SMOOTH QUASIREGULAR MAPS WITH BRANCHING IN $\mathbf{R}^n$

by Robert KAUFMAN, Jeremy T. TYSON*, *and* Jang-Mei WU**

ABSTRACT

According to a theorem of Martio, Rickman and Väisälä, all nonconstant $C^{n/(n-2)}$-smooth quasiregular maps in $\mathbf{R}^n$, $n \geq 3$, are local homeomorphisms. Bonk and Heinonen proved that the order of smoothness is sharp in $\mathbf{R}^3$. We prove that the order of smoothness is sharp in $\mathbf{R}^4$. For each $n \geq 5$ we construct a $C^{1+\epsilon(n)}$-smooth quasiregular map in $\mathbf{R}^n$ with nonempty branch set.

1. Introduction

Recall that a continuous mapping $f : D \to \mathbf{R}^n$ in the Sobolev space $W^{1,n}_{\mathrm{loc}}(D, \mathbf{R}^n)$ is called K-*quasiregular*, $K \geq 1$, if

$$|f'(x)|^n \leq K J_f(x), \quad \text{a.e. } x \in D. \tag{1.1}$$

Here $n \geq 2$, $D \subset \mathbf{R}^n$ is a domain, $|f'(x)|$ is the operator norm of the differential of f, and $J_f(x) = \det f'(x)$ denotes the Jacobian determinant. In the plane, 1-quasiregular maps are precisely analytic functions of a single complex variable.

Quasiregular mappings were first introduced and studied by Yu. G. Reshetnyak [18] under the name "mappings of bounded distortion". A deep theorem of Reshetnyak states that nonconstant quasiregular maps are discrete and open. Quasiregular maps were subsequently developed by Martio, Rickman, Väisälä, and their collaborators [15], [16]. See [19], [20] or [13] for a comprehensive account of the theory.

The *branch set* B_f of a continuous, discrete, and open mapping $f : D \to \mathbf{R}^n$ is the closed set of points in D where f does not define a local homeomorphism. By a theorem of Černavskiĭ [4], [5], the topological dimensions of the branch set and its image are equal and at most $n-2$. On the other hand, if B_f is not empty, then $\Lambda^{n-2}(f(B_f)) > 0$ by a theorem of Martio, Rickman and Väisälä [16]; moreover, $\Lambda^{n-2}(B_f) > 0$ when $n = 2$ (this is trivial) and when $n = 3$ (a result of Martio and Rickman [15]). Here Λ^r is the r-dimensional Hausdorff measure.

Branch sets of quasiregular mappings may exhibit complicated topological structure and may contain, for example, many wild Cantor sets of classical geometric topology [11], [21], [12], [10].

* Supported by the National Science Foundation under grant DMS-0228807.

** Supported by the National Science Foundation under the grants DMS-0070312 and 0400810.

DOI 10.1007/s10240-005-0031-4

Quasiregular mappings of $\mathbf{R}^2$ can be smooth without being locally homeomorphic. For example, $f(z) = z^2$ has branch set $B_f = \{0\}$. When $n \geq 3$, sufficiently smooth nonconstant quasiregular mappings are local homeomorphisms. In fact

1.2. *Theorem (Martio–Rickman–Väisälä). — Every nonconstant* $C^{n/(n-2)}$*-smooth quasiregular mapping must be locally homeomorphic when* $n \geq 3$.

A mapping $g : D \to \mathbf{R}^n$ is C^k-smooth, $k = 0, 1, 2, \ldots$, if all partial derivatives of the coordinate functions of g up to order k are continuous. If $k \in (0, \infty)$ is not an integer, g is said to be C^k-smooth if g is $C^{[k]}$ and all partial derivatives of the coordinate functions of g of order $[k]$ are $(k - [k])$-Hölder continuous on each compact subset of D.

Theorem 1.2 is essentially contained in [16]; an earlier version is due to Church [6]. The proof of Theorem 1.2 in [20, p. 12] uses the Morse-Sard theorem together with the discreteness and openness of quasiregular maps and a theorem of Martio–Rickman–Väisälä in [16]. In his 1978 ICM address [30], Väisälä asked whether the branch set of every C^1-smooth quasiregular map in $\mathbf{R}^n$, $n \geq 3$, must be empty.

Recently, Bonk and Heinonen [3] showed that the exponent $n/(n-2)$ is sharp when $n = 3$, and proved refined versions of Theorem 1.2 as well as a theorem of Sarvas [22].

1.3. *Theorem (Bonk–Heinonen [3]). — For every* $\epsilon > 0$ *and every integer* $d \geq 2$*, there exists a* $C^{3-\epsilon}$*-smooth quasiregular mapping* $F : \mathbf{R}^3 \to \mathbf{R}^3$ *of degree* d *whose branch set* B_F *is homeomorphic to* $\mathbf{R}$ *and has Hausdorff dimension* $3 - \delta(\epsilon)$ *with* $\delta(\epsilon) \to 0$ *as* $\epsilon \to 0$. *The map* F *has the Hölder property*

$$C^{-1}|x-y|^{3-\epsilon'} \leq |F(x) - F(y)| \leq C|x-y|^{3-\epsilon}, \quad \forall x, y \in B_F, |x-y| \leq 1,$$

for some $0 < \epsilon' \leq \epsilon$ *and* $C > 1$.

1.4. *Theorem (Bonk–Heinonen [3]). — Given* $n \geq 3$ *and* $K \geq 1$*, there exist constants* $\lambda = \lambda(n, K) > 0$ *and* $\delta = \delta(n, K) > 0$ *so that (i) the branch set of every* K*-quasiregular mapping* $f : D \to \mathbf{R}^n$ *has Hausdorff dimension at most* $n-\lambda$*, and (ii) every* $C^{n/(n-2)-\delta}$*-smooth* K*-quasiregular mapping* $f : D \to \mathbf{R}^n$ *is locally homeomorphic.*

We prove that Theorem 1.2 is sharp in $\mathbf{R}^4$.

1.5. *Theorem. — For every* $\epsilon > 0$ *and every integer* $d \geq 2$*, there exists a* $C^{2-\epsilon}$*-smooth quasiregular mapping* $F : \mathbf{R}^4 \to \mathbf{R}^4$ *of degree* d *whose branch set* B_F *is homeomorphic to* $\mathbf{R}^2$ *and has Hausdorff dimension* $4 - 2\epsilon$. *Moreover, the map* F *has the Hölder property*

$$C^{-1}|x-y|^{2-\epsilon} \leq |F(x) - F(y)| \leq C|x-y|^{2-\epsilon}, \quad \forall x, y \in B_F, |x-y| \leq 1,$$

for some $C > 1$.

The first step in Bonk and Heinonen's proof of Theorem 1.3 is the construction of a quasiconformal mapping g in $\mathbf{R}^3$ with uniformly expanding behavior on a line L. Then g is approximated off L by a C^∞-smooth quasiconformal mapping G by a smoothing procedure of Kiikka [14]. The map G^{-1} has the correct order of smoothness on $\mathbf{R}^3$; postcomposition with a winding map produces the desired quasiregular map F.

As explained in [3], it is not clear how to construct a quasiconformal mapping g in $\mathbf{R}^n$, $n \geq 4$, which is uniformly expanding on a codimension two linear subspace. Moreover, the smoothing procedure of Kiikka works in dimensions two and three only. Such approximation of general quasiconformal maps can fail to exist in dimensions higher than five [24], and is an open problem in dimension four [8].

The branch set for our map F in Theorem 1.5 is the product $\Gamma \times \Gamma$ of an infinite snowflake curve with itself. There is a canonical map f from $\mathbf{R}^2$ to $\Gamma \times \Gamma$, which can be written as the composition $f = f_{m-1} \circ \cdots \circ f_0$ of s-quasisymmetric maps with small s. These are quasisymmetric maps which are locally uniformly well-approximated by similarities. Following an extension process developed by Tukia and Väisälä [27], [31] for s-quasisymmetric maps with small s, we extend the maps f_j to quasiconformal maps g_j on $\mathbf{R}^4$. Smoothing off the products of snowflake curves via convolution with a variable kernel (see, e.g., [9]) produces smooth quasiconformal maps G_j. The composition of a winding map with the inverse of $G_{m-1} \circ G_{m-2} \circ \cdots \circ G_0$ yields the desired quasiregular map F.

In general, convolution does not preserve injectivity or quasiconformality. To obtain injectivity, quasiconformality, and the correct order of smoothness up to and including $\Gamma \times \Gamma$, convolution must be applied in conjunction with the special constructions of Tukia and Väisälä.

Our method does not apply to $\mathbf{R}^n$, $n \geq 5$, unless there exists an appropriate embedding

$$\underbrace{\Gamma \times \cdots \times \Gamma}_{n-2} \hookrightarrow \mathbf{R}^n$$

for some suitable snowflake curve Γ.

Recent results of Bishop [2] and David–Toro [7] provide snowflake-type embeddings $\mathbf{R}^{n-2} \hookrightarrow \Sigma \subset \mathbf{R}^{n-1}$ via global quasiconformal mappings of $\mathbf{R}^{n-1}$; by the Tukia–Väisälä extension theorem [26] these quasiconformal maps of $\mathbf{R}^{n-1}$ can be extended to quasiconformal maps of $\mathbf{R}^n$. The resulting codimension two snowflake-type surfaces $\Sigma \subset \mathbf{R}^n$ can be realized as the branch sets of $C^{1+\epsilon(n)}$-smooth branched quasiregular maps in $\mathbf{R}^n$, $n \geq 5$. Thus we answer Väisälä's question in the negative in all dimensions.

1.6. *Theorem. — Given integers* $n \geq 5$ *and* $d \geq 2$, *there exists* $\epsilon = \epsilon(n) > 0$ *(independent of* d*) and a* $C^{1+\epsilon}$*-smooth quasiregular map* $F : \mathbf{R}^n \to \mathbf{R}^n$ *of degree* d *whose branch set* B_F *is homeomorphic to* $\mathbf{R}^{n-2}$*. Moreover*

$$C^{-1}|x-y|^{1+\epsilon} \leq |F(x) - F(y)| \leq C|x-y|^{1+\epsilon}, \quad \forall x, y \in B_F, |x-y| \leq 1,$$

for some $C(n) > 1$.

In Sect. 2 we recall some preliminary material. In Sect. 3 we introduce a one-parameter family of snowflake surfaces in $\mathbf{R}^4$ which are mutually related by canonical quasisymmetric homeomorphisms. In Sect. 4 we extend these homeomorphisms to quasiconformal maps of $\mathbf{R}^4$, and in Sect. 5 we construct smooth approximations to the resulting maps via convolutions. Finally, Sect. 6 contains the proof of Theorem 1.5 and Sect. 7 contains the proof of Theorem 1.6.

1.7. *Acknowledgements*

The authors would like to thank Mario Bonk, Juha Heinonen, and Seppo Rickman for suggesting the problem of branch sets of smooth quasiregular maps on various occasions, and to acknowledge the referee for a careful reading of the paper and extensive comments which clarified both the exposition and the mathematics.

2. Preliminaries

2.1. *Notation*

We write $|x-y|$ for the distance between points x, y in any metric space, and we write $B(x, r)$ for the open ball centered at x of radius r. We denote by $\mathbf{R}^n$ the n-dimensional Euclidean space and by $e_1, \ldots, e_n$ the standard basis of $\mathbf{R}^n$. For x and y in $\mathbf{R}^n$ we write $[x, y]$ for the closed line segment with endpoints x and y. For $0 \leq s \leq n$ we write Λ^s for the s-dimensional Hausdorff measure. We reserve the notation "dim" for the Hausdorff dimension.

For a set $A \subset \mathbf{R}^n$ and $f : A \to \mathbf{R}$ we write $\|f\|_A := \sup\{|f(x)| : x \in A\}$.

A simplex in $\mathbf{R}^n$ is the closed convex hull of a set of $n+1$ points in gen-breakeral position. We write Δ^0 for the set of vertices of a simplex Δ.

For $x \in \mathbf{R}$ we write $[x]$ for the greatest integer less than or equal to x.

We denote by $C, c, \ldots$ various positive constants whose values may change from line to line.

2.2. *s-Quasisymmetric maps*

An embedding $f : X \to Y$ of metric spaces is called *s-quasisymmetric*, $s > 0$, if f is quasisymmetric and satisfies

$$|f(a) - f(x)| \leq (t + s)|f(b) - f(x)|$$

whenever $a, b, x \in X$ with $|a - x| \leq t|b - x|$ and $t \leq 1/s$. Recall that an embedding $f : X \to Y$ is *quasisymmetric* (for short, *QS*) if there is a homeomorphism $\eta : [0, \infty) \to [0, \infty)$ so that

$$|f(a) - f(x)| \leq \eta(t)|f(b) - f(x)|$$

whenever $a, b, x \in X$ with $|a - x| \leq t|b - x|$. We also say that f is η-*QS*.

Quasisymmetric maps on the real line were introduced by Beurling and Ahlfors [1] as the boundary functions for quasiconformal homeomorphisms of the upper half plane. A systematic study of quasisymmetric maps in metric spaces was begun by Tukia and Väisälä in [25]. s-Quasisymmetric maps were introduced by Tukia and Väisälä in [27] for the study of the extension problem for quasisymmetric maps.

In Euclidean spaces, s-quasisymmetric maps may be characterized as quasisymmetric maps which are locally uniformly close to similarities. A map $h : \mathbf{R}^p \to \mathbf{R}^n$ is *affine* if it is of the form $h(x) = \lambda B(x) + b$, where $\lambda > 0$, $b \in \mathbf{R}^n$, and B is an $n \times p$ matrix. If h is affine and B is orthogonal, we say that h is a *similarity*; in this case we write $L(h) = \lambda$.

In [27] and [31], Tukia and Väisälä proved the following theorems.

2.3. *Theorem (Väisälä [31], Theorem 3.1). — Let* $1 \leq p \leq n$ *be integers, let* A *be a compact set in* $\mathbf{R}^p$, *and let* $f : A \to \mathbf{R}^n$ *be an s-QS map. Then there is a similarity* $h : \mathbf{R}^p \to \mathbf{R}^n$ *so that*

$$\|h - f\|_A \leq \varkappa(s, p) L(h) \operatorname{diam} A,$$

where $s \mapsto \varkappa(s, p)$ *is an increasing function with* $\varkappa(s, p) \to 0$ *as* $s \to 0$.

2.4. *Theorem (Väisälä [31], Theorem 3.9). — Let* $1 \leq p \leq n$ *be integers, let* $0 < \varkappa \leq \frac{1}{25}$, *and let* $f : X \to \mathbf{R}^n$ *be a map from a connected set* $X \subset \mathbf{R}^p$ *such that for every bounded* $A \subset X$, *there is a similarity* $h : \mathbf{R}^p \to \mathbf{R}^n$ *with*

$$\|h - f\|_A \leq \varkappa L(h) \operatorname{diam} A.$$

Then f *is s-QS, where* $s = s(\varkappa) \to 0$ *as* $\varkappa \to 0$.

2.5. *Theorem (Tukia–Väisälä [27], Theorem 2.6). — Let* $f : \mathbf{R}^n \to \mathbf{R}^n$ *be an embedding,* $n \geq 2$. *If* f *is s-QS, then* f *is* K*-quasiconformal, where* $K = K(s, n) \to 1$ *as* $s \to 0$. *Conversely, if* f *is* K*-quasiconformal, then* f *is s-QS where* $s = s(K, n) \to 0$ *as* $K \to 1$. *Moreover,* $f(\mathbf{R}^n) = \mathbf{R}^n$.

2.6. *The quasisymmetric extension property*

A set $A \subset \mathbf{R}^n$ has the *quasisymmetric extension property* (*QSEP*) if there is $s_0 > 0$ so that if $0 < s \leq s_0$, then every s-QS $f : A \to \mathbf{R}^n$ has an s_1-QS extension $g : \mathbf{R}^n \to \mathbf{R}^n$, where $s_1 = s_1(s, n, A) \to 0$ as $s \to 0$. See [31, p. 239].

2.7. *Theorem (Tukia–Väisälä [27], Theorem 5.4). — Let $1 \leq p < n$ be integers. Then $\mathbf{R}^p$ has the quasisymmetric extension property in $\mathbf{R}^n$. The parameter s_0 may be chosen depending only on n.*

Following Väisälä [31], we say that a set $A \subset \mathbf{R}^p$ is *thick in* $\mathbf{R}^p$ if there are constants $r_0 > 0$ and $\beta > 0$ so that if $0 < r \leq r_0$ and $y \in A$, then there is a simplex Δ in $\mathbf{R}^p$ with $\Delta^0 \subset A \cap B(y, r)$ and $\Lambda^p(\Delta) \geq \beta r^p$.

The Cantor ternary set is thick in $\mathbf{R}^1$, while the von Koch snowflake curve is thick in $\mathbf{R}^2$ (see Proposition 3.5(a)). Thickness is not bi-Lipschitz invariant.

2.8. *Theorem (Väisälä [31], Theorem 6.2). — Suppose that A is closed and thick in $\mathbf{R}^p$, $1 \leq p \leq n$ are integers, and that either A or $\mathbf{R}^p \setminus A$ is bounded. Then A has the quasisymmetric extension property in $\mathbf{R}^n$. Moreover, $s_0 = s_0(A, n)$ depends only on n,* $\operatorname{diam}(\partial A)$ *and the thickness parameters r_0 and β.*

2.9. *Whitney triangulations*

Let A be a closed, nonempty, proper subset of $\mathbf{R}^n$, and let $\mathscr{K}$ be a Whitney decomposition of $\mathbf{R}^n \setminus A$ into closed dyadic n-cubes (see, e.g., [23, p. 16]). Following [31, p. 253], we define a triangulation $\mathscr{W}$ of $\mathscr{K}$ as follows. Let $\mathscr{W}^0 = \mathscr{K}^0$ consist of all vertices of $\mathscr{K}$. Suppose that a simplicial subdivision $\mathscr{W}^p$ of the p-skeleton $\mathscr{K}^p$ of $\mathscr{K}$ is given. Let Q be a $(p+1)$-cube of $\mathscr{K}$, and let v_Q be the center of Q. Since ∂Q is the underlying space of a subcomplex L_Q of $\mathscr{W}^p$, the cone construction $v_Q L_Q$ gives a simplicial subdivision of Q, and defines $\mathscr{W}^{p+1}$.

The complex $\mathscr{W}$ is called a *Whitney triangulation* of $\mathbf{R}^n \setminus A$.

2.10. *Remark.* — We assume, as we may, that

$$\frac{1}{9} \leq \frac{\operatorname{diam} Q}{\operatorname{dist}(Q, A)} \leq \frac{1}{4}$$

for all $Q \in \mathscr{K}$. Under this assumption, and by the construction, the simplices of $\mathscr{W}$ belong to a finite number of similarity classes. Therefore there exists a constant $C_1 > 1$ so that each n-simplex σ in $\mathscr{W}$ contains a ball of radius $C_1^{-1} \operatorname{diam} \sigma$.

2.11. *Regularized distance functions*

Let A be a closed, nonempty, and proper subset of $\mathbf{R}^n$, and let $\mathscr{W}$ be a Whitney triangulation of $\mathbf{R}^n \setminus \mathrm{A}$. Let δ_A be a positive C^∞-smooth function on $\mathbf{R}^n \setminus \mathrm{A}$ so that

$$\frac{1}{10^4 \mathrm{C}_1} \le \frac{\delta_\mathrm{A}(x)}{\mathrm{dist}(x, \mathrm{A})} \le \frac{1}{10^2 \mathrm{C}_1}, \tag{2.12}$$

$$\left|\frac{\partial \delta_\mathrm{A}}{\partial x_j}(x)\right| \le \mathrm{C}_2, \tag{2.13}$$

and

$$\left|\frac{\partial^2 \delta_\mathrm{A}}{\partial x_i \partial x_j}(x)\right| \le \frac{\mathrm{C}_2}{\delta_\mathrm{A}(x)} \tag{2.14}$$

for all $x \notin \mathrm{A}$, where C_1 is the value from Remark 2.10 and C_2 is a constant depending only on n. See, for example, [23, p. 170].

2.15. *Remark.* — Each n-simplex $\sigma \in \mathscr{W}$ contains a ball centered at some point x_σ of radius $2\delta_\mathrm{A}(x_\sigma)$.

2.16. *Smoothing with a variable kernel*

Fix a real valued function φ in $\mathrm{C}^\infty(\mathbf{R}^n)$ which is nonnegative, radial, supported in $\mathrm{B}(0, 1)$, and satisfies $\int_{\mathbf{R}^n} \varphi(x)\, dx = 1$ and

$$\sup_{\mathbf{R}^n}\left|\frac{\partial \varphi}{\partial x_i}\right|, \sup_{\mathbf{R}^n}\left|\frac{\partial^2 \varphi}{\partial x_i \partial x_j}\right| \le \mathrm{C}_3 \tag{2.17}$$

for some C_3 depending at most on n.

2.18. *Lemma.* — *Let $f(x) = \mathrm{B}x + b$ be an affine map with $\mathrm{B} \in \mathbf{R}^{n\times n}$ and $b \in \mathbf{R}^n$. Then*

$$f(x) = \int_{\mathbf{R}^n} f(y)\varphi(x - y)\, dy.$$

Proof. — Since φ is radial,

$$\int_{\mathbf{R}^n} (\mathrm{B}y + b)\varphi(x - y)\, dy = \int_{\mathbf{R}^n} (-\mathrm{B}y + \mathrm{B}x + b)\varphi(y)\, dy = \mathrm{B}x + b.$$

2.19. *Proposition.* — *Let* A *be a closed, nonempty, proper subset of* $\mathbf{R}^n$, *and let* $\delta \equiv \delta_A$ *be the regularized distance function on* $\mathbf{R}^n \setminus A$ *from the previous paragraph. Let* u *be a real-valued function on* $\mathbf{R}^n$, *and denote by* $\mathrm{Osc}(u, x, r)$ *the oscillation of* u *on* $B(x, r)$. *Set*

$$U(x) = \begin{cases} \frac{1}{\delta^n(x)} \int_{\mathbf{R}^n} u(y)\varphi\left(\frac{x-y}{\delta(x)}\right) dy, & x \in \mathbf{R}^n \setminus A, \\ u(x), & x \in A. \end{cases}$$

Then

(i) U *is* C^∞ *on* $\mathbf{R}^n \setminus A$;
(ii) if u *is continuous, then* U *is continuous;*
(iii) for $x \in \mathbf{R}^n \setminus A$,

$$\left|\frac{\partial U}{\partial x_j}(x)\right| \le \frac{C_4}{\delta(x)} \mathrm{Osc}(u, x, \delta(x)) \tag{2.20}$$

and

$$\left|\frac{\partial^2 U}{\partial x_i \partial x_j}(x)\right| \le \frac{C_5}{\delta^2(x)} \mathrm{Osc}(u, x, \delta(x)), \tag{2.21}$$

where C_4 *and* C_5 *are constants depending only on* n.

Smoothing by convolution with a variable kernel has occurred in the literature, cf. Arakelyan's approximation theorems [9].

The proof of Proposition 2.19 is by direct calculation and is omitted.

3. Quasisymmetrically equivalent snowflake surfaces in $\mathbf{R}^4$

In this section, we consider snowflake surfaces $\Gamma^\alpha \times \Gamma^\alpha$ in $\mathbf{R}^4$, $1 \le \alpha < 2$, where Γ^α is the periodic extension of a standard von Koch snowflake segment γ^α in $\mathbf{R}^2$. For each $\alpha \in [1, 2)$, $\Gamma^\alpha \times \Gamma^\alpha$ is canonically homeomorphic with $\Gamma^1 \times \Gamma^1 = \mathbf{R}^2$. We show that this homeomorphism is quasisymmetric, and factors as a composition of s-quasisymmetric maps. Moreover, $\Gamma^\alpha \times \Gamma^\alpha$ is thick in $\mathbf{R}^4$ for each $\alpha > 1$, with parameters $r_0 \equiv 1$ and $\beta = \beta(\alpha) \searrow 0$ as $\alpha \to 1$.

3.1. *von Koch snowflake curves in* $\mathbf{R}^2$

Fix $1 \le \alpha < 2$, and define γ^α to be the von Koch-type snowflake curve, homeomorphic with $\gamma^1 = [0, 1]$, consisting of four self-similar pieces scaled by factor

$$r = r_\alpha = 4^{-1/\alpha} \in [1/4, 1/2).$$

Precisely, let

$$\theta = \theta_\alpha = 2\arccos(2^{-1+1/\alpha}) \in [0, \pi/2),$$

(note that $r + r\cos\theta = \frac{1}{2}$), and define contractive similarities $\varphi_i^\alpha : \mathbf{R}^2 \to \mathbf{R}^2$ by

$$\varphi_1^\alpha(x) = rx, \quad \varphi_2^\alpha(x) = \mathrm{R}_\theta(rx) + re_1,$$
(**3.2**)
$$\varphi_3^\alpha(x) = \mathrm{R}_{-\theta}(rx) + \frac{1}{2}e_1 + r\sin\theta e_2, \quad \varphi_4^\alpha(x) = rx + (1-r)e_1,$$

where $\mathrm{R}_\theta : \mathbf{R}^2 \to \mathbf{R}^2$ is given by

$$\mathrm{R}_\theta(x_1, x_2) = (x_1\cos\theta - x_2\sin\theta, x_1\sin\theta + x_2\cos\theta).$$

The von Koch snowflake curve γ^α is the invariant set for the iterated function system $\mathscr{F}^\alpha = \{\varphi_1^\alpha, \varphi_2^\alpha, \varphi_3^\alpha, \varphi_4^\alpha\}$, i.e., the unique compact subset of $\mathbf{R}^2$ verifying

$$\gamma^\alpha = \bigcup_{i=1}^{4} \varphi_i^\alpha(\gamma^\alpha).$$

Since $\mathscr{F}^\alpha$ satisfies the open set condition [17],

$$\dim \gamma^\alpha = \frac{\log 4}{\log 1/r} = \alpha.$$

Denote by

$$\Gamma^\alpha = \bigcup_{n\in\mathbf{Z}} \left(\gamma^\alpha + ne_1\right)$$

the equivariant extension of γ^α with respect to the action of $\mathbf{Z}$ on $\mathbf{R}^2$ by translation in the first coordinate.

We write $\mathrm{S} = \{1, 2, 3, 4\}$, and we denote by $\Sigma = \mathrm{S}^{\mathbf{N}}$, respectively S^*, the space of all infinite, respectively finite, words with letters drawn from S. When Σ is endowed with the product topology arising from the discrete topology on S, the map $\pi^\alpha : \Sigma \to \gamma^\alpha$ given by

$$\pi^\alpha(w) = \lim_{m\to\infty} \varphi_{w_1}^\alpha \circ \cdots \circ \varphi_{w_m}^\alpha(0), \qquad w = (w_1, \ldots, w_m \ldots),$$

becomes a continuous map of compact sets. Thus the maps

$$f_\alpha^{\alpha'} := \pi^{\alpha'} \circ (\pi^\alpha)^{-1}$$

are well-defined homeomorphisms from γ^α to $\gamma^{\alpha'}$, $1 \le \alpha, \alpha' < 2$. We call $f_\alpha^{\alpha'}$ the *canonical homeomorphism* from γ^α to $\gamma^{\alpha'}$. Observe that $f_\alpha^{\alpha'}$ extends to a homeomorphism of Γ^α onto $\Gamma^{\alpha'}$ which is equivariant with respect to the action of $\mathbf{Z}$:

$$f_\alpha^{\alpha'}(x_1, x_2) = ([x_1], 0) + f_\alpha^{\alpha'}(x_1 - [x_1], x_2).$$

For $w = (w_1, \ldots, w_m) \in \mathrm{S}^*$ we let $\varphi_w^\alpha := \varphi_{w_1}^\alpha \circ \cdots \circ \varphi_{w_m}^\alpha$. We call the sets

$$\varphi_w^\alpha(\gamma^\alpha) + n \cdot e_1, \quad w \in \mathrm{S}^*, n \in \mathbf{Z},$$

the 4-*adic similarity pieces* of Γ^α; if the word w has length m we say that such a set is *of generation m*.

3.3. *Snowflake surfaces in* $\mathbf{R}^4$

Consider the product sets $\gamma^\alpha \times \gamma^\alpha$ and $\Gamma^\alpha \times \Gamma^\alpha$ in $\mathbf{R}^4$. Define

$$F_\alpha^{\alpha'} = f_\alpha^{\alpha'} \times f_\alpha^{\alpha'} : \Gamma^\alpha \times \Gamma^\alpha \to \Gamma^{\alpha'} \times \Gamma^{\alpha'}$$

and note that the image of $\gamma^\alpha \times \gamma^\alpha$ under $F_\alpha^{\alpha'}$ is $\gamma^{\alpha'} \times \gamma^{\alpha'}$.

3.4. *Remark.* — Observe that $\Gamma^\alpha \times \Gamma^\alpha$ and $F_\alpha^{\alpha'}$ are equivariant with respect to the action of $\mathbf{Z} \times \mathbf{Z}$ on $\mathbf{R}^4$ by translation in the first and third coordinates. That is,

$$\Gamma^\alpha \times \Gamma^\alpha + (i, 0, j, 0) = \Gamma^\alpha \times \Gamma^\alpha$$

and

$$F_\alpha^{\alpha'}(x_1 + i, x_2, x_3 + j, x_4) = (i, 0, j, 0) + F_\alpha^{\alpha'}(x_1, x_2, x_3, x_4)$$

for all $(i, j) \in \mathbf{Z} \times \mathbf{Z}$ and all $(x_1, x_2, x_3, x_4) \in \mathbf{R}^4$. For the remainder of the paper, we use the term "equivariant" to refer to this specific group action.

Our goal in this section is to prove the following proposition.

3.5. *Proposition.*

- *(a) For each* $\alpha \in (1, 2)$, $\Gamma^\alpha \times \Gamma^\alpha$ *is thick in* $\mathbf{R}^4$ *with parameters* $r_0 \equiv 1$ *and* $\beta(\alpha) \to 0$ *as* $\alpha \to 1$.
- *(b) For each* $\alpha, \alpha' \in [1, 2)$, *there exists* $C = C(\alpha, \alpha') < \infty$ *so that*

$$C^{-1}|x - y|^{\alpha/\alpha'} \le \left|F_\alpha^{\alpha'}(x) - F_\alpha^{\alpha'}(y)\right| \le C|x - y|^{\alpha/\alpha'} \tag{3.6}$$

  for all $x, y \in \Gamma^\alpha \times \Gamma^\alpha$, $|x - y| \le 1$.
- *(c) Given* $s > 0$ *and* $\alpha \in [1, 2)$, *there exists* $\delta(s, \alpha) > 0$ *so that for each* $\alpha' \in (\alpha - \delta(s, \alpha), \alpha + \delta(s, \alpha)) \cap [1, 2)$, *the canonical homeomorphism* $F_\alpha^{\alpha'}$ *is s-quasi-symmetric. Furthermore,* $\delta(s, \alpha)$ *can be chosen to be continuous in* α.

In the proof of Proposition 3.5(c) we use the following lemmas. For any $n \ge 1$, we denote by id the identity map on $\mathbf{R}^n$.

3.7. *Lemma.* — $\| \mathrm{id} - f_\alpha^{\alpha'} \|_{\Gamma^\alpha}$ *and* $\| \mathrm{id} - F_\alpha^{\alpha'} \|_{\Gamma^\alpha \times \Gamma^\alpha}$ *are continuous in* α *and* α', *and approach zero as* $\alpha' \to \alpha$.

This lemma is obvious; we omit the proof (but see Sect. 5 of [28]).

3.8. *Lemma. — Let $\alpha, \alpha' \in [1,2)$, and assume that $J \subset \Gamma^\alpha$ is either a 4-adic similarity piece of Γ^α or the union of two adjacent 4-adic similarity pieces of generation m. Then there is a similarity $h_J : \mathbf{R}^2 \to \mathbf{R}^2$ with*

$$L(h_J) = 4^{-(1/\alpha' - 1/\alpha)m} \tag{3.9}$$

so that

$$\left\| f_\alpha^{\alpha'} - h_J \right\|_J \leq \varkappa(\alpha, \alpha') L(h_J) \operatorname{diam} J, \tag{3.10}$$

where $\varkappa(\alpha, \alpha')$ can be chosen to be continuous in α and α', with $\varkappa(\alpha, \alpha') \to 0$ as $\alpha' \to \alpha$.

Proof. — The proof is essentially contained in [28]; we include a sketch for completeness. We consider two cases:

Case 1. — $J = \varphi_w^\alpha(\gamma^1) + ne_1$ for some $w \in S^*$ and $n \in \mathbf{Z}$;

Case 2. — $J = (\varphi_{w_L}^\alpha(\gamma^1) + n_L e_1) \cup (\varphi_{w_R}^\alpha(\gamma^1) + n_R e_1)$ for some $w_L, w_R \in S^*$ and $n_L, n_R \in \mathbf{Z}$, $n_L \leq n_R \leq n_L + 1$.

By equivariance, we may assume that $n = 0$ (case 1) or that $n_L = 0$ (case 2).

First, we consider case 1. Let

$$h_J = \varphi_w^{\alpha'} \circ \left(\varphi_w^\alpha\right)^{-1}$$

and observe that $L(h_J)$ is given by the formula in (3.9). Since

$$f_\alpha^{\alpha'} = \varphi_w^{\alpha'} \circ f_\alpha^{\alpha'} \circ \left(\varphi_w^\alpha\right)^{-1}$$

for all words $w \in S^*$, we have by Lemma 3.7 that

$$\left\| f_\alpha^{\alpha'} - h_J \right\|_J = (r_{\alpha'})^m \left\| f_\alpha^{\alpha'} - \mathrm{id} \right\|_{\gamma^\alpha} \leq c(\alpha, \alpha') 4^{-(1/\alpha' - 1/\alpha)m} \operatorname{diam} J$$

for some $c(\alpha, \alpha')$ continuous in α and α' with $c(\alpha, \alpha') \to 0$ as $\alpha' \to \alpha$. (Recall that $r_{\alpha'} = 4^{-1/\alpha'}$ and observe that $\operatorname{diam} J$ is comparable with $(r_\alpha)^m = 4^{-m/\alpha}$.)

Next we consider case 2. Let $J_L = \varphi_{w_L}^\alpha(\gamma^1) + n_L e_1$ and $J_R = \varphi_{w_R}^\alpha(\gamma^1) + n_R e_1$. The estimates

$$\left\| f_\alpha^{\alpha'} - \varphi_{w_L}^{\alpha'} \circ \left(\varphi_{w_L}^\alpha\right)^{-1} \right\|_{J_L} \leq c(\alpha, \alpha') 4^{-(1/\alpha' - 1/\alpha)m} \operatorname{diam} J$$

and

$$\left\| f_\alpha^{\alpha'} - \varphi_{w_R}^{\alpha} \circ \left(\varphi_{w_R}^\alpha\right)^{-1} \right\|_{J_R} \leq c(\alpha, \alpha') 4^{-(1/\alpha' - 1/\alpha)m} \operatorname{diam} J$$

hold as in case 1. Let

$$h_J = \varphi_{w_L}^{\alpha'} \circ \left(\varphi_{w_L}^\alpha\right)^{-1}. \tag{3.11}$$

We distinguish two subcases: (i) $n_R = 0$, and (ii) $n_R = 1$.

To relate w_R to w_L on J_R, we have in case 2(i),

$$\textbf{(3.12)} \qquad \left\| \varphi_{w_L}^{\alpha'} \circ \left(\varphi_{w_L}^{\alpha}\right)^{-1} - \varphi_{w_R}^{\alpha'} \circ \left(\varphi_{w_R}^{\alpha}\right)^{-1} \right\|_{J_R} \le c'(\alpha,\alpha') 4^{-(1/\alpha'-1/\alpha)m} \operatorname{diam} J$$

for some $c'(\alpha, \alpha')$ continuous in α and α' with $c'(\alpha, \alpha') \to 0$ as $\alpha' \to \alpha$. The inequality in (3.12) is a special case of an estimate which appears in a general theorem on s-quasisymmetry of the canonical homeomorphism between invariant sets of iterated function systems. See Sect. 5 of [28].

In the final case 2(ii), we have

$$\left\| \varphi_{w_L}^{\alpha'} \circ \left(\varphi_{w_L}^{\alpha}\right)^{-1} - \left(e_1 + \varphi_{w_R}^{\alpha'}\right) \circ \left(e_1 + \varphi_{w_R}^{\alpha}\right)^{-1} \right\|_{J_R} \le c'(\alpha,\alpha') 4^{-(1/\alpha'-1/\alpha)m} \operatorname{diam} J$$

by equivariance. We obtain (3.10) with $\varkappa(\alpha, \alpha') = \max\{c(\alpha, \alpha'), c'(\alpha, \alpha')\}$. This completes the proof of Lemma 3.8.

Proof of Proposition 3.5. — For part (a), see Väisälä [31, Example 6.13]. To prove (b), it suffices to show that

$$\textbf{(3.13)} \qquad \left| f_\alpha^{\alpha'}(x) - f_\alpha^{\alpha'}(y) \right| \simeq |x-y|^{\alpha/\alpha'}$$

for fixed $\alpha, \alpha' \in [1, 2)$ and all $x, y \in \Gamma^\alpha$, $|x-y| \le 1$. The notation $A \simeq B$ means that there exists a constant C, depending at most on α and α', so that $C^{-1}B \le A \le CB$.

We write $f = f_\alpha^{\alpha'}$ and let $x, y \in \Gamma^\alpha$, $|x-y| \le 1$, $x \ne y$. There is a set J which is either a 4-adic similarity piece of generation m on Γ^α or the union of two such adjacent pieces, so that $x, y \in J$ and $\operatorname{diam} J \simeq |x-y|$. Then $J' = f(J)$ is either a 4-adic similarity piece of generation m on $\Gamma^{\alpha'}$ or the union of two adjacent pieces, $f(x), f(y) \in J'$, and $\operatorname{diam} J' \simeq |f(x) - f(y)|$. Hence

$$|f(x) - f(y)| \simeq \operatorname{diam} J' \simeq (r_{\alpha'})^m = (r_\alpha)^{m\alpha/\alpha'} \simeq (\operatorname{diam} J)^{\alpha/\alpha'} \simeq |x-y|^{\alpha/\alpha'}.$$

This completes the proof of (3.13).

To verify part (c), we will show that the hypotheses of Theorem 2.4 hold for $f = F_\alpha^{\alpha'}$ and $X = \Gamma^\alpha \times \Gamma^\alpha$, with $\varkappa \to 0$ as $\alpha' \to \alpha$. Thus consider the canonical homeomorphism $F_\alpha^{\alpha'} : \Gamma^\alpha \times \Gamma^\alpha \to \Gamma^{\alpha'} \times \Gamma^{\alpha'}$. Suppose that A is a bounded set in $\Gamma^\alpha \times \Gamma^\alpha$. If $\operatorname{diam} A \ge 1$ then the hypotheses of Theorem 2.4 hold with $h = \mathrm{id}$, if α' is sufficiently close to α. See Lemma 3.7. Otherwise, choose $J_1 \times J_2 \supset A$ subject to the following constraints:

(i) each factor J_i, $i = 1, 2$, is either a 4-adic similarity piece of Γ^α or the union of two such 4-adic similarity pieces, and all of the similarity pieces comprising J_1 and J_2 are of the same generation;
(ii) $\operatorname{diam} J_1 \times J_2 \le 8\sqrt{2} \sec\theta_\alpha \operatorname{diam} A$.

By Lemma 3.8 there exist similarities $h_{J_1}, h_{J_2} : \mathbf{R}^2 \to \mathbf{R}^2$ with $L(h_{J_1}) = L(h_{J_2}) =: L$ and

$$(\mathbf{3.14}) \qquad \left\| f_\alpha^{\alpha'} - h_{J_i} \right\|_{J_i} \leq \varkappa(\alpha, \alpha') L \operatorname{diam} J_i$$

for some $\varkappa(\alpha, \alpha')$ continuous in α and α' with $\varkappa(\alpha, \alpha') \to 0$ as $\alpha' \to \alpha$. The product $h = h_{J_1} \times h_{J_2}$ is a similarity mapping of $\mathbf{R}^4$ with $L(h) = L$, and

$$\begin{aligned} \left\| F_\alpha^{\alpha'} - h \right\|_{J_1 \times J_2} &\leq \sqrt{2} \cdot \max\left\{ \left\| f_\alpha^{\alpha'} - h_{J_1} \right\|_{J_1}, \left\| f_\alpha^{\alpha'} - h_{J_2} \right\|_{J_2} \right\} \\ &\leq 16 \sec \theta_\alpha \varkappa(\alpha, \alpha') L(h) \operatorname{diam} A \end{aligned}$$

by (3.14) and constraint (ii) above. By Theorem 2.4, $F_\alpha^{\alpha'}$ is s-QS for α' sufficiently close to α, with $s \to 0$ as $\alpha' \to \alpha$. The existence of $\delta(s, \alpha) > 0$ as in the statement is obvious. Continuity of $\delta(s, \alpha)$ in α follows from continuity of $\varkappa(\alpha, \alpha')$ and continuity of the function $s(\varkappa)$ in Theorem 2.4.

4. Quasiconformal extension of equivariant s-quasisymmetric maps

For $\alpha \in [1, 2)$ we consider the snowflake surface $\Gamma^\alpha \times \Gamma^\alpha$ from the previous section. Set

$$I = [0, 1] \times [-2, 2] \times [0, 1] \times [-2, 2],$$

$$J = \mathbf{R} \times (-1, 1) \times \mathbf{R} \times (-1, 1),$$

and

$$A^\alpha = \Gamma^\alpha \times \Gamma^\alpha \cap [-4, 4] \times \mathbf{R} \times [-4, 4] \times \mathbf{R}.$$

Observe that $\Gamma^\alpha \times \Gamma^\alpha \subset \mathbf{R} \times [0, \sqrt{2}/2] \times \mathbf{R} \times [0, \sqrt{2}/2]$ and that $A^\alpha \subset B(0, 6)$.

Fix a Whitney triangulation $\mathscr{W}^\alpha$ of $\mathbf{R}^4 \setminus \Gamma^\alpha \times \Gamma^\alpha$ and a Whitney triangulation $\widetilde{\mathscr{W}}^\alpha$ of $\mathbf{R}^4 \setminus A^\alpha$ constructed by the procedures in Sect. 2. We require in addition the following:

- The interior of any n-simplex in $\mathscr{W}^\alpha$ or $\widetilde{\mathscr{W}}^\alpha$ does not meet ∂I;
- (Common triangulations for $\mathbf{R}^4 \setminus \Gamma^\alpha \times \Gamma^\alpha$ and $\mathbf{R}^4 \setminus A^\alpha$ near the origin) For each $\alpha \in [1, 2)$ and each n-simplex $\sigma \subset I$, we have $\sigma \in \mathscr{W}^\alpha$ if and only if $\sigma \in \widetilde{\mathscr{W}}^\alpha$.
- (Equivariant simplices) For each $\alpha \in [1, 2)$, if $\sigma \in \mathscr{W}^\alpha$ is an n-simplex with $\sigma \subset I$, then $\sigma + (i, 0, j, 0) \in \mathscr{W}^\alpha$ for all $i, j \in \mathbf{Z}$, and if $\sigma \in \widetilde{\mathscr{W}}^\alpha$ is an n-simplex with $\sigma \subset I$, then $\sigma + (i, 0, j, 0) \in \widetilde{\mathscr{W}}^\alpha$ for all integers $-2 \leq i, j \leq 1$;

– (Congruent simplices away from the snowflake surface) The triangulations $\mathscr{W}^\alpha$, $1 \le \alpha < 2$, all contain a common subcollection $\mathscr{W}^*$ of n-simplices satisfying

$$\bigcup\{\sigma : \sigma \in \mathscr{W}^*\} = \mathbf{R}^4 \setminus \mathrm{J}.$$

4.1. *Extension*

By Theorem 2.7, $\Gamma^1 \times \Gamma^1 = \mathbf{R}^2$ has the QSEP in $\mathbf{R}^4$. When $1 < \alpha < 2$, $\Gamma^\alpha \times \Gamma^\alpha$ is thick in $\mathbf{R}^4$. Since neither $\Gamma^\alpha \times \Gamma^\alpha$ nor its complement is bounded, Theorem 2.8 does not apply directly. However, the equivariance of the snowflake surfaces $\Gamma^\alpha \times \Gamma^\alpha$ and the corresponding canonical homeomorphisms $\mathrm{F}_\alpha^{\alpha'}$ substitutes for the assumption of boundedness. It suffices to establish the following proposition.

4.2. *Proposition. — For each* $\alpha \in [1, 2)$ *there exists* $s_0 = s_0(\alpha) > 0$ *so that every equivariant* s*-QS map* $f : \Gamma^\alpha \times \Gamma^\alpha \to \mathbf{R}^4$ *with* $0 < s \le s_0$ *has an equivariant* s_1*-QS extension* $g : \mathbf{R}^4 \to \mathbf{R}^4$*, where* $s_1 = s_1(s, \alpha) \to 0$ *as* $s \to 0$*. In fact,* g *is* K*-QC, with* $\mathrm{K} = \mathrm{K}(s, \alpha) \to 1$ *as* $s \to 0$.

Recall that the term "equivariant" refers to the specific group action in Remark 3.4.

In addition to the conclusion of Proposition 4.2, the process of constructing g and specific estimates which arise therein, play an essential role in our proof of Theorem 1.5. We emphasize that aside from the use of equivariance, the ideas in the following construction, and in particular, in the proof of Lemma 4.3, are due to Tukia and Väisälä. We follow closely the steps and notation from [27] and [31], and choose sets and constants that are, while not always the same, at least comparable to those from these references.

Fix $\alpha \in [1, 2)$. Following [27] and [31], we introduce an auxiliary parameter $q > 0$. To prove the extension property in Proposition 4.2, it suffices to find $q_0(\alpha) \in (0, 1)$ and, for each $q \in (0, q_0(\alpha)]$ a number $s = s(q, \alpha) > 0$ so that every equivariant s-QS map $f : \Gamma^\alpha \times \Gamma^\alpha \to \mathbf{R}^4$ has an equivariant s_1-QS extension $g : \mathbf{R}^4 \to \mathbf{R}^4$ with $s_1 = s_1(q, \alpha) > 0$ and $s(q, \alpha) \le s_1(q, \alpha) \to 0$ as $q \to 0$.

We will prove the following lemma on extension of the $s(q, \alpha)$-quasisymmetric map $f : \Gamma^\alpha \times \Gamma^\alpha \to \mathbf{R}^4$. Here and henceforth we abbreviate

$$d^\alpha(x) := \mathrm{dist}(x, \Gamma^\alpha \times \Gamma^\alpha).$$

4.3. *Lemma. — Let* $0 < q < 1/10$*, let* $b = q^{-1/3}$ *if* $\alpha > 1$ *and* $b = 20$ *if* $\alpha = 1$*, and assume that* $s = s(q, \alpha)$ *is sufficiently small. Then to each* $x \in \mathbf{R}^4 \setminus \Gamma^\alpha \times \Gamma^\alpha$ *there corresponds a similarity* $h_x^\alpha : \mathbf{R}^4 \to \mathbf{R}^4$ *so that the estimates*

$$\left\| h_x^\alpha - h_y^\alpha \right\|_{\mathrm{B}(y, d^\alpha(y))} \le \mathrm{M}(\alpha) q^{2/3} d^\alpha(x) \mathrm{L}\big(h_x^\alpha\big) \qquad (\mathbf{4.4})$$

and

$$\text{(4.5)} \qquad \mathrm{L}\big(h_y^\alpha\big)d^\alpha(y) \le \mathrm{M}(\alpha)b\mathrm{L}\big(h_x^\alpha\big)d^\alpha(x)$$

hold for all $x, y \in \mathbf{R}^4 \setminus \Gamma^\alpha \times \Gamma^\alpha$ *satisfying* $|y - x| < bd^\alpha(x)$. *Moreover, there exists an extension* g *of* f *to* $\mathbf{R}^4$ *so that* g *is affine on each n-simplex in* $\mathscr{W}^\alpha$, *and that for each* $x \in \mathbf{R}^4 \setminus \Gamma^\alpha \times \Gamma^\alpha$ *and each n-simplex* σ *in* $\mathscr{W}^\alpha$ *containing* x,

$$\text{(4.6)} \qquad \big\| g - h_x^\alpha \big\|_{\mathrm{B}(x, bd^\alpha(x))} \le \mathrm{M}(\alpha)q^{2/3}bd^\alpha(x)\mathrm{L}\big(h_x^\alpha\big)$$

and

$$\text{(4.7)} \qquad \big\| g - h_x^\alpha \big\|_\sigma \le \mathrm{M}(\alpha)q^{2/3}r(\sigma)\mathrm{L}\big(h_x^\alpha\big),$$

where $r(\sigma)$ *is the radius of the largest ball contained in* σ. *Here* $\mathrm{M}(\alpha) > 1$ *denotes a constant depending only on* α.

Compare [27, pp. 165–169] and [31, pp. 264–268].

Proof. — First, consider the case $1 < \alpha < 2$. Let $0 < q < 1/10$ and $b = q^{-1/3}$. We require s sufficiently small so that $\varkappa(s, 4) < q^2/12$ in Theorem 2.3. After replacing f by $h^{-1} \circ f$ for a suitable similarity h, we get

$$\| \mathrm{id} - f \|_{\mathrm{A}^\alpha} < q^2.$$

Assign, to each $x \in \mathbf{R}^4 \setminus \Gamma^\alpha \times \Gamma^\alpha$, a set $\mathrm{Q}(x) := \mathrm{B}(x, bd^\alpha(x)) \cap \Gamma^\alpha \times \Gamma^\alpha$ and a similarity $h_x^\alpha : \mathbf{R}^4 \to \mathbf{R}^4$ so that $h_x^\alpha = \mathrm{id}$ if $d^\alpha(x) \ge q$ and

$$\big\| h_x^\alpha - f \big\|_{\mathrm{Q}(x)} \le \varkappa(s)\mathrm{L}\big(h_x^\alpha\big) \operatorname{diam} \mathrm{Q}(x)$$

if $d^\alpha(x) < q$. We require in addition that

$$\text{(4.8)} \qquad h_x^\alpha \text{ is equivariant in } x,$$

i.e., $h_x^\alpha = h_{x'}^\alpha$ whenever $x' - x = (i, 0, j, 0)$ for some integers i and j. The existence of h_x^α follows from Theorem 2.3, the equivariance of the map f and the simplices in $\mathscr{W}^\alpha$, and the small value of q. We write $\varkappa(s)$ in place of $\varkappa(s, 4)$.

Define an extension g of f as follows. At each vertex v of an n-simplex $\sigma \in \mathscr{W}^\alpha$, set

$$g(v) = h_v^\alpha(v),$$

extend g to $\mathbf{R}^4 \setminus \Gamma^\alpha \times \Gamma^\alpha$ so that g is affine on each n-simplex in $\mathscr{W}^\alpha$, and let $g|_{\Gamma^\alpha \times \Gamma^\alpha} = f$. Observe that $g = \mathrm{id}$ on $\{d^\alpha > 2q\} \supset \mathbf{R}^4 \setminus \mathrm{J}$, and that g is equivariant.

The arguments of Väisälä for Theorem 2.8 cannot be applied directly to establish the quasiconformality of g since $\Gamma^\alpha \times \Gamma^\alpha$ is unbounded. However, the equivariance of the construction permits us to restrict our attention to the bounded set $A^\alpha \subset \Gamma^\alpha \times \Gamma^\alpha$, where the results of Theorem 2.8 can be applied.

Let $\tilde{f} = f|_{A^\alpha}$. Extend $\tilde{f}$ from A^α to $\mathbf{R}^4$ by the same procedure. To each $x \in \mathbf{R}^4 \setminus A^\alpha$, assign a set $\tilde{Q}(x) = B(x, bd^\alpha(x)) \cap A^\alpha$ and a similarity $\tilde{h}^\alpha_x : \mathbf{R}^4 \to \mathbf{R}^4$ so that $\tilde{h}^\alpha_x = \mathrm{id}$ if $\mathrm{dist}(x, A^\alpha) \geq q$ and

$$\left\| \tilde{h}^\alpha_x - f \right\|_{\tilde{Q}(x)} \leq \varkappa(s) L\left(\tilde{h}^\alpha_x\right) \operatorname{diam} \tilde{Q}(x)$$

if $\mathrm{dist}(x, A^\alpha) < q$. The common triangulations for $\mathbf{R}^4 \setminus \Gamma^\alpha \times \Gamma^\alpha$ and $\mathbf{R}^4 \setminus A^\alpha$ in I permit us to require in addition that

$$\tilde{h}^\alpha_x = h^\alpha_x \tag{4.9}$$

for all x with $\max\{|x|_1, |x|_3\} \leq 2$.

As in the previous paragraph, define $\tilde{g}(v) = \tilde{h}^\alpha_v(v)$ at each vertex v of an n-simplex $\sigma \in \widetilde{\mathscr{W}}^\alpha$, and extend $\tilde{g}$ affinely to all of σ. Conditions (4.8) and (4.9) ensure that

$$\tilde{g}(x_1 + i, x_2, x_3 + j, x_4) = \tilde{g}(x_1, x_2, x_3, x_4) + (i, 0, j, 0) \tag{4.10}$$

whenever $0 \leq x_1, x_3 \leq 1$ and $i, j \in \{-2, -1, 0, 1\}$, and that

$$g(x_1, x_2, x_3, x_4) = \tilde{g}(x_1, x_2, x_3, x_4) \tag{4.11}$$

whenever $\max\{|x_1|, |x_3|\} \leq 2$.

Because A^α is thick and bounded, it follows from the proof of Theorem 2.8 in [31] (see, in particular, the proofs of equations (6.3), (6.7), (6.8) and (6.9) therein) that functions $M(\alpha)$ and $s = s(q, \alpha)$ can be chosen so that the estimates (4.4)–(4.7) hold for $x, y \in \mathbf{R}^4 \setminus A^\alpha$, with $\tilde{h}^\alpha_x$, $\tilde{h}^\alpha_y$, $\tilde{g}$ and $\mathrm{dist}(x, A^\alpha)$ in place of h^α_x, h^α_y, g and $d^\alpha(x)$. Then we obtain (4.4)–(4.7) as stated for $\alpha \in (1, 2)$ and sufficiently small s by (4.11). The function $M(\alpha)$ may be taken to be continuous and decreasing in the thickness constant $\beta(\alpha)$, thus in α, while $s(q, \alpha)$ may be taken to be continuous in α, and increasing in q for fixed α.

Now assume $\alpha = 1$. We follow the notation and constructions from Sect. 5 of [27]. Assign, to each dyadic square $Q \subset \mathbf{R}^2$, a similarity $u_Q : \mathbf{R}^2 \to \mathbf{R}^4$ so that

$$\|u_Q - f\|_Q \leq \varkappa(s) L(u_Q) \operatorname{diam} Q;$$

the existence of $\varkappa(s)$ follows from Theorem 2.3. When $\{u_Q\}$ are chosen following certain additional rules, they can be extended to similarities $\{h_Q\}$ from $\mathbf{R}^4$ to $\mathbf{R}^4$ so that

$$\left\| h^1_Q - h^1_{Q'} \right\|_{Z_Q} \leq \epsilon(s) L(u_Q) \operatorname{diam} Q$$

for all dyadic squares Q and Q' satisfying $Q \cap Q' \neq \emptyset$ and

$$1/2 \leq \operatorname{diam} Q / \operatorname{diam} Q' \leq 2,$$

where Z_Q is the cube in $\mathbf{R}^4$ concentric with Q of diameter $b \operatorname{diam} Q$ and edges parallel to the coordinate axes. (To avoid lengthy definition, the set Z_Q described in the previous sentence is not the same as that in [27, p. 168], however, they both contain Q in their center half and have comparable diameters.) The function $\epsilon(s)$ is derived from $\varkappa(s)$ and satisfies $\lim_{s\to 0} \epsilon(s) = 0$. Proper choice of the planar similarities $\{u_Q\}$ and their extensions to similarities $\{h_Q\}$ on $\mathbf{R}^4$ requires considerable work. See [27, pp. 159–163]. Finally, to each $x \in \mathbf{R}^4 \setminus \mathbf{R}^2$, assign a dyadic square $Q(x) \subset \mathbf{R}^2$ following certain rules (as in pp. 164–169 of [27]). In particular, we require

$$1 \leq \frac{\operatorname{diam} Q(x)}{\operatorname{dist}(x, \mathbf{R}^2)} \leq 2$$

for all such x. Set $h_x^1 = h_{Q(x)}$.

As before, define $g(v) = h_v^1(v)$ at each vertex v on an n-simplex $\sigma \in \mathscr{W}^1$, extend g to be affine on each such σ, and let $g = f$ on $\mathbf{R}^2$. Now it follows from the proof of Theorem 2.7 in [27] (see in particular equations (5.9) and (5.10)) that $s = s(q, 1)$ and hence also $\varkappa(s)$ and $\epsilon(s)$ can be chosen small enough so that (4.4), (4.6) and (4.7) hold with $\alpha = 1$. The proof of the first inequality in [31, (6.7)] can be reproduced to give (4.5). This completes the proof of Lemma 4.3.

4.12. *Constants*

Let us pause to discuss the dependence of the various constants that have arisen. Recall that the constants C_i, $1 \leq i \leq 5$, from Remark 2.10, (2.13), (2.14), (2.17), (2.20) and (2.21) depend only on the dimension $n = 4$, and hence are absolute constants.

When $1 < \alpha < 2$, $M(\alpha)$ depends on $n = 4$, on the diameter of A^α, and on the numbers $r_0 = 1$ and $\beta(\alpha)$ describing the thickness of $\Gamma^\alpha \times \Gamma^\alpha$. It is important to observe that $M(\alpha)$ depends only on α, and is continuous and decreasing on the interval $(1, 2)$. In fact $M(\alpha) \to \infty$ as $\alpha \to 1$.

The constant $M(1)$ is derived from a different argument (Theorem 2.7) where thickness is not used. In fact, $M(1)$ depends on the dimension $n = 4$ only, and hence is an absolute constant.

We choose

$$(\mathbf{4.13}) \qquad q_0(\alpha) = \min\left\{\left(10^{10} C_1 C_4 C_5 M(\alpha)\right)^{-6}, \left(K_0(4)^{1/6} - 1\right)^3 \left(10^6 C_1 C_4 M(\alpha)\right)^{-3}\right\}$$

continuous and increasing in α, and assume from now on that $0 < q < q_0(\alpha)$, thus

$$(\mathbf{4.14}) \qquad 10^{10}C_1C_4C_5M(\alpha)q^{1/3} < q^{1/6}$$

and

$$(\mathbf{4.15}) \qquad \left(1 + 10^6C_1C_4M(\alpha)q^{1/3}\right)^6 < K_0(4).$$

Here $K_0(n)$ is the constant in Theorem 5.3.

We reiterate that the functions $s(q,\alpha)$ and $s_1(q,\alpha)$ are chosen to be continuous with respect to α and increasing with respect to q, with $s(q,\alpha) \le s_1(q,\alpha) \to 0$ as $q \to 0$. For fixed α, the inverse $q = q(s,\alpha)$ of $s = s(q,\alpha)$ is an increasing function of s, with $q(s,\alpha) \to 0$ as $s \to 0$. We define

$$(\mathbf{4.16}) \qquad s_0(\alpha) = s(q_0(\alpha),\alpha)$$

and observe that $s_0(\alpha)$ is continuous on the interval $(1,2)$ by the preceding remarks.

4.17. *Lemma. — The map g defined in Lemma 4.3 is a sense preserving quasiconformal map of $\mathbf{R}^4$, whose linear dilatation* $H(g)$ *satisfies*

$$(\mathbf{4.18}) \qquad H(g) \le (1 + 2M(\alpha)q^{2/3})^2$$

a.e. in $\mathbf{R}^4$.

For the definition of the linear dilatation, see [29, Definition 22.2].

Lemma 4.17 for g (when $\alpha = 1$) and for $\tilde{g}$ (when $1 < \alpha < 2$) can be proved by following the proofs of Theorems 2.7 and 2.8, together with Theorem 2.5. Lemma 4.17 then holds for g (when $1 < \alpha < 2$) by the equivariance, and the fact that g and $\tilde{g}$ coincide on $\{x : |x_1| \le 2 \text{ and } |x_3| \le 2\}$.

Let us sketch the proof of Lemma 4.17. As in the proofs of Theorems 2.7 and 2.8, the continuity of g is straightforward, and the sense preservation, injectivity and surjectivity can be deduced from the estimates in Lemma 4.3 and topological considerations.

On each n-simplex $\sigma \in \mathscr{W}^\alpha$, g is affine. It follows from (4.7) and Lemma 4.19 below that the linear dilatation $H(g)(x)$ is bounded by $(1 + 2M(\alpha)q^{2/3})^2$ on the union of the interiors of the n-simplices in $\mathscr{W}^\alpha$. Observing that the set $\cup\{\partial\sigma : \sigma \in \mathscr{W}^\alpha\}$ has σ-finite 3-dimensional measure, it follows from a removability theorem (Theorem 35.1 and Remark 34.2 of [29]) that g is quasiconformal in $\mathbf{R}^4 \setminus \Gamma^\alpha \times \Gamma^\alpha$. Hence g is quasiconformal in $\mathbf{R}^4$ when $\alpha = 1$, since $\mathbf{R}^2$ is removable. The verification of the boundedness of $H(g)$ when $1 < \alpha < 2$ requires some work; see the proof of a similar theorem in Väisälä [31, p. 260]. In all cases we deduce that g is quasiconformal on all of $\mathbf{R}^4$. This completes the proof of Lemma 4.17.

Proposition 4.2 follows from Lemma 4.17 and Theorem 2.5.

In the proof of Lemma 4.17 the following result was used.

4.19. *Lemma.* — *Let h be a similarity and let g be an affine map. If*

$$\|g - h\|_{B(x,r)} < \lambda r L(h),$$

for some $0 < \lambda < 1/100$, *then*

$$|g' - h'| \leq \lambda L(h)$$

and

$$H(g) \leq (1 + 2\lambda)^2,$$

moreover, g is sense-preserving if and only if h is sense-preserving.

Here we denote by $|A| := \sup_{v:|v|=1} |Av|$ the operator norm of a matrix A. In what follows, we will also make use of the quantity

$$\ell(A) := \inf_{v:|v|=1} |Av|.$$

Proof. — Note that $|g(x+y) - h(x+y)| < \lambda r L(h)$ when $|y| < r$. Replacing y by $-y$ and subtracting, we obtain by linearity the estimate $2|g'(y) - h'(y)| \leq 2\lambda r L(h)$ when $|y| < r$. Thus $|g' - h'| \leq \lambda L(h)$. The remaining statements can be found in [27, 3.5] and [31, 2.7].

4.20. *Estimates*

We now derive from Lemma 4.3 a few estimates for the extension g; some of these have been implicitly used in [27] and [31].

4.21. *Lemma.* — *For almost every x and y in $\mathbf{R}^4 \setminus \Gamma^\alpha \times \Gamma^\alpha$, the following estimates hold:*

(i) $|(h_x^\alpha)' - (h_y^\alpha)'| \leq 2M(\alpha)q^{2/3}L(h_x^\alpha)$ *and*

$$1 - 2M(\alpha)q^{2/3} \leq L\big(h_y^\alpha\big)/L(h_x^\alpha) \leq 1 + 2M(\alpha)q^{2/3}$$

if $|y - x| \leq d^\alpha(x)/2$;

(ii) $|g'(x) - (h_x^\alpha)'| \leq M(\alpha)q^{2/3}L(h_x^\alpha)$ *and*

$$1 - M(\alpha)q^{2/3} \leq \frac{\ell(g'(x))}{L\big(h_x^\alpha\big)} \leq \frac{|g'(x)|}{L\big(h_x^\alpha\big)} \leq 1 + M(\alpha)q^{2/3};$$

(iii) $|g'(x) - g'(y)| \leq 5M(\alpha)q^{2/3}L(h_x^\alpha)$ *if* $|y - x| \leq d^\alpha(x)/2$;

(iv)

$$1 - 4M(\alpha)q^{2/3} \le \frac{|g(x) - g(y)|}{|x-y|L(h_x^\alpha)} \le 1 + 4M(\alpha)q^{2/3}$$

if $|y - x| \le d^\alpha(x)/2$, *and*

$$|g(x) - g(y)| \le (|x-y| + 2M(\alpha)q^{2/3}bd^\alpha(x))L(h_x^\alpha)$$

if $|y - x| \le bd^\alpha(x)$;

(v)

$$\left(\frac{1}{2} - 3M(\alpha)q^{2/3}\right) \le \frac{\operatorname{dist}(g(x), f(\Gamma^\alpha \times \Gamma^\alpha))}{d^\alpha(x)L(h_x^\alpha)} \le (1 + 2M(\alpha)b\,q^{2/3}).$$

Proof. — Recall that $b = q^{-1/3}$ when $\alpha > 1$ and $b = 20$ when $\alpha = 1$. Assume that x and y are not on the boundary of any n-simplex in $\mathscr{W}^\alpha$. Thus (i) follows from (4.4) and Lemma 4.19, (ii) follows from (4.7) and Lemma 4.19, and (iii) follows from (i), (ii) and the triangle inequality.

If $|y - x| < d^\alpha(x)/2$, then

$$\begin{aligned}&\left|g(x) - g(y) - \left(h_x^\alpha\right)'(x-y)\right| \\ &\qquad \le \max_{z\in[x,y]} \left(\left|g'(z) - \left(h_z^\alpha\right)'\right| + \left|\left(h_z^\alpha\right)' - \left(h_x^\alpha\right)'\right|\right)|x-y|,\end{aligned}$$

which together with (i) and (ii) gives the first part of (iv). The second part of (iv) follows from the triangle inequality, and (4.6).

Let z be a point in $\Gamma^\alpha \times \Gamma^\alpha$ with $|x - z| = d^\alpha(x)$. Then

$$|g(x) - f(z)| = |g(x) - g(z)| \le (1 + 2M(\alpha)bq^{2/3})d^\alpha(x)L(h_x^\alpha)$$

by (iv); this proves the right hand inequality in (v). On the other hand, for any w with $|w - x| = d^\alpha(x)/2$,

$$|g(x) - g(w)| \ge \left(\frac{1}{2} - 3M(\alpha)q^{2/3}\right) d^\alpha(x)L(h_x^\alpha)$$

by (ii) and (iii); this proves the left hand inequality in (v).

5. Smoothing

For $\alpha \in [1, 2)$, let $A = \Gamma^\alpha \times \Gamma^\alpha$ and choose and fix a regularized distance function $\delta^\alpha = \delta_A$ to A which satisfies properties (2.12)–(2.14). Define

$$G(x) = \begin{cases} \delta^\alpha(x)^{-4} \int_{\mathbf{R}^4} g(y)\varphi\left(\frac{x-y}{\delta^\alpha(x)}\right) dy, & \text{on } \mathbf{R}^4 \setminus \Gamma^\alpha \times \Gamma^\alpha, \\ g(x), & \text{on } \Gamma^\alpha \times \Gamma^\alpha, \end{cases}$$

where g is the extension in the previous section.

5.1. *Lemma.* — *For almost every x and y in $\mathbf{R}^4 \setminus \Gamma^\alpha \times \Gamma^\alpha$, the following estimates hold:*

(i) $|G'(x) - g'(x)| \le 12 \cdot 10^4 C_1 C_4 M(\alpha) q^{1/3} L(h_x^\alpha)$;

(ii) $|G(x) - g(x)| \le 3 \cdot 10^4 C_1 C_4 M(\alpha) q^{1/3} d(x) L(h_x^\alpha)$;

(iii)

$$\frac{1}{2} - 10^5 C_1 C_4 M(\alpha) q^{1/3} \le \frac{\operatorname{dist}(G(x), f(\Gamma^\alpha \times \Gamma^\alpha))}{d^\alpha(x) L\left(h_x^\alpha\right)} \le 1 + 10^5 C_1 C_4 M(\alpha) q^{1/3}$$

and

$$1 - 10^6 C_1 C_4 M(\alpha) q^{1/3} \le \frac{\ell(G'(x))}{L\left(h_x^\alpha\right)} \le \frac{|G'(x)|}{L\left(h_x^\alpha\right)} \le 1 + 10^6 C_1 C_4 M(\alpha) q^{1/3};$$

(iv)

$$|G'(x) - G'(y)| \le 10^9 C_1^2 C_5 M(\alpha) |x - y| L\left(h_x^\alpha\right) / d^\alpha(x)$$

and

$$|G'(x)^{-1} - G'(y)^{-1}| \le 4 \cdot 10^9 C_1^2 C_5 M(\alpha) |x - y| / \left(L\left(h_x^\alpha\right) d^\alpha(x)\right)$$

if $|x - y| \le d^\alpha(x)/2$;

(v)

$$1 - 3 \cdot 10^5 C_1 C_4 M(\alpha) q^{1/3} \le \frac{|G(x) - G(y)|}{|x - y| L\left(h_x^\alpha\right)} \le 1 + 3 \cdot 10^5 C_1 C_4 M(\alpha) q^{1/3}$$

if $|x - y| \le d^\alpha(x)/2$.

Proof. — Since α is fixed, we write d, δ, h_x for d^α, δ^α and h_x^α. Suppose that x is in the interior of some n-simplex σ in $\mathscr{W}^\alpha$. Then $g^\sigma := g|_\sigma$ is an affine map, and we write $g^\sigma(z) = Bz + b$. By Lemma 2.18,

$$\begin{aligned} G(x) - g(x) &= G(x) - Bx - b \\ &= \delta^{-4}(x) \int_{\mathbf{R}^4} (g(y) - By - b) \varphi\left(\frac{x - y}{\delta(x)}\right) dy. \end{aligned}$$

Write $G = (G_1, G_2, G_3, G_4)$ and $g = (g_1, g_2, g_3, g_4)$. Then

$$\max_{i,j=1,\dots,4} \left| \frac{\partial}{\partial x_j} (G_i - g_i)(x) \right| \le \frac{C_4}{\delta(x)} \operatorname{Osc}(g - B - b, x, \delta(x))$$

by Proposition 2.19. Note that $g(x) = g^\sigma(x) = \mathrm{B}x + b$, so

$$\begin{aligned}|(g(y) - \mathrm{B}y - b) - (g(x) - \mathrm{B}x - b)| &= |g(y) - g^\sigma(y)|\\ &\le |g(y) - h_x(y)| + \left|h_x(y) - g^\sigma(y)\right|\end{aligned}$$

for $y \in \mathrm{B}(x, \delta(x))$. We get $|g(y) - h_x(y)| \le \mathrm{M}(\alpha)q^{1/3}d(x)\mathrm{L}(h_x^\alpha)$ from Lemma 4.3, and

$$\begin{aligned}\left|h_x(y) - g^\sigma(y)\right| &\le \left|h_x(y) - g^\sigma(y) - h_x(x) + g^\sigma(x)\right| + \left|h_x(x) - g^\sigma(x)\right|\\ &\le |h_x' - g'(x)| \cdot |y - x| + |h_x(x) - g(x)|\\ &\le \mathrm{M}(\alpha)q^{2/3}\delta(x)\mathrm{L}(h_x) + \mathrm{M}(\alpha)q^{1/3}d(x)\mathrm{L}(h_x)\\ &\le 2\mathrm{M}(\alpha)q^{1/3}d(x)\mathrm{L}(h_x)\end{aligned}$$

by Lemma 4.3 and Lemma 4.21(ii). Since $d(x) < 10^4\mathrm{C}_1\delta(x)$,

$$\left|\frac{\partial}{\partial x_j}(\mathrm{G}_i - g_i)(x)\right| \le 3 \cdot 10^4\mathrm{C}_1\mathrm{C}_4\mathrm{M}(\alpha)q^{1/3}\mathrm{L}(h_x).$$

This proves (i).

Again assume $x \in \sigma$ for some n-simplex $\sigma \in \mathscr{W}^\alpha$. By Remark 2.15, σ contains a ball $\mathrm{B}(x_\sigma, 2\delta(x))$. Since g is affine on σ, $\mathrm{G}(x_\sigma) = g(x_\sigma)$ (Lemma 2.18), and $|x - x_\sigma| \le \operatorname{diam}\sigma \le d(x)/4$, (i) implies that

$$\begin{aligned}|\mathrm{G}(x) - g(x)| &= |\mathrm{G}(x) - g(x) - (\mathrm{G}(x_\sigma) - g(x_\sigma))|\\ &\le 3 \cdot 10^4\mathrm{C}_1\mathrm{C}_4\mathrm{M}(\alpha)q^{1/3}\mathrm{L}(h_x)d(x).\end{aligned}$$

This proves (ii).

The first part of (iii) follows from (ii) and Lemma 4.21(v). The second part follows from (i) and Lemma 4.21(ii):

$$|\mathrm{G}'(x) - h_x'| \le |\mathrm{G}'(x) - g'(x)| + |g'(x) - h_x'| \le 10^6\mathrm{C}_1\mathrm{C}_4\mathrm{M}(\alpha)q^{1/3}\mathrm{L}\big(h_x^\alpha\big).$$

To prove (iv), we use the second derivative estimates for the convolution in Proposition 2.19 together with the second part of Lemma 4.21(iv) and (4.14). We get

$$\begin{aligned}\left|\frac{\partial}{\partial x_j}\mathrm{G}_i(x) - \frac{\partial}{\partial x_j}\mathrm{G}_i(y)\right| &\le \frac{\mathrm{C}_5}{\delta^2(x)}\operatorname{Osc}(g, x, 3d(x)/4)|x - y|\\ &\le \frac{\mathrm{C}_5}{\delta^2(x)}\left(\frac{3}{2}d(x) + 2\mathrm{M}(\alpha)bq^{2/3}d(x)\right)\mathrm{L}(h_x)|x - y|\\ &\le 2 \cdot 10^8\mathrm{C}_1^2\mathrm{C}_5|x - y|\mathrm{L}(h_x)/d(x),\end{aligned} \tag{5.2}$$

which gives the first inequality in (iv).

The second inequality follows from the first together with the second part of (iii) and Lemma 4.21(i):

$$\begin{aligned}|G'(x)^{-1} - G'(y)^{-1}| &= |G'(x)^{-1}(G'(y) - G'(x))G'(y)^{-1}| \\ &\leq \frac{|G'(y) - G'(x)|}{\ell(G'(x)) \cdot \ell(G'(y))} \\ &\leq 4 \cdot 10^9 C_1^2 C_5 M(\alpha) \frac{|x-y|}{L(h_x)d(x)}.\end{aligned}$$

Finally, (v) follows from (i) and Lemma 4.21(iv).

To verify the quasiconformality of G, we use some results from the general theory of quasiregular maps. Theorem 5.3 is due to Gol'dshteĭn and Martio–Rickman–Väisälä, and Theorem 5.4 is due to Zorich. See [20, Theorem VI.8.14 and Corollary III.3.8].

5.3. *Theorem (Martio–Rickman–Väisälä, Gol'dshteĭn).* — *For each* $n \geq 3$, *there exists a constant* $K_0(n) > 1$ *so that every nonconstant* $K_0(n)$*-quasiregular map* $f : U \to \mathbf{R}^n$, U *a domain in* $\mathbf{R}^n$, *is a local homeomorphism.*

5.4. *Theorem (Zorich).* — *Each locally homeomorphic quasiregular map* $f : \mathbf{R}^n \to \mathbf{R}^n$, $n \geq 3$, *is a homeomorphism, hence quasiconformal.*

5.5. *Proposition.* — G *is a quasiconformal homeomorphism on* $\mathbf{R}^4$, *and* G *and* G^{-1} *are* C^∞ *on* $\mathbf{R}^4 \setminus \Gamma^\alpha \times \Gamma^\alpha$ *and* $\mathbf{R}^4 \setminus f(\Gamma^\alpha \times \Gamma^\alpha)$, *respectively.*

Proof. — Continuity on $\mathbf{R}^4$ and C^∞-smoothness on $\mathbf{R}^4 \setminus \Gamma^\alpha \times \Gamma^\alpha$ for G follow from Proposition 2.19.

Lemma 5.1(v) provides a bound for the linear dilatation of G on the complement of $\Gamma^\alpha \times \Gamma^\alpha$:

$$H(G) \leq \left(1 + 10^6 C_1 C_4 M(\alpha) q^{1/3}\right)^2.$$

At each point $x \in \mathbf{R}^4 \setminus \Gamma^\alpha \times \Gamma^\alpha$, G is differentiable and

$$\max\left\{\frac{|G'(x)|^4}{\det G'(x)}, \frac{\det G'(x)}{\ell(G'(x))^4}\right\} \leq H(G)^3 \leq \left(1 + 10^6 C_1 C_4 M(\alpha) q^{1/3}\right)^6. \tag{5.6}$$

Suppose now that $y \in \Gamma^\alpha \times \Gamma^\alpha$ and $x \notin \Gamma^\alpha \times \Gamma^\alpha$; then

$$\begin{aligned}|G(y) - G(x)| &= |g(y) - G(x)| \\ &\geq |g(y) - g(z)| + |g(z) - g(x)| - |g(x) - G(x)|\end{aligned}$$

where z is a point on $\partial B(x, d(x)/2)$ with $g(z)$ on the line segment $[g(x), g(y)]$. By Lemma 5.1(ii) and Lemma 4.21(iv),

$$\begin{aligned}|g(x) - G(x)| &\leq 3 \cdot 10^4 C_1 C_4 M(\alpha) q^{1/3} d(x) L\big(h_x^\alpha\big) \\ &\leq 10^6 C_1 C_4 M(\alpha) q^{1/3} |g(z) - g(x)|;\end{aligned}$$

whence

$$|G(y) - G(x)| \geq \big(1 - 10^6 C_1 C_4 M(\alpha) q^{1/3}\big) |g(y) - g(x)|.$$

On the other hand,

$$\begin{aligned}|G(y) - G(x)| &\leq |g(y) - g(z)| + |g(z) - g(x)| + |g(x) - G(x)| \\ &\leq \big(1 + 10^6 C_1 C_4 M(\alpha) q^{1/3}\big) |g(y) - g(x)|.\end{aligned} \tag{5.7}$$

By Lemma 5.1(iii) and Lemma 4.21(ii), the ratios

$$\frac{\ell(G'(x))}{|g'(x)|} \quad \text{and} \quad \frac{|G'(x)|}{\ell(g'(x))}$$

are uniformly bounded away from zero and infinity on the union of the interiors of the n-simplices in $\mathscr{W}^\alpha$. Hence all directional derivatives of G and g are uniformly comparable on this set. Furthermore, g is ACL (absolutely continuous on lines) on $\mathbf{R}^4$. (See [29, §26] for the definition of absolute continuity on lines.) It follows that

$$G \text{ is ACL on the complement of } \Gamma^\alpha \times \Gamma^\alpha. \tag{5.8}$$

From (5.7), (5.8) and the fact that $G = g$ on $\Gamma^\alpha \times \Gamma^\alpha$, we obtain that G is ACL on $\mathbf{R}^4$. Finally, since $\Gamma^\alpha \times \Gamma^\alpha$ has 4-measure zero and g is ACL^4,

$$G \text{ is } ACL^4 \text{ on } \mathbf{R}^4. \tag{5.9}$$

(See [29, §26] for the definition of ACL^n.)

From (5.6) and (5.9), we conclude that G is K-quasiregular with

$$K := \big(1 + 10^6 C_1 C_4 M(\alpha) q^{1/3}\big)^6,$$

see [20, Definition I.2.1]. Thus

$$K \leq K_0(4) \tag{5.10}$$

by (4.15), where $K_0(n)$ is the constant in Theorem 5.3. Finally, it follows from Theorem 5.3 that G is a local homeomorphism, and then from Theorem 5.4 that G is a homeomorphism. Thus G is quasiconformal.

The C^∞-smoothness of G^{-1} on $\mathbf{R}^4 \setminus f(\Gamma^\alpha \times \Gamma^\alpha)$ follows from the injectivity of G.

Finally, surjectivity of G follows from the quasiconformality.

6. Proof of Theorem 1.5

Let $q_0(\alpha)$ and $s_0(\alpha)$ be the functions defined in (4.13) and (4.16) respectively, and recall that $s_0(\alpha)$ is continuous on the interval $(1, 2)$.

Given $\epsilon > 0$, fix $\alpha = 2 - \epsilon$. Let

$$\alpha_1 = 1 + \frac{1}{2}\delta(s_0(1), 1),$$

where $\delta(s, \alpha)$ is the function from Proposition 3.5(c). Let

$$s := \min\{s_0(t) : \alpha_1 \leq t \leq 2 - \epsilon\}$$

and observe that $s > 0$ since $s_0(\alpha)$ is continuous on $(1, 2)$. Next, let

$$\delta := \min\{\delta(s, t) : \alpha_1 \leq t \leq 2 - \epsilon\}$$

and observe that $\delta > 0$ since $\delta(s, \alpha)$ is continuous in α. Finally, choose

$$1 = \alpha_0 < \alpha_1 < \cdots < \alpha_m = \alpha$$

so that $|\alpha_{k+1} - \alpha_k| < \delta$ for each $k = 1, ..., m - 1$. From Proposition 3.5(c) and the choice of δ, the canonical map $f_k := F_{\alpha_k}^{\alpha_{k+1}}$ (in the notation of Sect. 3) from $\Gamma^{\alpha_k} \times \Gamma^{\alpha_k}$ to $\Gamma^{\alpha_{k+1}} \times \Gamma^{\alpha_{k+1}}$ is s-quasisymmetric for $k = 1, 2, ..., m - 1$, and is $s_0(1)$-quasisymmetric for $k = 0$. From the choice of s, each f_k admits a quasiconformal extension g_k to $\mathbf{R}^4$, following the construction in Proposition 4.2. Applying the smoothing procedure of Sect. 5 to each g_k, we obtain a function G_k smooth on $\mathbf{R}^4 \setminus \Gamma^{\alpha_k} \times \Gamma^{\alpha_k}$ and satisfying $G_k = f_k$ on $\Gamma^{\alpha_k} \times \Gamma^{\alpha_k}$. Finally, the choice of $q_0(\alpha)$ and $s_0(\alpha)$ and Proposition 5.5 guarantee that G_k is quasiconformal on $\mathbf{R}^4$. Let $H_k = G_k^{-1}$ and set

$$H = H_0 \circ \cdots \circ H_{m-2} \circ H_{m-1}.$$

Observe that the functions G_k and H_k, $0 \leq k \leq m - 1$, are quasiconformal with a uniform bound on the dilatation depending only on ϵ.

Let $U = \mathbf{R} \times (-2, 2) \times \mathbf{R} \times (-2, 2)$. From the extension construction in Sect. 4 and the smoothing process in Sect. 5, it follows that $G_k = \mathrm{id}$ on $\mathbf{R}^4 \setminus U$ for all k, whence also $H_k, H = \mathrm{id}$ on $\mathbf{R}^4 \setminus U$.

6.1. *Proposition.* — *Assume* $X, Y \in U$. *There exists* $\lambda(\alpha) > 1$ *so that*

$$\lambda(\alpha)^{-1} \leq \frac{|H'(X)|}{\text{dist}(X, \Gamma^\alpha \times \Gamma^\alpha)^{\alpha-1}} \leq \lambda(\alpha), \tag{6.2}$$

$$\lambda(\alpha)^{-1} \leq \frac{|H(X) - H(Y)|}{|X - Y|^\alpha} \leq \lambda(\alpha) \quad \textit{for all } X \in \Gamma^\alpha \times \Gamma^\alpha, |X - Y| \leq 1, \tag{6.3}$$

and

$$|H'(X) - H'(Y)| \leq \lambda(\alpha)|X - Y|^{\alpha-1} \quad \textit{for all } |X - Y| \leq 1. \tag{6.4}$$

The map H *is* C^α*-smooth (recall* $\alpha = 2 - \epsilon$*), quasiconformal on* $\mathbf{R}^4$*, and maps* $\Gamma^\alpha \times \Gamma^\alpha$ *onto* $\mathbf{R}^2$.

We postpone the proof of this proposition until we have completed the proof of Theorem 1.5.

Let $d \geq 2$ be an integer, and let $w : \mathbf{R}^4 \to \mathbf{R}^4$ be the winding map

$$w(x_1, x_2, x_3, x_4) = (x_1, r\cos d\theta, x_3, r\sin d\theta),$$

where (r, θ) denote polar coordinates in the x_2x_4-plane. We observe the following properties of w:

- w is quasiregular with branch set $\Pi := \{(x_1, 0, x_3, 0) : x_1, x_3 \in \mathbf{R}\}$ [20, p. 13].
- w is Lipschitz on $\mathbf{R}^4$,
- $w = (w_1, w_2, w_3, w_4)$ is C^∞-smooth on $\mathbf{R}^4 \setminus \Pi$ and

$$\max_{k,i,j} \left| \frac{\partial^2 w_k}{\partial x_i \partial x_j}(x) \right| \leq \frac{C}{\text{dist}(x, \Pi)} \quad \text{for } x \in \mathbf{R}^4 \setminus \Pi.$$

In what follows, we use the notation $A \preceq B$, respectively $A \simeq B$, to mean that there exists a constant C depending only on ϵ so that $A \leq CB$, respectively $C^{-1}B \leq A \leq CB$.

The composition $F := w \circ H$ is a quasiregular map on $\mathbf{R}^4$ of degree d whose branch set $\Gamma^\alpha \times \Gamma^\alpha$ has Hausdorff dimension $4 - 2\epsilon$. Its derivative is

$$F'(X) = \begin{cases} w'(H(X))H'(X), & \text{if } X \notin \Gamma^\alpha \times \Gamma^\alpha, \\ 0, & \text{if } X \in \Gamma^\alpha \times \Gamma^\alpha, \end{cases}$$

which shows that F is C^1 except possibly on $\Gamma^\alpha \times \Gamma^\alpha$. Since w is Lipschitz,

$$|F'(X)| \preceq |H'(X)| \simeq d^\alpha(X)^{\alpha-1}$$

for all $X \notin \Gamma^\alpha \times \Gamma^\alpha$, see (6.2). Thus F is C^1.

It remains to show that

$$(\mathbf{6.5}) \qquad |F'(X) - F'(Y)| \preceq |X - Y|^{\alpha-1}$$

for all X, Y.

If $Y \in \Gamma^\alpha \times \Gamma^\alpha$ and $X \notin \Gamma^\alpha \times \Gamma^\alpha$, then $F'(Y) = 0$ and

$$|F'(X)| \preceq |H'(X)| \preceq d^\alpha(X)^{\alpha-1} \leq |X - Y|^{\alpha-1}.$$

Finally, if $X, Y \notin \Gamma^\alpha \times \Gamma^\alpha$ write

$$|F'(X) - F'(Y)| \leq |w'(H(X)) - w'(H(Y))||H'(X)| \\ + |w'(H(Y))||H'(X) - H'(Y)|.$$

If $d^\alpha(X) + d^\alpha(Y) \leq 6|X - Y|$ then

$$|F'(X) - F'(Y)| \preceq |H'(X)| + |H'(Y)| \preceq d^\alpha(X)^{\alpha-1} + d^\alpha(Y)^{\alpha-1} \\ \preceq |X - Y|^{\alpha-1}$$

by (6.2).

Suppose instead that

$$(\mathbf{6.6}) \qquad d^\alpha(X) + d^\alpha(Y) > 6|X - Y|$$

and assume that $d^\alpha(X) \geq d^\alpha(Y)$. Then $d^\alpha(X) > 3|X - Y|$ so $d^\alpha(Z) \geq d^\alpha(X)/2$ and $|H'(Z)| \preceq d^\alpha(Z)^{\alpha-1}$ for all $Z \in [X, Y]$. The curve $C = H([X, Y])$ has length $\preceq |X - Y| d^\alpha(X)^{\alpha-1}$ and satisfies $\mathrm{dist}(C, \Pi) \succeq d^\alpha(X)^\alpha$. Using the estimate for the second derivative of w, we find

$$|w'(H(X)) - w'(H(Y))||H'(X)| \preceq d^\alpha(X)^{-\alpha}\,\mathrm{length}(C)|H'(X)| \\ \preceq |X - Y| d^\alpha(X)^{\alpha-2} \leq |X - Y|^{\alpha-1}.$$

On the other hand, $d^\alpha(X), d^\alpha(Y) \leq 3$ for all $X, Y \in U$. Thus $|X - Y| \leq 1$ for X, Y satisfying (6.6) and so

$$|w'(H(Y))||H'(X) - H'(Y)| \preceq |H'(X) - H'(Y)| \preceq |X - Y|^{\alpha-1}$$

by (6.4). This completes the proof of (6.5) and hence completes the proof of Theorem 1.5.

6.7. *Remark.* — The snowflake property (6.3) of H in Proposition 6.1 is essential in establishing the $C^{2-\epsilon}$-smoothness of F, since w is only Lipschitz continuous.

It remains to prove Proposition 6.1. We first give some estimates for G_k and H_k and their derivatives, and introduce the abbreviated notations $d_k = d^{\alpha_k}$ for the distance function to $\Gamma^{\alpha_k} \times \Gamma^{\alpha_k}$ and $h_{k,x} = h_x^{\alpha_k}$ for the similarities used in the construction of g_k. Recall that $U = \mathbf{R} \times (-2, 2) \times \mathbf{R} \times (-2, 2)$.

6.8. *Lemma.* — *Consider points* $x, y, X, Y \in U$.

(i) $|G_k(x) - G_k(y)| \simeq |x - y|^{\alpha_k/\alpha_{k+1}}$ *provided* $x \in \Gamma^{\alpha_k} \times \Gamma^{\alpha_k}$ *and* $|x - y| \leq 1$, *and*

$$|H_k(X) - H_k(Y)| \simeq |X - Y|^{\alpha_{k+1}/\alpha_k}$$

provided $X \in \Gamma^{\alpha_{k+1}} \times \Gamma^{\alpha_{k+1}}$ *and* $|X - Y| \leq 1$;

(ii) $d_{k+1}(G_k(x)) \simeq d_k(x)^{\alpha_k/\alpha_{k+1}}$ *provided* $x \notin \Gamma^{\alpha_k} \times \Gamma^{\alpha_k}$, *and*

$$d_k(H_k(X)) \simeq d_{k+1}(X)^{\alpha_{k+1}/\alpha_k}$$

provided $X \notin \Gamma^{\alpha_{k+1}} \times \Gamma^{\alpha_{k+1}}$;

(iii) G_k' *exists on the complement of* $\Gamma^{\alpha_k} \times \Gamma^{\alpha_k}$ *and*

$$|G_k'(x)| \simeq L(h_{k,x}) \simeq d_{k+1}(G_k(x))/d_k(x)$$

if $x \notin \Gamma^{\alpha_k} \times \Gamma^{\alpha_k}$, *while* H_k' *exists on all of* $\mathbf{R}^4$, $H_k'(X) = 0$ *if* $X \in \Gamma^{\alpha_{k+1}} \times \Gamma^{\alpha_{k+1}}$, *and*

$$|H_k'(X)| \simeq d_k(H_k(X))/d_{k+1}(X)$$

if $X \notin \Gamma^{\alpha_{k+1}} \times \Gamma^{\alpha_{k+1}}$;

(iv) $|H_k'(X) - H_k'(Y)| \preceq |X - Y|^{(\alpha_{k+1}/\alpha_k)-1}$ *for all* $|X - Y| \leq 1$ *and*

$$|H_k'(X) - H_k'(Y)| \preceq |X - Y| d_{k+1}(X)^{(\alpha_{k+1}/\alpha_k)-2}$$

for all $Y \in B(X, d_{k+1}(X)/2)$.

Proof. — To prove (i), we recall from Proposition 3.5 and the fact that $G_k = g_k = f_k$ on $\Gamma^{\alpha_k} \times \Gamma^{\alpha_k}$ that $|G_k(x) - G_k(y)| \simeq |x - y|^{\alpha_k/\alpha_{k+1}}$ for all $x, y \in \Gamma^{\alpha_k} \times \Gamma^{\alpha_k}$, $|x - y| \leq 1$. If $x \in \Gamma^{\alpha_k} \times \Gamma^{\alpha_k}$ and $y \notin \Gamma^{\alpha_k} \times \Gamma^{\alpha_k}$, choose $z \in \Gamma^{\alpha_k} \times \Gamma^{\alpha_k}$ with $|x - z| = |x - y|$; since the linear dilatations $H(G_k)$ are uniformly bounded on $\mathbf{R}^4$, we have

$$|G_k(x) - G_k(y)| \simeq |G_k(x) - G_k(z)| \simeq |x - z|^{\alpha_k/\alpha_{k+1}} = |x - y|^{\alpha_k/\alpha_{k+1}}.$$

The estimates on H_k follow by taking the inverse.

To prove (ii), let $X = G_k(x)$, choose $Y \in \Gamma^{\alpha_{k+1}} \times \Gamma^{\alpha_{k+1}}$ so that $|X - Y| = d_{k+1}(X)$, and let $y = G_k^{-1}(Y)$. Since $x \in U$, we have $X \in U$ and $|X - Y| \preceq 1$. Then

$$d_k(x) \leq |x - y| \simeq |X - Y|^{\alpha_{k+1}/\alpha_k} = d_{k+1}(X)^{\alpha_{k+1}/\alpha_k}$$

by (i). For the inverse, choose $z \in \Gamma^{\alpha_k} \times \Gamma^{\alpha_k}$ satisfying $|x - z| = d_k(x)$, and let $Z = G_k(z)$; then $|x - z| \preceq 1$ and

$$d_{k+1}(X) \leq |X - Z| \simeq |x - z|^{\alpha_k/\alpha_{k+1}} = d_k(x)^{\alpha_k/\alpha_{k+1}}.$$

This proves the first part of (ii). The second half of (ii) now follows by taking the inverse.

Part (iii) follows from Lemma 5.1(iii).

To prove (iv), we assume Lemma 6.9 below and let $x = H_k(X)$ and $y = H_k(Y)$. We consider three cases; any remaining cases are covered by interchanging X and Y.

Case 1. — $Y \in \Gamma^{\alpha_{k+1}} \times \Gamma^{\alpha_{k+1}}$. By Lemma 5.1(iii) and parts (ii) and (iii) of this proposition,

$$|H_k'(X) - H_k'(Y)| = |H_k'(X)| \simeq d_{k+1}(X)^{(\alpha_{k+1}/\alpha_k)-1} \preceq |X - Y|^{(\alpha_{k+1}/\alpha_k)-1}.$$

Case 2. — $Y \in B(X, a_\alpha d_{k+1}(X))$, where a_α is as in Lemma 6.9(ii). By Lemma 5.1(iv),(v) and part (ii) of this proposition,

$$\begin{aligned}|H_k'(X) - H_k'(Y)| &\preceq \frac{|x-y|}{d_k(x)} L(h_{k,x})^{-1} \preceq \frac{|X-Y|}{d_k(x)} L(h_{k,x})^{-2} \\ &\simeq |X - Y| d_{k+1}(X)^{(\alpha_{k+1}/\alpha_k)-2} \preceq |X - Y|^{(\alpha_{k+1}/\alpha_k)-1}.\end{aligned}$$

Case 3. — $Y \notin B(X, a_\alpha d_{k+1}(X)) \cup \Gamma^{\alpha_{k+1}} \times \Gamma^{\alpha_{k+1}}$. Choose $Z \in \Gamma^{\alpha_{k+1}} \times \Gamma^{\alpha_{k+1}}$ so that $|X - Z| = d_{k+1}(X)$. Then $|X - Z| \preceq |X - Y|$, $|Y - Z| \preceq |X - Y|$, and the first part of (iv) follows by applying Case 1 to X, Z and to Y, Z, and using the triangle inequality.

The second part of (iv) is essentially contained in the proof of Case 2 of the first part. This completes the proof of Lemma 6.8.

6.9. *Lemma.* — *(i) There exists* $A_\alpha > 1$ *so that whenever* $|X - Y| \leq 1$ *and* $0 \leq k \leq l \leq m-1$, *then*

$$|H_k \circ \cdots \circ H_l(X) - H_k \circ \cdots \circ H_l(Y)| \leq A_\alpha.$$

(ii) There exists $0 < a_\alpha < 1/2$ *so that whenever* $0 \leq k \leq l \leq m-1$, $|X - Y| \leq a_\alpha d_{l+1}(X)$ *and* $0 \leq k \leq m-1$, *then*

$$|H_k \circ \cdots \circ H_l(X) - H_k \circ \cdots \circ H_l(Y)| \leq \frac{1}{2} d_k(H_k \circ \ldots \circ H_l(X)).$$

Part (i) of this lemma is clear. Part (ii) follows from the fact that m depends only on α, and the maps G_k are K-quasiconformal for a common value of K depending only on α.

Proof of Proposition 6.1. — Note that (6.2) and (6.3) follow from Lemmas 6.8 and 6.9, and the chain rule.

To prove (6.4), we consider three cases as in Lemma 6.8(iv); any remaining cases are covered by interchanging X and Y. Assume that $|X-Y| \le 1$, and recall that $\alpha_m = \alpha$ and $d_m = d^\alpha$.

Case 1. — $Y \in \Gamma^{\alpha_m} \times \Gamma^{\alpha_m}$. Then $H'(Y) = 0$ and (6.4) follows from (6.2).

Case 2. — $Y \in B(X, a_\alpha d_m(X))$. We shall prove that

$$|(H_k \circ \cdots \circ H_{m-1})'(X) - (H_k \circ \cdots \circ H_{m-1})'(Y)| \preceq |X-Y| d_m(X)^{\alpha_m/\alpha_k - 2} \tag{\textbf{6.10}}$$

for $k = m-1, m-2, \ldots, 1, 0$ in succession. Then the Hölder continuity (6.4) follows from (6.10) by choosing $k=0$.

Note from Lemma 6.8(ii),(iii) and Lemma 6.9 that

$$d_k(H_k \circ \cdots \circ H_{m-1}(X)) \simeq d_m(X)^{\alpha_m/\alpha_k} \tag{\textbf{6.11}}$$

and

$$\begin{aligned} &|H_{k+1} \circ \cdots \circ H_{m-1}(X) - H_{k+1} \circ \cdots \circ H_{m-1}(Y)| \\ &\qquad \preceq |X-Y| \frac{d_{k+1}(H_{k+1} \circ \cdots \circ H_{m-1}(X))}{d_m(X)}. \end{aligned} \tag{\textbf{6.12}}$$

By Lemma 6.8(iv), the estimate (6.10) holds for $k = m-1$. Assume that (6.10) holds for $k+1$; we will show that it holds for k. By the chain rule, Lemmas 6.8 and 6.9, (6.11) and (6.12), and (6.10) for $k+1$, we get

$$\begin{aligned} &|(H_k \circ \cdots \circ H_{m-1})'(X) - (H_k \circ \cdots \circ H_{m-1})'(Y)| \\ &\preceq |H_k'(H_{k+1} \circ \cdots \circ H_{m-1}(X))| \\ &\quad \times |(H_{k+1} \circ \cdots \circ H_{m-1})'(X) - (H_{k+1} \circ \cdots \circ H_{m-1})'(Y)| \\ &\quad + |H_k'(H_{k+1} \circ \cdots \circ H_{m-1}(X)) - H_k'(H_{k+1} \circ \cdots \circ H_{m-1}(Y))| \\ &\quad \times |(H_{k+1} \circ \cdots \circ H_{m-1})'(Y)| \\ &\preceq \frac{d_k(H_k \circ \cdots \circ H_{m-1}(X))}{d_{k+1}(H_{k+1} \circ \cdots \circ H_{m-1}(X))} |X-Y| d_m(X)^{\alpha_m/\alpha_{k+1}-2} \\ &\quad + |H_{k+1} \circ \cdots \circ H_{m-1}(X) - H_{k+1} \circ \cdots \circ H_{m-1}(Y)| \\ &\qquad \times d_{k+1}(H_{k+1} \circ \cdots \circ H_{m-1}(X))^{\alpha_{k+1}/\alpha_k - 2} \frac{d_{k+1}(H_{k+1} \circ \cdots \circ H_{m-1}(X))}{d_m(X)} \\ &\preceq |X-Y| d_m(X)^{\alpha_m/\alpha_k - 2}. \end{aligned}$$

Thus (6.10) holds for all $k = m-1, m-2, \ldots, 1, 0$.

Case 3. — $Y \notin B(X, a_\alpha d_m(X)) \cup \Gamma^{\alpha_m} \times \Gamma^{\alpha_m}$. Choose $Z \in \Gamma^{\alpha_m} \times \Gamma^{\alpha_m}$ with $|X - Z| = d_m(X)$. Then (6.4) follows by applying Case 1 to X, Z and to Y, Z, and using the triangle inequality.

This completes the proof of Proposition 6.1.

7. Proof of Theorem 1.6

The proof of Theorem 1.6 relies on the construction by David and Toro of codimension one snowflake surfaces. See [7], specifically Theorem 2.10 with $Z = R^{n-2}$ and $f(r) = \max\{1, r^{-\epsilon/(1+\epsilon)}\}$ and the discussion following (13.33).

7.1. *Theorem (David–Toro).* — *For each* $n \geq 3$ *there exists* $\epsilon_0(n) > 0$ *and* $C = C(n) > 1$ *so that for each* $\epsilon \in (0, \epsilon_0(n))$ *there exists a* K*-quasiconformal map* $\Phi : \mathbf{R}^{n-1} \to \mathbf{R}^{n-1}$ *with*

$$C^{-1}|x-y|^{1/(1+\epsilon)} \leq |\Phi(x) - \Phi(y)| \leq C|x-y|^{1/(1+\epsilon)} \tag{7.2}$$

for all $x, y \in \mathbf{R}^{n-2}$, $|x-y| \leq 1$. *Furthermore,* $K \to 1$ *as* $\epsilon \to 0$.

David and Toro prove significantly stronger results; the domain $\mathbf{R}^{n-2}$ may be replaced by a metric space (Z, d) satisfying a Reifenberg flatness condition and the snowflake property in (7.2) may be replaced by Orlicz-type conditions

$$C^{-1}|x-y|f(|x-y|) \leq |\Phi(x) - \Phi(y)| \leq C|x-y|f(|x-y|)$$

for a variety of gauge functions $f(r)$.

By the celebrated quasiconformal extension theorem of Tukia–Väisälä [26], the map Φ of $\mathbf{R}^{n-1}$ in Theorem 7.1 may be further extended to a quasiconformal map of $\mathbf{R}^n$. We continue to denote this extension by Φ. Observe that the extension procedure in [26] is ostensibly different from that in [27] and [31] used in Sect. 4. It is therefore not obvious whether the smoothing procedure developed in Sect. 5 can be applied directly to the extended map Φ. We bypass the issue by an alternative argument. Choose ϵ sufficiently small so that the David–Toro map Φ in Theorem 7.1 is K-quasiconformal with K very close to one, hence s-quasisymmetric with s very close to zero (see Theorem 2.5). Then $\varphi = \Phi|_{\mathbf{R}^{n-2}}$ may be re-extended to a quasiconformal map $g : \mathbf{R}^n \to \mathbf{R}^n$ by the Tukia–Väisälä extension procedure from [27] and [31] (see Theorem 2.7). The smoothing procedure from Sect. 5 applies to g, yielding a quasiconformal map G on $\mathbf{R}^n$ whose inverse has the snowflake property in Proposition 6.1. The desired quasiregular map in Theorem 1.5 is obtained as the composition of a winding map with G^{-1}.

Proof of Theorem 1.6. — Let $n \geq 5$ and $d \geq 2$ be given. According to Theorem 2.7, $\mathbf{R}^{n-2}$ has the quasisymmetric extension property in $\mathbf{R}^n$; choose $s_0 > 0$ so that every s-quasisymmetric embedding $\varphi : \mathbf{R}^{n-2} \to \mathbf{R}^n$ with $s < s_0$ admits a quasisymmetric extension. Next, choose $K > 1$ so that every K-quasiconformal map of $\mathbf{R}^n$ is s-quasisymmetric for some $s \in (0, s_0)$ (Theorem 2.5), and choose $\epsilon > 0$ so that the map Φ from Theorem 7.1 is K-quasiconformal. Let $\varphi = \Phi|_{\mathbf{R}^{n-2}}$. Applying Theorem 2.7 (with $p = n - 2$) and following the procedure summarized in Sect. 4 (in the case $\alpha = 1$), extend φ to an s_1-quasisymmetric homeomorphism $g : \mathbf{R}^n \to \mathbf{R}^n$. When s is sufficiently small, the smoothing procedure in Sect. 5 applied to g yields a quasiconformal map $G : \mathbf{R}^n \to \mathbf{R}^n$, C^∞-smooth on $\mathbf{R}^n \setminus \mathbf{R}^{n-2}$, with $G = \varphi$ on $\mathbf{R}^{n-2}$, and as in Proposition 6.1, the snowflake property

$$C^{-1}|X - Y|^{1+\epsilon} \leq |G^{-1}(X) - G^{-1}(Y)| \leq C|X - Y|^{1+\epsilon} \tag{7.3}$$

holds for all $X \in \Sigma = \varphi(\mathbf{R}^{n-2})$ and all $Y \in \mathbf{R}^n$ with $|X - Y| \leq 1$. Here C denotes a suitable constant depending only on the dimension n.

Let $w : \mathbf{R}^n \to \mathbf{R}^n$ be the winding map of degree d

$$w(x_1, ..., x_{n-2}, r\cos\theta, r\sin\theta) = (x_1, ..., x_{n-2}, r\cos d\theta, r\sin d\theta),$$

where (r, θ) are polar coordinates in the $x_{n-1}x_n$-plane. The mapping w is quasiregular with branch set $\mathbf{R}^{n-2}$. Using (7.3) together with properties of w, it follows that $F = w \circ G^{-1}$ is a $C^{1+\epsilon}$-smooth quasiregular map on $\mathbf{R}^n$ of degree d with branch set $B_F = \Sigma$. The proof of Theorem 1.6 is complete.

REFERENCES

1. A. Beurling and L. V. Ahlfors, The boundary correspondence under quasiconformal mappings, *Acta Math.*, **96** (1956), 125–142.
2. C. J. Bishop, A quasisymmetric surface with no rectifiable curves, *Proc. Amer. Math. Soc.*, **127** (1999), 2035–2040.
3. M. Bonk and J. Heinonen, Smooth quasiregular mappings with branching, *Publ. Math., Inst. Hautes Études Sci.*, **100** (2004), 153–170.
4. A. V. Černavskiĭ, Finite-to-one open mappings of manifolds, *Mat. Sb. (N.S.)*, **65** (1964), 357–369.
5. A. V. Černavskiĭ, Addendum to the paper "Finite-to-one open mappings of manifolds", *Mat. Sb. (N.S.)*, **66** (1965), 471–472.
6. P. T. Church, Differentiable open maps on manifolds, *Trans. Amer. Math. Soc.*, **109** (1963), 87–100.
7. G. David and T. Toro, Reifenberg flat metric spaces, snowballs, and embeddings, *Math. Ann.*, **315** (1999), 641–710.
8. S. K. Donaldson and D. P. Sullivan, Quasiconformal 4-manifolds, *Acta Math.*, **163** (1989), 181–252.
9. W. H. J. Fuchs, *Théorie de l'approximation des fonctions d'une variable complexe*, Séminaire de Mathématiques Supérieures, no. 26 (Été 1967). Les Presses de l'Université de Montréal, Montréal, Que. (1968).
10. J. Heinonen, The branch set of a quasiregular mapping, in *Proceedings of the International Congress of Mathematicians, Vol. II (Beijing, 2002)*, Higher Ed. Press, Beijing (2002), pp. 691–700.
11. J. Heinonen and S. Rickman, Quasiregular maps $\mathbf{S}^3 \to \mathbf{S}^3$ with wild branch sets, *Topology*, **37** (1998), 1–24.
12. J. Heinonen and S. Rickman, Geometric branched covers between generalized manifolds, *Duke Math. J.*, **113** (2002), 465–529.

13. T. Iwaniec and G. Martin, *Geometric function theory and non-linear analysis*, Oxford Mathematical Monographs. The Clarendon Press, Oxford University Press, New York (2001).

14. M. Kiikka, Diffeomorphic approximation of quasiconformal and quasisymmetric homeomorphisms, *Ann. Acad. Sci. Fenn., Ser. A I, Math.*, **8** (1983), 251–256.

15. O. Martio and S. Rickman, Measure properties of the branch set and its image of quasiregular mappings, *Ann. Acad. Sci. Fenn., Ser. A I*, **541** (1973), 16.

16. O. Martio, S. Rickman and J. Väisälä, Topological and metric properties of quasiregular mappings, *Ann. Acad. Sci. Fenn., Ser. A I*, **488** (1971), 31.

17. P. Mattila, *Geometry of sets and measures in Euclidean spaces*, vol. 44 of Cambridge Studies in Advanced Mathematics, Cambridge University Press, Cambridge (1995).

18. Y. G. Reshetnyak, Space mappings with bounded distortion, *Sibirsk. Mat. Z.*, **8** (1967), 629–659.

19. Y. G. Reshetnyak, *Space mappings with bounded distortion*, vol. 73 of Translations of Mathematical Monographs, American Mathematical Society, Providence (1989). Translated from the Russian by H. H. McFadden.

20. S. Rickman, *Quasiregular Mappings*, Springer, Berlin (1993).

21. S. Rickman, Construction of quasiregular mappings, in *Quasiconformal mappings and analysis (Ann Arbor, MI 1995)*, Springer, New York (1998), pp. 337–345.

22. J. Sarvas, The Hausdorff dimension of the branch set of a quasiregular mapping, *Ann. Acad. Sci. Fenn., Ser. A I, Math.*, **1** (1975), 297–307.

23. E. M. Stein, *Singular integrals and differentiability properties of functions*, Princeton Mathematical Series, No. 30, Princeton University Press, Princeton, N.J. (1970).

24. D. Sullivan, Hyperbolic geometry and homeomorphisms, in *Geometric topology (Proc. Georgia Topology Conf., Athens, Ga. 1977)*, Academic Press, New York (1979), pp. 543–555.

25. P. Tukia and J. Väisälä, Quasisymmetric embeddings of metric spaces, *Ann. Acad. Sci. Fenn., Ser. A I, Math.*, **5** (1980), 97–114.

26. P. Tukia and J. Väisälä, Quasiconformal extension from dimension n to $n+1$, *Ann. of Math. (2),* **115** (1982), 331–348.

27. P. Tukia and J. Väisälä, Extension of embeddings close to isometries or similarities, *Ann. Acad. Sci. Fenn., Ser. A I, Math.*, **9** (1984), 153–175.

28. J. T. Tyson and J.-M. Wu, Quasiconformal dimensions of self-similar fractals, *Rev. Mat. Iberoamer.*, accepted for publication.

29. J. Väisälä, *Lectures on n-dimensional quasiconformal mappings*, no. 229 in Lecture Notes in Mathematics, Springer, Berlin (1971).

30. J. Väisälä, A survey of quasiregular maps in $\mathbf{R}^n$, in *Proceedings of the International Congress of Mathematicians (Helsinki 1978)*, Acad. Sci. Fennica, Helsinki (1980), pp. 685–691.

31. J. Väisälä, Bi-Lipschitz and quasisymmetric extension properties, *Ann. Acad. Sci. Fenn., Ser. A I, Math.*, **11** (1986), 239–274.

R. K., J. T. T., J.-M. W.
Department of Mathematics,
University of Illinois,
1409 West Green Street,
Urbana, IL 61801, USA
{rpkaufma, tyson, wu}@math.uiuc.edu

Manuscrit reçu le 19 avril 2004; révisé le 13 décembre 2004.