appear to suffice.

The significance of Theorem 0.1 is that even though C^* is not a full embedding, it is still sufficient for many classification problems about existence and

uniqueness of maps. For example, for any finite type space X and any finite type nilpotent space Y , Theorem 0.1 implies:

- (i) (Existence) Given any ϕ in $[C^*Y, C^*X]_{\mathfrak{E}}$, there exists f in $[X, Y]$ such that $H^*f = H^*\phi$.
- (ii) (Uniqueness) For f, g in $[X, Y]$, $f = g$ if and only if $C^*f = C^*g$ in the homotopy category of E_{∞} algebras.

We should also note that the Main Theorem is an immediate consequence of Theorem 0.1: By Dror's generalization of the Whitehead Theorem [2], a map between nilpotent spaces is a weak equivalence if and only if it induces an isomorphism on homology. For finite type spaces, a map induces an isomorphism on homology if and only if it induces an isomorphism on cohomology.

Finally, to show that the functor C^* is not full, we describe the set of maps $[C^*Y, C^*X]_{\mathfrak{E}}$ for Y finite type nilpotent. As we explain in the next section, the functor C^* has a contravariant right adjoint \mathbf{U} . That is, there exists a functor \mathbf{U} from the homotopy category of E_{∞} algebras to the homotopy category and a natural bijection

$$[X, \mathbf{U}A] \cong [A, C^*X]_{\mathfrak{E}}$$

for any space X and any E_{∞} algebra A . Thus, to understand $[C^*Y, C^*X]_{\mathfrak{E}}$, we just need to understand the space $\mathbf{U}C^*Y$. Let Y^{\wedge} denote the finite completion of Y and let $Y_{\mathbf{Q}}$ denote the rationalization of Y , as defined for example in [17]. Let ΛY^{\wedge} denote the free loop space of Y^{\wedge} , and consider the map $\Lambda Y^{\wedge} \rightarrow Y^{\wedge}$ given by evaluation at a point. Define $\Lambda_f Y$ to be the homotopy pullback of the map $\Lambda Y^{\wedge} \rightarrow Y^{\wedge}$ along the finite completion map $Y \rightarrow Y^{\wedge}$.

$$\begin{array}{ccc} \Lambda_f Y & \dashrightarrow & \Lambda Y^{\wedge} \\ \vdots & & \downarrow \\ Y & \longrightarrow & Y^{\wedge} \end{array}$$

We prove the following theorem.

Theorem 0.2. — *Let Y be a finite type nilpotent space. Then $\mathbf{U}C^*Y$ is weakly equivalent to $\Lambda_f Y$.*

We have that $\pi_n \Lambda_f Y = \pi_n Y \oplus \pi_{n+1} Y^{\wedge}$ at each base point of $\Lambda_f Y$. When Y is finite type nilpotent $\pi_{n+1} Y^{\wedge}$ is the pro-finite completion of $\pi_{n+1} Y$. When Y is connected and simply connected, $\Lambda_f Y$ is connected and simple (the fundamental group is abelian and acts trivially on the higher homotopy groups), and so $[S^n, \Lambda_f Y] \cong \pi_n \Lambda_f Y$.

It follows from Theorem 0.2 that $C^*: [X, Y] \rightarrow [C^*Y, C^*X]_{\mathfrak{E}}$ is usually not surjective. As a concrete example, when we take $X = S^2$ and $Y = S^3$, the set $[X, Y]$ is one point and the set $[C^*Y, C^*X]_{\mathfrak{E}}$ is uncountable.

It seems plausible that the finite type hypotheses may be dropped by considering a category of (flat) E_{∞} coalgebras in place of E_{∞} algebras. Although some authors have claimed this sort of result [15] or better results [16], as of this writing no one has found a correct proof.

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1. Outline of the argument

In this section we outline the proofs of Theorems 0.1 and 0.2. The underlying idea is to take advantage of the “arithmetic square” to reduce questions in terms of the integers to questions in terms of the rational numbers and the prime fields answered in [9]. Let \mathbf{Z}^{\wedge} denote the finite completion of the integers, i.e., the product of \mathbf{Z}_p^{\wedge} over all primes p , and let $\mathbf{Q}^{\wedge} = \mathbf{Z}^{\wedge} \otimes \mathbf{Q}$. The integers are the fiber product of the inclusions of \mathbf{Z}^{\wedge} and \mathbf{Q} in \mathbf{Q}^{\wedge} , and the square on the left below is called the arithmetic square.

$$\begin{array}{ccc}
 \mathbf{Z} & \longrightarrow & \mathbf{Z}^{\wedge} \\
 \downarrow & & \downarrow \\
 \mathbf{Q} & \longrightarrow & \mathbf{Q}^{\wedge}
 \end{array}
 \qquad
 \begin{array}{ccc}
 Y & \cdots\cdots\cdots & Y^{\wedge} \\
 \downarrow & & \downarrow \\
 Y_{\mathbf{Q}} & \longrightarrow & (Y^{\wedge})_{\mathbf{Q}}
 \end{array}$$

The arithmetic square in homotopy theory is the square on the right. When Y is a finite type nilpotent space, then Y is equivalent to the homotopy pullback of the rationalization map $Y^{\wedge} \rightarrow (Y^{\wedge})_{\mathbf{Q}}$ along the rationalization of the finite completion map $Y_{\mathbf{Q}} \rightarrow (Y^{\wedge})_{\mathbf{Q}}$. Looking at the definition of $\Lambda_f Y$, we obtain an arithmetic square of sorts for $\Lambda_f Y$ from the arithmetic square for Y .

Proposition 1.1. — *When Y is finite type nilpotent, $\Lambda_f Y$ is weakly equivalent to the homotopy pullback of $\Lambda Y^\wedge \rightarrow (Y^\wedge)_{\mathbf{Q}}$ along $Y_{\mathbf{Q}} \rightarrow (Y^\wedge)_{\mathbf{Q}}$.*

$$\begin{array}{ccc} \Lambda_f Y & \dashrightarrow & \Lambda Y^\wedge \\ \vdots \downarrow & & \downarrow \\ Y_{\mathbf{Q}} & \longrightarrow & (Y^\wedge)_{\mathbf{Q}} \end{array}$$

We would like to describe \mathbf{UC}^*Y as an analogous homotopy pullback. We begin by reviewing the construction of \mathbf{U} . We understand the category \mathfrak{E} of E_∞ algebras to be the category of \mathcal{E} -algebras for some cofibrant E_∞ operad \mathcal{E} (over the integers). We recall from [9, §2], as improved by [7], that \mathfrak{E} is a closed model category with weak equivalences the quasi-isomorphisms and fibrations the surjections. The cofibrations are determined by the weak equivalences and the fibrations, but a concrete description of them may be found in [9, 2.4–5]. We let \mathcal{S} denote the category of simplicial sets. The normalized cochain functor with coefficients in \mathbf{Z} naturally takes values in the category \mathfrak{E} , and defines a contravariant functor $C^*: \mathcal{S} \rightarrow \mathfrak{E}$; more generally, for any commutative ring \mathbf{R} , the normalized cochains with coefficients in \mathbf{R} defines a contravariant functor $C^*(-; \mathbf{R}): \mathcal{S} \rightarrow \mathfrak{E}$. These functors have contravariant right adjoints.

Proposition 1.2. — *Let \mathbf{R} be a commutative ring, let A be an E_∞ algebra and let $T(A; \mathbf{R})$ denote the simplicial set of maps $\mathfrak{E}(A, C^*(\Delta[\bullet]; \mathbf{R}))$, where $\Delta[n]$ denotes the standard n -simplex simplicial set. Then $T(-; \mathbf{R})$ is a contravariant functor $\mathfrak{E} \rightarrow \mathcal{S}$ and is right adjoint to $C^*(-; \mathbf{R})$.*

We omit the proof as it is identical to the proof of [9, 4.2]. The unit map $Y \rightarrow T(C^*(Y; \mathbf{R}); \mathbf{R})$ is easy to describe: It takes the n -simplex σ of Y to the n -simplex

$$C^*(f_\sigma; \mathbf{R}): C^*(Y; \mathbf{R}) \rightarrow C^*(\Delta[n]; \mathbf{R})$$

of $T(C^*(Y; \mathbf{R}); \mathbf{R})$, where $f_\sigma: \Delta[n] \rightarrow Y$ is the map of simplicial sets corresponding to σ (sending the non-degenerate n -simplex of $\Delta[n]$ to σ). For the reader familiar with Appendix A of [9] and its notation, we remark that $T(-; \mathbf{Z}) = U(-; \mathbf{Z})$. For more general \mathbf{R} , $U(-; \mathbf{R})$ was defined as a functor from the category of E_∞ \mathbf{R} -algebras to simplicial sets, so $T(-; \mathbf{R})$ is not the same as $U(-; \mathbf{R})$; rather, it satisfies $T(A; \mathbf{R}) \cong U(A \otimes \mathbf{R}; \mathbf{R})$.

The functor $C^*(-; \mathbf{R})$ sends weak equivalences of simplicial sets to quasi-isomorphisms of E_∞ algebras and sends cofibrations (injections) of simplicial sets to fibrations (surjections) of E_∞ algebras. This implies that $(C^*(-; \mathbf{R}), T(-; \mathbf{R}))$ is a Quillen adjoint pair, i.e., satisfies the conditions of [12, Theorem 4-3] or

[6, 9.7.(i)] (cf. [6, 9.8]) that ensure that the right derived functor of $T(-; \mathbf{R})$ exists and is adjoint to the derived functor of $C^*(-; \mathbf{R})$. We denote this right derived functor as $\mathbf{T}(-; \mathbf{R})$. We remark that since $T(-; \mathbf{R})$ is contravariant, $\mathbf{T}(A; \mathbf{R})$ is constructed by choosing a *cofibrant* approximation $A' \rightarrow A$, and setting $\mathbf{T}(A; \mathbf{R}) = T(A'; \mathbf{R})$. As a particular case, we get the functor \mathbf{U} as another name for $\mathbf{T}(-; \mathbf{Z})$. We summarize this as follows.

Proposition 1.3. — *The functor $T(-; \mathbf{R})$ converts cofibrations of E_∞ algebras to Kan fibrations of simplicial sets and converts quasi-isomorphisms of cofibrant E_∞ algebras to weak equivalences of simplicial sets. The right derived functor $\mathbf{T}(-; \mathbf{R})$ of $T(-; \mathbf{R})$ exists and is right adjoint to $C^*(-; \mathbf{R})$.*

Notation. — We write \mathbf{U} for $T(-; \mathbf{Z})$ and \mathbf{U} for $\mathbf{T}(-; \mathbf{Z})$.

The following theorem proved in Section 2 gives the arithmetic square we need.

Theorem 1.4. — *Let A be a cofibrant E_∞ algebra. Then the diagram*

$$\begin{array}{ccc} \mathbf{U}(A) & \longrightarrow & \mathbf{T}(A; \mathbf{Z}^\wedge) \\ \downarrow & & \downarrow \\ \mathbf{T}(A; \mathbf{Q}) & \longrightarrow & \mathbf{T}(A; \mathbf{Q}^\wedge) \end{array}$$

is homotopy cartesian, i.e., the induced map from $\mathbf{U}(A)$ to the homotopy pullback of $\mathbf{T}(A; \mathbf{Z}^\wedge) \rightarrow \mathbf{T}(A; \mathbf{Q}^\wedge) \rightarrow \mathbf{T}(A; \mathbf{Q})$ is a weak equivalence.

The proof of Theorem 0.2 involves showing that when we take A to be a cofibrant approximation of C^*Y , the square in Theorem 1.4 is equivalent to the square in Proposition 1.1 describing $\Lambda_f Y$. The rational part is straightforward to analyze. The adjunction of Proposition 1.2 for $\mathbf{U} = T(-; \mathbf{Z})$ gives us a map $Y \rightarrow \mathbf{U}(C^*Y)$, which we compose to obtain a map

$$(1.5) \quad \eta: Y \rightarrow \mathbf{U}(C^*Y) = T(C^*(Y); \mathbf{Z}) \rightarrow T(C^*Y; \mathbf{R}) \rightarrow T(A; \mathbf{R}).$$

Passing to the homotopy category, we obtain a map $Y \rightarrow \mathbf{T}(C^*Y; \mathbf{R})$ that is natural in Y . The following theorem is essentially [9, A.6]; we review the details and give a complete argument in Section 4.

Theorem 1.6. — *Let Y be a connected finite type nilpotent simplicial set. Then the natural map in the homotopy category $\eta: Y \rightarrow \mathbf{T}(C^*Y; \mathbf{Q})$ is rationalization.*

To analyze $\mathbf{T}(C^*Y; \mathbf{Z}^\wedge)$, we use the following theorem proved in Sections 3 and 4.

Theorem 1.7. — *Let R be a complete discrete valuation ring with maximal ideal \mathfrak{m} , such that the residue field R/\mathfrak{m} has finite characteristic. Then the natural map*

$$\mathbf{T}(\mathbf{C}^*Y; R) \rightarrow \mathbf{T}(\mathbf{C}^*Y; R/\mathfrak{m})$$

is a weak equivalence when Y is connected finite type nilpotent.

As a consequence, the map $\mathbf{T}(\mathbf{C}^*Y; \mathbf{Z}_p^\wedge) \rightarrow \mathbf{T}(\mathbf{C}^*Y; \mathbf{F}_p)$ is a weak equivalence. Appendix A of [9] proves (in our current notation) that when Y is connected finite type nilpotent, $\mathbf{T}(\mathbf{C}^*Y; \mathbf{F}_p)$ is weakly equivalent to ΛY_p^\wedge , the free loop space of the p -completion of Y . Since \mathbf{Z}^\wedge is the product of \mathbf{Z}_p^\wedge over all primes p , and ΛY^\wedge is equivalent to the product of ΛY_p^\wedge over all primes p , we obtain the following corollary.

Corollary 1.8. — *Let Y be a connected finite type nilpotent simplicial set. Then there is natural isomorphism in the homotopy category $\Lambda Y^\wedge \rightarrow \mathbf{T}(\mathbf{C}^*Y; \mathbf{Z}^\wedge)$.*

In order to describe the last corner, we need an observation on the natural map η when R contains \mathbf{Z}^\wedge . In the description of η in (1.5), we used the unit of the $(\mathbf{C}^*, \mathbf{U})$ adjunction; however, it is clear from the explicit description of the unit map above that when the commutative ring R contains \mathbf{Z}^\wedge , we obtain the same map as the following composite using the $(\mathbf{C}^*(-; \mathbf{Z}^\wedge), \mathbf{T}(-; \mathbf{Z}^\wedge))$ adjunction.

$$Y \rightarrow \mathbf{T}(\mathbf{C}^*(Y; \mathbf{Z}^\wedge); \mathbf{Z}^\wedge) \rightarrow \mathbf{T}(\mathbf{C}^*Y; R) \rightarrow \mathbf{T}(A; R)$$

Factoring the map $A \rightarrow \mathbf{C}^*(Y; \mathbf{Z}^\wedge)$ as a cofibration $A \rightarrow A^\wedge$ followed by an acyclic fibration $A^\wedge \rightarrow \mathbf{C}^*(Y; \mathbf{Z}^\wedge)$, for some E_∞ algebra A^\wedge , we see that the map η factors through the map $Y \rightarrow \mathbf{T}(\mathbf{C}^*(A^\wedge; \mathbf{Z}^\wedge); \mathbf{Z}^\wedge)$. In other words, η may be written as a composite

$$Y \rightarrow \mathbf{T}(\mathbf{C}^*(Y; \mathbf{Z}^\wedge); \mathbf{Z}^\wedge) \rightarrow \mathbf{T}(\mathbf{C}^*Y; R),$$

where the first map is the unit of the derived adjunction $(\mathbf{C}^*(-; \mathbf{Z}^\wedge), \mathbf{T}(-; \mathbf{Z}^\wedge))$. The significance of this is that the finite completion map $Y \rightarrow Y^\wedge$ induces a quasi-isomorphism $\mathbf{C}^*(Y^\wedge; \mathbf{Z}^\wedge) \rightarrow \mathbf{C}^*(Y; \mathbf{Z}^\wedge)$ when Y is finite type nilpotent (or when we take Y^\wedge to denote Bousfield finite completion [1]). It follows that the induced map $\mathbf{T}(\mathbf{C}^*(Y; \mathbf{Z}^\wedge); \mathbf{Z}^\wedge) \rightarrow \mathbf{T}(\mathbf{C}^*(Y^\wedge; \mathbf{Z}^\wedge); \mathbf{Z}^\wedge)$ is a weak equivalence, and hence that the map η factors in the homotopy category as the composite of finite completion $Y \rightarrow Y^\wedge$ and the natural map

$$(1.9) \quad \eta^\wedge: Y^\wedge \rightarrow \mathbf{T}(\mathbf{C}^*(Y^\wedge; \mathbf{Z}^\wedge); R) \simeq \mathbf{T}(\mathbf{C}^*(Y; \mathbf{Z}^\wedge); R) \rightarrow \mathbf{T}(\mathbf{C}^*Y; R).$$

The following theorem regarding this map is proved in Section 4.

Theorem 1.10. — *Let Y be a connected finite type nilpotent simplicial set. Then the natural map in the homotopy category $\eta^\wedge: Y^\wedge \rightarrow \mathbf{T}(C^*Y; \mathbf{Q}^\wedge)$ is rationalization.*

We note that the universal property of rationalization then implies that the following diagram (in the homotopy category) commutes.

$$\begin{array}{ccccc} Y & \longrightarrow & Y_{\mathbf{Q}} & \xrightarrow{\sim} & \mathbf{T}(C^*Y; \mathbf{Q}) \\ \downarrow & & \downarrow & & \downarrow \\ Y^\wedge & \longrightarrow & Y_{\mathbf{Q}}^\wedge & \xrightarrow{\sim} & \mathbf{T}(C^*Y; \mathbf{Q}^\wedge) \end{array}$$

This identifies one map in the diagram in Theorem 1.4. Identifying the other map in the diagram is significantly more difficult. The following theorem is proved in Section 6.

Theorem 1.11. — *Let Y be a connected finite type nilpotent simplicial set. The natural isomorphism in the homotopy category $\Lambda Y^\wedge \rightarrow \mathbf{T}(C^*Y; \mathbf{Z}^\wedge)$ makes the following diagram in the homotopy category commute,*

$$\begin{array}{ccc} \Lambda Y^\wedge & \xrightarrow{\sim} & \mathbf{T}(C^*Y; \mathbf{Z}^\wedge) \\ \downarrow & & \downarrow \\ Y^\wedge & \longrightarrow & Y_{\mathbf{Q}}^\wedge \xrightarrow{\sim} \mathbf{T}(C^*Y; \mathbf{Q}^\wedge). \end{array}$$

Theorem 0.2, which requires no naturality, is an easy consequence of Theorems 1.4, 1.6, 1.10, and 1.11 in the connected case; the non-connected case follows from the connected case and [10, 3.1] (for $k = \mathbf{Z}$). If we compose the inverse in the homotopy category of the equivalence $\Lambda_f Y \rightarrow \mathbf{U}C^*Y$ with the map $\Lambda_f Y \rightarrow Y$, we obtain a map in the homotopy category $\mathbf{U}(C^*Y) \rightarrow Y$. The preceding theorems are not strong enough to imply that we can arrange for this map to be natural in Y ; however, the proof of these theorems is. We prove the following theorem in Section 7.

Theorem 1.12. — *There is a natural map in the homotopy category $\epsilon: \mathbf{U}(C^*Y) \rightarrow Y$ for Y a finite type nilpotent simplicial set, such that the composite with the unit*

$$Y \rightarrow \mathbf{U}(C^*Y) \xrightarrow{\epsilon} Y$$

is the identity.

This theorem gives the first statement of Theorem 0.1, the natural retraction $\epsilon: [C^*Y, C^*X]_{\mathfrak{E}} \rightarrow [X, Y]$. The remainder of Theorem 0.1 requires us to show that

maps $f, g \in [C^*Y, C^*X]_{\mathfrak{e}}$ that satisfy $\epsilon(f) = \epsilon(g)$ must satisfy $H^*f = H^*g$. We do this by identifying in E_{∞} algebra terms when maps satisfy $\epsilon(f) = \epsilon(g)$. We need a ring that plays the same role for \mathbf{Z}_p^{\wedge} that $\bar{\mathbf{F}}_p$ plays for \mathbf{F}_p . This ring is the Witt vectors of $\bar{\mathbf{F}}_p$. For a proof of the uniqueness statement in the following definition, see [14, II§5]; for an elementary construction of the Witt vectors, see [14, II§6].

Definition 1.13. — Let $W(\bar{\mathbf{F}}_p)$ denote the p -typical Witt vectors of $\bar{\mathbf{F}}_p$, the unique complete discrete valuation ring with maximal ideal $(p) \neq (0)$ and residue field $\bar{\mathbf{F}}_p$. Let $\mathbf{W} = \prod_p W(\bar{\mathbf{F}}_p)$.

By Theorem 1.7 and the Main Theorem of [9], we have that the natural map $Y \rightarrow \mathbf{T}(C^*Y; W(\bar{\mathbf{F}}_p))$ is p -completion when Y is connected finite type nilpotent. It follows that the natural map $Y \rightarrow \mathbf{T}(C^*Y; \mathbf{W})$ is finite completion when Y is connected finite type nilpotent. Just as in [9], we may use the closure of \mathbf{F}_p under degree p extensions in place of $\bar{\mathbf{F}}_p$, i.e., the (infinite) algebraic extension that is the fixed field of $\prod_{\ell \neq p} \mathbf{Z}_{\ell}^{\wedge} < \mathbf{Z}^{\wedge} \cong \text{Gal}(\bar{\mathbf{F}}_p/\mathbf{F}_p)$.

In Section 6, we prove the following theorem.

Theorem 1.14. — Let Y be a connected finite type nilpotent simplicial set. There is an isomorphism in the homotopy category $Y^{\wedge} \rightarrow \mathbf{T}(C^*Y; \mathbf{W})$ such that the following diagram in the homotopy category commutes

$$\begin{array}{ccc} \Lambda Y^{\wedge} & \xrightarrow{\sim} & \mathbf{T}(C^*Y; \mathbf{Z}^{\wedge}) \\ \downarrow & & \downarrow \\ Y^{\wedge} & \xrightarrow{\sim} & \mathbf{T}(C^*Y; \mathbf{W}) \end{array}$$

where the top map is the map in Theorem 1.11.

Given a map $f: C^*Y \rightarrow C^*X$, write $f_{\mathbf{W}}$ and $f_{\mathbf{Q}}$ for the composite maps

$$C^*Y \xrightarrow{f} C^*X \rightarrow C^*(X; \mathbf{W}), \quad C^*Y \xrightarrow{f} C^*X \rightarrow C^*(X; \mathbf{Q}),$$

respectively. We have the following corollary of the previous theorem.

Corollary 1.15. — Let X be a simplicial set, let Y be a finite type nilpotent simplicial set, and let f and g be maps in $[C^*Y, C^*X]_{\mathfrak{e}}$ such that $\epsilon(f) = \epsilon(g)$ in $[X, Y]$. Then $f_{\mathbf{W}} = g_{\mathbf{W}}$ in $[C^*Y, C^*(X; \mathbf{W})]_{\mathfrak{e}}$ and $f_{\mathbf{Q}} = g_{\mathbf{Q}}$ in $[C^*Y, C^*(X; \mathbf{Q})]_{\mathfrak{e}}$.

Proof. — If $\epsilon(f) = \epsilon(g)$, then f and g are sent to the same element under the map

$$[C^*Y, C^*X]_{\mathfrak{e}} \cong [X, \mathbf{U}C^*Y] \rightarrow [X, Y^{\wedge}] \times [X, Y_{\mathbf{Q}}].$$

Since we can decompose the above map into a product over the components of \mathbf{X} , it suffices to consider the case when \mathbf{X} is connected. We can then decompose into a disjoint union over the components of \mathbf{Y} (by [10, 3.1]), and so it suffices to consider the case when \mathbf{Y} is also connected.

By the previous theorem, f and g are sent to the same element under the map above if and only if f and g are sent to the same element under the map

$$[\mathbf{C}^*\mathbf{Y}, \mathbf{C}^*\mathbf{X}]_{\mathfrak{E}} \cong [\mathbf{X}, \mathbf{UC}^*\mathbf{Y}] \rightarrow [\mathbf{X}, \mathbf{T}(\mathbf{C}^*\mathbf{Y}; \mathbf{W})] \times [\mathbf{X}, \mathbf{T}(\mathbf{C}^*\mathbf{Y}; \mathbf{Q})].$$

By the adjunction of Proposition 1.3, this is equivalent to the condition that $f_{\mathbf{W}} = g_{\mathbf{W}}$ and $f_{\mathbf{Q}} = g_{\mathbf{Q}}$.

We get the second statement of Theorem 0.2 as a consequence: When \mathbf{X} is finite type, the map $\mathbf{C}^*\mathbf{X} \rightarrow \mathbf{C}^*(\mathbf{X}; \mathbf{W}) \times \mathbf{C}^*(\mathbf{X}; \mathbf{Q})$ is injective on cohomology.

2. Simplicial and cosimplicial resolutions

In this section we prove Theorem 1.4, which establishes the arithmetic square fracturing the functor \mathbf{U} . The proof is a standard argument using the tools of simplicial and cosimplicial resolutions introduced in [5]. We review the basic definitions and terminology, which we use throughout the remainder of the paper.

Let \mathfrak{M} be a closed model category. For an object \mathbf{X} of \mathfrak{M} , a *cosimplicial resolution* of \mathbf{X} is a cosimplicial object \mathbf{X}^\bullet together with a weak equivalence $\mathbf{X}^0 \rightarrow \mathbf{X}$ such that \mathbf{X}^0 is cofibrant, each coface map in \mathbf{X}^\bullet is an acyclic cofibration, and each map $\mathbf{L}^n \rightarrow \mathbf{X}^{n+1}$ is a cofibration, where \mathbf{L}^n is the object denoted as (d^*, \mathbf{X}^n) in [5, 4.3]: It is defined to be the colimit of the diagram in \mathfrak{M} with objects

- for each i , $0 \leq i \leq n+1$, a copy of \mathbf{X}^n labeled (d^i, \mathbf{X}^n)
- for each (i, j) , $0 \leq i < j \leq n+1$, a copy of \mathbf{X}^{n-1} labeled $(d^j d^i, \mathbf{X}^{n-1})$ (we understand \mathbf{X}^{-1} to be the initial object)

and maps

- for each (i, j) , $0 \leq i < j \leq n+1$, a map $(d^j d^i, \mathbf{X}^{n-1}) \rightarrow (d^j, \mathbf{X}^n)$ given by the map $d^i: \mathbf{X}^{n-1} \rightarrow \mathbf{X}^n$
- for each (i, j) , $0 \leq i < j \leq n+1$, a map $(d^j d^i, \mathbf{X}^{n-1}) \rightarrow (d^i, \mathbf{X}^n)$ given by the map $d^{j-1}: \mathbf{X}^{n-1} \rightarrow \mathbf{X}^n$.

Simplicial resolutions are defined dually; a simplicial resolution in \mathfrak{M} is a cosimplicial resolution in \mathfrak{M}^{op} . The following proposition and its proof give enlightening examples of simplicial and cosimplicial resolutions.

Proposition 2.1. — *For any commutative ring \mathbf{R} , $\mathbf{C}^*(\Delta[\bullet]; \mathbf{R})$ is a simplicial resolution of \mathbf{R} in the category of \mathbf{E}_∞ algebras.*

Proof. — Clearly, $\Delta[\bullet]$ is a cosimplicial resolution of $*$ in the category of simplicial sets: The object L^n is just the boundary of $\Delta[n+1]$. If we view $C^*(-; \mathbf{R})$ as a functor to \mathcal{E}^{op} , then it is a Quillen left adjoint. In particular it preserves cofibrations, acyclic cofibrations, and colimits of finite diagrams. It follows that $C^*(-; \mathbf{R})$ takes cosimplicial resolutions of simplicial sets to cosimplicial resolutions in \mathcal{E}^{op} .

Cosimplicial resolutions have the following basic properties.

Proposition 2.2. — *Let X^\bullet be a cosimplicial resolution. The functor $\mathfrak{M}(X^\bullet, -)$ from \mathfrak{M} to simplicial sets preserves fibrations and acyclic fibrations, and preserves weak equivalences between fibrant objects.*

Proof. — If we write $L^{n,i}$ for the colimit of the diagram analogous to the one above but omitting (d^i, X^n) , a straightforward induction (written in detail in [8, §6]) shows that the map $L^{n,i} \rightarrow X^{n+1}$ is an acyclic cofibration. When $f: Y \rightarrow Z$ is a fibration, the left lifting property for the acyclic cofibrations $L^{n,i} \rightarrow X^{n+1}$ with respect to the fibration f translates under the universal property of the colimits defining the $L^{n,i}$ into the extension condition for $\mathfrak{M}(X^\bullet, f)$ to be a Kan fibration. Likewise, when f is an acyclic fibration, the left lifting property for the cofibration $L^n \rightarrow X^{n+1}$ with respect to f translates under the universal property of the colimit defining L^n into the extension condition for $\mathfrak{M}(X^\bullet, f)$ to be an acyclic Kan fibration. K. Brown's lemma [6, 9.9] then implies that $\mathfrak{M}(X^\bullet, -)$ preserves weak equivalences between fibrant objects.

Proposition 2.3. — *Let X^\bullet be a cosimplicial resolution of X and let Y_\bullet be a simplicial resolution of Y . If X is cofibrant and Y is fibrant, then the maps of simplicial sets*

$$\mathfrak{M}(X^\bullet, Y) \rightarrow \text{diag } \mathfrak{M}(X^\bullet, Y_\bullet) \leftarrow \mathfrak{M}(X, Y_\bullet)$$

are weak equivalences.

Proof. — When we regard $\mathfrak{M}(X^\bullet, Y)$ as a bisimplicial set, constant in the second simplicial direction, then the map of bisimplicial sets $\mathfrak{M}(X^\bullet, Y) \rightarrow \mathfrak{M}(X^\bullet, Y_\bullet)$ has the property that each map $\mathfrak{M}(X^\bullet, Y) \rightarrow \mathfrak{M}(X^\bullet, Y_n)$ is a weak equivalence by the previous proposition. It follows that the map $\mathfrak{M}(X^\bullet, Y) \rightarrow \text{diag } \mathfrak{M}(X^\bullet, Y_\bullet)$ is a weak equivalence. The map $\mathfrak{M}(X, Y_\bullet) \rightarrow \mathfrak{M}(X^\bullet, Y_\bullet)$ is a weak equivalence by the same argument in \mathfrak{M}^{op} .

Proposition 2.4. — *Every object has a simplicial resolution.*

Proof. — This is a straightforward factorization argument; see [5, 6.7] for details.

For the application of cosimplicial resolutions to the proof of Theorem 1.4, it is convenient to introduce the following terminology. Let

$$(2.5) \quad \begin{array}{ccc} W & \longrightarrow & X \\ \downarrow & & \downarrow \\ Y & \longrightarrow & Z \end{array}$$

be a commutative square in a closed model category \mathfrak{M} . By factoring maps by weak equivalences followed by fibrations, we can form a commutative square

$$\begin{array}{ccccc} X & \longrightarrow & Z & \longleftarrow & Y \\ \sim \downarrow & & \sim \downarrow & & \sim \downarrow \\ X' & \longrightarrow & Z' & \longleftarrow & Y' \end{array}$$

where Z' is fibrant, $X' \rightarrow Z'$ and $Y' \rightarrow Z'$ are fibrations, and the maps $X \rightarrow X'$, $Y \rightarrow Y'$, and $Z \rightarrow Z'$ are weak equivalences. We have an induced map from W to the pullback $X' \times_{Z'} Y'$. If this map is a weak equivalence for some choice of $X' \rightarrow Z' \leftarrow Y'$, it is a weak equivalence for any choice.

Definition 2.6. — We say that the square (2.5) is homotopy cartesian if the map $W \rightarrow X' \times_{Z'} Y'$ is a weak equivalence.

This is equivalent to the usual definition in the category of spaces or simplicial sets. Maps out of cosimplicial resolutions preserve the homotopy cartesian property when the objects in the square are fibrant.

Proposition 2.7. — Let A^\bullet be a cosimplicial resolution. If the square on the left is homotopy cartesian in \mathfrak{M} ,

$$\begin{array}{ccc} W \longrightarrow X & & \mathfrak{M}(A^\bullet, W) \longrightarrow \mathfrak{M}(A^\bullet, X) \\ \downarrow & & \downarrow \\ Y \longrightarrow Z & & \mathfrak{M}(A^\bullet, Y) \longrightarrow \mathfrak{M}(A^\bullet, Z) \end{array}$$

and W, X, Y, Z are all fibrant, then the square on the right is homotopy cartesian in \mathcal{S} .

Proof. — Let $X' \rightarrow Z' \leftarrow Y'$ be as in Definition 2.6 above, and let W' be the pullback $X' \times_{Z'} Y'$. Then the square

$$\begin{array}{ccc} \mathfrak{M}(A^\bullet, W') \longrightarrow \mathfrak{M}(A^\bullet, X') & & \\ \downarrow & & \downarrow \\ \mathfrak{M}(A^\bullet, Y') \longrightarrow \mathfrak{M}(A^\bullet, Z') & & \end{array}$$

is a pullback square with all maps fibrations and all objects Kan complexes. The vertical maps

$$\begin{array}{ccccc} \mathfrak{M}(A^\bullet, X) & \longrightarrow & \mathfrak{M}(A^\bullet, Z) & \longleftarrow & \mathfrak{M}(A^\bullet, Y) \\ \sim \downarrow & & \sim \downarrow & & \sim \downarrow \\ \mathfrak{M}(A^\bullet, X') & \longrightarrow & \mathfrak{M}(A^\bullet, Z') & \longleftarrow & \mathfrak{M}(A^\bullet, Y') \end{array}$$

are weak equivalences, as is the map $\mathfrak{M}(A^\bullet, W) \rightarrow \mathfrak{M}(A^\bullet, W')$. This proves that the square on the right in the statement is homotopy cartesian.

The following proposition is the last fact we need for the proof of Theorem 1.4.

Proposition 2.8. — *The arithmetic square of \mathbf{Z} , \mathbf{Z}^\wedge , \mathbf{Q} , \mathbf{Q}^\wedge is homotopy cartesian in the category of E_∞ algebras.*

Proof. — Factor $\mathbf{Z}^\wedge \rightarrow \mathbf{Q}^\wedge$ and $\mathbf{Q} \rightarrow \mathbf{Q}^\wedge$ through acyclic cofibrations $\mathbf{Z}^\wedge \rightarrow \mathbf{B}$ and $\mathbf{Q} \rightarrow \mathbf{C}$ followed by fibrations $\mathbf{B} \rightarrow \mathbf{Q}^\wedge$ and $\mathbf{C} \rightarrow \mathbf{Q}^\wedge$, and let \mathbf{D} be the pullback $\mathbf{B} \times_{\mathbf{Q}^\wedge} \mathbf{C}$. We get an induced long exact sequence on cohomology

$$\begin{aligned} \cdots \rightarrow H^{-1}(\mathbf{Q}^\wedge) \rightarrow H^0(\mathbf{D}) \rightarrow (H^0\mathbf{B} \times H^0\mathbf{C}) \rightarrow H^0(\mathbf{Q}^\wedge) \\ \rightarrow H^1(\mathbf{D}) \rightarrow \cdots \end{aligned}$$

Of course $H^{-1}\mathbf{Q}^\wedge = 0$ and the map

$$\mathbf{Z}^\wedge \times \mathbf{Q} = H^0\mathbf{B} \times H^0\mathbf{C} \rightarrow H^0(\mathbf{Q}^\wedge) = \mathbf{Q}^\wedge$$

is surjective, so $H^1\mathbf{D} = 0$. The long exact sequence above is therefore isomorphic to the lengthened short exact sequence

$$\cdots \rightarrow 0 \rightarrow \mathbf{Z} \rightarrow (\mathbf{Z}^\wedge \times \mathbf{Q}) \rightarrow \mathbf{Q}^\wedge \rightarrow 0 \rightarrow \cdots$$

and the map $\mathbf{Z} \rightarrow \mathbf{D}$ is a quasi-isomorphism.

We can now prove Theorem 1.4.

Proof of Theorem 1.4. — We choose a cosimplicial resolution A^\bullet of A . Then Propositions 2.8 and 2.7 imply that the square

$$\begin{array}{ccc} \mathfrak{E}(A^\bullet, \mathbf{Z}) & \longrightarrow & \mathfrak{E}(A^\bullet, \mathbf{Z}^\wedge) \\ \downarrow & & \downarrow \\ \mathfrak{E}(A^\bullet, \mathbf{Q}) & \longrightarrow & \mathfrak{E}(A^\bullet, \mathbf{Q}^\wedge) \end{array}$$

is homotopy cartesian. The theorem now follows from Propositions 2.1 and 2.3.

3. A reduction of Theorem 1.7

In this section, we use the theory of resolutions reviewed in the previous section to reduce Theorem 1.7 to a statement only involving the residue field $k = \mathbf{R}/\mathfrak{m}$. The basic idea is that a complete discrete valuation ring is the limit of a sequence of “square zero extensions” by the quotient field and that such extensions are obtained by base change from a trivial extension, denoted $k \oplus k[-1]$ below. As we explain below, this reduces the problem to understanding the simplicial sets of maps into $k \oplus k[-1]$ factoring a given map into k . The main reduction, Theorem 3.4 below, states that these are contractible; we prove this theorem in the next section.

Because of the structure of the argument that follows, it is convenient to reformulate Theorem 1.7 in the following form, where the basic construction is functorial in E_∞ algebra maps of \mathbf{R} instead of merely commutative ring maps of \mathbf{R} ; it is equivalent to the original statement by Proposition 2.3.

Theorem 3.1. — *Let Y be a connected finite type nilpotent simplicial set and let A^\bullet be a cosimplicial resolution of C^*Y . Let $(\mathbf{R}, \mathfrak{m})$ be a complete discrete valuation ring whose residue field $k = \mathbf{R}/\mathfrak{m}$ has finite characteristic. Then the map of simplicial sets $\mathfrak{E}(A^\bullet, \mathbf{R}) \rightarrow \mathfrak{E}(A^\bullet, k)$ is a weak equivalence.*

With notation as above, choose π to be an irreducible element of \mathfrak{m} ; then $\mathfrak{m} = (\pi)$. Since the valuation is complete, the canonical map

$$\mathbf{R} \rightarrow \operatorname{Lim} \mathbf{R}/(\pi^n)$$

is an isomorphism. Since the maps $\mathbf{R}/(\pi^{n+1}) \rightarrow \mathbf{R}/(\pi^n)$ are surjective, we have

$$\mathfrak{E}(A^\bullet, \mathbf{R}) \cong \operatorname{Lim} \mathfrak{E}(A^\bullet, \mathbf{R}/(\pi^n))$$

is the limit of Kan fibrations of Kan complexes. Thus, to prove Theorem 3.1, it suffices to show that each map

$$\mathfrak{E}(A^\bullet, \mathbf{R}/(\pi^{n+1})) \rightarrow \mathfrak{E}(A^\bullet, \mathbf{R}/(\pi^n))$$

is a weak equivalence.

Consider the following variant of the tower $\mathbf{R}/(\pi^n)$. Let \mathbf{R}_n be the Koszul complex associated to π^n : This is the commutative differential graded \mathbf{R} -algebra that is the exterior \mathbf{R} -algebra on an element x_n whose differential is π^n . We have a map of commutative differential graded \mathbf{R} -algebras $\mathbf{R}_{n+1} \rightarrow \mathbf{R}_n$ obtained by sending x_{n+1} to $\pi \cdot x_n$; this induces on homology the map of commutative rings $\mathbf{R}/(\pi^{n+1}) \rightarrow \mathbf{R}/(\pi^n)$.

Let $k \oplus k[-1]$ denote the graded commutative k -algebra, exterior on an element of degree -1 (where differentials raise degree). We regard this as a commutative differential graded \mathbf{R} -algebra with zero differential, augmented to k by the projection map sending the exterior generator to zero. (We think of $k \oplus k[-1]$ as the trivial “square zero extension” of k .) We have a map of commutative differential graded \mathbf{R} -algebras $\mathbf{R}_n \rightarrow k \oplus k[-1]$ that sends x_n to the exterior generator; this map is surjective.

Proposition 3.2. — *The commutative diagram*

$$\begin{array}{ccc} \mathbf{R}_{n+1} & \longrightarrow & k \\ \downarrow & & \downarrow \\ \mathbf{R}_n & \longrightarrow & k \oplus k[-1] \end{array}$$

is a pullback square and homotopy cartesian square in the category of \mathbf{E}_∞ algebras.

By Proposition 2.7, it follows that the square

$$\begin{array}{ccc} \mathfrak{E}(\mathbf{A}^\bullet, \mathbf{R}_{n+1}) & \longrightarrow & \mathfrak{E}(\mathbf{A}^\bullet, k) \\ \downarrow & & \downarrow \\ \mathfrak{E}(\mathbf{A}^\bullet, \mathbf{R}_n) & \longrightarrow & \mathfrak{E}(\mathbf{A}^\bullet, k \oplus k[-1]) \end{array}$$

is homotopy cartesian in the category of simplicial sets. Since the maps

$$\mathfrak{E}(\mathbf{A}^\bullet, \mathbf{R}_{n+1}) \rightarrow \mathfrak{E}(\mathbf{A}^\bullet, \mathbf{R}/(\pi^{n+1})), \quad \mathfrak{E}(\mathbf{A}^\bullet, \mathbf{R}_n) \rightarrow \mathfrak{E}(\mathbf{A}^\bullet, \mathbf{R}/(\pi^n))$$

are weak equivalences, we are reduced to showing that the map

$$\mathfrak{E}(\mathbf{A}^\bullet, k) \rightarrow \mathfrak{E}(\mathbf{A}^\bullet, k \oplus k[-1])$$

is a weak equivalence, or equivalently, that the retraction

$$r: \mathfrak{E}(\mathbf{A}^\bullet, k \oplus k[-1]) \rightarrow \mathfrak{E}(\mathbf{A}^\bullet, k)$$

(induced by the augmentation $k \oplus k[-1] \rightarrow k$) is a weak equivalence.

Since the map $k \oplus k[-1] \rightarrow k$ is surjective, the map r is a Kan fibration. It therefore suffices to show that the fiber of r at each vertex is contractible. Choose and fix a map $b: \mathbf{A}^0 \rightarrow k$. If we write \mathfrak{E}/k for the category of \mathbf{E}_∞ algebras lying over k , then the fiber of r over the point b is exactly the simplicial set

$$(\mathfrak{E}/k)(\mathbf{A}^\bullet, k \oplus k[-1]),$$

where the map $A^s \rightarrow k$ is the composite of the degeneracy $A^s \rightarrow A^0$ and the map $b: A^0 \rightarrow k$. Thus, Theorem 1.7 reduces to showing that this simplicial set is contractible. We reformulate this as Theorem 3.4 below, using Proposition 2.3 to switch from using a cosimplicial resolution of C^*Y to using a simplicial resolution of $k \oplus k[-1]$.

We construct a simplicial resolution for $k \oplus k[-1]$ as follows. For a simplicial set X , let $C^*(X; k[-1])$ denote the differential graded k -module obtained from $C^*(X; k)$ by shifting one degree down,

$$C^n(X; k[-1]) = C^{n+1}(X; k).$$

We make $k \oplus C^*(X; k[-1])$ an E_∞ algebra lying over k by giving it the *square zero multiplication*. This is an E_∞ algebra structure coming from the augmented commutative differential graded k -algebra structure where any pair of elements in the augmentation ideal $C^*(X; k[-1])$ multiply to zero (i.e., $C^*(X; k[-1])$ is a square zero ideal). The elements in k multiply normally and multiply with elements of $C^*(X; k[-1])$ by the usual k -module action. Now consider the simplicial object of \mathfrak{E}/k given by

$$k \oplus C^*(\Delta[\bullet]; k[-1]);$$

clearly, this is a simplicial resolution of $k \oplus k[-1]$ in \mathfrak{E}/k .

Definition 3.3. — *Let A be an E_∞ algebra lying over k . Let*

$$D(A; k) = (\mathfrak{E}/k)(A, k \oplus C^*(\Delta[\bullet]; k[-1])).$$

The work above together with Proposition 2.3 reduces Theorem 1.7 to the following theorem.

Theorem 3.4. — *Let k be a field of positive characteristic. Let Y be a connected finite type nilpotent simplicial set, let $C \rightarrow C^*Y$ be a cofibrant approximation, and let $C \rightarrow k$ be a map of E_∞ algebras. Then $D(C; k)$ is contractible.*

As stated, the theorem above requires us to prove contractibility for all cofibrant approximations C . The following lemma allows us to replace the implicit “for every” with a “there exists” and work with a convenient cofibrant approximation instead of an arbitrary one. It is an immediate consequence of [9, 2.12] or the fact that in the model category \mathfrak{E} all objects are fibrant.

Lemma 3.5. — *Let $C \rightarrow C^*Y$ be a given cofibrant approximation. If $A \rightarrow C^*Y$ is any cofibrant approximation, then there exists a quasi-isomorphism $A \rightarrow C$.*

4. The proof of Theorems 1.6, 1.10, and 3.4

In this section we prove Theorems 1.6 and 1.10 from Section 1, and Theorem 3.4 from the previous section. What these theorems have in common is that they are proved by induction up a principally refined Postnikov tower of a connected finite type nilpotent simplicial set.

Recall that a principally refined Postnikov tower is a finite or infinite tower of fibrations

$$\cdots \rightarrow Y_i \rightarrow \cdots \rightarrow Y_1 \rightarrow Y_0 = *$$

such that each Y_i is formed as the pullback of a map $Y_i \rightarrow \mathbf{K}(G_i, n_i + 1)$ along the fibration $L(G_i, n_i + 1) \rightarrow \mathbf{K}(G_i, n_i + 1)$, and n_1, n_2, \dots is a non-decreasing sequence of positive integers, taking on a given value at most finitely many times. Here $\mathbf{K}(G_i, n_i + 1)$ denotes the standard Eilenberg–Mac Lane complex with $\pi_{n_i+1} = G_i$, the map $L(G_i, n_i + 1) \rightarrow \mathbf{K}(G_i, n_i + 1)$ is a Kan fibration, and $L(G_i, n_i + 1)$ is a contractible Kan complex. See for example [11, §23].

A connected simplicial set Y is finite type nilpotent if and only if there exists a principally refined Postnikov tower $\{Y_i \rightarrow \mathbf{K}(G_i, n_i + 1)\}$ and a weak equivalence $Y \rightarrow \text{Lim } Y_i$ such that each G_i is \mathbf{Z} or $\mathbf{Z}/p\mathbf{Z}$ for some p (depending on i). If we let G_i^\wedge denote the pro-finite completion of G_i , then Y^\wedge , the finite completion of Y , admits a principally refined Postnikov tower $\{Y_i^\wedge \rightarrow \mathbf{K}(G_i^\wedge, n_i + 1)\}$. Specifically, each Y_i^\wedge is the finite completion of Y_i and the maps $Y_i^\wedge \rightarrow \mathbf{K}(G_i^\wedge, n_i + 1)$ represent the finite completion of the maps $Y_i \rightarrow \mathbf{K}(G_i, n_i + 1)$, i.e., the diagram

$$\begin{array}{ccc} Y_i & \longrightarrow & \mathbf{K}(G_i, n_i + 1) \\ \downarrow & & \downarrow \\ Y_i^\wedge & \longrightarrow & \mathbf{K}(G_i^\wedge, n_i + 1) \end{array}$$

commutes in the homotopy category. The following proposition gives the first reduction.

Proposition 4.1. — *Let $\cdots \rightarrow Y_1 \rightarrow Y_0$ be a principally refined Postnikov tower for the connected finite type nilpotent simplicial set Y . If each Y_i satisfies Theorem 1.6, 1.10, or 3.4, then so does Y .*

Proof. — Let $A_0 \rightarrow C^*Y_0$ be a cofibrant approximation, and inductively construct $A_i \rightarrow C^*Y_i$ by factoring the composite map $A_{i-1} \rightarrow C^*Y_{i-1} \rightarrow C^*Y_i$ as a cofibration $A_{i-1} \rightarrow A_i$ followed by an acyclic fibration $A_i \rightarrow C^*Y_i$. Let $A = \text{Colim } A_i$; then the canonical map $A \rightarrow C^*(\text{Lim } Y) \rightarrow C^*Y$ is a quasi-isomorphism since the canonical map $\text{Colim } H^*(\text{Lim } Y_i) \rightarrow H^*Y$ is an isomorphism.

Let

$$T_i = T(A_i, \mathbf{Q}), \quad T_i^\wedge = T(A_i, \mathbf{Q}^\wedge), \quad D_i = D(A_i, k),$$

and let T , T^\wedge , and D be the corresponding constructions for A . In the last case we choose the augmentations for the $A_i \rightarrow k$ by choosing an augmentation for $A \rightarrow k$ using Lemma 3.5 to find a quasi-isomorphism $A \rightarrow C$ from A to our given cofibrant approximation C and composing with the given augmentation $C \rightarrow k$. Then we have

$$T \cong \text{Lim } T_i, \quad T^\wedge \cong \text{Lim } T_i^\wedge, \quad D \cong \text{Lim } D_i,$$

the limits of towers of fibrations of Kan complexes. We have a commuting diagram of towers $Y_i \rightarrow T_i$, and the natural maps in the homotopy category η^\wedge give us a homotopy commuting diagram of towers $Y_i^\wedge \rightarrow T_i^\wedge$, where $\{Y_i^\wedge\}$ is the Postnikov tower for Y^\wedge corresponding to $\{Y_i\}$. By hypothesis the T_i , T_i^\wedge , or D_i are all connected and the sequence $\pi_n T_i$, $\pi_n T_i^\wedge$, or $\pi_n D_i$ is Mittag-Leffler, and so we have

$$\pi_n T \cong \text{Lim } \pi_n T_i, \quad \pi_n T^\wedge \cong \text{Lim } \pi_n T_i^\wedge, \quad \pi_n D \cong \text{Lim } \pi_n D_i.$$

The proposition now follows.

Next we need the following proposition that explains the effect on cochain E_∞ algebras of pullback along a Kan fibration. It is not formal; it is proved by the same argument as [9, 5.2], which we omit. In the statement, *homotopy cocartesian* is the dual of homotopy cartesian (Definition 2.6): A square is homotopy cocartesian if the corresponding square in the opposite category is homotopy cartesian.

Proposition 4.2. — *If the square on the left is a homotopy cartesian square of connected finite type simplicial sets with \mathbf{K} simply connected,*

$$\begin{array}{ccc} W & \longrightarrow & L \\ \downarrow & & \downarrow \\ X & \longrightarrow & K \end{array} \qquad \begin{array}{ccc} C^*W & \longleftarrow & C^*L \\ \uparrow & & \uparrow \\ C^*X & \longleftarrow & C^*K \end{array}$$

then the square on the right is a homotopy cocartesian square of E_∞ algebras.

This proposition allows us to prove the following reduction.

Proposition 4.3. — *Theorems 1.6, 1.10, and 3.4 hold for an arbitrary connected finite type nilpotent simplicial set if they hold for $\mathbf{K}(\mathbf{Z}, n)$ and $\mathbf{K}(\mathbf{Z}/p\mathbf{Z}, n)$ for all p , $n \geq 2$.*

Proof. — Choose a principally refined Postnikov tower $\{Y_i \rightarrow K(G_i, n_i + 1)\}$ with each $G_i = \mathbf{Z}$ or $\mathbf{Z}/p\mathbf{Z}$ for some p . By induction, we suppose the theorem holds for Y_{i-1} . We choose cofibrant approximations and cofibrations, making the following diagram commute.

$$\begin{array}{ccccc} A_{i-1} & \longleftarrow & B & \longrightarrow & C \\ \sim \downarrow & & \sim \downarrow & & \sim \downarrow \\ C^*Y_{i-1} & \longleftarrow & C^*K(G_i, n_i + 1) & \longrightarrow & C^*L(G_i, n_i + 1) \end{array}$$

If we let A_i be the pushout $A_{i-1} \amalg_B C$, then the previous proposition tells us that the map $A_i \rightarrow C^*Y_i$ is a quasi-isomorphism. The squares

$$\begin{array}{ccc} T(A_i, \mathbf{Q}) \longrightarrow T(C, \mathbf{Q}) & & T(A_i, \mathbf{Q}^\wedge) \longrightarrow T(C, \mathbf{Q}^\wedge) \\ \downarrow & & \downarrow \\ T(A_{i-1}, \mathbf{Q}) \longrightarrow T(B, \mathbf{Q}) & & T(A_{i-1}, \mathbf{Q}^\wedge) \longrightarrow T(B, \mathbf{Q}^\wedge) \\ & & \\ D(A_i, k) \longrightarrow D(C, k) & & \\ \downarrow & & \downarrow \\ D(A_{i-1}, k) \longrightarrow D(B, k) \end{array}$$

are pullbacks of Kan fibrations of Kan complexes. Here, in the last square, we choose the augmentations by choosing an augmentation $A_i \rightarrow k$ using Lemma 3.5 and the given augmentation on our given cofibrant approximation. We have a commutative diagram comparing the fibration square defining Y_i with the square on the left and a homotopy commutative diagram comparing the fibration square defining Y_i^\wedge with the square in the middle. Inspection of the long exact sequence of homotopy groups associated to these fibration squares then gives the result.

To prove the theorems for $K(G, n)$'s, it is convenient to change coefficients. For any commutative ring R , we can consider the category \mathfrak{E}_R of E_∞ R -algebras over the operad $(\mathcal{E} \otimes R)$. We have an extension of scalars functor obtained by tensoring an E_∞ algebra with the commutative ring R . Extension of scalars is the left adjoint of the forgetful functor $\mathfrak{E}_R \rightarrow \mathfrak{E}$: We have a bijection

$$\mathfrak{E}(A, B) \cong \mathfrak{E}_R(A \otimes R, B),$$

natural in the E_∞ algebra A and the E_∞ R -algebra B . The category \mathfrak{E}_R is a model category with fibrations the surjections and weak equivalences the quasi-isomorphisms. In particular, the forgetful functor $\mathfrak{E}_R \rightarrow \mathfrak{E}$ preserves fibrations and quasi-isomorphisms, and it follows formally that the extension of scalars functor preserves cofibrations and quasi-isomorphisms between cofibrant objects. We can now give the proofs of Theorems 1.6, 1.10, and 3.4.

Proof of Theorems 1.6 and 1.10. — By Proposition 4.3, it suffices to consider the case when $Y = K(G, n)$ for $G = \mathbf{Z}$ or $\mathbf{Z}/p\mathbf{Z}$ and $n \geq 2$. Let $Y^\wedge = K(G^\wedge, n)$. Choose a cofibrant approximation $A \rightarrow C^*Y$. The map $C^*(Y^\wedge; \mathbf{Z}^\wedge) \rightarrow C^*(Y; \mathbf{Z}^\wedge)$ is an acyclic fibration, and so we can choose a lift of the map $A \rightarrow C^*Y \rightarrow C^*(Y; \mathbf{Z}^\wedge)$ to a map $A \rightarrow C^*(Y^\wedge; \mathbf{Z}^\wedge)$. Factor the map $A \rightarrow C^*(Y^\wedge; \mathbf{Z}^\wedge)$ as a cofibration $A \rightarrow A^\wedge$ followed by an acyclic fibration $A^\wedge \rightarrow C^*(Y^\wedge; \mathbf{Z}^\wedge)$. Then the composite map $A^\wedge \rightarrow C^*(Y; \mathbf{Z}^\wedge)$ is a cofibrant approximation.

In the case when $G = \mathbf{Z}/p\mathbf{Z}$, let E be the initial object $\mathcal{E}(0)$ in \mathfrak{E} , and let $E \rightarrow A$ be the unique map; the map $E \otimes \mathbf{Q} \rightarrow A \otimes \mathbf{Q}$ is then a quasi-isomorphism since the cohomology of $Y = K(G, n)$ is torsion. In the case when $G = \mathbf{Z}$, let E be the free E_∞ algebra on one generator in degree n ; we choose a map of E_∞ algebras $E \rightarrow A$ that sends the generator to any cocycle that represents in cohomology the fundamental class of $H^*(K(\mathbf{Z}, n))$. Again the map $E \otimes \mathbf{Q} \rightarrow A \otimes \mathbf{Q}$ is a quasi-isomorphism; this time because the rational cohomology of $K(\mathbf{Z}, n)$ is the free graded commutative \mathbf{Q} -algebra on the fundamental class and so is the cohomology of $E \otimes \mathbf{Q}$. For R containing \mathbf{Q} , the map

$$\begin{aligned} T(A; R) &= \mathfrak{E}(A, C^*(\Delta[\bullet]; R)) \cong \mathfrak{E}_{\mathbf{Q}}(A \otimes \mathbf{Q}, C^*(\Delta[\bullet]; R)) \\ &\rightarrow \mathfrak{E}_{\mathbf{Q}}(E \otimes \mathbf{Q}, C^*(\Delta[\bullet]; R)) \cong \mathfrak{E}(E, C^*(\Delta[\bullet]; R)) = T(E; R) \end{aligned}$$

is then a homotopy equivalence by the dual form of Proposition 2.2. On the other hand, we can identify $T(E; R)$ as follows.

When $G = \mathbf{Z}/p\mathbf{Z}$ and $E = \mathcal{E}(0)$, we have that $T(E; R)$ is a single point. Thus, $T(A; R)$ is contractible. It follows that map $\eta: Y \rightarrow T(A; \mathbf{Q})$ and the map $\eta^\wedge: Y^\wedge \rightarrow T(A; \mathbf{Q})$ are rationalizations.

When $G = \mathbf{Z}$ and E is the free E_∞ algebra on a generator in degree n with zero differential, the set of s -simplices of $T(E; R)$ is the set of degree n cocycles of $C^*(\Delta[s]; R)$; in other words, $T(E; R)$ is the standard Eilenberg–Mac Lane complex $K(R, n)$ (see for example [11, §23]). The models for $K(\mathbf{Z}, n)$ and $K(\mathbf{Z}^\wedge, n)$ we have chosen for Y and Y^\wedge are strictly $(n-1)$ -connected (have only one vertex and no non-degenerate i -simplices for $1 \leq i \leq n-1$) and have a unique cocycle representing each cohomology class of $H^n Y$ and of $H^n(Y^\wedge; \mathbf{Z}^\wedge)$. Looking at the explicit description of the unit of the (C^*, T) adjunction of Section 1, we see that the maps $Y \rightarrow T(E; \mathbf{Q})$ and $Y^\wedge \rightarrow T(E; \mathbf{Q}^\wedge)$ are rationalization, and it follows that the maps $\eta: Y \rightarrow T(A; \mathbf{Q})$ and $\eta^\wedge: Y^\wedge \rightarrow T(A; \mathbf{Q}^\wedge)$ are rationalization.

Proof of Theorem 3.4. — By Proposition 4.3, we are reduced to proving the theorem for $Y = K(\mathbf{Z}, n)$ or $Y = K(\mathbf{Z}/p\mathbf{Z}, n)$ for $n \geq 2$. We start with $C \rightarrow C^*Y$ a given cofibrant approximation, and $C \rightarrow k$ a given map of E_∞ algebras. If E is any cofibrant E_∞ k -algebra and $E \rightarrow C \otimes k$ is a quasi-isomorphism, then the

map

$$\begin{aligned} D(C; k) &= (\mathfrak{E}/k)(C, k \oplus C^*(\Delta[\bullet]; k[-1])) \\ &\cong (\mathfrak{E}_k/k)(C \otimes k, k \oplus C^*(\Delta[\bullet]; k[-1])) \\ &\rightarrow (\mathfrak{E}_k/k)(E, k \oplus C^*(\Delta[\bullet]; k[-1])) \end{aligned}$$

is a homotopy equivalence by the dual form of Proposition 2.2; here \mathfrak{E}_k/k denotes the category of E_∞ k -algebras lying over k . We denote the last simplicial set in the display above as $D(E \setminus k; k)$. In this notation, it suffices to show that for some such E , $D(E \setminus k; k)$ is contractible.

In the case when $G = \mathbf{Z}/p\mathbf{Z}$ and p is different from the characteristic of k , then $H^*(K(\mathbf{Z}/p\mathbf{Z}, n); k)$ is trivial and we can take E to be the initial object $\mathcal{E}(0) \otimes k$; then $D(E \setminus k; k)$ consists of a single point and is therefore contractible. We now set p to be the characteristic of k . In the case when $G = \mathbf{Z}$, the map $C^*(K(\mathbf{Z}_p^\wedge, n), k) \rightarrow C^*(K(\mathbf{Z}, n); k)$ is a quasi-isomorphism. We can write $K(\mathbf{Z}_p^\wedge, n)$ as the limit of a tower

$$\cdots \rightarrow K(\mathbf{Z}/p^i\mathbf{Z}, n) \rightarrow \cdots K(\mathbf{Z}/p\mathbf{Z}, n)$$

of principal fibrations $K(\mathbf{Z}/p^i\mathbf{Z}, n) \rightarrow K(\mathbf{Z}/p\mathbf{Z}, n+1)$, and the canonical map $\text{Colim } H^*(K(\mathbf{Z}/p^i\mathbf{Z}, n); k) \rightarrow H^*(K(\mathbf{Z}_p^\wedge, n); k)$ is a quasi-isomorphism. The arguments for Propositions 4.1 and 4.3 then reduce the case $G = \mathbf{Z}$ to the case $G = \mathbf{Z}/p\mathbf{Z}$.

We use the work of [9, §6], which constructs an explicit cofibrant approximation of $C^*(K(\mathbf{Z}/p\mathbf{Z}, n); k)$ for a field k of characteristic p . Write \mathbf{E} for the free functor from differential graded k -modules to E_∞ k -algebras; it takes a differential graded k -module M to the E_∞ k -algebra

$$\mathbf{E}M = \bigoplus_{j \geq 0} (\mathcal{E}(j) \otimes k) \otimes_{k[\Sigma_j]} M^{(j)}.$$

Let $k[n]$ denote the differential graded k -module free on one generator x in degree n , and let $Ck[n]$ be the differential graded k -module free on generators in degrees $n-1$ and n with the differential taking the lower generator to the higher generator. In this notation, [9, §6] describes a map $\wp: \mathbf{E}k[n] \rightarrow \mathbf{E}Ck[n]$ such that the pushout over the inclusion $\mathbf{E}k[n] \rightarrow \mathbf{E}Ck[n]$,

$$\begin{array}{ccc} \mathbf{E}k[n] & \xrightarrow{\quad} & \mathbf{E}Ck[n] \\ \wp \downarrow & & \downarrow \\ \mathbf{E}k[n] & \xrightarrow{\quad} & \mathbf{E} \end{array}$$

$E = \mathbf{E}k[n] \sqcup_{\varphi} \mathbf{E}Ck[n]$ (which was denoted as B_n in [9, 6.2]) is quasi-isomorphic to $C^*(\mathbf{K}(\mathbf{Z}/p\mathbf{Z}, n); k)$. All we need about the map φ is that it takes the generator x of $\mathbf{E}k[n]$ to the generator x of $\mathbf{E}k[n]$ minus a class ρ in

$$(\mathcal{E}(\rho) \otimes k) \otimes_{k[\Sigma_p]} (k[n])^{(\rho)} \subset \mathbf{E}k[n].$$

(The class ρ represents P^0x in $H^*(\mathbf{E}k[n])$.)

Choosing a quasi-isomorphism $E \rightarrow C \otimes k$, we obtain an augmentation $E \rightarrow k$. From the pushout square above, we get a pullback square.

$$\begin{array}{ccc} D(E \setminus k; k) & \longrightarrow & D(\mathbf{E}Ck[n] \setminus k; k) \\ \downarrow & & \downarrow \\ D(\mathbf{E}k[n] \setminus k; k) & \xrightarrow{\varphi^*} & D(\mathbf{E}k[n] \setminus k; k) \end{array}$$

The vertical arrows are fibrations and the simplicial set in the upper right hand corner is contractible. Since $k \oplus C^*(\Delta[\bullet]; k[-1])$ has the square zero multiplication and the degree of $x - \rho$ is bigger than 0, any map $\alpha: \mathbf{E}k[n] \rightarrow k \oplus C^*(\Delta[\bullet]; k[-1])$ sends $x - \rho$ to the same element it sends x ; it follows that the bottom horizontal map φ^* is an isomorphism. Thus, $D(E \setminus k; k)$ is contractible.

5. Function complexes and continuous functors

The purpose of this section is to set up the machinery we need to prove Theorems 1.11 and 1.14. The main difficulty is that the identification of $\mathbf{T}(C^*(Y); \mathbf{Z}^\wedge)$ as ΛY^\wedge required comparing with $\mathbf{T}(C^*(Y); \mathbf{R})$ for $\mathbf{R} = \mathbf{F}_p$ and this makes it difficult to compare with $\mathbf{T}(C^*(Y); \mathbf{R})$ for $\mathbf{R} = \mathbf{Q}$ and $\mathbf{R} = \mathbf{Q}^\wedge$. The basic idea is to produce a version of ΛY^\wedge that is a representable functor and to produce a version of $\mathbf{T}(C^*Y; \mathbf{R})$ that is a continuous functor. We then can able to apply the Yoneda Lemma to construct *natural* maps and to identify natural transformations. As a side benefit this gives us sufficient naturality to prove Theorem 1.12 in Section 7.

To carry out this strategy, we use the theory of “function complexes” developed in the papers [3–5] of W. G. Dwyer and D. M. Kan. This theory works best when we apply it to a small category. The set-theoretic technicalities involved in using a category that is not small are usually treated by ignoring them. This is harmless for most applications; essentially the only time these technicalities become an issue is in the context of mapping space adjunctions. Unfortunately, this is the context in which we are working, and so we are forced to deal with them.

To address these issues, we arrange to work in a small category by limiting the size of the E_∞ algebras and simplicial sets we consider. For unrelated reasons

we explain below, in addition to the category \mathfrak{E} of E_∞ \mathbf{Z} -algebras, we also need to work in the category $\mathfrak{E}_{\mathbf{Z}^\wedge}$ of E_∞ \mathbf{Z}^\wedge -algebras (over the operad $\mathfrak{E} \otimes \mathbf{Z}^\wedge$). The following definition describes the precise categories we use.

Definition 5.1. — For a simplicial set Y , write $\#Y$ for the cardinality of the set of non-degenerate simplices of Y , and for a differential graded module M , write $\#M$ for the product of the cardinalities of the modules of all degrees. Let \varkappa be a cardinal at least as big as $2^{\mathfrak{c}}$, where \mathfrak{c} is the cardinality of the continuum, and at least as big as $\prod \#\mathcal{E}(n)$. For a commutative ring R with $\#R \leq \mathfrak{c}$, let \mathfrak{E}'_R be a skeleton of the full subcategory of E_∞ R -algebras A satisfying $\#A \leq \varkappa$; we write \mathfrak{E}' for $\mathfrak{E}'_{\mathbf{Z}}$. Let \mathcal{S}' be a skeleton of the full subcategory of simplicial sets Y satisfying $\#Y \leq \varkappa$.

Recall that a skeleton of a category is a full subcategory with one object in each isomorphism class; the inclusion of a skeleton is an equivalence of categories. We can assume without loss of generality that $\varkappa^{\mathfrak{c}} = \varkappa$ (by replacing \varkappa with 2^\varkappa if necessary); then for any commutative ring R with cardinality at most \mathfrak{c} and any simplicial set with at most \varkappa non-degenerate simplices, $C^*(X; R)$ has cardinality at most \varkappa in each degree. The normalized cochain functor with coefficients in R therefore defines a functor $C^*(-; R): \mathcal{S}' \rightarrow \mathfrak{E}'_R$. In addition, we have the following homotopy theoretic observations; the proofs are straightforward cardinality arguments given at the end of this section.

Theorem 5.2. — The category \mathcal{S}' admits the following closed model structures:

- (i) Weak equivalences the usual weak equivalences, cofibrations the injections, and fibrations the Kan fibrations.
- (ii) Weak equivalences the rational equivalences, cofibrations the injections, and fibrations defined by the right lifting property.
- (iii) Weak equivalences the finite equivalences (maps that induce isomorphisms on homology with finite coefficients), cofibrations the injections, and fibrations defined by the right lifting property.

Moreover, every finite type nilpotent simplicial set is weakly equivalent to an object in \mathcal{S}' .

Theorem 5.3. — Let R be a commutative ring with $\#R \leq \mathfrak{c}$. The category \mathfrak{E}' is a closed model category with weak equivalences the quasi-isomorphisms, fibrations the surjections, and cofibrations as described in [9, 2.4].

The version of the Dwyer–Kan theory we use is called the “hammock localization” [4, §2]. It is defined for any small category with a suitable notion of weak equivalence, but we only need to apply it to small closed model categories. In [4], the hammock localization is defined to be a simplicial category with the

same set of objects as the original category. The simplicial sets of maps in this category are virtually never Kan complexes. To avoid this inconvenience in the next two sections, we convert this into a topological category.

Definition 5.4. — Let \mathfrak{M} be a small closed model category, and let $L_{\bullet}^H \mathfrak{M}$ be the simplicial category obtained by hammock localization as defined in [4, §2]. Let $L\mathfrak{M}$ be the topological category obtained from $L_{\bullet}^H \mathfrak{M}$ by geometric realization.

As mentioned above, the simplicial category $L_{\bullet}^H \mathfrak{M}$ and therefore the topological category $L\mathfrak{M}$ has the same object set as \mathfrak{M} . The discrete category \mathfrak{M} includes in $L\mathfrak{M}$ as a subcategory, and so a map in \mathfrak{M} gives us a map in $L\mathfrak{M}$. We need the following additional properties of this localization.

Proposition 5.5 ([4, 3.1]). — The category $\pi_0 L\mathfrak{M}$ is equivalent to the homotopy category of \mathfrak{M} .

Proposition 5.6 ([4, 3.3]). — If $f: X \rightarrow Y$ is a weak equivalence in \mathfrak{M} , then for any object Z in \mathfrak{M} , the maps

$$f_*: L\mathfrak{M}(Z, X) \rightarrow L\mathfrak{M}(Z, Y), \quad f^*: L\mathfrak{M}(Y, Z) \rightarrow L\mathfrak{M}(X, Z)$$

are homotopy equivalences.

Proposition 5.7 ([4, 3.4]). — A functor $F: \mathfrak{M}_1 \rightarrow \mathfrak{M}_2$ that preserves weak equivalences has a canonical extension to a continuous functor $LF: L\mathfrak{M}_1 \rightarrow L\mathfrak{M}_2$. If $G: \mathfrak{M}_2 \rightarrow \mathfrak{M}_3$ preserves weak equivalences, then $L(G \circ F) = LG \circ LF$.

We warn the reader that L does not preserve natural transformations; a natural transformation $F \rightarrow G$ generally does not extend to a natural transformation $LF \rightarrow LG$. See for example [4, 3.5] where the diagram commutes only up to homotopy. In the terminology of category theory, L is a functor but not a 2-functor.

The theory of resolutions described in Section 2 gives a different way of assigning a simplicial set of maps. The following proposition, proved in [8, 6.1] compares these two constructions. In the statement, we denote the geometric realization of a simplicial set by $|\cdot|$.

Proposition 5.8. — Let X^\bullet be a cosimplicial resolution of X and let Y be a fibrant object in \mathfrak{M} . The inclusions

$$|\mathfrak{M}(X^\bullet, Y)| \rightarrow |L\mathfrak{M}(X^\bullet, Y)| \leftarrow L\mathfrak{M}(X, Y)$$

are homotopy equivalences.

The dual statement for simplicial resolutions and cofibrant objects also holds since $(\mathbf{LM})^{\text{op}}$ and $\mathbf{L}(\mathfrak{M}^{\text{op}})$ are isomorphic topological categories. Alternatively, it is readily deducible from Propositions 2.3, 5.6, and 5.8. Combining this with Proposition 2.7, we get the following proposition, which requires no fibrancy or cofibrancy assumptions.

Proposition 5.9. — *If the square on the left is homotopy cartesian in \mathfrak{M} ,*

$$\begin{array}{ccc} W & \longrightarrow & X \\ \downarrow & & \downarrow \\ Y & \longrightarrow & Z \end{array} \qquad \begin{array}{ccc} \mathbf{LM}(A, W) & \longrightarrow & \mathbf{LM}(A, X) \\ \downarrow & & \downarrow \\ \mathbf{LM}(A, Y) & \longrightarrow & \mathbf{LM}(A, Z) \end{array}$$

then for any A in \mathfrak{M} , the square on the right is homotopy cartesian in \mathfrak{Top} .

We now apply the previous observations to the case at hand, the model structure on \mathfrak{E}' and the model structures on \mathcal{S}' . It is convenient to use \mathcal{S}' to denote the usual model structure 5.2.(i) on \mathcal{S}' ; we write $\mathcal{S}'_{\mathbf{Q}}$ for the model structure 5.2.(ii) with weak equivalences the rational equivalences, and \mathcal{S}'_f for the model structure 5.2.(iii) with weak equivalences the finite equivalences. The following theorem is an immediate consequence of Proposition 5.8. For part (iv), let S^1 denote the simplicial model for the circle with one vertex and one non-degenerate 1-simplex.

Proposition 5.10. — *We have natural isomorphisms in $\text{Ho}(\mathfrak{Top})$*

- (i) $|Y| \simeq \mathbf{L}_{\mathcal{S}'}(*, Y)$
- (ii) $|Y|_{\mathbf{Q}} \simeq \mathbf{L}_{\mathcal{S}'_{\mathbf{Q}}}(*, Y)$
- (iii) $|Y|^{\wedge} \simeq \mathbf{L}_{\mathcal{S}'_f}(*, Y)$
- (iv) $\Lambda|Y|^{\wedge} \simeq \mathbf{L}_{\mathcal{S}'_f}(S^1, Y)$.

Thus, we have produced a version of ΛY^{\wedge} that is a representable functor on $\mathbf{L}_{\mathcal{S}'_f}$. The following proposition in particular produces a version of $\mathbf{T}(C^*Y; \mathbf{R})$ that is a continuous functor of $\mathbf{L}_{\mathcal{S}'}$; it is an immediate consequence of the dual statement to Proposition 5.8.

Proposition 5.11. — *Let \mathbf{R} be a commutative ring with $\#\mathbf{R} \leq \mathfrak{c}$. We have a natural isomorphism in the homotopy category $\text{Ho}(\mathfrak{Top})$, $|\mathbf{T}(A, \mathbf{R})| \simeq \mathbf{L}\mathfrak{E}'(A, \mathbf{R})$.*

Unfortunately, when we apply the Yoneda Lemma for natural transformations out of the representable functor $\mathbf{L}_{\mathcal{S}'_f}(S^1, -)$, we need the target to be a continuous functor of $\mathbf{L}_{\mathcal{S}'_f}$, but $\mathbf{L}C^*$ is merely a continuous functor of $\mathbf{L}_{\mathcal{S}'}$ since C^* does not preserve general finite equivalences. The functor $C^*(-; \mathbf{Z}^{\wedge})$ does preserve finite equivalences, and here is where we need the category $\mathfrak{E}'_{\mathbf{Z}^{\wedge}}$. We have in mind $\mathbf{R} = \mathbf{Z}^{\wedge}$, \mathbf{W} , or \mathbf{Q}^{\wedge} in the following proposition.

Proposition 5.12. — *Let \mathbf{R} be a commutative ring containing \mathbf{Z}^\wedge and satisfying $\#\mathbf{R} \leq \mathfrak{c}$. Then the map*

$$\begin{aligned} \alpha_{\mathbf{R}} : \mathbf{L}\mathcal{S}'_f(*, \mathbf{Y}) &\xrightarrow{\text{LC}^*(-; \mathbf{Z}^\wedge)} \mathbf{L}\mathcal{E}'_{\mathbf{Z}^\wedge}(\mathbf{C}^*(\mathbf{Y}; \mathbf{Z}^\wedge), \mathbf{Z}^\wedge) \\ &\xrightarrow{i_*} \mathbf{L}\mathcal{E}'_{\mathbf{Z}^\wedge}(\mathbf{C}^*(\mathbf{Y}; \mathbf{Z}^\wedge), \mathbf{R}) \end{aligned}$$

is natural in maps of \mathbf{Y} in $\mathbf{L}\mathcal{S}'_f$. The map

$$\beta_{\mathbf{R}} : \mathbf{L}\mathcal{E}'_{\mathbf{Z}^\wedge}(\mathbf{C}^*(\mathbf{Y}; \mathbf{Z}^\wedge), \mathbf{R}) \rightarrow \mathbf{L}\mathcal{E}'(\mathbf{C}^*(\mathbf{Y}; \mathbf{Z}^\wedge), \mathbf{R}) \rightarrow \mathbf{L}\mathcal{E}'(\mathbf{C}^*(\mathbf{Y}), \mathbf{R})$$

is natural in maps of \mathbf{Y} in \mathcal{S}' (but is not natural for maps of \mathbf{Y} in $\mathbf{L}\mathcal{S}'$ or $\mathbf{L}\mathcal{S}'_f$), and is a homotopy equivalence when \mathbf{Y} is finite type.

Proof. — The first statement follows from Proposition 5.7. In the second display, the first map is natural in $\mathbf{L}\mathcal{S}'$ (again by Proposition 5.7), and the second map is natural in \mathcal{S}' (but not natural in $\mathbf{L}\mathcal{S}'$ or $\mathbf{L}\mathcal{S}'_f$). When \mathbf{Y} is finite type and $\mathbf{A} \rightarrow \mathbf{C}^*\mathbf{Y}$ is a cofibrant approximation, then $\mathbf{A} \otimes \mathbf{Z}^\wedge \rightarrow \mathbf{C}^*(\mathbf{Y}; \mathbf{Z}^\wedge)$ is a cofibrant approximation in $\mathcal{E}_{\mathbf{Z}^\wedge}$ and the composite map

$$\begin{aligned} \mathcal{E}_{\mathbf{Z}^\wedge}(\mathbf{A} \otimes \mathbf{Z}^\wedge, \mathbf{C}^*(\Delta[\bullet]; \mathbf{R})) &\rightarrow \mathcal{E}(\mathbf{A} \otimes \mathbf{Z}^\wedge, \mathbf{C}^*(\Delta[\bullet]; \mathbf{R})) \\ &\rightarrow \mathcal{E}(\mathbf{A}, \mathbf{C}^*(\Delta[\bullet]; \mathbf{R})) \end{aligned}$$

is an isomorphism. Proposition 5.8 then implies that $\beta_{\mathbf{R}}$ is a homotopy equivalence for \mathbf{Y} finite type.

To provide a bridge between the statements in Section 1 and the propositions above, we offer the following propositions; they are immediate consequences of Proposition 5.8 and its dual.

Proposition 5.13. — *The following diagram in $\text{Ho}(\mathfrak{Top})$ commutes where the top row is (1.5) and the vertical maps are the equivalences of Propositions 5.10 and 5.11.*

$$\begin{array}{ccccc} |\mathbf{Y}| & \longrightarrow & |\mathbf{T}(\mathbf{C}^*(\mathbf{Y}); \mathbf{Z})| & \longrightarrow & |\mathbf{T}(\mathbf{C}^*(\mathbf{Y}); \mathbf{R})| \\ \simeq \Big| & & \simeq \Big| & & \simeq \Big| \\ \mathbf{L}\mathcal{S}'_f(*, \mathbf{Y}) & \xrightarrow{\text{LC}^*} & \mathbf{L}\mathcal{E}'(\mathbf{C}^*(\mathbf{Y}), \mathbf{Z}) & \longrightarrow & \mathbf{L}\mathcal{E}'(\mathbf{C}^*(\mathbf{Y}), \mathbf{R}) \end{array}$$

For the next proposition, let \mathbf{Y}_f be a fibrant approximation of \mathbf{Y} in \mathcal{S}'_f ; then \mathbf{Y}_f is the Bousfield finite completion of \mathbf{Y} and the map $\mathbf{Y} \rightarrow \mathbf{Y}_f$ is a model for the finite completion map $\mathbf{Y} \rightarrow \mathbf{Y}^\wedge$ when \mathbf{Y} is finite type nilpotent. The map $\mathbf{Y} \rightarrow \mathbf{Y}_f$ induces a quasi-isomorphism $\mathbf{C}^*(\mathbf{Y}_f; \mathbf{Z}^\wedge) \rightarrow \mathbf{C}^*(\mathbf{Y}; \mathbf{Z}^\wedge)$. As mentioned above, the following proposition is an immediate consequence of Proposition 5.8 and its dual. Note that the top composite is η^\wedge from Proposition 1.10 and the bottom composite is $\beta_{\mathbf{R}} \circ \alpha_{\mathbf{R}}$ from Proposition 5.12.

Proposition 5.14. — *Let \mathbf{R} be as in Proposition 5.12. The following diagram in $\text{Ho}(\mathfrak{Top})$ commutes.*

$$\begin{array}{ccccc}
 Y_f & \longrightarrow & \mathbf{T}(C^*(Y_f; \mathbf{Z}^\wedge); \mathbf{R}) & \xleftarrow{\cong} & \mathbf{T}(C^*(Y; \mathbf{Z}^\wedge); \mathbf{R}) & \longrightarrow & \mathbf{T}(C^*Y; \mathbf{R}) \\
 \cong \downarrow & & \cong \downarrow & & \downarrow \cong & & \downarrow \cong \\
 L_{\mathcal{S}'_f}(*, Y_f) & \longrightarrow & L\mathfrak{E}'(C^*(Y_f; \mathbf{Z}^\wedge), \mathbf{R}) & & & & \\
 \cong \uparrow & & \swarrow \cong & & & & \\
 L_{\mathcal{S}'_f}(*, Y) & \longrightarrow & L\mathfrak{E}'(C^*(Y; \mathbf{Z}^\wedge), \mathbf{R}) & \longrightarrow & L\mathfrak{E}'(C^*Y, \mathbf{R}) & &
 \end{array}$$

Finally, we close this section with the proof of Theorems 5.2 and 5.3.

Proof of Theorems 5.2 and 5.3. — Using basic cardinal arithmetic, it is easy to see that \mathcal{S}' is closed under finite limits and colimits and that $\mathfrak{E}'_{\mathbf{R}}$ is closed under finite limits. To see that $\mathfrak{E}'_{\mathbf{R}}$ is closed under finite colimits, note that the free E_∞ \mathbf{R} -algebra on a differential graded \mathbf{R} -module \mathbf{M} , \mathbf{EM} , satisfies

$$\#\mathbf{EM} \leq \prod (\mathfrak{c} \cdot \#\mathcal{E}(n) \cdot (\#\mathbf{M})^n).$$

If $F: \mathcal{D} \rightarrow \mathfrak{E}'_{\mathbf{R}}$ is a diagram of E_∞ \mathbf{R} -algebras, the colimit of E_∞ \mathbf{R} -algebras, $\text{Colim}_{\mathcal{D}}^{\mathfrak{E}} F$ may be described as a differential graded module as the coequalizer

$$\mathbf{E}(\text{Colim}_{\mathcal{D}} \mathbf{E} \circ F) \rightrightarrows \mathbf{E}(\text{Colim}_{\mathcal{D}} F) \longrightarrow \text{Colim}_{\mathcal{D}}^{\mathfrak{E}} F$$

(where “ $\text{Colim}_{\mathcal{D}}$ ” denotes colimit of differential graded modules). Thus, when \mathcal{D} is a finite diagram and F factors through $\mathfrak{E}'_{\mathbf{R}}$, the colimit satisfies $\#(\text{Colim}_{\mathcal{D}}^{\mathfrak{E}} F) \leq \varkappa$, and so $\mathfrak{E}'_{\mathbf{R}}$ is closed under finite colimits.

It remains to see that the factorization axioms hold, since $\mathfrak{E}'_{\mathbf{R}}$ and \mathcal{S}' inherit the remaining axioms from the model structure on $\mathfrak{E}'_{\mathbf{R}}$ and the corresponding model structures on \mathcal{S}' . In the usual model structure on simplicial sets and the Bousfield local model structures [1, §10–11] the factorizations are constructed by the small object argument [12, II.3.3–4]. Let $f: \mathbf{X} \rightarrow \mathbf{Y}$ be a map, and consider either of the standard factorizations $\mathbf{X} \rightarrow \mathbf{Z} \rightarrow \mathbf{Y}$. By inspection in the case of (i) and by [1, 11.1, 11.5] in the case of (ii) and (iii), an easy cardinality argument shows that

$$\#\mathbf{Z} \leq \mathfrak{c} \cdot \#\mathbf{X} \cdot \#\mathbf{Y}.$$

It follows that for a map in \mathcal{S}' the usual factorizations in \mathcal{S} may be performed in \mathcal{S}' , and so \mathcal{S}' inherits the listed closed model structures. The argument in $\mathfrak{E}'_{\mathbf{R}}$ for Theorem 5.3 is entirely similar.

6. The proof of Theorems 1.11 and 1.14

This section is devoted to the proof of Theorems 1.11 and 1.14. The argument is to use the basic tools of function complexes described in the previous section to reinterpret the statements in terms of continuous functors and to apply the Yoneda Lemma. We begin by stating the following theorem that combines and refines Theorems 1.11 and 1.14.

Theorem 6.1. — *There is a natural (in \mathcal{S}') transformation*

$$\lambda: \mathbf{L}\mathcal{S}'_f(\mathbb{S}^1, \mathbf{Y}) \rightarrow \mathbf{L}\mathcal{E}'(\mathbf{C}^*(\mathbf{Y}); \mathbf{Z}^\wedge)$$

that is a weak equivalence when \mathbf{Y} is connected finite type nilpotent. Moreover, the following diagrams in \mathfrak{Top} commute up to natural homotopy.

$$\begin{array}{ccc} \mathbf{L}\mathcal{S}'_f(\mathbb{S}^1, \mathbf{Y}) & \xrightarrow{\lambda} & \mathbf{L}\mathcal{E}'(\mathbf{C}^*(\mathbf{Y}), \mathbf{Z}^\wedge) \\ \downarrow & & \downarrow \\ \mathbf{L}\mathcal{S}'_f(*, \mathbf{Y}) & \xrightarrow{\gamma_{\mathbf{W}}} & \mathbf{L}\mathcal{E}'(\mathbf{C}^*(\mathbf{Y}), \mathbf{W}) \\ \mathbf{L}\mathcal{S}'_f(\mathbb{S}^1, \mathbf{Y}) & \xrightarrow{\lambda} & \mathbf{L}\mathcal{E}'(\mathbf{C}^*(\mathbf{Y}), \mathbf{Z}^\wedge) \\ \downarrow & & \downarrow \\ \mathbf{L}\mathcal{S}'_f(*, \mathbf{Y}) & \xrightarrow{\gamma_{\mathbf{Q}^\wedge}} & \mathbf{L}\mathcal{E}'(\mathbf{C}^*(\mathbf{Y}), \mathbf{Q}^\wedge) \end{array}$$

Here $\gamma_{\mathbf{W}} = \beta_{\mathbf{W}} \circ \alpha_{\mathbf{W}}$ and $\gamma_{\mathbf{Q}^\wedge} = \beta_{\mathbf{Q}^\wedge} \circ \alpha_{\mathbf{Q}^\wedge}$ for the maps $\alpha_{\mathbf{R}}, \beta_{\mathbf{R}}$ of Proposition 5.12.

It is convenient to use a formulation of Theorem 1.7 in terms of $\mathbf{L}\mathcal{S}'_f$ and $\mathbf{L}\mathcal{E}'$. The precise formulation we need is the following proposition. It is an immediate consequence of Theorem 1.7, Proposition 5.14, and the Main Theorem of [9].

Proposition 6.2. — *Let \mathbf{Y} be a connected finite type nilpotent simplicial set in \mathcal{S}' . The natural map $\beta_{\mathbf{W}} \circ \alpha_{\mathbf{W}}: \mathbf{L}\mathcal{S}'_f(*, \mathbf{Y}) \rightarrow \mathbf{L}\mathcal{E}'(\mathbf{C}^*\mathbf{Y}, \mathbf{W})$ is a weak equivalence.*

Using Propositions 5.13 and 5.14, after passing to the homotopy category and dropping naturality, Theorem 6.1 becomes Theorems 1.11 and 1.14.

We now begin the proof of Theorem 6.1. Recall from [14, II§5] that there is a unique ring automorphism Φ_p of the Witt vectors $W(\bar{\mathbf{F}}_p)$ that induces the Frobenius on the residue field $\bar{\mathbf{F}}_p$. We have that the ring of p -adic integers \mathbf{Z}_p^\wedge is the subring of $W(\bar{\mathbf{F}}_p)$ of elements a satisfying $\Phi_p a = a$. Another property of the

Frobenius is that for every element a of $W(\bar{\mathbf{F}}_p)$ there is some element x in $W(\bar{\mathbf{F}}_p)$ that satisfies $\Phi_p x - x = a$. We let Φ be the automorphism of \mathbf{W} that performs Φ_p on the factor $W(\bar{\mathbf{F}}_p)$ for each p . We then have that \mathbf{Z}^\wedge is the subring of \mathbf{W} of elements a satisfying $\Phi a = a$, and for every element a of \mathbf{W} , there is some element x in \mathbf{W} that satisfies $\Phi x - x = a$. It follows that the square

$$\begin{array}{ccc} \mathbf{Z}^\wedge & \longrightarrow & \mathbf{W} \\ \downarrow & & \downarrow (\text{id}, \Phi) \\ \mathbf{W} & \xrightarrow{(\text{id}, \text{id})} & \mathbf{W} \times \mathbf{W} \end{array}$$

is a pullback square and a simple calculation shows that it is homotopy cartesian, but note that none of the maps in the square are surjections. We can factor the bottom map through a fibration very easily, as

$$\mathbf{W} \rightarrow C^*(\mathbf{I}, \mathbf{W}) \rightarrow \mathbf{W} \times \mathbf{W},$$

where $\mathbf{I} = \Delta[1]$ is the standard 1-simplex. The maps are the induced maps on $C^*(-; \mathbf{W})$ of the projection $\mathbf{I} \rightarrow *$ and the inclusions of the vertices $\{0, 1\} \rightarrow \mathbf{I}$. Let Z' be the E_∞ \mathbf{Z}^\wedge -algebra that makes the following square a pullback.

$$\begin{array}{ccc} Z' & \longrightarrow & \mathbf{W} \\ \downarrow & & \downarrow (\text{id}, \Phi) \\ C^*(\mathbf{I}; \mathbf{W}) & \longrightarrow & \mathbf{W} \times \mathbf{W} \end{array}$$

It follows that the induced map $j: \mathbf{Z}^\wedge \rightarrow Z'$ is a quasi-isomorphism. As a consequence we get the following proposition.

Proposition 6.3. — *The natural map*

$$j_*: L\mathcal{E}'_{\mathbf{Z}^\wedge}(C^*(Y; \mathbf{Z}^\wedge), \mathbf{Z}^\wedge) \rightarrow L\mathcal{E}'_{\mathbf{Z}^\wedge}(C^*(Y; \mathbf{Z}^\wedge), Z')$$

is a homotopy equivalence.

The endomorphism Φ of \mathbf{W} restricts to the identity endomorphism of \mathbf{Z}^\wedge , and so replacing \mathbf{W} with \mathbf{Z}^\wedge , the analogue of the square defining Z' as a pullback is the square

$$\begin{array}{ccc} C^*(S^1; \mathbf{Z}^\wedge) & \longrightarrow & \mathbf{Z}^\wedge \\ \downarrow & & \downarrow (\text{id}, \text{id}) \\ C^*(\mathbf{I}; \mathbf{Z}^\wedge) & \longrightarrow & \mathbf{Z}^\wedge \times \mathbf{Z}^\wedge \end{array}$$

describing $C^*(S^1; \mathbf{Z}^\wedge)$ as a pullback. The inclusion $\mathbf{Z}^\wedge \rightarrow \mathbf{W}$ and the universal property of the pullback induce the diagonal maps in the following commutative diagram.

$$\begin{array}{ccccc}
 C^*(S^1; \mathbf{Z}^\wedge) & \longrightarrow & \mathbf{Z}^\wedge & & \\
 \downarrow & \searrow & \downarrow & \searrow & \\
 & & \mathbf{Z}' & \longrightarrow & \mathbf{W} \\
 & & \downarrow & & \downarrow \\
 C^*(I; \mathbf{Z}^\wedge) & \longrightarrow & \mathbf{Z}^\wedge \times \mathbf{Z}^\wedge & & \\
 \searrow & & \downarrow & \searrow & \\
 & & C^*(I; \mathbf{W}) & \longrightarrow & \mathbf{W} \times \mathbf{W}
 \end{array}$$

We obtain the following commutative diagram by composing $LC^*(-; \mathbf{Z}^\wedge)$ with the maps induced by the diagonal maps above.

$$\begin{array}{ccccc}
 L_{\mathcal{S}'_f}(S^1, Y) & \longrightarrow & L_{\mathcal{S}'_f}(*, Y) & & \\
 \downarrow & \dashrightarrow^{\theta} & \downarrow & \searrow & \\
 & & L_{\mathcal{E}'_{\mathbf{Z}^\wedge}}(C^*(Y; \mathbf{Z}^\wedge), Z') & \longrightarrow & L_{\mathcal{E}'_{\mathbf{Z}^\wedge}}(C^*(Y; \mathbf{Z}^\wedge), \mathbf{W}) \\
 & & \downarrow & & \downarrow \\
 L_{\mathcal{S}'_f}(I, Y) & \longrightarrow & L_{\mathcal{S}'_f}(\{0, 1\}, Y) & & \\
 \searrow & & \downarrow & \searrow & \\
 & & L_{\mathcal{E}'_{\mathbf{Z}^\wedge}}(C^*(Y; \mathbf{Z}^\wedge), C^*(I; \mathbf{W})) & \longrightarrow & L_{\mathcal{E}'_{\mathbf{Z}^\wedge}}(C^*(Y; \mathbf{Z}^\wedge), \mathbf{W} \times \mathbf{W})
 \end{array}$$

Note that all the diagonal maps in this diagram are natural for maps of Y in $L_{\mathcal{S}'_f}$. By Proposition 5.9 and its dual, both squares in the diagram above are homotopy cartesian in the category of spaces. Proposition 6.2 implies that when Y is connected finite type nilpotent, the solid diagonal arrows are homotopy equivalences. The dashed arrow θ is then a weak equivalence; we obtain the following proposition.

Proposition 6.4. — *There is a natural transformation*

$$\theta : L_{\mathcal{S}'_f}(S^1, Y) \rightarrow L_{\mathcal{E}'_{\mathbf{Z}^\wedge}}(C^*(Y; \mathbf{Z}^\wedge), Z')$$

of functors $L_{\mathcal{S}'_f} \rightarrow \mathfrak{Top}$ that is a weak equivalence whenever Y is connected finite type nilpotent.

Let f_0 denote the image under θ of the identity map of S^1 . Let f_1 be a point in $\mathbf{L}\mathcal{E}'(\mathbf{C}^*(S^1; \mathbf{Z}^\wedge), \mathbf{Z}^\wedge)$ sent to the component of f_0 under the map j_* in Proposition 6.3. Choose a path $F: I \rightarrow \mathbf{L}\mathcal{E}'(\mathbf{C}^*(S^1; \mathbf{Z}^\wedge), \mathbf{Z}')$ connecting f_0 and j_*f_1 . The Yoneda Lemma then gives us a natural map

$$\zeta: \mathbf{L}\mathcal{S}'_f(S^1, Y) \rightarrow \mathbf{L}\mathcal{E}'(\mathbf{C}^*(S^1; \mathbf{Z}^\wedge), \mathbf{Z}^\wedge)$$

and a natural homotopy

$$\Theta: \mathbf{L}\mathcal{S}'_f(S^1, Y) \times I \rightarrow \mathbf{L}\mathcal{E}'(\mathbf{C}^*(S^1; \mathbf{Z}^\wedge), \mathbf{Z}')$$

between θ and $j_* \circ \zeta$. Combining this with Propositions 6.3 and 6.4, we obtain the following proposition.

Proposition 6.5. — *There is a natural transformation*

$$\zeta: \mathbf{L}\mathcal{S}'_f(S^1, Y) \rightarrow \mathbf{L}\mathcal{E}'_{\mathbf{Z}^\wedge}(\mathbf{C}^*(Y; \mathbf{Z}^\wedge), \mathbf{Z}^\wedge)$$

of functors $\mathbf{L}\mathcal{S}'_f \rightarrow \mathfrak{Top}$ that is a weak equivalence whenever Y is connected finite type nilpotent.

We now prove Theorem 6.1.

Proof of Theorem 6.1. — Since $\mathbf{L}\mathcal{E}'_{\mathbf{Z}^\wedge}(\mathbf{C}^*(S^1; \mathbf{Z}^\wedge), \mathbf{W})$ and $\mathbf{L}\mathcal{E}'_{\mathbf{Z}^\wedge}(\mathbf{C}^*(S^1; \mathbf{Z}^\wedge), \mathbf{Q}^\wedge)$ are connected, just as above, the Yoneda Lemma allows us to choose natural (in $\mathbf{L}\mathcal{S}'_f$) homotopies for the following diagrams in \mathfrak{Top} for $\mathbf{R} = \mathbf{W}, \mathbf{Q}^\wedge$.

$$\begin{array}{ccc} \mathbf{L}\mathcal{S}'_f(S^1, Y) & \xrightarrow{\zeta} & \mathbf{L}\mathcal{E}'_{\mathbf{Z}^\wedge}(\mathbf{C}^*(Y; \mathbf{Z}^\wedge), \mathbf{Z}^\wedge) \\ \downarrow & & \downarrow \\ \mathbf{L}\mathcal{S}'_f(*, Y) & \xrightarrow{\alpha_{\mathbf{R}}} & \mathbf{L}\mathcal{E}'_{\mathbf{Z}^\wedge}(\mathbf{C}^*(Y; \mathbf{Z}^\wedge), \mathbf{R}) \end{array}$$

Let $\lambda = \beta_{\mathbf{Z}^\wedge} \circ \zeta$. Then λ is natural in \mathcal{S}' and is a weak equivalence for connected finite type nilpotent Y by Propositions 5.12 and 6.5. We compose the homotopies above with the maps $\beta_{\mathbf{W}}$ and $\beta_{\mathbf{Q}^\wedge}$ to get the natural (in \mathcal{S}') homotopies for the statement.

7. The proof of Theorem 1.12

In this section we deduce Theorem 1.12 from Theorem 6.1. The argument consists exclusively of manipulating homotopy pullbacks.

Define BY to be the homotopy pullback of $L\mathcal{E}'(C^*Y, \mathbf{Z}^\wedge) \rightarrow L\mathcal{E}'(C^*Y, \mathbf{Q}^\wedge)$ along $L\mathcal{E}'(C^*Y, \mathbf{Q}) \rightarrow L\mathcal{E}'(C^*Y, \mathbf{Q}^\wedge)$, and let DY be the homotopy pullback of λ along $L\mathcal{E}'(C^*Y, \mathbf{Q}) \rightarrow L\mathcal{E}'(C^*Y, \mathbf{Q}^\wedge)$.

$$\begin{array}{ccc}
 BY & \cdots\cdots\cdots\rightarrow & L\mathcal{E}'(C^*Y, \mathbf{Z}^\wedge) \\
 \vdots & & \downarrow \\
 L\mathcal{E}'(C^*Y, \mathbf{Q}) & \longrightarrow & L\mathcal{E}'(C^*Y, \mathbf{Q}^\wedge) \\
 & & \downarrow \\
 DY & \cdots\cdots\cdots\rightarrow & L\mathcal{S}'_f(S^1, Y) \\
 \downarrow \lambda & & \downarrow \lambda \\
 L\mathcal{E}'(C^*Y, \mathbf{Q}) & \longrightarrow & L\mathcal{E}'(C^*Y, \mathbf{Q}^\wedge)
 \end{array}$$

Then B and D are functors of Y (in \mathcal{S}') and we have natural transformations

$$L\mathcal{E}'(C^*Y, \mathbf{Z}) \rightarrow BY \leftarrow DY$$

that are weak equivalences when Y is connected finite type nilpotent. Let AY be the homotopy pullback of $\gamma_{\mathbf{Q}^\wedge}$ along $L\mathcal{E}'(C^*Y, \mathbf{Q}) \rightarrow L\mathcal{E}'(C^*Y, \mathbf{Q}^\wedge)$.

$$\begin{array}{ccc}
 AY & \cdots\cdots\cdots\rightarrow & L\mathcal{S}'_f(*, Y) \\
 \vdots & & \downarrow \gamma_{\mathbf{Q}^\wedge} \\
 L\mathcal{E}'(C^*Y, \mathbf{Q}) & \longrightarrow & L\mathcal{E}'(C^*Y, \mathbf{Q}^\wedge)
 \end{array}$$

Then A is a functor of Y (in \mathcal{S}') and we have a natural transformation $D \rightarrow A$ induced by the (second) natural homotopy in Theorem 6.1.

When we restrict to the case when Y is connected finite type nilpotent, we get a natural transformation in the homotopy category

$$\delta: |Y| \simeq L\mathcal{S}'_f(*, Y) \rightarrow L\mathcal{E}'(C^*Y, \mathbf{Z}) \rightarrow BY \simeq DY \rightarrow AY.$$

We can see from Proposition 5.10.(iii) and Theorems 1.6 and 1.10 that the square defining AY is equivalent in the homotopy category to the arithmetic square for Y , and that AY is abstractly weakly equivalent to Y , but for our argument below we need to see that this particular map is a weak equivalence.

Theorem 7.1. — *For Y connected finite type nilpotent, the natural map in the homotopy category $\delta: Y \rightarrow AY$ constructed above is a weak equivalence.*

Proof. — Since Y and AY are both finite type nilpotent spaces, to see that δ is a weak equivalence, it suffices to check that it is a weak equivalence after finite

completion and that it is a weak equivalence after rationalization. To do this, it suffices to check that the composite maps in the homotopy category

$$|Y| \rightarrow AY \rightarrow L_{\mathcal{L}'_f}(*, Y), \quad |Y| \rightarrow AY \rightarrow L\mathcal{E}'(C^*Y, \mathbf{Q})$$

are a finite equivalence and a rational equivalence respectively. We can use the fact that the maps comparing $L\mathcal{E}'(C^*Y, \mathbf{Z})$, BY , DY , and AY are induced by maps of squares to analyze these composite maps. The map from $DY \rightarrow AY$ sends $L_{\mathcal{L}'}(S^1, Y)$ to $L_{\mathcal{L}'_f}(*, Y)$ by the map induced by the inclusion of the vertex in S^1 . We therefore see that the composite map $|Y| \rightarrow L_{\mathcal{L}'_f}(*, Y)$ is a finite equivalence because the first diagram in Theorem 6.1 commutes in the homotopy category and the composite map $|Y| \rightarrow L\mathcal{E}'(C^*Y, \mathbf{W})$ is finite completion by Proposition 6.2. Theorem 1.6 and Proposition 5.13 identify the composite map $|Y| \rightarrow L\mathcal{E}'(C^*Y, \mathbf{Q})$ as a rational equivalence.

Theorem 7.1 is the main result we need for the proof of Theorem 1.12.

Proof of Theorem 1.12. — For Y connected finite type nilpotent, we define ϵ to be the composite of the natural map in the homotopy category

$$L\mathcal{E}'(C^*Y, \mathbf{Z}) \rightarrow BY \simeq DY \rightarrow AY$$

and δ^{-1} . When Y is not connected, [10, 3.1] allows us to break up $L\mathcal{E}'(C^*Y, \mathbf{Z})$ naturally into a disjoint union of $L\mathcal{E}'(C^*Y_0, \mathbf{Z})$ over the components Y_0 of Y . We then define ϵ componentwise.

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