INDECOMPOSABLE PARABOLIC BUNDLES

and the Existence of Matrices in Prescribed Conjugacy Class Closures with Product Equal to the Identity

by WILLIAM CRAWLEY-BOEVEY

Dedicated to Claus Michael Ringel on the occasion of his sixtieth birthday

ABSTRACT

We study the possible dimension vectors of indecomposable parabolic bundles on the projective line, and use our answer to solve the problem of characterizing those collections of conjugacy classes of $n \times n$ matrices for which one can find matrices in their closures whose product is equal to the identity matrix. Both answers depend on the root system of a Kac-Moody Lie algebra. Our proofs use Ringel's theory of tubular algebras, work of Mihai on the existence of logarithmic connections, the Riemann-Hilbert correspondence and an algebraic version, due to Dettweiler and Reiter, of Katz's middle convolution operation.

CONTENTS

1. Introduction

Given some matrices in $GL_n(\mathbf{C})$, if one knows their conjugacy classes, what can one say about the conjugacy class of their product? This is called the 'recognition problem' by Neto and Silva [32, §3]. Inverting one of the matrices, one obtains the following more symmetrical formulation: given conjugacy classes $C_1, \ldots, C_k$, determine whether or not one can solve the equation

$$(\mathbf{1}) \qquad A_1 A_2 \ldots A_k = 1$$

with $A_i \in C_i$. We solve a weaker version of this, determining whether or not one can solve equation (1) with matrices A_i in the closures $\overline{C_i}$ of the conjugacy classes. Of course if the C_i are diagonalizable, they are already closed, and the recognition problem is solved.

Our solution depends on properties of indecomposable parabolic bundles. Let X be a connected Riemann surface, $D = (a_1, \ldots, a_k)$ a collection of distinct points of X,

DOI 10.1007/s10240-004-0025-7

and $w = (w_1, ..., w_k)$ a collection of positive integers. By a *parabolic bundle* on X of *weight type* (D, w) we mean a collection $\mathbf{E} = (\mathrm{E}, \mathrm{E}_{ij})$ consisting of a holomorphic vector bundle E on X and flags of vector subspaces

$$\mathrm{E}_{a_i} = \mathrm{E}_{i0} \supseteq \mathrm{E}_{i1} \supseteq \cdots \supseteq \mathrm{E}_{i,w_i-1} \supseteq \mathrm{E}_{i,w_i} = 0$$

of its fibres at the points in D. The parabolic bundles naturally form a category $\mathrm{par}_{\mathrm{D},w}(\mathrm{X})$ in which the morphisms $\mathbf{E} \to \mathbf{F}$ are the vector bundle homomorphisms $f : \mathrm{E} \to \mathrm{F}$ with

$$f_{a_i}(\mathrm{E}_{ij}) \subseteq \mathrm{F}_{ij}$$

for all i, j. Clearly any points a_i with weighting $w_i = 1$ can be omitted from D without changing the category. Therefore, one may if one wishes assume that all $w_i \geq 2$. This also shows that in order to specify a weight type, it suffices to fix the divisor $\sum_{i=1}^{k}(w_i - 1)a_i$.

The category $\mathrm{par}_{\mathrm{D},w}(\mathrm{X})$ has been much studied before. It is equivalent to the natural category of vector bundles for an orbifold Riemann surface [13, Theorem 5.7], or, in case X is the projective line, the category of vector bundles on a weighted projective line in the sense of Geigle and Lenzing [14] (see [26, Theorem 4.4]). Analogous categories arise when one studies one-dimensional smooth stacks or sheaves of torsion-free $\mathscr{H}$-modules, where $\mathscr{H}$ is a sheaf of classical hereditary orders on an algebraic curve, see [6].

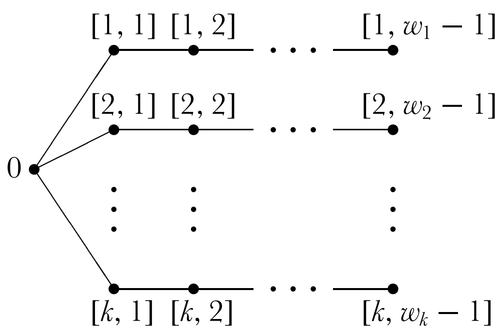
The usual notion of a parabolic bundle due to Mehta and Seshadri [28, 35] also associates 'weights' in [0, 1) to the terms in the flags. These weights are used to define the notion of semistability for parabolic bundles, and Mehta and Seshadri relate the moduli space of semistable parabolic bundles on a compact Riemann surface of genus $g \geq 2$ with equivalence classes of unitary representations of a Fuchsian group. Agnihotri and Woodward [1] and Belkale [3] independently observed that one can in the same way relate semistable parabolic bundles on the projective line with unitary representations of the fundamental group of the punctured projective line. In this way they were able to solve the recognition problem for conjugacy classes in $\mathrm{SU}(n)$ in terms of quantum Schubert calculus. In this paper semistability will play no role, so it is convenient to ignore the weights, and actually, we will need to associate arbitrary complex numbers to the terms in the flags. Note that the term 'quasi-parabolic structure' [28, 35] is sometimes used for the flags without the weights.

The basic invariant of a parabolic bundle $\mathbf{E}$ is its *dimension vector* α, which consists of the numbers $\alpha_0 = \mathrm{rank}\,\mathrm{E}$ and $\alpha_{ij} = \dim \mathrm{E}_{ij}$ ($1 \leq i \leq k$, $1 \leq j \leq w_i - 1$). Assuming that X is compact, one also has the degree, $d = \deg \mathrm{E}$. What can one say about the invariants of an *indecomposable* parabolic bundle? Clearly the dimension vector must be *strict*, by which we mean that

$$\alpha_0 \geq \alpha_{i1} \geq \alpha_{i2} \geq \cdots \geq \alpha_{i,w_i-1} \geq 0 \tag{2}$$

for all i. If X has genus $g \geq 1$, then there are indecomposable parabolic bundles of all possible strict dimension vectors and degrees, for one knows that there is an indecomposable vector bundle of any rank and degree, and it can be turned into an indecomposable parabolic bundle using arbitrary flags. Thus we restrict to the case when $X = \mathbf{P}^1$, the complex projective line.

Let Γ_w be the star-shaped graph



with vertex set $I = \{0\} \cup \{[i, j] : 1 \leq i \leq k, 1 \leq j \leq w_i - 1\}$. For ease of notation, if $\alpha \in \mathbf{Z}^I$, we write its components as α_0 and α_{ij}. In this way one sees that dimension vectors, as defined above, are naturally elements of $\mathbf{Z}^I$. Associated to Γ_w there is a Kac-Moody Lie algebra, and hence a root system, which we consider as a subset of $\mathbf{Z}^I$. We briefly recall its combinatorial definition. There is a symmetric bilinear form $(-, -)$ on $\mathbf{Z}^I$ given by

$$(\epsilon_v, \epsilon_{v'}) = \begin{cases} 2 & (\text{if } v = v') \\ -1 & (\text{if an edge joins } v \text{ and } v') \\ 0 & (\text{otherwise}) \end{cases}$$

where ϵ_v denotes the coordinate vector at a vertex $v \in I$. The associated quadratic form is $q(\alpha) = \frac{1}{2}(\alpha, \alpha)$, and we also define $p(\alpha) = 1 - q(\alpha)$. For $v \in I$, the reflection $s_v : \mathbf{Z}^I \to \mathbf{Z}^I$, is defined by $s_v(\alpha) = \alpha - (\alpha, \epsilon_v)\epsilon_v$, the *Weyl group* W is the subgroup of $\mathrm{Aut}(\mathbf{Z}^I)$ generated by the s_v, and the *real roots* are the images under elements of W of the coordinate vectors. The *fundamental region* consists of the nonzero elements $\alpha \in \mathbf{N}^I$ which have connected support and which are not made smaller by any reflection; its closure under the action of W and change of sign is, by definition, the set of *imaginary roots*. Note that $p(\alpha) = 0$ for real roots, and $p(\alpha) > 0$ for imaginary roots. Recall that any root is *positive* ($\alpha \in \mathbf{N}^I$), or *negative* ($\alpha \in -\mathbf{N}^I$).

Note that when determining whether or not α is a root, all that matters is α_0 and for each arm i, the unordered collection of numbers

$$(\mathbf{3}) \qquad \alpha_0 - \alpha_{i1},\ \alpha_{i1} - \alpha_{i2},\ \ldots,\ \alpha_{i,w_i-2} - \alpha_{i,w_i-1},\ \alpha_{i,w_i-1},$$

since the reflection at vertex $[i,j]$ has the effect of exchanging the jth and $(j+1)$st terms in this collection.

We say that an element of $\mathbf{Z}^{\mathrm{I}}$ is *strict* if it satisfies the inequalities (2). Any root with $\alpha_0 > 0$ must be strict, for if some term in the collection (3) is negative, it can be moved to the last place, and then α has both positive and negative components. It follows that the only positive roots which are not strict are those of the form

$$(\mathbf{4}) \qquad \alpha_0 = 0, \quad \alpha_{ij} = \begin{cases} 1 & (\text{if } i = \ell \text{ and } r \le j \le s) \\ 0 & (\text{otherwise}), \end{cases}$$

for some $1 \le \ell \le k$ and $1 \le r \le s \le w_\ell - 1$.

Our first main result is as follows.

Theorem **1.1.** — *The dimension vector of an indecomposable parabolic bundle on* $\mathbf{P}^1$ *of weight type* (D, w) *is a strict root for* Γ_w. *Conversely, if* α *is a strict root,* $d \in \mathbf{Z}$, *and* d *and* α *are coprime, then there is an indecomposable parabolic bundle* $\mathbf{E}$ *on* $\mathbf{P}^1$ *of weight type* (D, w), *with dimension vector* α *and* $\deg \mathrm{E} = d$. *Moreover, if* α *is a real root, this indecomposable parabolic bundle is unique up to isomorphism.*

Here we say that d and α are *coprime* provided that there is no integer ≥ 2 which divides d and all components of α. We conjecture that this coprimality assumption is unnecessary.

Note that in case Γ_w is an extended Dynkin diagram, the corresponding weighted projective line has genus one, and Lenzing and Meltzer [29, Theorem 4.6] have given a complete classification of indecomposable coherent sheaves on the weighted projective line in terms of the roots of a quadratic form.

Corollary **1.2.** — *The dimension vectors of indecomposable parabolic bundles on* $\mathbf{P}^1$ *of weight type* (D, w) *are exactly the strict roots for* Γ_w.

We now return to the problem about matrices. Given a collection of positive integers $w = (w_1, \ldots, w_k)$, and a set Ω, we denote by Ω^w the set of collections $\xi = (\xi_{ij})$ with $\xi_{ij} \in \Omega$ for $1 \le i \le k$ and $1 \le j \le w_i$. To fix conjugacy classes $\mathrm{C}_1, \ldots, \mathrm{C}_k$ in $\mathrm{GL}_n(\mathbf{C})$ we fix w and $\xi \in (\mathbf{C}^*)^w$ with

$$(\mathrm{A}_i - \xi_{i1}1)(\mathrm{A}_i - \xi_{i2}1) \ldots (\mathrm{A}_i - \xi_{i,w_i}1) = 0$$

for $A_i \in C_i$. Clearly, if one wishes one can take w_i to be the degree of the minimal polynomial of A_i, and $\xi_{i1}, \ldots, \xi_{i,w_i}$ to be its roots, counted with multiplicity. The conjugacy class C_i is then determined by the ranks of the partial products

$$\alpha_{ij} = \operatorname{rank}(A_i - \xi_{i1} 1) \ldots (A_i - \xi_{ij} 1)$$

for $A_i \in C_i$ and $1 \le j \le w_i - 1$. Setting $\alpha_0 = n$, we obtain a dimension vector α for the graph Γ_w. An obvious necessary condition for a solution to equation (1) is that

$$\det A_1 \times \cdots \times \det A_k = 1.$$

For $A_i \in C_i$ this is equivalent to the condition $\xi^{[\alpha]} = 1$, where we define

$$\xi^{[\alpha]} = \prod_{i=1}^{k} \prod_{j=1}^{w_i} \xi_{ij}^{\alpha_{i,j-1} - \alpha_{ij}}$$

using the convention that $\alpha_{i0} = \alpha_0$ and $\alpha_{i,w_i} = 0$ for all i. Our other main result is the following.

Theorem **1.3.** — *There is a solution to* $A_1 \ldots A_k = 1$ *with* $A_i \in \overline{C_i}$ *if and only if* α *can be written as a sum of positive roots for* Γ_w, *say* $\alpha = \beta + \gamma + \ldots$, *with* $\xi^{[\beta]} = \xi^{[\gamma]} = \cdots = 1$.

The most basic example, familiar from the hypergeometric equation, is when $k = 3$ and the matrices are 2×2. If C_i has distinct eigenvalues λ_i, μ_i, one can take $w = (2, 2, 2)$, $\xi_{i1} = \lambda_i$ and $\xi_{i2} = \mu_i$, and the conjugacy classes are then given by the dimension vector

$$\begin{array}{ccc} & & 1 \\ & \diagup & \\ 2 & — & 1 \\ & \diagdown & \\ & & 1 \end{array}$$

for the Dynkin diagram D_4. Since this is a positive root, the only requirement for solvability is $\xi^{[\alpha]} = 1$, that is, $\prod_i \lambda_i \mu_i = 1$. On the other hand, if the matrices are 3×3 and C_i is diagonalizable, with λ_i having multiplicity 2 and μ_i having multiplicity 1, the relevant dimension vector is

$$\begin{array}{ccc} & & 1 \\ & \diagup & \\ 3 & — & 1 \\ & \diagdown & \\ & & 1 \end{array}$$

which is not a root. Its possible decompositions as a sum of positive roots are

$$\begin{array}{ccc} & & 1\\ & \diagup & \\ 1 & - & 0\\ & \diagdown & \\ & & 0\end{array} \quad + \quad \begin{array}{ccc} & & 0\\ & \diagup & \\ 1 & - & 1\\ & \diagdown & \\ & & 0\end{array} \quad + \quad \begin{array}{ccc} & & 0\\ & \diagup & \\ 1 & - & 0\\ & \diagdown & \\ & & 1\end{array}$$

$$\begin{array}{ccc} & & 0\\ & \diagup & \\ 1 & - & 0\\ & \diagdown & \\ & & 0\end{array} \quad + \quad \begin{array}{ccc} & & 1\\ & \diagup & \\ 2 & - & 1\\ & \diagdown & \\ & & 1\end{array}$$

and refinements of these, so the condition for solvability is that $\mu_1\lambda_2\lambda_3 = \lambda_1\mu_2\lambda_3 = \lambda_1\lambda_2\mu_3 = 1$ or $\prod_i \lambda_i = \prod_i \lambda_i\mu_i = 1$.

Although we have not been able to solve the full recognition problem, we believe that the approach using indecomposable parabolic bundles will be of assistance in solving this and related problems. We now make some remarks about these. A variation of the recognition problem asks whether or not there is a solution to (1) which is *irreducible*, meaning that the A_i have no common invariant subspace. This is called the 'Deligne-Simpson problem' by Kostov [21–24]. Here we have a conjecture.

Conjecture **1.4.** — *There is an irreducible solution to* $A_1 \dots A_k = 1$ *with* $A_i \in C_i$ *if and only if* α *is a positive root,* $\xi^{[\alpha]} = 1$*, and* $p(\alpha) > p(\beta) + p(\gamma) + \dots$ *for any nontrivial decomposition of* α *as a sum of positive roots* $\alpha = \beta + \gamma + \dots$ *with* $\xi^{[\beta]} = \xi^{[\gamma]} = \cdots = 1$.

A solution to (1) with $A_i \in C_i$ is said to be *rigid* if it is the unique such solution, up to simultaneous conjugacy. Deligne (see [36, Lemma 6]) observed that if there is an irreducible solution, it is rigid if and only if

$$\sum_{i=1}^{k} \dim C_i = 2n^2 - 2. \tag{5}$$

In our formulation this becomes the condition $p(\alpha) = 0$. In [20], Katz introduced an operation, called 'middle convolution', which enabled him to give an algorithm for studying rigid irreducible solutions. In [12], Detweiller and Reiter gave a purely algebraic version of middle convolution, called simply 'convolution', and the corresponding algorithm. In Sections 2–4 we use the methods of Detweiller and Reiter to prove the rigid case of our conjecture.

Theorem **1.5.** — *There is a rigid irreducible solution to* $A_1 \dots A_k = 1$ *with* $A_i \in C_i$ *if and only if* α *is a positive real root for* Γ_w, $\xi^{[\alpha]} = 1$*, and there is no nontrivial decomposition of* α *as a sum of positive roots* $\alpha = \beta + \gamma + \dots$ *with* $\xi^{[\beta]} = \xi^{[\gamma]} = \cdots = 1$.

In Section 5 we study parabolic bundles in the case when the dimension vector is in the fundamental region. In Sections 6–7 we show how to pass between parabolic bundles and solutions of equation (1), and then in Section 8 we combine all this with the methods used in the rigid case to solve the Deligne-Simpson problem for generic eigenvalues. The main results, Theorems 1.1 and 1.3, are then deduced from the generic case in Section 9.

Finally we remark that instead of equation (1), one can consider the additive equation

$$A_1 + A_2 + \cdots + A_k = 0. \tag{6}$$

We have already solved the corresponding additive Deligne-Simpson problem in [9]. In this case one fixes conjugacy classes $D_1, \ldots, D_k$ in $M_n(\mathbf{C})$ using an element $\zeta \in \mathbf{C}^w$, and dimension vector α. Defining

$$\zeta * [\alpha] = \sum_{i=1}^{k} \sum_{j=1}^{w_i} \zeta_{ij}(\alpha_{i,j-1} - \alpha_{ij}),$$

we showed that there is an irreducible solution to (6) with $A_i \in D_i$ if and only if α is a positive root for Γ_w, $\zeta * [\alpha] = 0$, and $p(\alpha) > p(\beta) + p(\gamma) + \ldots$ for any nontrivial decomposition of α as a sum of positive roots $\alpha = \beta + \gamma + \ldots$ with $\zeta * [\beta] = \zeta * [\gamma] = \cdots = 0$. Note that the method of [9] together with [7, Theorem 4.4] and Theorem 2.1 already gives the additive analogue of Theorem 1.3. Namely, there is a solution to $A_1 + \cdots + A_k = 0$ with $A_i \in \overline{D_i}$ if and only if α can be written as a sum of positive roots $\alpha = \beta + \gamma + \ldots$ with $\zeta * [\beta] = \zeta * [\gamma] = \cdots = 0$.

The crucial Theorem 7.1, which should be of independent interest, was proved during a visit to the Center for Advanced Study at the Norwegian Academy of Science and Letters in September 2001, and I would like to thank my hosts for their hospitality. I would also like to thank C. Geiß and H. Lenzing for some invaluable discussions.

2. Fixing conjugacy classes and their closures

In Sections 2–5 we work over an algebraically closed field K.

We first deal with a single conjugacy class. Let V be a vector space of dimension n, let $d \geq 1$, and fix a collection $\xi = (\xi_1, \ldots, \xi_d)$ with $\xi_j \in K$. We say that an endomorphism $A \in \mathrm{End}(V)$ has *type* ξ if $(A - \xi_1 1) \ldots (A - \xi_d 1) = 0$. In this case its *dimension vector* is the sequence of integers $(n_0, n_1, \ldots, n_{d-1})$ where n_j is the rank of the partial product $(A - \xi_1 1) \ldots (A - \xi_j 1)$, so that $n_0 = n$. By convention we define $n_d = 0$.

Clearly the type and dimension vector only depend on the conjugacy class of A. Now A has type ξ if and only if for all $\lambda \in K$, the Jordan normal form of A only involves Jordan blocks with eigenvalue λ up to size r_λ, where

$$r_\lambda = |\{\ell \mid 1 \le \ell \le d, \xi_i = \lambda\}|.$$

Moreover $n_{j-1} - n_j$ is the number of Jordan blocks of eigenvalue ξ_j of size at least m_j involved in A, where

$$m_j = |\{\ell \mid 1 \le \ell \le j, \xi_\ell = \xi_j\}|.$$

It follows that $A \in \mathrm{End}(V)$ of type ξ is determined up to conjugacy by its dimension vector. Clearly given any $j < \ell$ with $\xi_j = \xi_\ell$ we have $m_j < m_\ell$, so that

$$n_{j-1} - n_j \ge n_{\ell-1} - n_\ell. \tag{7}$$

Conversely any sequence $(n_0, \ldots, n_{d-1})$ of integers with $n = n_0 \ge n_1 \cdots \ge n_{d-1} \ge n_d = 0$ and satisfying (7) arises as the dimension vector of some $A \in \mathrm{End}(V)$ of type ξ.

Theorem **2.1.** — *Let* $A \in \mathrm{End}(V)$ *have type* ξ *and dimension vector* $(n_0, n_1, \ldots, n_{d-1})$. *If* $B \in \mathrm{End}(V)$, *then the following conditions are equivalent.*

(i) B *is in the closure of the conjugacy class of* A.

(ii) *There is a flag of subspaces*

$$V = V_0 \supseteq V_1 \supseteq \cdots \supseteq V_d = 0$$

with $\dim V_j = n_j$ *and such that* $(B - \xi_j 1)(V_{j-1}) \subseteq V_j$ *for all* $1 \le j \le d$.

(iii) *There are vector spaces* V_j *and linear maps* ϕ_j, ψ_j,

$$V = V_0 \underset{\psi_1}{\overset{\phi_1}{\rightleftarrows}} V_1 \underset{\psi_2}{\overset{\phi_2}{\rightleftarrows}} V_2 \underset{\psi_3}{\overset{\phi_3}{\rightleftarrows}} \cdots \underset{\psi_d}{\overset{\phi_d}{\rightleftarrows}} V_d = 0$$

where V_j *has dimension* n_j, *and satisfying*

$$B - \psi_1\phi_1 = \xi_1 1,$$
$$\phi_j\psi_j - \psi_{j+1}\phi_{j+1} = (\xi_{j+1} - \xi_j)1, \qquad (1 \le j < d).$$

In case K has characteristic 0, the equivalence of (i) and (iii) follows from [8, Lemma 9.1].

Proof. — (i)⇒(ii). Let F be the corresponding flag variety, and let Z be the closed subset of $F \times \mathrm{End}(V)$ consisting of all flags V_j and endomorphisms B with $(B - \xi_j 1)(V_{j-1}) \subseteq V_j$ for all $1 \le j \le d$. By definition the set of B which satisfy condition (ii) is the image I of Z under the projection to End(V). Now I contains A, since one can take $V_j = \mathrm{Im}(A - \xi_1 1) \ldots (A - \xi_j 1)$, it is stable under the conjugation action of $\mathrm{GL}(V)$, and it is closed since F is projective. Thus it contains the closure of the conjugacy class of A.

(ii)⇒(iii). Take ϕ_j to be the restriction of $B-\xi_j 1$ to V_{j-1} and ψ_j to be the inclusion.

(iii)⇒(i). We prove this by induction on d. If $d = 1$ it is trivial, so suppose that $d > 1$. Identify V_1 with $\mathrm{Im}(A-\xi_1 1)$, and let A_1 be the restriction of A to V_1. Observe that A_1 has type $(\xi_2, \ldots, \xi_d)$ and dimension vector $(n_1, \ldots, n_d)$. Now letting $B_1 = \phi_1\psi_1 + \xi_1 1$, we have

$$B_1 - \psi_2\phi_2 = \xi_2 1,$$
$$\phi_j\psi_j - \psi_{j+1}\phi_{j+1} = (\xi_{j+1} - \xi_j)1, \qquad (2 \le j < d)$$

so by the inductive hypothesis B_1 is in the closure of the conjugacy class in $\mathrm{End}(V_1)$ containing A_1.

Recall that the Gerstenhaber-Hesselink Theorem [15, Theorem 1.7] says that if $A, B \in \mathrm{End}(V)$, then B is in the closure of the conjugacy class of A if and only if

$$\mathrm{rank}(B - \lambda 1)^m \le \mathrm{rank}(A - \lambda 1)^m$$

for all $\lambda \in K$ and $m \ge 1$. Applying this to A_1 and B_1, we have

$$\mathrm{rank}(B_1 - \lambda 1)^m \le \mathrm{rank}(A_1 - \lambda 1)^m = \mathrm{rank}(A - \xi_1 1)(A - \lambda 1)^m.$$

Now if $\lambda = \xi_1$ we have

$$(B - \lambda 1)^{m+1} = (\psi_1\phi_1)^{m+1} = \psi_1(B_1 - \lambda 1)^m\phi_1$$

so that

$$\mathrm{rank}(B - \lambda 1)^{m+1} \le \mathrm{rank}(B_1 - \lambda 1)^m \le \mathrm{rank}(A - \lambda 1)^{m+1}.$$

On the other hand, if $\lambda \ne \xi_1$, then

$$\mathrm{rank}(\psi_1\phi_1 + (\xi_1 - \lambda)1)^m = n_0 - n_1 + \mathrm{rank}(\phi_1\psi_1 + (\xi_1 - \lambda)1)^m$$

by Lemma 2.2 below, so $\mathrm{rank}(B-\lambda 1)^m = n_0 - n_1 + \mathrm{rank}(B_1 - \lambda 1)^m$. However, we also have

$$\mathrm{rank}(A_1 - \lambda)^m = \mathrm{rank}(A - \xi_1 1)(A - \lambda 1)^m = n_1 - n_0 + \mathrm{rank}(A - \lambda 1)^m$$

using that

$$\mathrm{Ker}(A - \xi_1 1)(A - \lambda 1)^m = \mathrm{Ker}(A - \xi_1 1) \oplus \mathrm{Ker}(A - \lambda 1)^m,$$

and it follows that $\mathrm{rank}(B-\lambda 1)^m \le \mathrm{rank}(A-\lambda 1)^m$. Since this inequality holds for all λ and m, B is in the closure of the conjugacy class of A by the Gerstenhaber-Hesselink Theorem. □

Lemma **2.2.** — *If* $\phi : V \to W$ *and* $\psi : W \to V$ *are linear maps, then*

$$\dim V - \operatorname{rank}(\psi\phi + \mu 1)^m = \dim W - \operatorname{rank}(\phi\psi + \mu 1)^m$$

for any $0 \neq \mu \in K$ *and* $m \geq 0$.

Proof. — From the formula $\psi(\phi\psi + \mu 1) = (\psi\phi + \mu 1)\psi$ one deduces that

$$\psi(\phi\psi + \mu 1)^m = (\psi\phi + \mu 1)^m\psi,$$

so that ψ induces a map from $\operatorname{Ker}(\phi\psi + \mu 1)^m$ into $\operatorname{Ker}(\psi\phi + \mu 1)^m$. This map is injective, since if $\psi(x) = 0$, then $(\phi\psi + \mu 1)^m(x) = \mu^m x$, so $x = 0$. Thus

$$\dim \operatorname{Ker}(\phi\psi + \mu 1)^m \leq \dim \operatorname{Ker}(\psi\phi + \mu 1)^m.$$

The reverse inequality holds by symmetry, and the result follows. □

We consider the following operation on a sequence $\mathbf{n} = (n_0, n_1, \ldots, n_{d-1})$. Suppose that $1 \leq r \leq s \leq d-1$, that $\xi_r = \xi_{s+1}$, and that the sequence

$$\mathbf{n}' = (n_0, \ldots, n_{r-1}, n_r - 1, n_{r+1} - 1, \ldots n_s - 1, n_{s+1}, \ldots)$$

still has all terms non-negative. In this case we say that the sequence $\mathbf{n}'$ is obtained from $\mathbf{n}$ by *reduction* with respect to r, s (and ξ).

Lemma **2.3.** — *Suppose that* $A \in \operatorname{End}(V)$ *has type* ξ *and dimension vector* $\mathbf{n} = (n_0, n_1, \ldots, n_{d-1})$. *Suppose that* $\mathbf{m} = (m_0, m_1, \ldots, m_{d-1})$ *is obtained from* $\mathbf{n}$ *by a finite number of reductions. Let* $B \in \operatorname{End}(V)$. *If there are vector spaces* V_j^0 *and linear maps* ϕ_j^0, ψ_j^0,

$$V = V_0^0 \underset{\psi_1^0}{\overset{\phi_1^0}{\rightleftarrows}} V_1^0 \underset{\psi_2^0}{\overset{\phi_2^0}{\rightleftarrows}} V_2^0 \underset{\psi_3^0}{\overset{\phi_3^0}{\rightleftarrows}} \cdots \underset{\psi_d^0}{\overset{\phi_d^0}{\rightleftarrows}} V_d^0 = 0$$

where V_j^0 *has dimension* m_j, *and satisfying*

$$\begin{aligned} B - \psi_1^0\phi_1^0 &= \xi_1 1, \\ \phi_j^0\psi_j^0 - \psi_{j+1}^0\phi_{j+1}^0 &= (\xi_{j+1} - \xi_j)1, \qquad (1 \leq j < d), \end{aligned}$$

then B *is in the closure of the conjugacy class of* A.

Proof. — Suppose that $\mathbf{m}$ is obtained from $\mathbf{n}$ by reducing with respect to r_i and s_i for $1 \leq i \leq p$. For each $1 \leq i \leq p$, define vector spaces

$$V_j^i = \begin{cases} K & (r_i \leq j \leq s_i) \\ 0 & (\text{otherwise}) \end{cases}$$

and linear maps

$$0 = V_0^i \underset{\psi_1^i}{\overset{\phi_1^i}{\rightleftarrows}} V_1^i \underset{\psi_2^i}{\overset{\phi_2^i}{\rightleftarrows}} V_2^i \underset{\psi_3^i}{\overset{\phi_3^i}{\rightleftarrows}} \cdots \underset{\psi_d^i}{\overset{\phi_d^i}{\rightleftarrows}} V_d^0 = 0$$

by $\phi_j^i = (\xi_j - \xi_{r_i})1$ and $\psi_j^i = 1$ for $r_i < j \leq s_i$ and $\phi_j^i = 0$ and $\psi_j^i = 0$ otherwise. Bearing in mind that many of the spaces are zero, and using the fact that $\xi_{r_i} = \xi_{s_i+1}$, one easily sees that

$$-\psi_1^i\phi_1^i = \xi_1 1,$$
$$\phi_j^i\psi_j^i - \psi_{j+1}^i\phi_{j+1}^i = (\xi_{j+1} - \xi_j)1, \qquad (1 \leq j < d).$$

Now the vector spaces

$$V_j = \bigoplus_{i=0}^{p} V_p^i$$

have dimension n_j, and the maps ϕ_j and ψ_j given by the direct sums of the ϕ_j^i and the ψ_j^i satisfy condition (iii) in Theorem 2.1. Thus B is in the closure of the conjugacy class of A. □

Now we introduce notation for dealing with collections of conjugacy classes. Let $k \geq 0$ and let $V_1, \ldots, V_k$ be n-dimensional vector spaces. We consider k-tuples $A_1, \ldots, A_k$ with $A_i \in \mathrm{End}(V_i)$. Allowing the possibility that each $V_i = K^n$, this may just be a k-tuple of $n \times n$-matrices.

If $w = (w_1, \ldots, w_k)$ is a collection of positive integers, and $\xi \in K^w$, we say that a k-tuple $A_1, \ldots, A_k$ has *type* ξ provided that $\prod_{j=1}^{w_i}(A_i - \xi_{ij}1) = 0$ for all i. If so, then as in the introduction we define the *dimension vector* of the k-tuple (with respect to ξ) to be the element $\alpha \in \mathbf{Z}^I$, where I is the vertex set of the graph Γ_w, given by $\alpha_0 = n$ and $\alpha_{ij} = \mathrm{rank} \prod_{\ell=1}^{j}(A_i - \xi_{i\ell}1)$.

Note that not all dimension vectors $\alpha \in \mathbf{N}^I$ arise this way. By (7), the α which can arise are the strict elements with

$$(\mathbf{8}) \qquad \alpha_{i,j-1} - \alpha_{ij} \geq \alpha_{i,\ell-1} - \alpha_{i\ell}$$

for all i and $j < \ell$ with $\xi_{ij} = \xi_{i\ell}$.

Clearly the type and dimension vector of a k-tuple are unchanged by conjugation of any of the terms. Thus one can speak about the type of a k-tuple of conjugacy classes in $\mathrm{End}(V_i)$ or $M_n(K)$.

3. Convolution

In order to study local systems on a punctured Riemann sphere, Katz [20] defined a 'middle convolution' operator. A purely algebraic version, called simply 'convolution', was found by Dettweiler and Reiter [12]. In this section we describe the effect of this operator, using our notation for conjugacy classes. (An alternative algebraic version of middle convolution was found by Völklein [38].)

Let G be the group $\langle g_1, \ldots, g_k : g_1 \ldots g_k = 1\rangle$. If $\xi \in (K^*)^w$, we say that a representation $\rho : G \to GL_n(K)$ is of type ξ if the k-tuple $\rho(g_1), \ldots, \rho(g_k)$ is of type ξ, and we define the dimension vector correspondingly. Note that the dimension vector is additive on direct sums, but not necessarily on short exact sequences. The following fact was mentioned in the introduction.

Lemma **3.1.** — *If ρ is a representation of* G *of type ξ and dimension vector α, then* $\xi^{[\alpha]} = 1$.

We say that an irreducible representation of G is *ξ-collapsing* if it is of type ξ, 1-dimensional, and at least $k-1$ of the g_i act as $\xi_{i1}1$. Thus its dimension vector α is either ϵ_0, or there are $1 \le \ell \le k$ and $1 \le r \le w_\ell - 1$ such that

$$\alpha_0 = 1, \quad \alpha_{ij} = \begin{cases} 1 & (i = \ell \text{ and } j \le r) \\ 0 & (\text{otherwise}). \end{cases} \tag{9}$$

Observe that $s_0(\alpha)$ is either negative, or not strict, hence the name 'collapsing'. We say that a representation is *ξ-noncollapsing* if it is of type ξ, and it has no subrepresentation or quotient which is ξ-collapsing.

Define $r_0' : (K^*)^w \to (K^*)^w$ by

$$r_0'(\xi)_{ij} = \begin{cases} 1/\xi_{i1} & (\text{if } j = 1) \\ \frac{\xi_{ij}}{\xi_{i1}^2} \prod_{s=1}^k \xi_{s1} & (\text{if } j > 1). \end{cases}$$

Observe that $r_0'(\xi)^{[s_0(\alpha)]} = \xi^{[\alpha]}$ for any α.

Our formulation of convolution is as follows.

Theorem **3.2** *(Dettweiler and Reiter).* — *Given $\xi \in (K^*)^w$, if $\xi^{[\epsilon_0]} \neq 1$, then there is an equivalence* R_0 *from the category of ξ-noncollapsing representations of* G *to the category of $r_0'(\xi)$-noncollapsing representations of* G*. It acts as on dimension vectors as the reflection s_0. Moreover* $R_0(\rho)$ *is irreducible if and only if ρ is irreducible.*

Proof. — Let $\lambda = \xi^{[\epsilon_0]} = \prod_i \xi_{i1} \neq 1$. For simplicity we write ξ' for $r_0'(\xi)$. Given a representation $\rho : G \to GL(V)$, define $A_1, \ldots, A_k$ by

$$A_{k+1-i} = \frac{1}{\xi_{i1}} \rho(g_i) \in GL(V),$$

so that $\lambda A_k \dots A_2 A_1 = 1$. As in [12], define $G_i \in GL(V^k)$ by the block matrix

$$G_i = \begin{pmatrix} 1 & \dots & 0 & 0 & 0 & \dots & 0 \\ \vdots & \ddots & \vdots & \vdots & \vdots & & \vdots \\ 0 & \dots & 1 & 0 & 0 & \dots & 0 \\ A_1 - 1 & \dots & A_{i-1} - 1 & \lambda A_i & \lambda(A_{i+1} - 1) & \dots & \lambda(A_k - 1) \\ 0 & \dots & 0 & 0 & 1 & \dots & 0 \\ \vdots & & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & \dots & 0 & 0 & 0 & \dots & 1 \end{pmatrix}.$$

Let also $D \in GL(V^k)$ be given by

$$D = \begin{pmatrix} A_1 & 0 & \dots & 0 \\ 0 & A_2 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & A_k \end{pmatrix}.$$

Define subspaces of V^k,

$$\mathscr{K}_V = \mathrm{Ker}(D - 1), \quad \text{and} \quad \mathscr{L}_V = \mathrm{Ker}(G_1 - 1) \cap \dots \cap \mathrm{Ker}(G_k - 1).$$

These are invariant subspaces for the G_i. We define $R_0(\rho)$ to be the representation $\rho' : \langle g_1, \dots, g_k \rangle \to GL(V')$, where

$$V' = V^k / (\mathscr{K}_V + \mathscr{L}_V) \quad \text{and} \quad \rho'(g_{k+1-i}) = \frac{1}{\xi_{k+1-i,1}} \overline{G_i}$$

where $\overline{G_i}$ denotes the endomorphism of V' induced by G_i. This defines R_0 on objects, but it is clear how to extend this to a functor, see [12, Proposition 2.6].

In [12, §3], Dettweiler and Reiter consider the two conditions

(*) $$\bigcap_{j \neq i} \mathrm{Ker}(A_j - 1) \cap \mathrm{Ker}(\tau A_i - 1) = 0, \quad (i = 1, \dots, k, \ \tau \in K^*), \ \text{and}$$

(**) $$\sum_{j \neq i} \mathrm{Im}(A_j - 1) + \mathrm{Im}(\tau A_i - 1) = V, \quad (i = 1, \dots, k, \ \tau \in K^*).$$

Observe that the A_i satisfy condition (*) if and only if V has no nonzero subspace on which at least $k - 1$ of the g_i act as multiplication by ξ_{i1}, and the remaining g_i acts as multiplication by some scalar. Thus the A_i satisfy condition (*) if and only if V has no ξ-collapsing subrepresentation. Dually, it is easy to see that the A_i satisfy (**) if and only if V has no ξ-collapsing quotient representation. Thus V is ξ-noncollapsing if and only if the A_i satisfy (*) and (**).

Now suppose that V is ξ-noncollapsing. Clearly

$$U_\infty = \mathrm{Im}(A_1 - 1) + \cdots + \mathrm{Im}(A_k - 1)$$

is an invariant subspace for the A_i. All the A_i act trivially on the quotient V/U_∞, but $\lambda A_k \dots A_1 = 1$, and $\lambda \neq 1$, and hence $U_\infty = V$. Thus [12, Lemma 4.1(b)] implies that $\overline{G_k} \dots \overline{G_1} = \lambda 1$, so that $\rho'(g_1) \dots \rho'(g_k) = 1$, and $R_0(V)$ is actually a representation of G.

We show that $R_0(V)$ has type ξ' and that if V has dimension α, then V' has dimension $\alpha' = s_0(\alpha)$. First, by [12, Lemma 2.7], $\mathscr{L}_V$ is isomorphic to V, as a vector space. Also

$$\rho(g_i) - \xi_{i1} 1 = \xi_{i1}(A_{k+1-i} - 1)$$

so that

$$\dim V' = \sum_{i=1}^{k} \mathrm{rank}(\rho(g_i) - \xi_{i1} 1) - \dim \mathscr{L} = \sum \alpha_{i1} - \alpha_0 = \alpha'_0.$$

Also,

$$\rho'(g_i) - \xi'_{i1} 1 = \frac{1}{\xi_{i1}}(\overline{G_{k+1-i}} - 1),$$

and

$$\mathrm{rank}(\overline{G_{k+1-i}} - 1) = \mathrm{rank}(A_{k+1-i} - 1)$$

by [12, Lemma 4.2(a)], so that $\alpha'_{i1} = \alpha_{i1}$. Now we have

$$\begin{aligned}
\mathrm{rank} \prod_{j=1}^{\ell} (\rho(g_i) - \xi_{ij} 1) &= \mathrm{rank} \prod_{j=2}^{\ell} (\rho(g_i) - \xi_{ij} 1)|_{\mathrm{Im}(\rho(g_i) - \xi_{i1} 1)} \\
&= \mathrm{rank} \prod_{j=2}^{\ell} \left(A_{k+1-i} - \frac{\xi_{ij}}{\xi_{i1}} 1 \right)\Big|_{\mathrm{Im}(A_{k+1-i} - 1)} \\
&= \mathrm{rank} \prod_{j=2}^{\ell} \left(\overline{G_{k+1-i}} - \frac{\xi_{ij}\lambda}{\xi_{i1}} 1 \right)\Big|_{\mathrm{Im}(\overline{G_{k+1-i}} - 1)} \quad \text{by [12, Lemma 4.1(a)]} \\
&= \mathrm{rank} \prod_{j=2}^{\ell} \left(\rho'(g_i) - \frac{\xi_{ij}\lambda}{\xi_{i1}^2} 1 \right)\Big|_{\mathrm{Im}(\rho'(g_i) - \frac{1}{\xi_{i1}} 1)} \\
&= \mathrm{rank} \prod_{j=1}^{\ell} (\rho'(g_i) - \xi'_{ij} 1).
\end{aligned}$$

It follows that $R_0(V)$ is of type ξ' and $\alpha'_{ij} = \alpha_{ij}$ for $j > 1$. Thus $\alpha' = s_0(\alpha)$, as claimed.

Note also that $R_0(V)$ is ξ'-noncollapsing if and only if the $\overline{G_i}$ satisfy (*) and (**), so [12, Proposition] shows that R_0 sends ξ-noncollapsing representations of G to ξ'-noncollapsing representations of G.

If we temporarily denote this functor by R_0^{ξ}, then there is also a functor $R_0^{\xi'}$ sending ξ'-noncollapsing representations of G to ξ-noncollapsing representations of G (because clearly $r_0'(\xi') = \xi$). By [12, Theorem 3.5 and Proposition 3.2], these functors are inverse equivalences. Now [12, Corollary 3.6] completes the proof. □

4. Rigid case of the Deligne-Simpson Problem

Let $G = \langle g_1, \ldots, g_k : g_1 \ldots g_k = 1\rangle$. If $\xi \in (K^*)^w$ and $\alpha \in \mathbf{N}^I$, we say that a representation of G of type ξ and dimension vector α is *rigid* if, up to isomorphism, it is the only representation of type ξ and dimension vector α.

Given $\xi \in (K^*)^w$, we write S_ξ for the set of strict real roots α for Γ_w with $\xi^{[\alpha]} = 1$, and such that there is no nontrivial decomposition of α as a sum of positive roots $\alpha = \beta + \gamma + \ldots$ with $\xi^{[\beta]} = \xi^{[\gamma]} = \cdots = 1$. We show that S_ξ is the set of dimension vectors of rigid representations of G of type ξ.

Lemma **4.1.** — *An irreducible representation of* G *of type* ξ *and dimension vector* α *is rigid if and only if* $p(\alpha) = 0$.

Proof. — We mentioned this in the introduction. The equivalence of equation (5) and $p(\alpha) = 0$ is easy. Thus the lemma follows from Deligne's observation in [36]. Note that this holds for an arbitrary algebraically closed field – see [37]. □

Lemma **4.2** *(Scott).* — *If an irreducible representation of* G *has type* ξ, *dimension* $\alpha \neq \epsilon_0$, *and* $\xi^{[\epsilon_0]} = 1$, *then* $2\alpha_0 \leq \sum_{i=1}^k \alpha_{i1}$, *so that* $s_0(\alpha) \geq \alpha$.

Proof. — Let the representation be $\rho : G \to GL_n(K)$, with $n = \alpha_0$. The condition on ξ implies that one obtains another irreducible representation $\sigma : G \to GL_n(K)$ by defining $\sigma(g_i) = \frac{1}{\xi_{i1}}\rho(g_i)$. Now the condition that $\alpha \neq \epsilon_0$ ensures that σ is not the 1-dimensional trivial representation. Then, since it is irreducible, a result of Scott [34, Theorem 1] gives

$$2n \leq \sum_{i=1}^{k} \operatorname{rank}(\sigma(g_i) - 1) = \sum_{i=1}^{k} \operatorname{rank}(\rho(g_i) - \xi_{i1} 1) = \sum_{i=1}^{k} \alpha_{i1},$$

as required. □

The convolution functor acts on dimension vectors as the reflection s_0 corresponding to the vertex 0 in Γ_w. We show that reflections at other vertices can be obtained by permuting the components of $\xi \in (K^*)^w$. Suppose $1 \leq i \leq k$ and $1 \leq j \leq$

$w_i - 1$, and let $v = [i,j]$ be the corresponding vertex in Γ_w. Define $r'_v : (K^*)^w \to (K^*)^w$ by

$$r'_v(\xi)_{pq} = \begin{cases} \xi_{ij} & \text{(if } p = i \text{ and } q = j+1) \\ \xi_{i,j+1} & \text{(if } p = i \text{ and } q = j) \\ \xi_{pq} & \text{(otherwise).} \end{cases}$$

Note that $r'_v(\xi)^{[s_v(\alpha)]} = \xi^{[\alpha]}$ for any α.

Lemma **4.3.** — *Let* $v = [i,j]$ *and let* $\alpha \in \mathbf{N}^I$.

(i) *If there is a representation of type* ξ *and dimension vector* α, *and* $s_v(\alpha) < \alpha$, *then* $\xi_{ij} \neq \xi_{i,j+1}$, *that is,* $\xi^{[\epsilon_v]} \neq 1$.

(ii) *If* $\xi^{[\epsilon_v]} \neq 1$ *then representations of type* ξ *with dimension vector* α *are exactly the same as representations of type* $r'_v(\xi)$ *with dimension vector* $s_v(\alpha)$.

Proof. — (i) This follows from equation (8).

(ii) If $\xi^{[\epsilon_v]} \neq 1$, then $\xi_{ij} \neq \xi_{i,j+1}$. Since $r'_v(\xi)$ is obtained from ξ by exchanging ξ_{ij} and $\xi_{i,j+1}$, it follows from the definition that a representation ρ has type ξ if and only if it has type $r'_v(\xi)$. Suppose that ρ has dimension vector α with respect to ξ and α' with respect to $r'_v(\xi)$. Clearly we have $\alpha_{i\ell} = \alpha'_{i\ell}$ for $\ell \neq j$, since the products $(\rho(g_i) - \xi_{i1}1)\dots(\rho(g_i) - \xi_{i\ell}1)$, and $(\rho(g_i) - r'_v(\xi)_{i1}1)\dots(\rho(g_i) - r'_v(\xi)_{i\ell}1)$ are the same, except possibly that two terms are exchanged. To relate α_{ij} and α'_{ij}, let $W = \operatorname{Im}(\rho(g_i) - \xi_{i1}1)\dots(\rho(g_i) - \xi_{i,j-1}1)$, so $\dim W = \alpha_{i,j-1}$. Then

$$\alpha_{ij} = \dim(\rho(g_i) - \xi_{ij}1)(W) = \alpha_{i,j-1} - x,$$

where x is the multiplicity of ξ_{ij} as an eigenvalue of $\rho(g_i)$ on W. Moreover, since $\xi_{ij} \neq \xi_{i,j+1}$, we have

$$\alpha_{i,j+1} = \dim(\rho(g_i) - \xi_{ij}1)(\rho(g_i) - \xi_{i,j+1}1)(W) = \alpha_{i,j-1} - x - y,$$

where y is the multiplicity of $\xi_{i,j+1}$ as an eigenvalue of $\rho(g_i)$ on W. On the other hand

$$\alpha'_{ij} = \dim(\rho(g_i) - \xi_{i,j+1}1)(W) = \alpha_{i,j-1} - y = \alpha_{i,j-1} + \alpha_{i,j+1} - \alpha_{ij}.$$

Thus $\alpha' = s_v(\alpha)$. □

Lemma **4.4.** — *Let* $\xi \in (K^*)^w$ *and* $\alpha \in \mathbf{N}^I$. *Let* v *be a vertex in* Γ_w.

(i) *If* $\alpha \in S_\xi$, $\alpha \neq \epsilon_0$ *and* $s_v(\alpha) < \alpha$, *then* $\xi^{[\epsilon_v]} \neq 1$.

(ii) *If* α *and* $s_v(\alpha)$ *are strict and* $\xi^{[\epsilon_v]} \neq 1$, *then* $\alpha \in S_\xi \Leftrightarrow s_v(\alpha) \in S_{r'_v(\xi)}$.

Proof. — (i) Since α is strict and $\alpha \neq \epsilon_0$, we have $\alpha \neq \epsilon_v$, so that $s_v(\alpha)$ is a positive root. But now there is a decomposition

$$\alpha = s_v(\alpha) + \epsilon_v + \cdots + \epsilon_v.$$

Since $\xi^{[\alpha]} = 1$, if $\xi^{[\epsilon_v]} = 1$, then also $\xi^{[s_v(\alpha)]} = 1$, and this decomposition contradicts the fact that $\alpha \in S_\xi$.

(ii) By symmetry it suffices to prove one implication. Applying s_v to any decomposition of α as a sum of positive roots, one obtains a decomposition of $s_v(\alpha)$ as a sum of roots. The only difficulty is to ensure that they are all positive. But the only positive root which changes sign is ϵ_v, and this is ruled out since $\xi^{[\epsilon_v]} \neq 1$. □

Theorem 1.5 can now be formulated as follows.

Theorem **4.5.** — *If* $\xi \in (K^*)^w$, *then there is a rigid irreducible representation of* G *of type* ξ *and dimension* α *if and only if* $\alpha \in S_\xi$.

Proof. — We prove this for all ξ by induction on α. Either condition requires α to be strict, so the smallest possible dimension vector is the coordinate vector ϵ_0. For the equivalence in this case, observe that $\xi^{[\alpha]} = \prod_{i=1}^{k} \xi_{i1}$, and if this is equal to 1, then the representation $\rho(g_i) = \xi_{i1} 1$ is clearly irreducible and rigid. Thus suppose that $\alpha \neq \epsilon_0$.

By Lemma 4.1, either the existence of a rigid irreducible representation, or the assumption that $\alpha \in S_\xi$ implies that $p(\alpha) = 0$. Then

$$\sum_{v \in I} \alpha_v(\alpha, \epsilon_v) = (\alpha, \alpha) = 2,$$

so there must be a vertex v with $(\alpha, \epsilon_v) > 0$, and therefore $s_v(\alpha) < \alpha$.

If there is a vertex of the form $v = [i, j]$ with $s_v(\alpha) < \alpha$, then Lemmas 4.3 and 4.4, show that either condition ensures that $\xi^{[\epsilon_v]} \neq 1$, and they then reduce the problem to the corresponding one for $r'_v(\xi)$ and $s_v(\alpha)$, so one can use induction.

Thus suppose that the only vertex v with $s_v(\alpha) < \alpha$ is $v = 0$. In particular α is not of the form (9), which is reduced by the reflection at $v = [\ell, r]$.

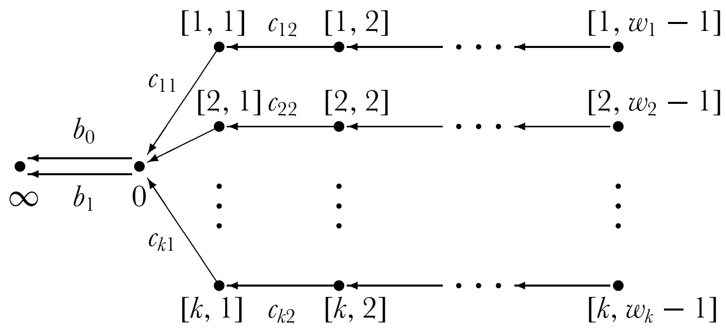
If there is a rigid irreducible representation of G of type ξ and dimension vector α, then $\xi^{[\epsilon_0]} \neq 1$ by Lemma 4.2. Moreover, since α is not ϵ_0 or of the form (9), the representation is not ξ-collapsing, so since it is irreducible, it is ξ-noncollapsing. Thus by convolution, Theorem 3.2, there is an irreducible representation of type $r'_0(\xi)$ and dimension vector $s_0(\alpha)$. It is rigid by Lemma 4.1, since $p(s_0(\alpha)) = p(\alpha)$. Then by induction $s_0(\alpha) \in S_{r'_0(\xi)}$, and so $\alpha \in S_\xi$ by Lemma 4.4.

Conversely, suppose that $\alpha \in S_\xi$. Since α is a positive root and $\alpha \neq \epsilon_0$, $s_0(\alpha)$ is a positive root. Moreover $s_0(\alpha)$ is strict, for otherwise it must be of the form (4), and hence α must be of the form (9). Thus by Lemma 4.4, $\xi^{[\epsilon_0]} \neq 1$ and $s_0(\alpha) \in S_{r'_0(\xi)}$. Now by induction there is a rigid irreducible representation of G of type $r'_0(\xi)$ and dimension vector $s_0(\alpha)$. This representation is not $r'_0(\xi)$-collapsing, since $\alpha = s_0(s_0(\alpha))$ is strict, so since it is irreducible it is $r'_0(\xi)$-noncollapsing. Thus by Theorem 3.2 there is an irreducible representation of G of type ξ and dimension vector α. It is rigid by Lemma 4.1. □

5. Squids and the fundamental region

To parametrize families of parabolic bundles on $\mathbf{P}^1$ it is possible to use a quot-scheme construction, but it is equivalent to study a space of representations of a certain algebra $S_{D,w}$, called a *squid*. By combining general position arguments with known results about the representation theory of the squid, we will show the existence of indecomposable parabolic bundles in some cases.

Given a collection $D = (a_1, \ldots, a_k)$ of distinct points of $\mathbf{P}^1$, and a collection $w = (w_1, \ldots, w_k)$ of positive integers, we define $S_{D,w}$ to be the finite-dimensional associative algebra given by the quiver Q_w



modulo the relations

$$(\mathbf{10}) \qquad (\lambda_{i0} b_0 + \lambda_{i1} b_1) c_{i1} = 0,$$

where $a_i = [\lambda_{i0} : \lambda_{i1}] \in \mathbf{P}^1$ for $i = 1, \ldots, k$. Squids were first studied by Brenner and Butler [5] in connection with tilting theory. Recall that a representation X of $S_{D,w}$ is the same as a representation of the quiver Q_w, by vector spaces and linear maps, which satisfies the relations (10). The quiver with vertex set $\{\infty, 0\}$ and arrows $\{b_0, b_1\}$ is a so-called *Kronecker quiver*, since the classification of its representations is the same as Kronecker's classification of pencils of matrices. We say that a representation of $S_{D,w}$ is *Kronecker-preinjective*, if its restriction to this Kronecker quiver is preinjective. That is, if $\lambda_0 b_0 + \lambda_1 b_1$ is a surjective linear map for all $[\lambda_0 : \lambda_1] \in \mathbf{P}^1$.

We define the *dimension type* of a representation X of $S_{D,w}$ (or Q_w) in a nonstandard way. If X_v denotes the vector space at vertex v, then the dimension type is defined to be the pair (α, d) where $d = \dim X_\infty$, and α is the dimension vector for the graph Γ_w given by $\alpha_{ij} = \dim X_{[i,j]}$ and $\alpha_0 = \dim X_0 - \dim X_\infty$.

Representations of Q_w of dimension type (α, d) are given by elements of the set $B \times C$, where

$$B = M_{d\times(\alpha_0+d)}(K) \times M_{d\times(\alpha_0+d)}(K)$$

and

$$C = \prod_{i=1}^{k} \left(M_{(\alpha_0+d)\times\alpha_{i1}}(K) \times \prod_{j=2}^{w_i-1} M_{\alpha_{i,j-1}\times\alpha_{ij}}(K) \right).$$

We denote elements of B and C by $b = (b_0, b_1)$ and $c = (c_{ij})$. Representations of the squid $S_{D,w}$ are given by elements of the closed subset

$$S(\alpha, d) = \{(b, c) \in B \times C \mid (\lambda_{i0}b_0 + \lambda_{i1}b_1)c_{i1} = 0 \text{ for all } i\},$$

and the Kronecker-preinjective representations form a subset $KI(\alpha, d)$ of $S(\alpha, d)$. The algebraic group

$$GL(\alpha, d) = GL_d(K) \times GL_{\alpha_0+d}(K) \times \prod_{i,j} GL_{\alpha_{ij}}(K)$$

acts on each of these, with the orbits corresponding to isomorphism classes.

Lemma **5.1.** — $KI(\alpha, d)$ *is an open subset of* $S(\alpha, d)$. *It is an irreducible and smooth variety of dimension*

$$\dim KI(\alpha, d) = \dim GL(\alpha, d) - q(\alpha).$$

Proof. — Representations of the Kronecker quiver of dimension vector (d, α_0+d) are parametrized by the set B. We show that the preinjective representations form an open subset $U \subseteq B$. Let $\mathbf{P}((K^d)^*)$ denote the projective space of nonzero linear forms ξ on K^d, up to rescaling. Let Z be the closed subset of $B\times\mathbf{P}^1\times\mathbf{P}((K^d)^*)$ consisting of the elements $((b_0, b_1), [\lambda_0 : \lambda_1], \xi)$ with $\xi(\lambda_0 b_0 + \lambda_1 b_1) = 0$. Clearly $B \setminus U$ is the image of Z under the projection to B, so by projectivity it is closed. (We thank a referee for this argument, which is shorter than our original one.)

Now $KI(\alpha, d) = S(\alpha, d) \cap (U \times C)$, so $KI(\alpha, d)$ is open in $S(\alpha, d)$.

Let $V = \prod_{i=1}^{k} M_{d\times\alpha_{i1}}(K)$ and let $h : U \times C \to U \times V$ be the map sending a pair (b, c) to to the element $(b, (\lambda_{i0}b_0 + \lambda_{i1}b_1)c_{i1})$. Clearly h is a homomorphism of trivial vector bundles over U, and using the fact that $\lambda_{i0}b_0 + \lambda_{i1}b_1$ is a surjective linear map for all $b \in \mathrm{U}$, it is easy to see that h is surjective on fibres. It follows that the kernel of h is a vector bundle over U of rank

$$r = \sum_{i=1}^{k} \left(\alpha_0 \alpha_{i1} + \sum_{j=2}^{w_i - 1} \alpha_{i,j-1} \alpha_{ij} \right).$$

However, this kernel is clearly equal to $\mathrm{KI}(\alpha, d)$. The assertions follow. □

Because $\mathrm{KI}(\alpha, d)$ is irreducible, one can ask about the properties of a general element of it. Is it indecomposable? What is the dimension of its endomorphism ring? And so on. Note that the closure of $\mathrm{KI}(\alpha, d)$ is an irreducible component of $\mathrm{S}(\alpha, d)$, so such questions fit into the theory initiated in [10]. Given dimension types (β, e) and (γ, f), we define $\mathrm{ext}((\beta, e), (\gamma, f))$ to be the general dimension of $\mathrm{Ext}^1(\mathrm{X}, \mathrm{Y})$ with X in $\mathrm{KI}(\beta, e)$ and Y in $\mathrm{KI}(\gamma, f)$. By the main theorems of [10], one has the following canonical decomposition.

Theorem **5.2.** — *Given a dimension type* (α, d), *there are decompositions*

$$\alpha = \beta^{(1)} + \beta^{(2)} + \dots, \qquad d = d^{(1)} + d^{(2)} + \dots,$$

unique up to simultaneous reordering, such that the general element of $\mathrm{KI}(\alpha, d)$ *is a direct sum of indecomposables of dimension types* $(\beta^{(i)}, d^{(i)})$. *These decompositions of* α *and* d *are characterized by the property that the general elements of* $\mathrm{KI}(\beta^{(i)}, d^{(i)})$ *are indecomposable, and* $\mathrm{ext}((\beta^{(i)}, d^{(i)}), (\beta^{(j)}, d^{(j)})) = 0$ *for* $i \neq j$.

Following the argument of [25, Theorem 3.3], we have the following result.

Lemma **5.3.** — *If* α *is in the fundamental region, nonisotropic* (*that is* $q(\alpha) \neq 0$), *or indivisible* (*that is, its components have no common divisor*), *then the general element of* $\mathrm{KI}(\alpha, d)$ *is indecomposable.*

Proof. — If the general element of $\mathrm{KI}(\alpha, d)$ decomposes, there are decompositions $\alpha = \beta + \gamma$ and $d = e + f$, such that the map

$$\phi : \mathrm{GL}(\alpha, d) \times \mathrm{KI}(\beta, e) \times \mathrm{KI}(\gamma, f) \to \mathrm{KI}(\alpha, d), \quad (g, x, y) \mapsto g \cdot (x \oplus y)$$

is dominant. But ϕ is constant on the orbits of the free action of $\mathrm{GL}(\beta, e) \times \mathrm{GL}(\gamma, f)$ on the domain of ϕ given by

$$(g_1, g_2) \cdot (g, x, y) = (g \begin{pmatrix} g_1 & 0 \\ 0 & g_2 \end{pmatrix}^{-1}, g_1 x, g_2 y).$$

Thus

$$\dim \mathrm{KI}(\alpha, d) \leq \big(\dim \mathrm{GL}(\alpha, d) + \dim \mathrm{KI}(\beta, e) + \dim \mathrm{KI}(\gamma, f)\big) \\ - \big(\dim \mathrm{GL}(\beta, e) + \dim \mathrm{GL}(\gamma, f)\big)\,,$$

so $q(\alpha) \geq q(\beta) + q(\gamma)$. However, since α is in the fundamental region, the lemmas in [25, §3.1] now imply that α is isotropic and divisible. □

Let A_0 be the path algebra of the Kronecker quiver with vertex set $\{\infty, 0\}$ and arrows $\{b_0, b_1\}$. The regular modules for A_0 form a stable separating family $\mathscr{T}$, see [33, §3.2]. For $1 \leq i \leq k$, let E_i be the regular module of dimension vector (1,1) in which $\lambda_{i0} b_0 + \lambda_{i1} b_1 = 0$, and let K_i be the full subquiver of Q_w on the vertices $[i, 1], \ldots, [i, w_i - 1]$. Clearly $S_{D,w}$ is identified with the tubular extension $A_0[E_i, K_i]_{i=1}^k$ in the sense of Ringel [33, §4.7].

Note that since the root of the branch K_i is the unique sink in K_i, the length function ℓ_{K_i} (see [33, §4.4]) is equal to $w_i - j$ at vertex $[i, j]$. This is the same as the dimension vector of the direct sum of all indecomposable projective representations of K_i. Thus, if M is an indecomposable module whose support is contained in K_i, then $\langle \ell_{K_i}, \underline{\dim} M\rangle > 0$.

We define module classes $\mathscr{P}_0$, $\mathscr{T}_0$ and $\mathscr{Q}_0$ in the category $S_{D,w}$-mod as follows. Let $\mathscr{P}_0$ be the class of preprojective A_0-modules, let $\mathscr{T}_0$ be the class given by all indecomposable modules M such that the underlying Kronecker-module is nonzero and regular, and let $\mathscr{Q}_0$ be the class of all Kronecker-preinjective modules.

By Ringel [33, Theorem 4.7(1)], every indecomposable $S_{D,w}$-module belongs to $\mathscr{P}_0$, $\mathscr{T}_0$ or $\mathscr{Q}_0$, and $\mathscr{T}_0$ is a separating tubular family, separating $\mathscr{P}_0$ from $\mathscr{Q}_0$. In particular, there are no nonzero maps from modules in $\mathscr{Q}_0$ to modules in $\mathscr{P}_0$ or $\mathscr{T}_0$. Now since all indecomposable projective modules belong to $\mathscr{P}_0$ or $\mathscr{T}_0$, Ringel [33, 2.4(1*)] shows that all Kronecker-preinjective representations have injective dimension ≤ 1.

Lemma **5.4.** — *If α is in the fundamental region and $d \geq 0$, then there are infinitely many isomorphism classes of indecomposable Kronecker-preinjective representations of* $S_{D,w}$ *of dimension type* (α, d).

Proof. — First suppose that α is nonisotropic. Thus the general element of $\mathrm{KI}(\alpha, d)$ is indecomposable. Now isomorphism classes correspond to orbits of the group $\mathrm{GL}(\alpha, d)$, or actually its quotient $\mathrm{GL}(\alpha, d)/K^*$. If there were only finitely many isomorphism classes of indecomposables, one would have

$$\dim \mathrm{GL}(\alpha, d) - q(\alpha) = \dim \mathrm{KI}(\alpha, d) \leq \dim \mathrm{GL}(\alpha, d) - 1$$

so $q(\alpha) \geq 1$, which is impossible for α in the fundamental region.

Now suppose that α is isotropic. By deleting any vertices v of Γ_w with $\alpha_v = 0$, we may suppose that Γ_w is an extended Dynkin diagram. Thus, up to reordering, and omitting weights which are equal to 1, we have that w is of the form (2,2,2,2), (3,3,3), (4,4,2) or (6,3,2). Thus the squid $A = S_{D,w}$ is a tubular algebra [33, §5].

Let χ_A be the quadratic form on $K_0(A)$ for the algebra A, see [33, §2.4]. If S_v denotes the simple A-module corresponding to vertex v in Q_w, then

$$\dim \operatorname{Ext}^1(S_0, S_\infty) = 2, \quad \dim \operatorname{Ext}^1(S_{[i,1]}, S_0) = 1,$$
$$\dim \operatorname{Ext}^1(S_{[i,j]}, S_{[i,j-1]}) = 1, \quad \dim \operatorname{Ext}^2(S_{[i,1]}, S_\infty) = 1$$

for $1 \le i \le k$ and $1 < j \le w_i - 1$, and all other Ext spaces between simples are zero. If (α, d) is a dimension type, the corresponding element of $K_0(A)$ is

$$c = d[S_\infty] + (\alpha_0 + d)[S_0] + \sum_{i=1}^{k} \sum_{j=1}^{w_i - 1} \alpha_{ij}[S_{[i,j]}],$$

and using the homological formula for χ_A, one computes

$$\begin{aligned}\chi_A(c) &= d^2 + (\alpha_0 + d)^2 + \sum_{i,j} \alpha_{ij}^2 - 2d(\alpha_0 + d) - \sum_i (\alpha_0 + d)\alpha_{i1} \\ &\quad - \sum_{i,\, j>1} \alpha_{i,j-1}\alpha_{ij} + \sum_i d\alpha_{i1} \\ &= q(\alpha).\end{aligned}$$

Thus if α is isotropic, then $\chi_A(c) = 0$, so by [33, Theorem 5.2.6] there are infinitely many indecomposable A-modules of dimension type (α, d). □

We now consider parabolic bundles $\mathbf{E}$ on $\mathbf{P}^1$ of weight type (D, w), where the underlying vector bundle E is algebraic, or equivalently, if the base field K is the field of complex numbers, holomorphic.

Tilting theory [5] led first to the theory of derived equivalences between algebras, and then to derived equivalences between categories of coherent sheaves on weighted projective lines and certain algebras, the canonical algebras [26], or more generally quasi-tilted algebra [17], including squids. The following lemma is a special case of the theory.

Lemma **5.5.** — *There is an equivalence between the category of parabolic bundles* $\mathbf{E}$ *on* $\mathbf{P}^1$ *of weight type* (D, w) *such that* E^* *is generated by global sections, and the category of Kronecker-preinjective representations of the squid* $S_{D,w}$ *in which the linear maps* c_{ij} *are all injective.*

Under this equivalence, parabolic bundles of dimension vector α *and degree* $-d$ *correspond to representations of the squid of dimension type* (α, d).

Proof. — This follows immediately from the equivalence between the category of vector bundles E on $\mathbf{P}^1$ such that E^* is generated by global sections, and the category of preinjective representations of the Kronecker quiver, which we now recall.

Let $\mathcal{O}$ be the trivial bundle on $\mathbf{P}^1$ and $\mathcal{O}(1)$ the universal quotient bundle. There is a natural map $\mathcal{O}^2 \to \mathcal{O}(1)$, and let $f_0, f_1 \in \mathrm{Hom}(\mathcal{O}, \mathcal{O}(1))$ be its two components.

The equivalence sends a preinjective representation X of the Kronecker quiver,

$$X_0 \overset{b_0}{\underset{b_1}{\rightrightarrows}} X_\infty,$$

to the kernel E of the map of vector bundles

$$f_1 \otimes b_0 - f_0 \otimes b_1 : \mathcal{O} \otimes X_0 \to \mathcal{O}(1) \otimes X_\infty.$$

The induced map on fibres over $a = [\lambda_0 : \lambda_1]$ is the map

$$\begin{aligned} &X_0 \to [K^2/K(\lambda_0, \lambda_1)] \otimes X_\infty, \\ &x \mapsto [K(\lambda_0, \lambda_1) + (0,1)] \otimes b_0(x) - [K(\lambda_0, \lambda_1) + (1,0)] \otimes b_1(x), \end{aligned}$$

which up to a scalar is the map $\lambda_0 b_0 + \lambda_1 b_1 \in \mathrm{Hom}(X_0, X_\infty)$. Since X is preinjective, $f_1 \otimes b_0 - f_0 \otimes b_1$ is surjective on fibres, and hence E is a vector bundle. Clearly E^* is generated by global sections. Moreover, using the fact that $\mathrm{Hom}(\mathcal{O}(1), \mathcal{O}) = \mathrm{Ext}^1(\mathcal{O}(1), \mathcal{O}) = 0$, it is easy to see that this functor is full and faithful. It is also dense. Namely, recall that for each $i \geq 0$ there is an indecomposable preinjective Kronecker representation with dimension vector

$$i+1 \rightrightarrows i$$

(see for example [33, §3.2]). Under this functor it gets sent to a vector bundle of rank 1 and degree $-i$, which must therefore be $\mathcal{O}(-i)$.

Finally, to see the connection between parabolic bundles and squids, observe that if X is a Kronecker-preinjective representations of $S_{D,w}$ in which the linear maps c_{ij} are all injective, and if E is the corresponding vector bundle, then the squid relation $(\lambda_{i0} b_0 + \lambda_{i1} b_1) c_{i1} = 0$ ensures that the spaces

$$\mathrm{Im}(c_{i1}) \supseteq \mathrm{Im}(c_{i1} c_{i2}) \supseteq \dots$$

are all contained in the fibre E_{a_i}, so define a parabolic bundle $\mathbf{E}$. Conversely, given a parabolic bundle $\mathbf{E}$ in which the underlying vector bundle E arises from a preinjective Kronecker module, one can extend this to a module for $S_{D,w}$ by letting the vector space at $[i, j]$ be E_{ij}, and letting the maps c_{ij} be the natural inclusions. □

A parabolic bundle $\mathbf{E} = (E, E_{ij})$ can be tensored with any vector bundle F, giving the parabolic bundle $\mathbf{E} \otimes F$, with underlying vector bundle $E \otimes F$ and subspaces $E_{ij} \otimes F_{a_i}$. Clearly the operation of tensoring with a line bundle is invertible, so must preserve indecomposability.

Lemma **5.6.** — *If $d \in \mathbf{Z}$ and α is in the fundamental region, then there are infinitely many isomorphism classes of indecomposable parabolic bundles on $\mathbf{P}^1$ of weight type (D, w) with dimension vector α and degree d.*

Proof. — If α is in the fundamental region, then certainly $\alpha_0 > 0$, and hence one can find an integer N such that $d - N\alpha_0 < 0$. By Lemmas 5.5 and 5.4 one can find infinitely many indecomposable parabolic bundles of dimension vector α and degree $d - N\alpha_0$. Now by tensoring with $\mathcal{O}(N)$ one obtains indecomposable parabolic bundles of dimension vector α and degree d. □

Note that, instead of using Ringel's work on tubular algebras, it would have been possible to use the work of Lenzing and Meltzer [29].

6. Riemann-Hilbert correspondence

Let X be a connected Riemann surface and $D = (a_1, \ldots, a_k)$ a collection of distinct points of X. We denote by $X \setminus D$ the Riemann surface obtained by deleting the points of D from X. We write $\mathcal{O}_X$ for the sheaf of holomorphic functions on X, and $\Omega^1_X(\log D)$ for the sheaf of *logarithmic 1-forms* on X, that is, the subsheaf of $j_*\Omega^1_{X\setminus D}$ generated by Ω^1_X and dx_i/x_i, where j is the inclusion of $X \setminus D$ in X and x_i is a local coordinate centred at a_i. See [11, §II.3]. If E is a vector bundle on X, a *logarithmic connection*

$$\nabla : E \to E \otimes \Omega^1_X(\log D)$$

is a morphism of sheaves of vector spaces which satisfies the Leibnitz rule

$$\nabla(fe) = e \otimes df + f\nabla(e)$$

for local sections f of $\mathcal{O}_X$ and e of E. The connection ∇ has residues

$$\mathrm{Res}_{a_i} \nabla \in \mathrm{End}(E_{a_i}).$$

Recall that a *transversal* to $\mathbf{Z}$ in $\mathbf{C}$ is a subset $T \subset \mathbf{C}$ with the property that for any $z \in \mathbf{C}$ there is a unique element $t \in T$ such that $z - t \in \mathbf{Z}$. Equivalently, T is a transversal if the function $t \mapsto \exp(-2\pi\sqrt{-1}\,t)$ is a bijection from T to $\mathbf{C}^*$. An example is given by $T = \{z \in \mathbf{C} \mid 0 \leq \Re e(z) < 1\}$, but clearly any subset of $\mathbf{C}$ whose elements never differ by a nonzero integer can be extended to a transversal.

Let $\mathbf{T} = (T_1, \ldots, T_k)$ be a collection of transversals. We say that a logarithmic connection $\nabla : E \to E \otimes \Omega^1_X(\log D)$ *has eigenvalues in* $\mathbf{T}$ if the eigenvalues of $\mathrm{Res}_{a_i} \nabla$ are in T_i for all i. We write $\mathrm{conn}_{D,\mathbf{T}}(X)$ for the category whose objects are the pairs (E, ∇) consisting of a vector bundle E on X and a logarithmic connection $\nabla : E \to E \otimes \Omega^1_X(\log D)$ having residues in $\mathbf{T}$, and whose morphisms are the vector bundle homomorphisms commuting with the connections.

We need the following version of the Riemann-Hilbert correspondence, which is presumably already implicit in the work of Deligne [11] and others.

Theorem **6.1.** — *Monodromy gives an equivalence from the category* $\mathrm{conn}_{D,\mathbf{T}}(X)$ *to the category of representations of* $\pi_1(X \setminus D)$. *Moreover, any morphism in* $\mathrm{conn}_{D,\mathbf{T}}(X)$ *has constant rank.*

Proof. — If M is a connected complex manifold, then monodromy gives an equivalence from the category of vector bundles on M equipped with an integrable holomorphic connection to the category of representations of $\pi_1(M)$, see [11, §I.1,I.2]. In particular this applies to $X \setminus D$, and since it has complex dimension 1, integrability is automatic. It thus suffices to prove that restriction defines an equivalence from $\mathrm{conn}_{D,\mathbf{T}}(X)$ to the category of vector bundles on $X \setminus D$ equipped with a holomorphic connection.

Clearly the restriction functor is faithful. To see that it is dense one uses Manin's construction – a good reference is Malgrange [27, Theorem 4.4]. It thus remains to check that any morphism

$$\theta : (G'|_{X\setminus D}, \nabla'|_{X\setminus D}) \to (G|_{X\setminus D}, \nabla|_{X\setminus D})$$

extends to a morphism $(G', \nabla') \to (G, \nabla)$ which has constant rank. Now this problem is local on X, so we may assume that X is a disk, D is its centre and G and G' are trivial bundles. Moreover, we may assume that ∇ and ∇' have connection forms $\omega = \Gamma \frac{dx}{x}$, $\omega' = \Gamma' \frac{dx}{x}$ given by constant matrices Γ, Γ' (see [27]). Then θ is given by a holomorphic invertible matrix S on $X \setminus D$ which satisfies $dS = S\omega' - \omega S$. Developing S in its Laurent series $S = \sum_{i=-\infty}^{\infty} S_i x^i$, we obtain

$$\sum_i i S_i x^{i-1} dx = \sum_i (S_i \Gamma' - \Gamma S_i) x^{i-1} dx,$$

and hence

$$(\Gamma + i1)S_i = S_i \Gamma'.$$

By assumption the eigenvalues of Γ and Γ' belong to the same transversal to $\mathbf{Z}$ in $\mathbf{C}$, which implies that $S_i = 0$ for $i \neq 0$. The result follows. □

In particular Theorem 6.1 shows that the category $\mathrm{conn}_{D,\mathbf{T}}(X)$ is abelian. The following fact is well-known. See for example [16, Proposition 1.4.1].

Lemma **6.2.** — *If* E *is a vector bundle on* X *and* ∇ *is a connection with eigenvalues in* $\mathbf{T}$*, then the monodromy around* a_i *is conjugate to* $\exp(-2\pi\sqrt{-1}\,\mathrm{Res}_{a_i}\nabla)$.

Now if $X = \mathbf{P}^1$, we have

$$\pi_1(\mathbf{P}^1 \setminus D) = G = \langle g_1, \ldots, g_k : g_1 \ldots g_k = 1\rangle$$

where the g_i are suitable loops from a fixed base point about the a_i. Thus, given any representation $\rho : \pi_1(\mathbf{P}^1 \setminus D) \to \mathrm{GL}_n(\mathbf{C})$, one obtains matrices $\rho(g_1), \ldots, \rho(g_k)$ whose product is the identity.

7. Connections on parabolic bundles

Let X be a connected Riemann surface, $D = (a_1, \ldots, a_k)$ a collection of distinct points of X, $w = (w_1, \ldots, w_k)$ a collection of positive integers and $\zeta \in \mathbf{C}^w$.

If $\mathbf{E} = (E, E_{ij})$ is a parabolic bundle on X of weight type (D, w), we say that a logarithmic connection $\nabla : E \to E \otimes \Omega^1_X(\log D)$ is a ζ*-connection* on $\mathbf{E}$ if

$$(\mathbf{11}) \qquad (\mathrm{Res}_{a_i}\nabla - \zeta_{ij}1)(E_{i,j-1}) \subseteq E_{ij}$$

for all $1 \le i \le k$ and $1 \le j \le w_i$, where by convention E_{i0} is the fibre E_{a_i} and $E_{i,w_i} = 0$. Equivalently, if the E_{ij} are invariant subspaces for $\mathrm{Res}_{a_i}\nabla$, and $\mathrm{Res}_{a_i}\nabla$ acts on $E_{i,j-1}/E_{ij}$ as multiplication by ζ_{ij}.

If $\nabla : E \to E \otimes \Omega^1_X(\log D)$ is a logarithmic connection on a vector bundle E, then the residues $\mathrm{Res}_{a_1}\nabla, \ldots, \mathrm{Res}_{a_k}\nabla$ form a k-tuple of endomorphisms of vector spaces of the same dimension, and so one can talk about those logarithmic connections whose residues have type ζ, in the sense of Section 2, and the associated dimension vector α with $\alpha_0 = \mathrm{rank}\, E$. We write $\mathrm{conn}_{D,w,\zeta}(X)$ for the category whose objects are the pairs (E, ∇) consisting of a vector bundle E on X and a logarithmic connection $\nabla : E \to E \otimes \Omega^1_X(\log D)$ whose residues have type ζ. Provided that the elements ζ_{ij} for fixed i never differ by a nonzero integer, one can find transversals $\mathbf{T} = (T_1, \ldots, T_k)$ such that $\zeta_{ij} \in T_i$ for all i, j, and then $\mathrm{conn}_{D,w,\zeta}(X)$ is a full subcategory of $\mathrm{conn}_{D,\mathbf{T}}(X)$, and is an abelian category.

If (E, ∇) is an object in $\mathrm{conn}_{D,w,\zeta}(X)$, then ∇ is a ζ-connection on the parabolic bundle $\mathbf{E} = (E, E_{ij})$, where

$$(\mathbf{12}) \qquad E_{ij} = \mathrm{Im}(\mathrm{Res}_{a_i}\nabla - \zeta_{i1}1) \ldots (\mathrm{Res}_{a_i}\nabla - \zeta_{ij}1).$$

Moreover the dimension vector of $\mathbf{E}$ is the same as the dimension vector of the residues of ∇.

Conversely, it is clear that if ∇ is a ζ-connection on a parabolic bundle $\mathbf{E}$, then (E, ∇) is an object in $\mathbf{conn}_{D,w,\zeta}(X)$. However, the dimension vector of $\mathbf{E}$ will not be the same as the dimension vector associated to the residues of ∇ if any of the inclusions in (11) are strict.

By analogy with the 'parabolic degree' in [28], we define the ζ*-degree* of a parabolic bundle $\mathbf{E}$ of dimension vector α to be

$$\deg_\zeta \mathbf{E} = \deg E + \zeta * [\alpha].$$

Recall that the notation $\zeta * [\alpha]$ was defined in the introduction. The main result of this section is as follows.

Theorem **7.1.** — *A parabolic bundle* $\mathbf{E}$ *on* X *of weight type* (D, w), *has a* ζ*-connection if and only if the dimension vector* α' *of any indecomposable direct summand* $\mathbf{E}'$ *of* $\mathbf{E}$ *satisfies* $\deg_\zeta \mathbf{E}' = 0$.

The case when D is empty is a theorem of Weil [39] (see also [2, Theorem 10]). The case when the ζ_{ij} are strictly increasing rational numbers in the interval $[0, 1)$ was proved independently by Biswas [4]. Compare the theorem also with [7, Theorem 3.3].

Our proof of Theorem 7.1 relies on a result of Mihai [30, 31]. Note that Mihai uses the notion of an 's-connection' on a vector bundle E over a complex manifold, where s is a section with simple zeros of a line bundle S. However, taking s to be the natural section of $S = \mathcal{O}(D)$, where D now denotes the divisor $\sum_{i=1}^{k} a_i$, this coincides for a Riemann surface with the notion of a logarithmic connection. Bearing in mind that we use the opposite sign convention for the residues, Theorem 1 of [30, 31] can be written for a compact Riemann surface X as follows.

Theorem **7.2** *(Mihai).* — *Suppose that* E *is a vector bundle on* X, *and let* $\rho_i \in \mathrm{End}(E_{a_i})$ *for* $1 \leq i \leq k$. *Then there is a logarithmic connection* $\nabla : E \to E \otimes \Omega^1_X(\log D)$ *with* $\mathrm{Res}_{a_i}(\nabla) = \rho_i$ *for all* i, *if and only if*

$$\sum_{i=1}^{k} \mathrm{tr}(\rho_i f_{a_i}) = \frac{1}{2\pi\sqrt{-1}} \langle b(E), f \rangle$$

for all $f \in \mathrm{End}(E)$.

Here $\langle -, - \rangle : \mathrm{Ext}^1(E, E \otimes \Omega^1_X) \times \mathrm{End}(E) \to \mathbf{C}$ is the pairing corresponding to Serre duality, and $b(E)$ is the class of the Atiyah sequence

$$0 \to E \otimes \Omega^1_X \to B(E) \to E \to 0.$$

In particular, Atiyah [2, Proposition 18] showed that

$$\langle b(\mathrm{E}), 1_{\mathrm{E}}\rangle = -2\pi\sqrt{-1}\,\deg \mathrm{E},$$

and $\langle b(\mathrm{E}), f\rangle = 0$ if f is nilpotent. Moreover, the Corollary to [2, Proposition 7] shows that the construction of B(E) is additive on direct sums, which implies the following:

Lemma **7.3.** — *If* $f \in \mathrm{End}(\mathrm{E})$ *is the projection onto a direct summand* E' *of* E, *then* $\langle b(\mathrm{E}), f\rangle = -2\pi\sqrt{-1}\,\deg \mathrm{E}'$.

If $\mathbf{E}$ is a parabolic bundle, we write $\iota^{\mathbf{E}}_{ij}$ for the inclusions $\mathrm{E}_{ij} \to \mathrm{E}_{i,j-1}$.

Lemma **7.4.** — *Given a parabolic bundle* $\mathbf{E}$, *there is an exact sequence*

$$\bigoplus_{i=1}^{k}\bigoplus_{j=1}^{w_i-1}\mathrm{Hom}(\mathrm{E}_{i,j-1}, \mathrm{E}_{ij}) \xrightarrow{\mathrm{F}} \mathrm{End}(\mathrm{E})^* \oplus \bigoplus_{i=1}^{k}\bigoplus_{j=1}^{w_i-1}\mathrm{End}(\mathrm{E}_{ij}) \xrightarrow{\mathrm{G}} \mathrm{End}(\mathbf{E})^* \to 0,$$

where F *sends* $\phi = (\phi_{ij})$ *with* $\phi_{ij} \in \mathrm{Hom}(\mathrm{E}_{i,j-1}, \mathrm{E}_{ij})$ *to the element* (ξ, η_{ij}) *with* $\xi \in \mathrm{End}(\mathrm{E})^*$ *and* $\eta_{ij} \in \mathrm{End}(\mathrm{E}_{ij})$ *given by* $\xi(f) = \sum_{i=1}^{k} \mathrm{tr}(\iota^{\mathbf{E}}_{i1}\phi_{i1} f_{a_i})$ *for* $f \in \mathrm{End}(\mathrm{E})$ *and*

$$\eta_{ij} = \begin{cases} \iota^{\mathbf{E}}_{i,j+1}\phi_{i,j+1} - \phi_{ij}\iota^{\mathbf{E}}_{ij} & (\text{if } 1 \le j < w_i - 1) \\ -\phi_{ij}\iota^{\mathbf{E}}_{ij} & (\text{if } j = w_i - 1), \end{cases}$$

and G *sends* (ξ, η_{ij}) *to the map sending* $\theta \in \mathrm{End}(\mathbf{E})$ *to*

$$\xi(\theta) + \sum_{i=1}^{k}\sum_{j=1}^{w_i-1} \mathrm{tr}(\eta_{ij}\theta_{a_i}|_{\mathrm{E}_{ij}}).$$

Proof. — Clearly there is an exact sequence

$$0 \to \mathrm{End}(\mathbf{E}) \xrightarrow{\mathrm{G}'} \mathrm{End}(\mathrm{E}) \oplus \bigoplus_{i=1}^{k}\bigoplus_{j=1}^{w_i-1}\mathrm{End}(\mathrm{E}_{ij}) \xrightarrow{\mathrm{F}'} \bigoplus_{i=1}^{k}\bigoplus_{j=1}^{w_i-1}\mathrm{Hom}(\mathrm{E}_{ij}, \mathrm{E}_{i,j-1})$$

where G' sends an endomorphism $\theta \in \mathrm{End}(\mathbf{E})$ to the collection $(\theta, \theta_{a_i}|_{\mathrm{E}_{ij}})$, and F' sends a collection of endomorphisms (θ, θ_{ij}) to the element whose component in $\mathrm{Hom}(\mathrm{E}_{ij}, \mathrm{E}_{i,j-1})$ is $\theta_{i,j-1}\iota^{\mathbf{E}}_{ij} - \iota^{\mathbf{E}}_{ij}\theta_{ij}$ for $j \ge 2$, and $\theta_{a_i}\iota^{\mathbf{E}}_{i1} - \iota^{\mathbf{E}}_{i1}\theta_{i1}$ for $j = 1$. The required sequence is obtained by dualizing this sequence, and using the trace pairing to identify $\mathrm{Hom}(\mathrm{U}, \mathrm{V})^*$ with $\mathrm{Hom}(\mathrm{V}, \mathrm{U})$ for any vector spaces U, V. □

If E is a vector bundle on X, any endomorphism $f \in \mathrm{End}(E)$, induces an endomorphism f_a of each fibre E_a. Clearly the assignment $X \to \mathbf{C}$, $a \mapsto \mathrm{tr}(f_a)$ is a globally defined holomorphic function on X, hence constant, since X is compact. We denote its common value by $\mathrm{tr}(f)$.

Proof of Theorem 7.1. — Given $\rho_i \in \mathrm{End}(E_{a_i})$, one has

$$(\mathbf{13}) \qquad (\rho_i - \zeta_{ij}1)(E_{i,j-1}) \subseteq E_{ij}$$

for all j if and only if there are maps $\phi_{ij} : E_{i,j-1} \to E_{ij}$ for $1 \le j \le w_i - 1$ satisfying $\rho_i = \iota^{\mathbf{E}}_{i1}\phi_{i1} + \zeta_{i1}1$ and

$$(\mathbf{14}) \qquad (\zeta_{ij} - \zeta_{i,j+1})1 = \begin{cases} \iota^{\mathbf{E}}_{i,j+1}\phi_{i,j+1} - \phi_{ij}\iota^{\mathbf{E}}_{ij} & (\text{if } 1 \le j < w_i - 1) \\ -\phi_{ij}\iota^{\mathbf{E}}_{ij} & (\text{if } j = w_i - 1). \end{cases}$$

Namely, if there are ϕ_{ij}, then an induction on j shows that

$$(\rho_i - \zeta_{ij}1)|_{E_{i,j-1}} = \phi_{ij},$$

so it has image contained in E_{ij}, and on the other hand, if ρ_i satisfies (13) one can take ϕ_{ij} to be the map $E_{i,j-1} \to E_{ij}$ induced by $\rho_i - \zeta_{ij}1$. Thus by Theorem 7.2, there is a ζ-connection on $\mathbf{E}$ if and only if there are maps ϕ_{ij} satisfying (14) and

$$(\mathbf{15}) \qquad \sum_{i=1}^{k} \mathrm{tr}\left(\left(\iota^{\mathrm{E}}_{i1}\phi_{i1} + \zeta_{i1}1\right)f_{a_i}\right) = \frac{1}{2\pi\sqrt{-1}} \langle b(E), f\rangle$$

for all $f \in \mathrm{End}(E)$. Now if one defines

$$(\xi, \eta_{ij}) \in \mathrm{End}(E)^* \oplus \bigoplus_{i=1}^{k} \bigoplus_{j=1}^{w_i-1} \mathrm{End}(E_{ij})$$

by

$$\xi(f) = \frac{1}{2\pi\sqrt{-1}} \langle b(E), f\rangle - (\sum_{i=1}^{k} \zeta_{i1})\ \mathrm{tr}(f),$$

for $f \in \mathrm{End}(E)$, and $\eta_{ij} = (\zeta_{ij} - \zeta_{i,j+1})1$, then $(\xi, \eta_{ij}) \in \mathrm{Im}(F)$ if and only if there are elements $\phi_{ij} \in \mathrm{Hom}(E_{i,j-1}, E_{ij})$ satisfying (14) and (15).

Thus by exactness, there is a ζ-connection on $\mathbf{E}$ if and only if $(\xi, \eta_{ij}) \in \mathrm{Ker}(G)$, that is,

$$(\mathbf{16}) \qquad \frac{1}{2\pi\sqrt{-1}} \langle b(E), \theta\rangle - (\sum_{i=1}^{k} \zeta_{i1})\ \mathrm{tr}(\theta) + \sum_{i=1}^{k} \sum_{j=1}^{w_i-1} (\zeta_{ij} - \zeta_{i,j+1})\ \mathrm{tr}(\theta_{a_i}|_{E_{ij}}) = 0$$

for all $\theta \in \mathrm{End}(\mathbf{E})$.

Suppose that there is a ζ-connection on $\mathbf{E}$, and let θ be the projection onto an indecomposable direct summand $\mathbf{E}' = (\mathrm{E}', \mathrm{E}'_{ij})$ of $\mathbf{E}$ of dimension vector α'. By Lemma 7.3, equation (16) becomes

$$-\deg \mathrm{E}' - (\sum_{i=1}^{k} \zeta_{i1})\,\alpha'_0 + \sum_{i=1}^{k}\sum_{j=1}^{w_i-1}(\zeta_{ij} - \zeta_{i,j+1})\alpha'_{ij} = 0.$$

That is, $\deg \mathrm{E}' + \zeta * [\alpha'] = 0$.

For the converse it suffices to show that an indecomposable parabolic bundle $\mathbf{E}$ of dimension α with $\deg \mathrm{E} + \zeta * [\alpha] = 0$ has a ζ-connection. Now $\mathrm{par}_{\mathrm{D},w}(\mathrm{X})$ is an additive category with split idempotents whose Hom spaces are finite-dimensional vector spaces over $\mathbf{C}$, so any endomorphism of an indecomposable parabolic bundle can be written as $\theta = \lambda 1 + \phi$ with $\lambda \in \mathbf{C}$ and ϕ nilpotent. Thus by linearity it suffices to check equation (16) for $\theta = 1$ and for θ nilpotent. The first holds because of the degree condition, the second because all terms are zero. Thus equation (16) holds for all θ, and hence there is a ζ-connection on E. □

Sometimes we shall use the following special case of Theorem 7.1. It also follows directly from [31, Corollaire 3].

Corollary **7.5.** — *If a parabolic bundle* $\mathbf{E}$ *on* X *of weight type* (D, w) *has a* ζ*-connection, then* $\deg_\zeta \mathbf{E} = 0$.

8. Generic eigenvalues

In this section $\mathrm{X} = \mathbf{P}^1$, $\mathrm{D} = (a_1, \dots, a_k)$ is a collection of distinct points of $\mathbf{P}^1$ and $w = (w_1, \dots, w_k)$ is a collection of positive integers. Note that in some of the results below D is explicitly given, while in others, such as Theorem 8.3, D does not occur in the statement, and it may be chosen arbitrarily.

Given $\alpha \in \mathbf{Z}^{\mathrm{I}}$, we say that $\xi \in (\mathbf{C}^*)^w$ is a *generic solution* to $\xi^{[\alpha]} = 1$ provided that for $\beta \in \mathbf{Z}^{\mathrm{I}}$,

$$\xi^{[\beta]} = 1 \Leftrightarrow \beta \text{ is an integer multiple of } \alpha.$$

Given $\alpha \in \mathbf{Z}^{\mathrm{I}}$ and $d \in \mathbf{Z}$, we say that $\zeta \in \mathbf{C}^w$ is a *generic solution* to $\zeta * [\alpha] = -d$ provided that $\zeta * [\alpha] = -d$ and for $\beta \in \mathbf{Z}^{\mathrm{I}}$,

$$\zeta * [\beta] \in \mathbf{Z} \Leftrightarrow \beta \text{ is an integer multiple of } \alpha.$$

In this section we study the existence of representations of G of type ξ, or the existence of connections with residues of type ζ, in these generic cases. The results will be used in the next section to prove the main theorems.

Lemma **8.1.** — *Suppose that* $\alpha \in \mathbf{Z}^I$, $\zeta \in \mathbf{C}^w$ *and* $\zeta * [\alpha] = -d \in \mathbf{Z}$. *Define* $\xi \in (\mathbf{C}^*)^w$ *by* $\xi_{ij} = \exp(-2\pi\sqrt{-1}\,\zeta_{ij})$. *Then* ξ *is a generic solution to* $\xi^{[\alpha]} = 1$ *if and only if* ζ *is a generic solution to* $\zeta * [\alpha] = -d$.

Proof. — $\xi^{[\beta]} = \exp(-2\pi\sqrt{-1}\,\zeta * [\beta])$, so $\xi^{[\beta]} = 1 \Leftrightarrow \zeta * [\beta] \in \mathbf{Z}$. □

Lemma **8.2.** — *Suppose* α *is nonzero.*
(i) *If* α, d *are coprime, there is always a generic solution to* $\zeta * [\alpha] = -d$.
(ii) *There is always a generic solution to* $\xi^{[\alpha]} = 1$.

Proof. — For (i), one just needs to remove countably many hyperplanes from the set $\{\zeta \in \mathbf{C}^w : \zeta * [\alpha] = -d\}$. For (ii), we use part (i) to choose a generic solution to $\zeta * [\alpha] = -1$, and then the previous lemma gives a generic solution to $\xi^{[\alpha]} = 1$. □

Theorem **8.3.** — *Let* $\alpha \in \mathbf{N}^I$ *be strict, and suppose that* ξ *is a generic solution to* $\xi^{[\alpha]} = 1$. *Then there is a representation of* G *of type* ξ *and dimension* α *if and only if* α *is a root. If* α *is a real root, this representation is unique up to isomorphism.*

Note that any such representation must be irreducible, for any subrepresentation also has type ξ, and if it has dimension vector β then $\xi^{[\beta]} = 1$ by Lemma 3.1. By genericity, β is a multiple of α, and in particular the dimension of the subrepresentation is a multiple of the dimension of the representation.

Proof. — First note that in case α is a real root, we have already shown the existence of a representation of G of type ξ and dimension α, unique up to isomorphism, in Theorem 4.5.

Suppose that α is a root. We show the existence of a representation of type α by induction.

To get started, this is clear if $\alpha = \epsilon_0$. Suppose that α is in the fundamental region. Choose a collection of transversals $\mathbf{T} = (T_1, \dots, T_k)$, and define $\zeta \in \mathbf{C}^w$ with $\zeta_{ij} \in T_i$ and $\exp(-2\pi\sqrt{-1}\,\zeta_{ij}) = \xi_{ij}$. The condition that $\xi^{[\alpha]} = 1$ implies that $d = -\zeta * [\alpha] \in \mathbf{Z}$. By Lemma 5.6 there is an indecomposable parabolic bundle $\mathbf{E}$ of dimension vector α and degree d. Now by Theorem 7.1 this parabolic bundle has a ζ-connection ∇. Clearly the residues of ∇ have type ζ (but possibly a different dimension vector). By the Riemann-Hilbert correspondence, Theorem 6.1 and Lemma 6.2, one obtains a representation of G of dimension α_0 and type ξ. Now by genericity it must have dimension vector α.

Now suppose that α is not ϵ_0 or in the fundamental region. Since it is a root, some reflection $\alpha' = s_v(\alpha)$ is smaller than α. Moreover, we can choose v so that α' is still strict. Namely, if α' is not strict then $v = 0$ and α' is of the form (4), so α must be of the form (9), in which case $s_u(\alpha) < \alpha$ for $u = [\ell, r]$, and $s_u(\alpha)$ is strict.

Observe that $\xi' = r'_v(\xi)$ is a generic solution to $(\xi')^{[\alpha']} = 1$, so by induction there is a representation ρ' of G of type ξ' and dimension vector α'. By genericity, $\xi^{[\epsilon_v]} \neq 1$. Thus by convolution, Theorem 3.2, in case $v = 0$ or by Lemma 4.3 in case $v = [i, j]$, one obtains a representation ρ of G of type ξ and dimension vector α. (To see that ρ' is ξ'-noncollapsing, note that since it is irreducible, if it weren't, it would be ξ'-collapsing. But this is impossible since $s_v(\alpha') = \alpha > \alpha'$.)

It remains to prove that if there is a representation of G of type ξ and dimension α then α is a positive root. This is a special case of the next result. □

Theorem **8.4.** — *Suppose that* $\alpha \in \mathbf{N}^{\mathrm{I}}$ *is strict and that* ξ *is a generic solution to* $\xi^{[\alpha]} = 1$. *If there is an indecomposable representation of* G *of type* ξ *and dimension* $r\alpha$, *with* $r \geq 1$, *then* $r\alpha$ *is a root.*

Proof. — We prove this by induction on α. Let $\rho : \mathrm{G} \to \mathrm{GL}(\mathrm{V})$ be the representation.

If α is in the fundamental region, then we're done, as any multiple of it is a root. Thus suppose that α is not in the fundamental region. Since it is strict, it has connected support. Thus, by the definition of the fundamental region, there is some vertex v with $\alpha' = s_v(\alpha) < \alpha$. Clearly $\xi' = r'_v(\xi)$ is a generic solution to $(\xi')^{[\alpha']} = 1$.

Suppose one can take $v \neq 0$; say $v = [i, j]$. In this case α' is strict. Now by Lemma 4.3, ρ is an indecomposable representation of G of type ξ' and dimension $r\alpha'$. By induction, $r\alpha'$ is a root, and hence so is $r\alpha$.

Thus we may suppose the α is only reduced by the reflection s_0.

If α is a multiple of the coordinate vector ϵ_0, then $\rho(g_i) = \xi_{i1} 1$ for all i, so the indecomposability of ρ implies that it is 1-dimensional, and hence $r\alpha = \epsilon_0$, which is a root. Thus we may suppose that α is not a multiple of ϵ_0, and hence by genericity $\xi^{[\epsilon_0]} \neq 1$.

We show that ρ is ξ-noncollapsing. Suppose it has a ξ-collapsing representation ρ' in its top or socle. Now ρ' also has type ξ, and if its dimension vector is β, then $\xi^{[\beta]} = 1$, so by genericity β is a multiple of α. Now since ρ' is ξ-collapsing, it is 1-dimensional, so its dimension vector must actually be equal to α. However, the dimension vector of a ξ-collapsing representation is either ϵ_0, which we have already eliminated, or it is reduced by the reflection at a vertex $v \neq 0$, which we have also dealt with. Thus ρ is ξ-noncollapsing.

Thus by convolution, Theorem 3.2, ρ corresponds to an indecomposable representation of G of type ξ' and dimension $r\alpha'$. By induction $r\alpha'$ is a root, and hence so is $r\alpha$. □

These results have the following consequences.

Corollary **8.5.** — *Suppose that* $\alpha \in \mathbf{N}^{\mathrm{I}}$ *is strict,* $d \in \mathbf{Z}$, *and* ζ *is a generic solution to* $\zeta * [\alpha] = -d$. *Then there is a vector bundle* E *of degree* d *and a logarithmic connection* $\nabla : \mathrm{E} \to$

$\mathrm{E} \otimes \Omega^1_{\mathbf{P}^1}(\log \mathrm{D})$ *with residues of type* ζ *and dimension vector* α *if and only if* α *is a root. If* α *is a real root, the pair* (E, ∇) *is unique up to isomorphism.*

Corollary **8.6.** — *Suppose that* $\alpha \in \mathbf{N}^{\mathrm{I}}$ *is strict,* $d \in \mathbf{Z}$*, and* ζ *is a generic solution to* $\zeta * [\alpha] = -d$*. Let* $r \geq 1$*. If there is a vector bundle* E *of degree* rd*, a logarithmic connection* $\nabla : \mathrm{E} \to \mathrm{E} \otimes \Omega^1_{\mathbf{P}^1}(\log \mathrm{D})$ *with residues of type* ζ *and dimension vector* $r\alpha$*, and the pair* (E, ∇) *is indecomposable, then* $r\alpha$ *is a root.*

In both cases one wants to apply the Riemann-Hilbert correspondence. In order to do so, one needs to find transversals $\mathbf{T} = (\mathrm{T}_1, \ldots, \mathrm{T}_k)$ such that $\zeta_{ij} \in \mathrm{T}_i$ for all i, j. The next lemma shows that this is possible.

Lemma **8.7.** — *If* ζ *is a generic solution to* $\zeta * [\alpha] = -d$*, with* α *strict, then for fixed* i*, the elements* ζ_{ij} *never differ by an integer.*

Proof. — If β is the dimension vector given by (4), then $\zeta * [\beta] = \zeta_{\ell,s+1} - \zeta_{\ell r}$. Now β is not strict, so not a multiple of α, and hence this is never an integer by genericity. □

In applying the corollaries, the following lemma will be of use.

Lemma **8.8.** — *Suppose that* $\alpha \in \mathbf{N}^{\mathrm{I}}$ *is strict,* $d \in \mathbf{Z}$*, and* ζ *is a generic solution to* $\zeta * [\alpha] = -d$*. Let* $r \geq 1$*. If* $\mathbf{E}$ *is a parabolic bundle of dimension vector* $r\alpha$ *and degree* rd*, and* ∇ *is a* ζ*-connection on* $\mathbf{E}$*, then*

$$(\mathrm{Res}_{a_i} \nabla - \zeta_{ij} 1)(\mathrm{E}_{i,j-1}) = \mathrm{E}_{ij}$$

for all i, j*, so that the residues of* ∇ *have dimension vector* $r\alpha$*.*

Proof. — If not, then the residues of ∇ have type ζ, but dimension vector $\beta \neq r\alpha$. Now equation (12) defines a parabolic bundle of dimension vector β which has a ζ-connection. Thus by Corollary 7.5 we have $\zeta * [\beta] \in \mathbf{Z}$. Thus by genericity β is a multiple of α. But since $\beta_0 = \operatorname{rank} \mathrm{E} = r\alpha_0 \neq 0$, this implies that $\beta = r\alpha$. □

9. Proofs of the main theorems

Let X, D and w be as in the previous section.

Theorem **9.1.** — *Suppose that* α *is a strict root for* Γ_w *and* $d \in \mathbf{Z}$*. If* d *and* α *are coprime, then there is an indecomposable parabolic bundle on* $\mathbf{P}^1$ *of weight type* (D, w) *with dimension vector* α *and degree* d*. Moreover, if* α *is a real root, this indecomposable is unique up to isomorphism.*

Proof. — By Lemma 8.2 we can fix a generic solution ζ of $\zeta * [\alpha] = -d$. By Corollary 8.5 there is a vector bundle E of degree d and a logarithmic connection $\nabla : \mathrm{E} \to \mathrm{E} \otimes \Omega^1_{\mathbf{P}^1}(\log \mathrm{D})$ with residues of type ζ and dimension vector α. By (12) this defines a parabolic bundle $\mathbf{E}$ of dimension vector α equipped with a ζ-connection ∇. Now $\mathbf{E}$ must be indecomposable by Theorem 7.1 and the genericity of ζ.

Now suppose that α is a real root. By Theorem 7.1, any indecomposable parabolic bundle $\mathbf{E}$ of dimension vector α and degree d can be equipped with a ζ-connection. The residues of ∇ have dimension vector α by Lemma 8.8. The uniqueness thus follows from Corollary 8.5. □

Theorem **9.2.** — *If there is an indecomposable parabolic bundle* $\mathbf{E}$ *on* $\mathbf{P}^1$ *of weight type* (D, w) *with dimension vector* α *and degree* d*, then* α *is a strict root for* Γ_w.

Proof. — Write $\alpha = r\alpha'$ and $d = rd'$ for some $r \geq 1$ with α', d' coprime. By Lemma 8.2 we can fix a generic solution ζ of $\zeta * [\alpha'] = -d'$.

Since $\mathbf{E}$ is indecomposable and $\zeta * [\alpha] = -d = -\deg \mathrm{E}$, Theorem 7.1 gives the existence of a ζ-connection ∇ on $\mathbf{E}$. Moreover the residues of ∇ satisfy

$$(\mathrm{Res}_{a_i} \nabla - \zeta_{ij} 1)(\mathrm{E}_{i,j-1}) = \mathrm{E}_{ij}$$

for all i, j by Lemma 8.8, so the spaces E_{ij} can be reconstructed from ∇, and hence, since $\mathbf{E}$ is indecomposable, (E, ∇) is an indecomposable object of $\mathrm{conn}_{\mathrm{D},w,\zeta}(\mathbf{P}^1)$. Thus by Corollary 8.6, α is a root. □

Together, these last two results prove Theorem 1.1.

Theorem **9.3.** — *Suppose given conjugacy classes* $\mathrm{D}_1, \ldots, \mathrm{D}_k$ *of type* ζ *and dimension vector* α. *The following are equivalent*

(i) *There is a vector bundle* E *on* $\mathbf{P}^1$ *and a logarithmic connection* $\nabla : \mathrm{E} \to \mathrm{E} \otimes \Omega^1_{\mathbf{P}^1}(\log \mathrm{D})$ *such that* $\mathrm{Res}_{a_i} \nabla \in \overline{\mathrm{D}_i}$

(ii) *One can write* α *as a sum of strict roots* $\alpha = \beta^{(1)} + \beta^{(2)} + \ldots$ *with* $\zeta * [\beta^{(p)}] \in \mathbf{Z}$ *for all* p.

(iii) *One can write* α *as a sum of positive roots* $\alpha = \beta^{(1)} + \beta^{(2)} + \ldots$ *with* $\zeta * [\beta^{(p)}] \in \mathbf{Z}$ *for all* p*, and* $\zeta * [\beta^{(p)}] = 0$ *for those* p *with* $\beta^{(p)}$ *non-strict.*

Proof. — (i)⇒(ii). Suppose there are E and ∇ with $\mathrm{Res}_{a_i} \nabla \in \overline{\mathrm{D}_i}$. By the implication (i)⇒(ii) of Theorem 2.1, ∇ is a ζ-connection on some parabolic bundle $\mathbf{E}$ of dimension vector α. Writing $\mathbf{E}$ as a direct sum of indecomposable parabolic bundles $\mathbf{E} = \mathbf{E}^{(1)} \oplus \mathbf{E}^{(2)} \oplus \ldots$, one obtains a corresponding decomposition $\alpha = \beta^{(1)} + \beta^{(2)} + \ldots$. Now the $\beta^{(p)}$ are strict roots by the previous theorem, and $\zeta * [\beta^{(p)}] = -\deg \mathrm{E}^{(p)} \in \mathbf{Z}$ by Theorem 7.1.

(ii)⇒(iii) is vacuous.

(iii)⇒(i). Suppose that α can be written as a sum of positive roots $\alpha = \beta^{(1)} + \beta^{(2)} + \ldots$ with $d_p = -\zeta * [\beta^{(p)}] \in \mathbf{Z}$ for all p, and $d_p = 0$ in case $\beta^{(p)}$ is non-strict. We may assume that $\beta^{(p)}$ and d_p have no common divisor, for if $\beta^{(p)} = r\beta'$ and $d_p = rd'$, one can replace $\beta^{(p)}$ by r copies of β' in the decomposition of α.

Suppose that $\beta^{(1)}, \ldots, \beta^{(t)}$ are the strict roots involved in the decomposition, and $\beta^{(t+1)}, \ldots$ are the non-strict roots. By Theorem 9.1 there are indecomposable parabolic bundles of dimensions $\beta^{(p)}$ and degrees d_p for $p \leq t$, so by Theorem 7.1 there is a parabolic bundle of dimension vector $\gamma = \beta^{(1)} + \cdots + \beta^{(t)}$ equipped with a ζ-connection.

For a non-strict root $\beta^{(p)}$ of the form (4), the condition $\zeta * [\beta^{(p)}] = 0$ says that $\zeta_{\ell r} = \zeta_{\ell,s+1}$. Thus for any i the sequence $(\gamma_0, \gamma_{i1}, \gamma_{i2}, \ldots, \gamma_{i,w_i-1})$ is obtained from $(\alpha_0, \alpha_{i1}, \alpha_{i2}, \ldots, \alpha_{i,w_i-1})$ by a finite number of reductions in the sense of Section 2, with respect to various r, s and $(\xi_{i1}, \xi_{i2}, \ldots, \xi_{i,w_i})$. Thus by Lemma 2.3 the residues of ∇ are in the $\overline{\mathrm{D}_i}$. □

Finally we can prove Theorem 1.3.

Theorem **9.4.** — *Suppose given conjugacy classes* $\mathrm{C}_1, \ldots, \mathrm{C}_k$ *of type* ξ *and dimension vector* α. *The following are equivalent.*

(i) *There is a solution to* $\mathrm{A}_1 \ldots \mathrm{A}_k = 1$ *with* $\mathrm{A}_i \in \overline{\mathrm{C}_i}$.

(ii) *One can write* α *as a sum of strict roots* $\alpha = \beta^{(1)} + \beta^{(2)} + \ldots$ *with* $\xi^{[\beta^{(p)}]} = 1$ *for all* p.

(iii) *One can write* α *as a sum of positive roots* $\alpha = \beta^{(1)} + \beta^{(2)} + \ldots$ *with* $\xi^{[\beta^{(p)}]} = 1$ *for all* p.

Proof. — Fix a collection of transversals $\mathbf{T} = (\mathrm{T}_1, \ldots, \mathrm{T}_k)$, and define ζ by $\xi_{ij} = \exp(-2\pi\sqrt{-1}\,\zeta_{ij})$ and $\zeta_{ij} \in \mathrm{T}_i$. Since α satisfies condition (8) with respect to ξ, it also satisfies it with respect to ζ, so let $\mathrm{D}_1, \ldots, \mathrm{D}_k$ be the k-tuple of conjugacy classes given by ζ, α. Clearly $\mathrm{C}_i = \exp(-2\pi\sqrt{-1}\,\mathrm{D}_i)$. By monodromy, there is a solution to $\mathrm{A}_1 \ldots \mathrm{A}_k = 1$ with $\mathrm{A}_i \in \overline{\mathrm{C}_i}$ if and only if there is a vector bundle E on $\mathbf{P}^1$ and a logarithmic connection $\nabla : \mathrm{E} \to \mathrm{E} \otimes \Omega^1_{\mathbf{P}^1}(\log \mathrm{D})$ with $\mathrm{Res}_{a_i} \nabla \in \overline{\mathrm{D}_i}$. Observe that $\xi^{[\beta]} = \exp(-2\pi\sqrt{-1}\,\zeta * [\beta])$, so $\xi^{[\beta]} = 1$ if and only if $\zeta * [\beta] \in \mathbf{Z}$. Note also that if β is a non-strict positive root and $\zeta * [\beta] \in \mathbf{Z}$ then actually $\zeta * [\beta] = 0$. Namely, if β is given by (4), then $\zeta * [\beta] = \zeta_{\ell,s+1} - \zeta_{\ell r}$, which cannot be a nonzero integer since $\zeta_{\ell,s+1}$ and $\zeta_{\ell r}$ belong to the same transversal. The result thus follows from Theorem 9.3. □

REFERENCES

1. S. Agnihotri and C. Woodward, Eigenvalues of products of unitary matrices and quantum Schubert calculus, *Math. Res. Lett.*, **5** (1998), 817–836.
2. M. F. Atiyah, Complex analytic connections in fibre bundles, *Trans. Am. Math. Soc.*, **85** (1957), 181–207.

3. P. Belkale, Local systems on $\mathbf{P}^1 - S$ for S a finite set, *Compos. Math.,* **129** (2001), 67–86.
4. I. Biswas, A criterion for the existence of a flat connection on a parabolic vector bundle, *Adv. Geom.,* **2** (2002), 231–241.
5. S. Brenner and M. C. R. Butler, Generalizations of the Bernstein-Gelfand-Ponomarev reflection functors, *Representation Theory II* (Ottawa, 1979), V. Dlab and P. Gabriel (eds.), *Lect. Notes Math.,* **832**, Springer, Berlin (1980), 103–169.
6. D. Chan and C. Ingalls, Non-commutative coordinate rings and stacks, *Proc. London Math. Soc.,* **88** (2004), 63–88.
7. W. Crawley-Boevey, Geometry of the moment map for representations of quivers, *Compos. Math.,* **126** (2001), 257–293.
8. W. Crawley-Boevey, Normality of Marsden-Weinstein reductions for representations of quivers, *Math. Ann.,* **325** (2003), 55–79.
9. W. Crawley-Boevey, On matrices in prescribed conjugacy classes with no common invariant subspace and sum zero, *Duke Math. J.,* **118** (2003), 339–352.
10. W. Crawley-Boevey and J. Schröer, Irreducible components of varieties of modules, *J. Reine Angew. Math.,* **553** (2002), 201–220.
11. P. Deligne, Equations différentielles à points singuliers réguliers, *Lect. Notes Math.,* **163**, Springer, Berlin (1970).
12. M. Dettweiler and S. Reiter, An algorithm of Katz and its application to the inverse Galois problem, *J. Symb. Comput.,* **30** (2000), 761–798.
13. M. Furuta and B. Steer, Seifert fibred homology 3-spheres and the Yang-Mills equations on Riemann surfaces with marked points, *Adv. Math.,* **96** (1992), 38–102.
14. W. Geigle and H. Lenzing, A class of weighted projective curves arising in representation theory of finite dimensional algebras, *Singularities, representations of algebras, and vector bundles* (Lambrecht, 1985), G.-M. Greuel and G. Trautmann (eds.), *Lect. Notes Math.,* **1273**, Springer, Berlin (1987), 265–297.
15. M. Gerstenhaber, On nilalgebras and linear varieties of nilpotent matrices, III, *Ann. Math.,* **70** (1959), 167–205.
16. A. Haefliger, Local theory of meromorphic connections in dimension one (Fuchs theory), chapter III of A. Borel et al., *Algebraic D-modules*, Acad. Press, Boston (1987), 129–149.
17. D. Happel, I. Reiten and S. O. Smalø, Tilting in abelian categories and quasitilted algebras, *Mem. Am. Math. Soc.,* **120**, no. 575 (1996).
18. V. G. Kac, Infinite root systems, representations of graphs and invariant theory, *Invent. Math.,* **56** (1980), 57–92.
19. V. G. Kac, Root systems, representations of quivers and invariant theory, *Invariant theory* (Montecatini, 1982), F. Gherardelli (ed.), *Lect. Notes Math.,* **996**, Springer, Berlin (1983), 74–108.
20. N. M. Katz, *Rigid local systems*, Princeton University Press, Princeton, NJ (1996).
21. V. P. Kostov, On the existence of monodromy groups of Fuchsian systems on Riemann's sphere with unipotent generators, *J. Dynam. Control Systems,* **2** (1996), 125–155.
22. V. P. Kostov, On the Deligne-Simpson problem, *C. R. Acad. Sci., Paris, Sér. I, Math.,* **329** (1999), 657–662.
23. V. P. Kostov, On some aspects of the Deligne-Simpson problem, *J. Dynam. Control Systems,* **9** (2003), 393–436.
24. V. P. Kostov, The Deligne-Simpson problem – a survey, preprint math.RA/0206298.
25. H. Kraft and Ch. Riedtmann, Geometry of representations of quivers, *Representations of algebras* (Durham, 1985), P. Webb (ed.) *Lond. Math. Soc. Lect. Note Ser.,* **116**, Cambridge Univ. Press (1986), 109–145.
26. H. Lenzing, Representations of finite dimensional algebras and singularity theory, *Trends in ring theory* (Miskolc, Hungary, 1996), *Canadian Math. Soc. Conf. Proc.,* **22** (1998), Am. Math. Soc., Providence, RI (1998), 71–97.
27. B. Malgrange, Regular connections, after Deligne, chapter IV of A. Borel et al., *Algebraic D-modules*, Acad. Press, Boston (1987), 151–172.
28. V. B. Mehta and C. S. Seshadri, Moduli of vector bundles on curves with parabolic structure, *Math. Ann.,* **248** (1980), 205–239.
29. H. Lenzing and H. Meltzer, Sheaves on a weighted projective line of genus one, and representations of a tubular algebra, *Representations of algebras* (Ottawa, 1992), *Can. Math. Soc. Conf. Proc.,* **14** (1993), Am. Math. Soc., Providence, RI (1993), 313–337.
30. A. Mihai, Sur le résidue et la monodromie d'une connexion méromorphe, *C. R. Acad. Sci., Paris, Sér. A,* **281** (1975), 435–438.
31. A. Mihai, Sur les connexions méromorphes, *Rev. Roum. Math. Pures Appl.,* **23** (1978), 215–232.
32. O. Neto and F. C. Silva, Singular regular differential equations and eigenvalues of products of matrices, *Linear Multilinear Algebra,* **46** (1999), 145–164.

33. C. M. RINGEL, Tame algebras and integral quadratic forms, *Lect. Notes Math.*, **1099**, Springer, Berlin (1984).
34. L. L. SCOTT, Matrices and cohomology, *Ann. Math.*, **105** (1977), 473–492.
35. C. S. SESHADRI, Fibrés vectoriels sur les courbes algébriques, *Astérisque*, **98** (1982), 1–209.
36. C. T. SIMPSON, Products of Matrices, *Differential geometry, global analysis, and topology* (Halifax, NS, 1990), *Can. Math. Soc. Conf. Proc.*, **12** (1992), Am. Math. Soc., Providence, RI (1991), 157–185.
37. K. STRAMBACH and H. VÖLKLEIN, On linearly rigid tuples, *J. Reine Angew. Math.*, **510** (1999), 57–62.
38. H. VÖLKLEIN, The braid group and linear rigidity, *Geom. Dedicata*, **84** (2001), 135–150.
39. A. WEIL, Generalization de fonctions abeliennes, *J. Math. Pures Appl.*, **17** (1938), 47–87.

W. C.-B.
Department of Pure Mathematics,
University of Leeds,
Leeds LS2 9JT, UK

Manuscrit reçu le 21 juillet 2003.