MÉMOIRES DE LA S. M. F.

JEAN-PIERRE LABESSE

Noninvariant base change identities

Mémoires de la S. M. F. 2e série, tome 61 (1995)

<http://www.numdam.org/item?id=MSMF_1995_2_61__1_0>

Société Mathématique de France
Mémoire 61
Supplément au Bulletin de la S.M.F.
Tome 123, 1995, fascicule 2

NONINVARIANT BASE CHANGE IDENTITIES

Jean-Pierre LABESSE

Abstract. – We prove, in the cyclic base change situation for the group $GL(n)$, an identity between noninvariant trace formulas for pairs of strongly associated functions. We construct sufficiently many such pairs of functions in order to get a new proof of the existence of base change for automorphic representations of $GL(n)$ over a number field. Our proof is more direct and elementary than Arthur and Clozel's one, although based on a similar method: a trace formula identity.

Résumé. – On établit, dans le cas du changement de base cyclique pour le groupe $GL(n)$, une identité entre les formules des traces non-invariantes pour les paires de fonctions fortement associées. Nous construisons assez de telles paires de fonctions pour en déduire une preuve nouvelle de l'existence du changement de base cyclique pour les représentations automorphes de $GL(n)$ sur un corps de nombre. Notre preuve est plus directe et plus élémentaire que celle d'Arthur et Clozel quoique basée sur une méthode analogue: une identité de formule de traces.

A.M.S. Subject Classification Index (1991) : 11F72, 11R39, 22E55.

Texte reçu le 17 juillet 1992, révisé le 14 février 1995
Laboratoire de Mathématiques de l'École Normale Supérieure,
URA 762 du CNRS
45, rue d'Ulm
F-75230 Paris Cedex 05

TABLE OF CONTENTS

INTRODUCTION

Let G be an inner form of a reductive quasi-split group H , defined over a global field F . Let E/F be a finite field extension. According to the Langlands philosophy there should exist a base change correspondence between automorphic representations of $H(\mathbb{A}_F)$ and $G(\mathbb{A}_E)$. To prove the existence of such a correspondence when E/F is a cyclic extension of degree ℓ , one may use a technique due to Saito Shintani and Langlands : a term by term comparison of two trace formulas.

In the case of number fields, this has been worked out for inner forms of $GL(n)$ in [AC] and for unitary group in three variables attached to a quadratic extension E/F in [Rog]. Let θ be a generator for the Galois group of E over F . Roughly speaking, one first shows the equality of the geometric expansions of the stable trace formula for H and of the stable trace formula for $L = \mathrm{Res}_{E/F}\, G \rtimes \theta$ when applied to pairs (f, ϕ) of *associated functions* $f \in \mathcal{C}_c^\infty(H(\mathbb{A}_F))$ and $\phi \in \mathcal{C}_c^\infty(L(\mathbb{A}_F))$. The correspondence $\phi \mapsto f$ is a particular case of twisted endoscopic transfer whose existence has to be established; moreover one has to show that *association* is compatible with base change for functions in the unramified Hecke algebras. This is the *fundamental lemma* for the stable base change, now proved in general in [Clo] for fields of zero characteristic, and in [Lab2]. This allows to separate unramified infinitesimal characters (i.e. characters of the unramified Hecke algebras) and one deduces from this the matching of the various terms in the spectral expansions of the two trace formulas; this yields the base change correspondence for automorphic representations.

Even in the case $H = GL(n)$ which is particularly simple since, for such a group, conjugacy and stable conjugacy coincide, the term by term comparison of the geometric expansions in the two trace formulas applied to pairs of associated functions (f, ϕ) is not straightforward. The main difficulty arises from the following fact : the trace formula is obtained by a truncation process which is noninvariant under conjugacy, while the concept of association allows only comparison between invariant distributions. The standard procedure is to put the trace formula into an invariant form. The existence of such an invariant form is proved in [A8] but uses long and difficult prerequisites ([A6], [A9], etc.) Moreover, it is not easy to compare the invariant distributions $I_M(\gamma, f)$ and $I_{M^L}(\delta, \phi)$ – constructed from the weighted orbital integrals $J_M(\gamma, f)$ and $J_{M^L}(\delta, \phi)$ – that show up in the invariant trace formula, since they are defined in a rather implicit way if $M \neq H$. Another difficulty is that the contributions (invariant or not) of non-semisimple conjugacy classes are very complicated and a direct comparison seems hopeless. These are the reasons for the quite intricated and difficult arguments in [AC] chapter 2.

Our aim is to suggest a way to bypass these difficulties and to test this program in the case of $GL(n)$. The main simplification is that we compare directly the primitive – noninvariant – form of the two trace formulas. This is made possible by using a noninvariant endoscopic transfer we call *strong association.*

Another simplification is that we do not use any analysis, locally or globally, of the behaviour of orbital integrals near the singular set. Globally this is because we may use, at some place, pairs of functions with regular support: doing so we kill the singular terms in the geometric expansion of the trace formula, but fortunately we do not lose any spectral information. Locally, besides the noninvariant fundamental lemma for units in the unramified Hecke algebras, we only need the noninvariant endoscopic transfer for functions with regular support; this is enough thanks to the very strong finiteness results which follow from the rigidity of cuspidal automorphic representations of $G = GL(n)$. Unfortunately, for other groups, such finiteness results may not be available right away and it might turnout that one would have to rely more on noninvariant harmonic analysis for groups over local fields.

This paper is an expanded version of a preprint [Lab3] that has been circulated in 1992. We have strived to make the paper self-contained from our starting point: the trace formula as obtained in the early papers by Arthur. To make the paper more accessible we even review the definition of the distributions that show up in the trace formula and we sketch the proof of the properties we need. As a result, most of the material in chapter I and a large part of chapter II is borrowed from Arthur's papers, but we believe it more convenient for the reader to have it reviewed in some detail here. Many techniques are borrowed from [AC], this is acknowledged case by case, but we have tried not to rely on references to [AC]. This is so with few exceptions, where we have only quoted some results whose proof do not depend of the main body of [AC]: in I.8.2, the first step in the construction of a function on a Cartan subalgebra is borrowed from the chapter 2 of [AC] but this is an elementary result; in III.1.5 we refer to the first few pages of the first chapter of [AC] for the classical properties of the norm map; the most significant borrowed result is the compatibility of local L-functions with the local base change, the proof of which occupies a large part of the last two sections of the first chapter of [AC]; this is our proposition VI.5.2. Let us now describe the contents of the paper.

In chapter I we give the definitions and review the basic properties of the distributions that show up in the geometric and the spectral expansions of the noninvariant trace formula. The last two sections contain new material.

In chapter II we review the noninvariant trace formula itself. The absolute convergence of the spectral expansion of the trace formula is stated as a conjecture (Conjecture A) in section II.2. We hope that conjecture A will follow from work in progress by W. Müller. We recall an estimate, due to Arthur, that can be used to separate infinitesimal characters, via multipliers, at archimedean places. This estimate is a weak form of the conjectural absolute convergence of the spectral expansion. In section II.4, a particular case of conjecture A which is enough for our needs is established.

In chapter III we begin the study of base change; to avoid stabilization problems we restrict ourselves to groups G that may show up as Levi subgroups of inner forms of $GL(n)$. We introduce a refined version of the concept of association: we consider pairs of functions f and ϕ such that not only orbital integrals but also weighted orbital integrals $J_M(\gamma, f)$ and $J_{M^L}(\delta, \phi)$ are equal, if γ is the norm of δ , at least when these elements are regular semisimple. Moreover the weighted orbital integrals of f have to satisfy some vanishing properties if γ is not a norm. Such pairs of functions will be called *strongly associated.* The best we hope, as regards this noninvariant endoscopic transfer, is stated as conjecture B. The existence of pairs of strongly associated functions with regular support is easy to establish. At the end of chapter III we prove the conjecture B for split places.

In chapter IV we study unramified places: we have to show that the noninvariant endoscopic transfer is compatible with the base change map between unramified Hecke algebras. The key observation is that, thanks to a result of Kottwitz, a noninvariant fundamental lemma holds for units in the unramified Hecke algebras and yield pairs of strongly associated functions. We first recall the definition of elementary functions and we show that they are closely related to functions bi-invariant under an Iwahori subgroup. We show that elementary functions give rise to pairs of strongly associated functions. Moreover, strong association of elementary functions is compatible with base change for weighted characters; this allows to prove a noninvariant form of the fundamental lemma for all functions in the unramified Hecke algebra. Most of the proof of these last two results is postponed to chapter V.

In chapter V we state our base change identity. The matching of the regular semisimple terms in the two trace formulas for pairs of strongly associated functions is obvious. For pairs of strongly associated functions (f, ϕ) , with regular support at one place, the contributions of non-semisimple conjugacy classes vanish and we get the equality of two noninvariant trace formulas :

$$J^H(f) = J^L(\phi) \ .$$

As a first consequence of this identity we prove a twisted version of a noninvariant form of Kazdan's density theorem. Then we show how to use conjecture B2 to refine the spectral identity for pairs of strongly associated functions by separating infinitesimal characters at archimedean places. This is applied to the proof of the noninvariant fundamental lemma. The proof is based on a refinement of the local-global argument used in [Lab2].

In chapter VI, we deal with the base change of automorphic representations. We first refine the spectral identity for pairs of strongly associated functions by separating infinitesimal characters at unramified places. If conjecture B2 holds (in particular if $G = GL(n)$ and E/F splits over archimedean places) we may first separate the archimedean infinitesimal characters and we are left, for a given conductor, with a finite set of automorphic representations; using pairs of associated elementary functions or the noninvariant fundamental lemma, we may separate finite sum of unramified infinitesimal characters. In general, since we do not know that strong association at archimedean places is compatible with multipliers, we have to separate infinite families of unramified infinitesimal characters. This could be done directly, using the fundamental lemma, if we knew that the spectral expansion of trace formula is absolutely convergent (conjecture A); the particular case established in chapter II is enough to conclude if we may choose the normalizing factors for intertwining operators to be compatible with the weak base change. To finish the proof of the existence of base change and of his properties for $GL(n)$ we use in an essential way, as in [AC], the strong finiteness properties that follow from Jacquet-Shalika's theorem on L-functions of pairs, in particular the rigidity (or strong multiplicity one) for cuspidal automorphic representations of $GL(n)$. Thus we obtain a new proof of Arthur-Clozel's theorem. Our result is slightly more general since, thanks to Mœglin-Waldspurger's description of the discrete spectrum, it is no more necesary to restrict oneself to automorphic representations "induced from cuspidal". For inner forms we cannot use a priori the rigidity, although it can be deduced from the properties of the endoscopic correspondence. Hence, to extract the expected informations on the endoscopic correspondence from our noninvariant trace formula identity, without using the rigidity, one would need either a weaker form of it, namely some a priori finiteness result (conjecture C), or further local results.

We observe that, if the trace formula for groups over function fields were available, our proof should extend easily to the case where ℓ is prime to the characteristic of the function field.

Acknowledgements. It is my pleasure to thank the "Université du Québec à Montréal" and the "Katholische Universität Eichstätt", for their hospitality during the preparation of a preliminary version of this paper. I must also thank Guy Henniart and the referee for many very useful critical remarks and suggestions.

I. – SOME NONINVARIANT DISTRIBUTIONS

In this chapter we review results of J. Arthur on weighted orbital integrals and weighted characters. Notation and conventions for the nonconnected situation, in particular the concepts of Levi subsets, parabolic subsets and of regular elements, are borrowed from the first paragraph of [A6]. But we shall not adopt systematically Arthur's notation. The reader should be warned that our definitions of $L(\mathbb{A}_F)^1$ (the kernel of the H_L map) and of the normalized weighted orbital integrals $J_M(\gamma, f)$, do not coincide with those of Arthur in the nonconnected case. We need the notion of (L, M)-family introduced in [A2] sections 6 and 7, as well as the descent and splitting formulas to be found in [A7] section 7, but we shall not use the invariant distributions defined there. In the last section we construct multipliers that will be used to separate infinitesimal characters at archimedean places.

I.1 – (L, M)-families.

Unless otherwise stated, F is a local or a global field of zero characteristic. Let $\widetilde{L}$ be a reductive group defined over F , and L^0 the connected component of the neutral element. Let L be a connected component of $\widetilde{L}$ defined over F ; assume that $L(F)$ is nonempty. We denote by L^+ the group generated by L and let ℓ be the order of the cyclic group $L^0\backslash L^+$. Let P^0 be a parabolic subgroup of L^0 , denote by $\widetilde{P}$ the normalizer of P^0 in $\widetilde{L}$; if the intersection $P = L \cap \widetilde{P}$ is nonempty we say that P is a parabolic subset of L ; we denote by P^+ the subgroup generated by P . Let M^0 be a Levi subgroup of a parabolic subgroup P^0 of L^0 ; denote by $\widetilde{M}$ the normalizer of M^0

in $\widetilde{L}$. The intersection $M = P \cap \widetilde{M}$ is by definition a Levi subset of P ; we denote by M^+ the subgroup generated by M . The maximal split torus in the center of M^+ is denoted by A_M : this is the split component of M . Let Q be a parabolic subset. We denote by $\mathcal{L}^Q(M)$ the set of Levi subsets M_1 contained in Q and containing M . We denote by $\mathcal{P}^Q(M)$ the set of parabolic subsets $P \subset Q$ with Levi subset M . We fix a minimal Levi subset M_0 . The Levi subsets containing M_0 are called semistandard. A parabolic subset P containing M_0 has a unique Levi subset M containing M_0 ; it will be called *the* Levi subset of P .

Let P be a parabolic subset. Denote by $X(P)_F$ the group of F-rational characters of P^+ and let

$$\mathfrak{a}_P = \mathrm{Hom}(X(P)_F, \mathbb{R}).$$

Its dimension equals the dimension of A_M if M is a Levi subset of P and $\mathfrak{a}_P = \mathfrak{a}_M$. Its dual is $\mathfrak{a}_P^* = X(P)_F \otimes \mathbb{R}$. Given $\xi \in X(P)_F$ we denote by $\mu(\xi)$ its image in $\mathfrak{a}_P^*$.

Let F be a global field. One defines $L(\mathbb{A}_F)^+$ as the subgroup of $\prod_v L^+(F_v)$ generated by $L^0(\mathbb{A}_F)$ and $L^+(F)$, endowed with the topology such that the inclusion $L^0(\mathbb{A}_F) \to L(\mathbb{A}_F)^+$ is an open map. There is a map

$$H_L \ : \ L(\mathbb{A}_F)^+ \to \mathfrak{a}_L$$

such that, for any $\xi \in X(P)_F$ and any $x \in L(\mathbb{A}_F)^+$:

$$|\xi(x)| = e^{<\mu(\xi), H_L(x)>} .$$

We denote by $L(\mathbb{A}_F)^1$ the kernel of the restriction of H_L to $L^0(\mathbb{A}_F)$. Observe that, in general, $L^0(\mathbb{A}_F)^1 \subsetneqq L(\mathbb{A}_F)^1$. If M is a Levi subset one has a natural direct sum decomposition ([A6] p. 228-229)

$$\mathfrak{a}_M = \mathfrak{a}_M^L \oplus \mathfrak{a}_L .$$

An (L, M)-family is a collection of smooth functions $c_P(\Lambda)$ for $P \in \mathcal{P}^L(M)$ and $\Lambda \in i\mathfrak{a}_M^*$ such that $c_{P_1}(\Lambda) = c_{P_2}(\Lambda)$ if P_1 and P_2 are defined by adjacent chambers and Λ lies in the wall between the two chambers. For each $P \in \mathcal{P}^Q(M)$ one defines a function θ_P^Q on $\mathfrak{a}_M^* \otimes \mathbb{C}$:

$$\theta_P^Q(\Lambda) = (a_M^Q)^{-1} \prod_{\alpha \in \Delta_P^Q} \Lambda(\check{\alpha})$$

where Δ_P^Q is the set of simple roots defined by P in $(\mathfrak{a}_M^Q)^*$ and a_M^Q is the covolume of the coroot lattice in $\mathfrak{a}_M^Q$ (see [A6] §1 p. 229).

I.1.1. Lemma. – *The function*

$$c_M^Q(\Lambda) = \sum_{P \in \mathcal{P}^Q(M)} c_P(\Lambda)\ \theta_P^Q(\Lambda)^{-1}$$

defined for Λ not in a wall, extends to a continuous function on $i\mathfrak{a}_M^*$. The value of $c_M^Q(\Lambda)$ at $\Lambda = 0$ is denoted c_M^Q :

$$c_M^Q = \lim_{\Lambda \to 0} \sum_{P \in \mathcal{P}^Q(M)} c_P(\Lambda)\ \theta_P^Q(\Lambda)^{-1}.$$

Let $p = \dim \mathfrak{a}_M^Q$, then for any regular Λ one has

$$c_M^Q = \frac{1}{p!} \sum_{P \in \mathcal{P}^Q(M)} \left(\lim_{t \to 0} \left(\frac{d}{dt} \right)^p c_P(t\Lambda) \right)\ \theta_P^Q(\Lambda)^{-1} .$$

Proof. The first assertion is lemma 6.2 p. 37 of [A2]. The second is also quoted from [A2] p. 37.

□

Let L_Q be the Levi subset of Q which contains M . For any $R \in \mathcal{P}^{L_Q}(M)$ there is a unique parabolic subset $Q(R) \in \mathcal{P}^Q(M)$ such that $Q(R) \cap L_Q = R$; the functions $e_R = c_{Q(R)}$ define an (L_Q, M) family and numbers $e_M^{L_Q} = c_M^Q$. We shall sometimes write $c_M^{L_Q}$ instead of c_M^Q if this number is independent of the parabolic subset Q with Levi subset L_Q . On the other hand if $P \subset Q$ the restriction of c_P to $i\mathfrak{a}_Q$, is independent of P and will be denoted c_Q . This gives rise to an (L, L_Q)-family and to numbers $c_{L_Q}^L$.

Given L_1 and L_2 two Levi subsets in $\mathcal{L}^Q(M)$, Arthur introduces in [A7] p. 356 numbers $d_M^Q(L_1,L_2)$. They are nonzero if and only if $\mathfrak{a}_M^{L_1}\oplus\mathfrak{a}_M^{L_2}=\mathfrak{a}_M^Q$. They show up in the descent and splitting formulas.

Each Weyl chamber in $\mathfrak{a}_{L_1}^Q$ defines a parabolic subset Q_1 with Levi subset L_1 . Let $\mathfrak{b}$ be a subspace of $\mathfrak{a}_M$, such that $\mathfrak{a}_M=\mathfrak{a}_{L_1}^Q\oplus\mathfrak{b}$. A point $\zeta\in\mathfrak{a}_M$, in general position, projects along $\mathfrak{b}$ inside some Weyl chamber of $\mathfrak{a}_{L_1}^Q$. Hence ζ and $\mathfrak{b}$ define a section $L_1\mapsto Q_1$ of the natural map which associates to a parabolic subset Q_1 contained in Q and containing M , its Levi subset $L_1\in\mathcal{L}^Q(M)$ ([A7] p. 355-357).

I.1.2. Lemma. – *Let $L_1\in\mathcal{L}^Q(M)$. Fix a point $\zeta\in\mathfrak{a}_M^Q$ in general position; one has :*

$$c_{L_1}^Q=\sum_{L_2\in\mathcal{L}^Q(M)} d_M^Q(L_1,L_2)\; c_M^{Q_2}$$

where Q_2 is the parabolic subset with Levi subset L_2 corresponding to the Weyl chamber containing the intersection of $\zeta+\mathfrak{a}_{L_1}$ with $\mathfrak{a}_{L_2}^Q$. The Weyl chamber is well defined if $d_M^Q(L_1,L_2)\neq 0$.

Proof. This is Corollary 7.2 p. 357 of [A7] in the particular case where $\mathfrak{b}=\mathfrak{a}_{L_1}$.

□

Given two (L,M)-families $\{c_P|P\in\mathcal{P}^L(M)\}$ and $\{e_P|P\in\mathcal{P}^L(M)\}$ one has a splitting formula :

I.1.3. Lemma. – *Let c and e be two (L,M)-families. Fix $\zeta\in\mathfrak{a}_M^Q$ in general position, one has*

$$(c\,e)_M^Q=\sum_{L_1,L_2\in\mathcal{L}^Q(M)} d_M^Q(L_1,L_2)\; c_M^{Q_1}\, e_M^{Q_2}$$

where the parabolic subsets Q_i with Levi subsets L_i correspond to the Weyl chambers containing the points $\zeta_i\in\mathfrak{a}_{L_i}^Q$ such that $\zeta=\zeta_1-\zeta_2$; they are well defined whenever $d_M^Q(L_1,L_2)\neq 0$.

Proof. This is Corollary 7.4 p. 358 of [A7].

□

I.2 – Maximal compact subgroups.

Let F be a global field. We fix, for each place v , a maximal subgroup $K_v^{L^0}$ of L_v^0 – often simply denoted by K_v – assumed to be *special* (cf. [T]) for all finite places v . The algebraic variety L is obtained from a scheme, again denoted by L , over $\mathfrak{O}_F^\Sigma$ the ring of elements of F that are integer outside Σ , some finite set of places. We assume that $K_v = L^0(\mathfrak{O}_v)$ for almost all places $v \notin \Sigma$. We say that a pair (L_v, K_v) is unramified if

(i) L_v^0 is quasi-split, split over an unramified extension

(ii) K_v is an hyperspecial subgroup of L_v^0 ,

(iii) the normalizer K_v^+ of K_v in L_v^+ is such that $K_v^L := K_v^+ \cap L(F_v)$ is nonempty.

I.2.1. Lemma. – *At almost all places the pair (L_v, K_v) is unramified.*

Proof. The first two conditions are well known to hold almost everywhere. Consider $\varepsilon \in L(F)$, and let ε_v be the image of ε via the injection of $L(F)$ in L_v . Observe that $\varepsilon_v \in L(\mathfrak{O}_v)$ and $K_v = L^0(\mathfrak{O}_v)$ for almost all places v . For such places the group K_v^+ is generated by K_v and ε_v and hence (iii) holds almost everywhere.

□

The groups L_v^0 , K_v , M_v^0 and N_v , are endowed with Haar measures normalized so that $\mathrm{vol}\,(K_v) = 1$ and such that $dx = dm\,dn\,dk$ if $x = mnk$ is an Iwasawa decomposition. Note that if v is a finite place K_v is open; this will be used to normalize the Haar measures on L_v^0 and N_v .

Let S be a set of places of a global field F . We shall use a lower index S to denote objects over S i.e. restricted products (with respect to some family of open compact subsets) over places $v \in S$ of local objects; for example $\mathbb{A}_{F,S}$ (or F_S if S is finite) is the restricted product over places $v \in S$ of the local fields F_v . We shall use an upper index S for objects outside of S : the restricted product over places $v \notin S$. For example K^S is the product of the maximal compact subgroups K_v for all $v \notin S$. We shall omit the upper index S when S is the empty set. For example K will denote the product of K_v for all v .

For each parabolic subset P of L with Levi decomposition $P = MN$, the Iwasawa decomposition $L_v^0 = N_v M_v^0 K_v$ allows one to extend the map H_M from $M^0(\mathbb{A}_F)$ to $\mathfrak{a}_M$ to a function

$$H_P \,:\, L^0(\mathbb{A}_F) \to \mathfrak{a}_M$$

such that $H_P(nmk) = H_M(m)$ for $k \in K$, $m \in M^0(\mathbb{A}_F)$ and $n \in N(\mathbb{A}_F)$.

I.3 – Weighted orbital integrals.

The weights v_M^Q, that are used to construct weighted orbital integrals, are functions on $L^0(\mathbb{A}_{F,S})$ defined by the (L,M)-family :

$$v_P(\Lambda, x) = e^{-<\Lambda\,,\, H_P(x)>}$$

([A2] p. 40-41); and hence

$$v_M^Q(x) \;=\; \lim_{\Lambda\to 0} \sum_{P\in\mathcal{P}^Q(M)} v_P(\Lambda,x)\;\theta_P^Q(\Lambda)^{-1}.$$

Notice that $v_M^Q \equiv 1$ if M is the Levi subset of Q ; these weights will be called *trivial weights.*

I.3.1. Lemma. – *For all $m \in M^0(\mathbb{A}_{F,S})$ and $k \in K_S$*

$$v_M^Q(mxk) = v_M^Q(x);$$

moreover $v_M^Q(k) = 0$ for all $k \in K_S$ unless M is the Levi subset of Q. In particular the nontrivial weights are linearly independent from the trivial one.

Proof. The first assertion is proved in [A2] p. 41. The second follows from I.1.1 applied to the trivial (L,M) family $c_P = 1$, since $H_P(k) = 0$ for $k \in K_S$.

□

Let F be a local or a global field. Consider $\gamma \in L(F)$; the connected component of 1 in the centralizer $Z_L(\gamma)$ of γ in L^+ will be denoted by L_γ or L_γ^0 since it is also the connected component of 1 in the centralizer $Z_{L^0}(\gamma)$ of γ in L^0. We denote by $\iota(\gamma)$ the index of L_γ in $Z_{L^0}(\gamma)$. Recall that $\gamma \in L(F)$ is said to be regular if the number of eigenvalues equal to 1 for the adjoint action of γ in the Lie algebra of L^0 is minimal. Such a γ is semisimple, and the connected component T containing γ in $Z_L(\gamma)$ is by definition a *maximal torus in L*. It is the translate of a torus in the usual sense : $T = \gamma L_\gamma$. The group L_γ is a torus in L^0 whose centralizer $\widetilde{L}_\gamma$ in L^0 is a maximal torus, in the usual sense, in L^0 (cf. [A6] p. 227-228). Denote by $A(\gamma)$ the maximal split torus in the center of L_γ . Let M be a Levi subset. If $\gamma \in M(F)$ is such that $A(\gamma) = A_M$ we say that γ is elliptic in M . Let γ be regular in L , then γ is elliptic in M if and only if M is minimal among Levi subsets containing a conjugate of γ .

Let F be a global field. The basic ingredients of the geometric side of the trace formula are the weighted orbital integrals. Let M be a Levi subset, Q a parabolic subset containing M . Let $\gamma \in M(F)$ be regular in $L(F)$. Given $f_S \in \mathcal{C}_c^\infty(\mathbb{A}_{F,S})$, a smooth compactly supported function on the groups of S-points of L , its weighted orbital integral for the triple (γ, M, Q) is the integral

$$\Phi_M^Q(\gamma, f_S) = \int_{L_\gamma(\mathbb{A}_{F,S})\backslash L^0(\mathbb{A}_{F,S})} f_S(x^{-1}\gamma x)\ v_M^Q(x)\ d\dot{x}.$$

This integral makes sense more generally for $\gamma \in M(F_S)$ if S is finite. We shall sometimes omit the upper index if $Q = L$. It is convenient to normalize the weighted orbital integrals so that they satisfy simple compatibility formulas with constant terms (see I.6.4). It is classical to use $|D^L(\gamma)|_S^{1/2}$ as normalizing factor with

$$D^L(\gamma) = \det\ \{(1 - \mathrm{Ad}\ \gamma)|\mathfrak{l}/\mathfrak{l}_\gamma\}$$

where $\mathfrak{l}$ is the Lie algebra of L^0 and $\mathfrak{l}_\gamma$ the Lie algebra of L_γ . But, in the base change situation, $D^L(\gamma)$ may contain parasitic powers of ℓ ([AC] chap. 2 lemma 1.1 p. 80). Instead, we shall use

$$\widetilde{D}^L(\gamma) = \det\ \{(1 - \mathrm{Ad}\ \gamma)|\mathfrak{l}/\tilde{\mathfrak{l}}_\gamma\}$$

where $\tilde{\mathfrak{l}}_\gamma$ is the Lie algebra of $\widetilde{L}_\gamma$. This discriminant has a simpler behaviour with respect to norm maps (III.1.8). Our normalized orbital integrals are the following distributions

$$J_M^Q(\gamma, f_S) = |\widetilde{D}^L(\gamma)|_S^{1/2}\Phi_M^Q(\gamma, f_S)\ .$$

We hope that the use of Arthur's notation $J_M^Q(\gamma, f_S)$ for a slightly different object is harmless and will cause no confusion.

I.4 – Weighted characters.

Let F be a global field. The basic ingredients of the spectral side of the trace formula are the weighted characters; they are constructed with an (L,M)-family introduced and studied first in [A4] p. 1313-1323. Denote by $\Pi(M(\mathbb{A}_{F,S}))$ the set of equivalence classes of admissible irreducible representations of

$$M(\mathbb{A}_{F,S})^{+} = M^{0}(\mathbb{A}_{F,S})\, M^{+}(F)$$

that remain irreducible when restricted to $M^{0}(\mathbb{A}_{F,S})$. Any representation π in $\Pi(M(\mathbb{A}_{F,S}))$ has a restriction to $M^{0}(\mathbb{A}_{F,S})$ which is a tensor product of representations π_v of $M^{0}(F_v)$. Since $M^{+}(F_v)/M^{0}(F_v)$ is cyclic, π_v can be extended, in a noncanonical way, to a representation of $M^{+}(F_v)$. Given π in $\Pi(M(\mathbb{A}_{F,S}))$ and $\Lambda \in \mathfrak{a}_M^{*} \otimes \mathbb{C}$, one defines as usual a representation π_Λ by :

$$\pi_\Lambda(m) = e^{<\Lambda\,,\,H_M(m)>}\pi(m).$$

Let P be a parabolic subset with Levi subset M ; the representation π_Λ , extended trivially on the unipotent radical, defines a representation still denoted π_Λ of $P(\mathbb{A}_{F,S})^{+}$. Denote by $\mathcal{I}_P^L(\pi_\Lambda,\cdot)$ the representation of $L(\mathbb{A}_{F,S})^{+}$ unitarily induced from the representation π_Λ of $P(\mathbb{A}_{F,S})^{+}$. This procedure is called parabolic induction.

Assume for a while S to be finite. Given P and P' two parabolic subset with a common Levi subset M ; as in [A9] we denote by $J_{P'|P}(\pi_\Lambda)$ the usual intertwining operator between $\mathcal{I}_P^L(\pi_\Lambda,\cdot)$ and $\mathcal{I}_{P'}^L(\pi_\Lambda,\cdot)$. It will be written as the product of a meromorphic scalar function $r_{P'|P}(\pi_\Lambda)$, the normalizing factor, and of a normalized meromorphic operator $R_{P'|P}(\pi_\Lambda)$:

$$J_{P'|P}(\pi_\Lambda) = r_{P'|P}(\pi_\Lambda)\, R_{P'|P}(\pi_\Lambda)\ .$$

The two factors have to satisfy various requirements discussed in [CLL] lecture 15 (see [A9] p. 28-29). In particular if (L_v, K_v) is an unramified pair, normalized intertwining operators $R_{P'|P}(\pi_\Lambda)$ are scalars independent of Λ on the K_v-fixed vectors. The existence of normalizing factors has been first proved by Langlands in [CLL] Lecture 15 (theorem 2.1 in [A9]). The operator $J_{P'|P}$ and the normalizing factor $r_{P'|P}$ have global analogues when π is automorphic (i.e. when the restriction of π to $M^0(\mathbb{A}_F)$ is an irreducible automorphic representation). It should be possible to define canonical normalizing factors in terms of L-functions; at unramified places for unramified representations this is a part of the definition; this has been checked in some other cases, in particular for archimedean places in [A9] section 3. If L^0 is a product of groups $GL(n_i)$ this possible thanks to results of Shahidi ([Shah2] [Shah3]).

The normalized intertwining operators $R_{P'|P}(\pi_\Lambda)$ define an (L, M) family :

$$\mathcal{R}_{P'}(\Lambda, \pi, P) = R_{P'|P}(\pi)^{-1} R_{P'|P}(\pi_\Lambda) \ .$$

This allows one to define (see [A7] p. 335) a generalized logarithmic derivative of normalized intertwining operators :

$$\mathcal{R}_M^Q(\pi, P) = \lim_{\Lambda \to 0} \sum_{P' \in \mathcal{P}^Q(M)} \mathcal{R}_{P'}(\Lambda, \pi, P) \, \theta_{P'}^Q(\Lambda)^{-1} \ .$$

I.4.1. Lemma. – *The operator $\mathcal{R}_M^Q(\pi, P)$ commute with $\mathcal{I}_P^L(\pi_\Lambda, k)$ for $k \in K_S$.*

Proof. The operator $\mathcal{I}_P^L(\pi_\Lambda, k)$ for $k \in K_S$ is independent of Λ and commutes with $R_{P'}(\Lambda, \pi, P)$.

□

The weighted characters are the following distributions :

$$J_M^Q(\pi_S, f_S) = \text{ trace } (\mathcal{R}_M^Q(\pi_S, P) \mathcal{I}_P^L(\pi_S, f_S)) \ .$$

To make sense it is not necessary to assume S to be finite. In fact, outside a finite set $\Sigma(f_S, \pi_S)$ of places in S , the functions f_v are the characteristic functions of $K_v^+ \cap L_v$ and the normalized intertwining operators leave invariant the K_v-fixed vector in the space of π_v . If Σ is any finite subset of S of places containing $\Sigma(f_S, \pi_S)$ the distribution $J_M^Q(\pi_\Sigma, f_\Sigma)$ is independent of Σ and one let

$$J_M^Q(\pi_S, f_S) = J_M^Q(\pi_\Sigma, f_\Sigma) \ .$$

These distributions are known to be independent of the parabolic subset $P \subset Q$ with Levi subset M ([A2] p. 43).

If the restriction of π_Λ to $M^0(\mathbb{A}_F)$ is an irreducible automorphic representation, the product over all places of the local normalizing factors can be defined by meromorphic continuation and we get global meromorphic normalizing factors $r_{P'|P}(\pi_\Lambda)$. As above we form the corresponding (L,M)-family

$$r_{P'}(\Lambda,\pi,P) = r_{P'|P}(\pi)^{-1} r_{P'|P}(\pi_\Lambda)$$

and one can define ([A8] p. 519)

$$r_M^Q(\pi) = \lim_{\Lambda\to 0} \sum_{P'\in\mathcal{P}^Q(M)} r_{P'}(\Lambda,\pi,P)\ \theta_{P'}^Q(\Lambda)^{-1} \ .$$

In II.2 we shall define numbers $a_{\text{disc}}^M(\pi)$. In the spectral expansion of the trace formula the following distributions, that are a mixture of S-local and global objects, will show up :

$$\mathcal{J}_M^Q(\pi,f_S) = a_{\text{disc}}^M(\pi) \sum_{M'\in\mathcal{L}(M)} r_M^{M'}(\pi)\ J_{M'}^Q(\pi_S,f_S) \ .$$

I.5 – Unramified characters.

Let F be a non archimedean local field and let (L,K) be an unramified pair. We say that a representation $\pi\in\Pi(L(F))$ is unramified if the space of π contains a nonzero K-invariant vector.

I.5.1. Lemma. – *Let h be a compactly supported function bi-invariant by K . Then*

$$J_M^Q(\pi,h) = 0$$

unless M is the Levi subset of Q and π is unramified.

Proof. By definition of normalizing factors, since (L,K) is an unramified pair, the normalized intertwining operators $R_{P'|P}(\pi_\Lambda)$ are scalars independent of Λ on the K-fixed vectors; hence on the K-fixed vectors the derivatives that occur in I.1.1 vanish (cf. [A7] lemma 2.1 p. 334).

□

Let us denote by $\Pi_{nr}(L(F))$ the set of equivalence classes of irreducible representations ψ of $L^0(F)$ that are restrictions of unramified representations π of $L^+(F)$. A given ψ has a unique extension π with a nonzero K^+-invariant vector. Denote by $\mathcal{H}_F^L$ the unramified Hecke algebra i.e. the convolution algebra of compactly supported function on $L^0(F)$, bi-invariant by K . The group K^+/K acts on the unramified Hecke algebra. Given $h \in \mathcal{H}_F^L$ we denote by $\hat{h}$ its scalar Fourier transform :

$$\hat{h}(\psi) = \text{ trace } (\psi(h)) \ .$$

An element $\psi \in \Pi_{nr}(L(F))$ defines a character

$$h \mapsto \hat{h}(\psi)$$

of the unramified Hecke algebra. Conversely any K^+/K-invariant character of the unramified Hecke algebra is obtained in this way. Such characters will also be called unramified infinitesimal characters (since they define characters of Bernstein's center). There is a natural topology on $\Pi_{nr}(L(F))$ for which the functions

$$\hat{h} \, : \, \psi \mapsto \hat{h}(\psi)$$

are continuous. The subset of unitary unramified representations $\Pi_{nr,u}(L(F))$ is a compact subspace of $\Pi_{nr}(L(F))$. For $h \in \mathcal{H}_F^L$ let

$$||\hat{h}||_L = \sup_{\psi \in \Pi_{nr,u}(L(F))} |\hat{h}(\psi)| \ .$$

I.6 – Formal properties.

Let F be a global field, and let S be a set of places of F .

I.6.1. Lemma. – *Let $w \in K_S$, such that w normalizes M_0 . Then*

$$J_M^Q(\gamma, f_S) = J_{M^w}^{Q^w}(\gamma^w, f_S)$$
$$J_M^Q(\pi_S, f_S) = J_{M^w}^{Q^w}(\pi_S^w, f_S)$$
$$\mathcal{J}_M^Q(\pi, f_S) = \mathcal{J}_{M^w}^{Q^w}(\pi^w, f_S)$$

where $\gamma^w = w^{-1}\gamma w$, $M^w = w^{-1}Mw$, $Q^w = w^{-1}Qw$ and $\pi^w(x) = \pi(wxw^{-1})$.

Proof. Remark first that all distributions under consideration are invariant under K_S-conjugacy of f_S : the weights are right K_S-invariants (lemma I.3.1) and the logarithmic derivatives commute with K_S (lemma I.4.1). On the other hand if at the same time we replace f_S , M , P , Q , π_S and γ by their conjugates under some $w \in K_S$ that normalizes M_0 the distributions are preserved by this transport of structure. The lemma follows from these two remarks.

□

The descent and splitting formulas for weighted orbital integrals and weighted characters are particular cases of the first step ([A7] (8.2) p. 362) in the proof of theorem 8.1 of [A7] and of the first step ([A7] (8.6) p. 367) in the proof of theorem 8.4 of [A7].

I.6.2. Lemma. – *Given $\gamma \in M(\mathbb{A}_{F,S})$ regular locally everywhere in $L(\mathbb{A}_{F,S})$, one has for any Levi subset $L_1 \in \mathcal{L}^Q(M)$*

$$J_{L_1}^Q(\gamma, f_S) = \sum_{L_2 \in \mathcal{L}^Q(M)} d_M^Q(L_1, L_2)\ J_M^{Q_2}(\gamma, f_S)\ .$$

If π_1 is a representation in $\Pi(L_1(\mathbb{A}_{F,S}))$ obtained, via parabolic induction, from π in $\Pi(M(\mathbb{A}_{F,S}))$ one has

$$J_{L_1}^Q(\pi_1, f_S) = \sum_{L_2 \in \mathcal{L}^Q(M)} d_M^Q(L_1, L_2)\ J_M^{Q_2}(\pi, f_S)\ .$$

The section $L_2 \to Q_2$ depends on the choice of a generic element ζ in $\mathfrak{a}_M$.

Proof. This is an immediate consequence of the descent formula for (L, M)-families (lemma I.1.2).

□

In particular, if Q is a parabolic subset with M as Levi subset

$$J_L(\gamma, f_S) = J_M^Q(\gamma, f_S) \ .$$

I.6.3. Lemma. – *Choose ζ a generic point in $\mathfrak{a}_M$. If $S = S_1 \cup S_2$ and $f_S = f_{S_1} \otimes f_{S_2}$*

$$J_M^Q(\gamma, f_S) = \sum_{L_1, L_2} d_M^Q(L_1, L_2) \ J_M^{Q_1}(\gamma, f_{S_1}) \ J_M^{Q_2}(\gamma, f_{S_2})$$

$$J_M^Q(\pi_S, f_S) = \sum_{L_1, L_2} d_M^Q(L_1, L_2) \ J_M^{Q_1}(\pi_{S_1}, f_{S_1}) \ J_M^{Q_2}(\pi_{S_2}, f_{S_2})$$

and if π is automorphic on M

$$\mathcal{J}_M^Q(\pi, f_S) = \sum_{L_1, L_2} d_M^Q(L_1, L_2) \ \mathcal{J}_M^{Q_1}(\pi, f_{S_1}) \ J_M^{Q_2}(\pi_{S_2}, f_{S_2}) \ .$$

The parabolic subsets Q_i with Levi subsets L_i are attached to points $\zeta_i \in \mathfrak{a}_{L_i}^Q$ such that $\zeta = \zeta_1 - \zeta_2$.

Proof. This is an immediate consequence of the splitting formula for (L, M)-families (lemma I.1.3).

□

If M is a Levi subset of a parabolic subset $P = MN$ in L one defines the *constant term* of f_S along P as the function on M defined by

$$f_{S,P}(m) = \delta_{S,P}(m)^{1/2} \int_{K_S} \int_{N_S} f_S(k^{-1}mnk) \ dn \ dk$$

where $\delta_{S,P}$ is the usual modulus function for P :

$$\delta_{S,P}(m) = |\det \ (Ad(m)|LieN)|_S \ .$$

I.6.4. Lemma. – *Given $L_1 \supset M_1 \supset M$ consider $Q \in \mathcal{P}^L(L_1)$ and $R \in \mathcal{P}^{L_1}(M_1)$. There is a parabolic subset $Q(R) \in \mathcal{P}^Q(M_1)$ such that $Q(R) \cap L_1 = R$.*

(i) If $\gamma \in M(\mathbb{A}_{F,S})$ is regular in $L(\mathbb{A}_{F,S})$

$$J_M^R(\gamma, f_{S,Q}) = J_M^{Q(R)}(\gamma, f_S) \ .$$

(ii) If $\pi_S \in \Pi(M(\mathbb{A}_{F,S}))$

$$J_M^R(\pi_S, f_{S,Q}) = J_M^{Q(R)}(\pi_S, f_S) \ .$$

(iii) If π *is automorphic on* M

$$\mathcal{J}_M^R(\pi, f_{S,Q}) = \mathcal{J}_M^{Q(R)}(\pi, f_S) \ .$$

Proof. Our assertions follow from standard changes of variables : cf. [A2] (8.1) p. 46-47 and lemma 7.1 p. 44.

□

Let I be an invariant distribution. We shall sometimes write $I(f_M)$ instead of $I(f_Q)$ to emphasize that its value is independent of the choice of the parabolic subset Q with Levi subset M . For example, with such a convention, we may write

$$J_M^M(\pi, f_Q) = \text{ trace } \pi(f_M) \ .$$

I.6.5. Corollary. – *Let* S *be a set of places of* F *outside of which* (L_v, K_v) *is unramified. Let* $f = f_S \otimes h$ *with* $f_S \in \mathcal{C}_c^\infty(L(\mathbb{A}_{F,S}))$ *and* $h \in \mathcal{C}_c^\infty(L(\mathbb{A}_F^S))$ *bi-invariant under* $K^{L,S}$ *. Then for* $\pi \in \Pi(M(\mathbb{A}_F))$

$$J_M^Q(\pi, f) = J_M^Q(\pi_S, f_S) \ \text{ trace } (\pi^S(h_M))$$

and if π *is automorphic on* M

$$\mathcal{J}_M^Q(\pi, f) = \mathcal{J}_M^Q(\pi, f_S) \ \text{ trace } (\pi^S(h_M)) \ .$$

Proof. This is a particular case of the lemma I.6.3 using that $J_M^R(\pi^S, h) = 0$ unless M is the Levi subset of R (lemma I.5.1) in which case

$$J_M^R(\pi^S, h) = J_M^M(\pi^S, h_R) = \text{ trace } (\pi^S(h_M)) \ .$$

□

I.6.6. Proposition. – *If S is a finite set of places of a global field F . Consider a pair of functions f_S and ϕ_S that have the same weighted characters*

$$J_M^Q(\pi,\phi_S)=J_M^Q(\pi,f_S)$$

for all representations $\pi\in\Pi(M(F_S))$, all Levi subset M and all parabolic subset Q , then f_S and ϕ_S have the same weighted orbital integrals :

$$J_M^Q(\gamma,\phi_S)=J_M^Q(\gamma,f_S)$$

for regular elements $\gamma\in L(F_S)$.

Proof. This follow from II.2.2 and II.1.3.

□

Remark. – Using the local trace formula J. Arthur has established, for connected reductive groups, that the weighted orbital integrals have a spectral expansion in term of weighted characters ([A10] Corollary 4.2). In particular this gives, for connected groups, a local proof of this noninvariant Kazdan's density theorem. Conversely, weighted characters have a geometric expansion ([A10] Corollary 4.4), but we shall not use this fact.

I.7 – Functions with vanishing weighted orbital integrals.

Let F be a local field. Two smooth compactly supported functions f and f' on $L(F)$ are said to be *equivalent* if they have the same normalized (ordinary) orbital integrals for regular semisimple elements γ :

$$J_L(\gamma,f)=J_L(\gamma,f').$$

I.7.1. Lemma. – *Given f a smooth function on $L(F)$, with compact support in the set of regular semisimple elements, there exists a smooth compactly supported function f' with vanishing weighted orbital integrals for nontrivial weights, and which is equivalent to f .*

Proof. Let $\mathcal{T}_F$ be a set of representatives of $L^0(F)$-conjugacy classes of F-maximal tori in L . Given $T\in\mathcal{T}_F$ and $\gamma\in T(F)$ we have $T(F)=T^0(F).\gamma$ where T^0 is a torus in L^0 . The set $L(F)_{reg}$ of regular elements in $L(F)$ is a disjoint union of open sets $L(T,F)$ where T runs through $\mathcal{T}_F$ and where $L(T,F)$ is the set of conjugacy classes of regular semisimple elements $\gamma \in L(F)$ whose conjugacy class meets $T(F)$. Let $T(F)_{reg} = L(F)_{reg} \cap T(F)$ and consider

$$\widetilde{L}(T,F) := T(F)_{reg} \times (T^0(F)\backslash L^0(F)) .$$

The map

$$\gamma \times x \mapsto x^{-1}\gamma x$$

from $\widetilde{L}(T,F)$ to $L(T,F)$ is an étale covering whose fibers are orbits under a finite group $W^{L^0}(T)$: the quotient of the normalizer of $T(F)$ in $L^0(F)$ by its centralizer $T^0(F)$ (i.e. the Weyl group of T if $L = L^0$). The weighted orbital integrals of f , are indexed by triples (γ, M, Q) where γ is an L-regular semisimple element in $M(F)$. A triple will be said to be primitive if γ is M-elliptic. This is equivalent to say that T is M-elliptic; given T such a Levi subset M is unique. By the descent formula I.6.2 one can express any weighted orbital integrals as a sum of weighted integrals attached to primitive triples. The weights v_M^Q are functions on $M^0(F)\backslash L^0(F)/K$ and lemma I.3.1 shows that $v_M^Q(mk) = 0$ for any $k \in K$ and $m \in M^0(F)$ unless M is the Levi subset of Q in which case $v_M^Q = 1$. There exist a compactly supported function α_T on $T^0(F)\backslash L^0(F)/K$, leftinvariant under $W^{L^0}(T)$ and whose integrals against all nontrivial weights vanish :

$$\int_{T^0(F)\backslash L^0(F)} \alpha_T(x)\ v_M^Q(x)\ d\dot{x} = 0$$

unless M is the Levi subset of Q , in which case it has an integral equal to 1 :

$$\int_{T^0(F)\backslash L^0(F)} \alpha_T(x)\ d\dot{x} = 1.$$

The ordinary orbital integral of f defines, for each $T \in \mathcal{T}_F$, a $W^{L^0}(T)$-invariant smooth compactly supported function on $T(F)_{reg}$:

$$\gamma \mapsto J_L(\gamma, f) .$$

The function on $\widetilde{L}(T,F)$:

$$\gamma \times x \mapsto \alpha_T(x)\, J_L(\gamma, f)$$

is $W^{L^0}(T)$-invariant. Hence there exist a smooth compactly supported functions f' whose restriction to $L(T,F)$ is such that

$$f'(x^{-1}\gamma x) = \alpha_T(x)\, J_L(\gamma, f)\ .$$

It has the same ordinary orbital integrals as f but has vanishing weighted orbital integrals for all nontrivial weights i.e. for triples (γ, M, Q) if M is not the Levi subset of Q .

□

Remark. – This lemma, which appeared as lemma 2.1 in [Lab3], is quoted at the end of section 3 in [A10].

I.8 – Infinitesimal characters and multipliers.

In this section F is an archimedean field. Let $\mathfrak{h}_0$ be the Lie algebra of a maximal split torus in $L^0(F)$ considered as a real Lie group. Let $\mathfrak{t}_0$ be a Cartan subalgebra in a maximal compact subgroup of the centralizer of $\mathfrak{h}_0$ in $L^0(F)$. The abelian algebra

$$\mathfrak{h}(L) = \mathfrak{h}_0 \oplus i\mathfrak{t}_0$$

(simply denoted by $\mathfrak{h}$ if no confusion may arise) is a real form of a Cartan subalgebra in the complexified Lie algebra of $L^0(F)$. The infinitesimal character of an irreducible unitary representation π of $L^0(F)$ is an orbit under $W_{\mathbb{C}} = W_{\mathbb{C}}^{L^0}$, the complex Weyl group of $L^0(F)$, of an element ν_π in $\mathfrak{h}_{\mathbb{C}}^* = \mathfrak{h}^* \otimes \mathbb{C}$ the complex dual of $\mathfrak{h}$. Given a Levi subset M there is a natural map from $\mathfrak{h}$ onto $\mathfrak{a}_M$ and $\mathfrak{a}_M^* \otimes \mathbb{C}$ acts by translations in $\mathfrak{h}_{\mathbb{C}}^*$.

Unitary representations have infinitesimal characters defined by $W_{\mathbb{C}}$-orbits in $\mathfrak{h}_u$, the set of $\nu \in \mathfrak{h}_{\mathbb{C}}^*$ such that $\bar{\nu} = -s\nu$ for some $s \in W_{\mathbb{C}}$ of order 2. (Note that the minus sign is forgotten in [A8] p. 356). The subset $\mathfrak{h}_u$ has a natural description via Chevalley's theorem: there is a real vector space V , and a surjective polynomial map

$$\varphi\,:\,\mathfrak{h}_{\mathbb{C}}^* \to V \otimes \mathbb{C}$$

whose fibers are the $W_{\mathbb{C}}$-orbits, and such that $\varphi^{-1}(V) = i\mathfrak{h}_u$.

Given a K-finite function $f \in \mathcal{C}_c^\infty(L(F))$ and $\alpha \in \mathcal{E}'(\mathfrak{h})^{W_{\mathbb{C}}}$ a compactly supported $W_{\mathbb{C}}$-invariant distribution on $\mathfrak{h}$, the theory of multipliers ([A5], [Delorme]) shows that there exist a function $f_\alpha \in \mathcal{C}_c^\infty(L(F))$ such that, for any irreducible admissible representation π of $L^+(F)$ whose restriction to $L^0(F)$ is irreducible and whose infinitesimal character is the $W_{\mathbb{C}}$-orbit of $\nu_\pi \in \mathfrak{h}_{\mathbb{C}}^*$, one has

$$\pi(f_\alpha) = \hat{\alpha}(\nu_\pi)\pi(f) \tag{1}$$

where $\hat{\alpha}$ is the Fourier transform of α . Similarly given f , a compactly supported K-finite distribution on $L(F)$, and $\alpha \in C_c^\infty(\mathfrak{h})^{W_{\mathbb{C}}}$ there exist a function $f_\alpha \in \mathcal{C}_c^\infty(L(F))$ satisfying (1) .

I.8.1. Lemma. – *Given a K-finite function $f \in \mathcal{C}_c^\infty(L(F))$ there exists a K-finite function $f' \in \mathcal{C}_c^\infty(L(F))$ and functions $\alpha_j \in C_c^\infty(\mathfrak{h})^{W_{\mathbb{C}}}$ with $j = 1,2$ such that*

$$f = f'_{\alpha_1} + f_{\alpha_2} .$$

Proof. Let Δ be the laplacian for a $W_{\mathbb{C}}$-invariant metric on $\mathfrak{h}$ and δ the Dirac measure. By Dixmier-Malliavin's key lemma in the proof of their factorization theorem ([DM] lemma 2.5) there exists sequences of real numbers a_n and functions $\alpha_j \in C_c^\infty(\mathfrak{h})^{W_{\mathbb{C}}}$, such that

$$\delta = \lim_n \sum_{p=0}^{p=n} a_p \Delta^p * \alpha_1 + \alpha_2 \ ; \tag{2}$$

moreover the sequence a_n may be chosen sufficiently rapidly decreasing so that, given f , there exists a function $f' \in \mathcal{C}_c^\infty(L(F))$ such that

$$\lim_n \sum_{p=0}^{p=n} a_p \Omega^p * f = f'$$

where Ω denotes the operator in the center of the envelopping algebra corresponding to Δ , via Harish Chandra's isomorphism. The α_j define multipliers and (2) shows that

$$f = f'_{\alpha_1} + f_{\alpha_2} .$$

□

Remark. – This is an elaboration using [DM] of lemma 6.4 in [A8].

I.8.2. Lemma. – *Let* $\nu_0 \in \mathfrak{h}_u$ *and* $r > 0$. *There exist a function* $\alpha \in \mathcal{C}_c^\infty(\mathfrak{h})^{W_\mathbb{C}}$ *whose Fourier transform* $\hat{\alpha}$ *is real valued on* $\mathfrak{h}_u$ *and such that on the subset* $\nu \in \mathfrak{h}_u$, $|Re(\nu)| < r$ *one has :*

(i) $\hat{\alpha}(\nu) \leq 1$.

(ii) $\hat{\alpha}(\nu) = 1$ *if and only if* ν *belongs to the orbit of* ν_0 *under the complex Weyl group.*

(iii) *If* $\hat{\alpha}(\nu) = 1$ *and* $M \in \mathcal{L}$ *is such that* $\nu + i\mathfrak{a}_M \subset \mathfrak{h}_u$ *then*

$$\hat{\alpha}(\nu + i\Lambda) = 1 - Q_{M,\alpha,\nu}(\Lambda) + o(||\Lambda||)^2 \qquad \text{for} \quad \Lambda \in \mathfrak{a}_M^*$$

where $Q_{M,\alpha,\nu}$ *is a positive definite quadratic form on* $\mathfrak{a}_M^*$.

Proof. Let $\mathfrak{b}$ be a subspace of $\mathfrak{h}^*$ such that $\nu_0 + i\mathfrak{b}$ is a subset of $\mathfrak{h}_u$. A function $\alpha_1 \in \mathcal{C}_c^\infty(\mathfrak{h})^{W_\mathbb{C}}$ that satisfies (i) and (ii) is constructed in [AC] chapter 2, lemma 15.2 p. 182; the construction is based on Chevalley's theorem. By hypothesis ν_0 is a maximum for $\hat{\alpha}_1$ in an open neighbourhood of ν_0 in $\mathfrak{h}_u$, hence for $\Lambda \in \mathfrak{b}$:

$$\hat{\alpha}_1(\nu_0 + i\Lambda) = 1 - Q_1(\Lambda) + o(||\Lambda||)^2$$

where Q_1 is a quadratic form positive on $\mathfrak{b}$. We are to construct a new function α such that the corresponding quadratic form is positive definite. Consider a positive definite $W_\mathbb{C}$-invariant quadratic form Q_0 on $\mathfrak{h}^*$. Choose a polynomial P_0 on $\mathfrak{h}^*$ such that

(i) $P_0(\nu_0 + i\Lambda) = Q_0(\Lambda) + o(||\Lambda||^2)$ for $\Lambda \in \mathfrak{h}^*$

(ii) $P_0(w\nu_0) = 1$ if $w \notin W_\mathbb{C}(\nu_0)$, the stabilizer of ν_0 in $W_\mathbb{C}$.

Let

$$P_1(\nu) = 1 - \prod_{w \in W_\mathbb{C}/W_\mathbb{C}(\nu_0)} \frac{1}{|W_\mathbb{C}(\nu_0)|} \sum_{s \in W_\mathbb{C}(\nu_0)} P_0(sw^{-1}\nu) .$$

This is a $W_\mathbb{C}$-invariant polynomial on $\mathfrak{h}^*$ such that

$$P_1(\nu_0 + i\Lambda) = 1 - Q_0(\Lambda) + o(||\Lambda||^2) \qquad \text{for} \quad \Lambda \in \mathfrak{h}^* .$$

Consider now

$$P(\nu) = \left(P_1(\nu)\overline{P_1(-\overline{\nu})}\right)^2 .$$

Since P_1 is $W_{\mathbb{C}}$-invariant the polynomial P is $W_{\mathbb{C}}$-invariant and takes positive real values on $\mathfrak{h}_u$. Since $-\overline{\nu}_0 = s\nu_0$ for some $s \in W_{\mathbb{C}}$ one has

$$P(\nu_0 + i\Lambda) = 1 - 4Q_0(\Lambda) + o(||\Lambda||^2) \qquad \text{for} \quad \Lambda \in \mathfrak{h}^* \cap \overline{\mathfrak{h}^*} \quad .$$

Let n be an integer; since P is a $W_{\mathbb{C}}$-invariant polynomial on $\mathfrak{h}^*$, there exist a function $\alpha \in \mathcal{C}_c^\infty(\mathfrak{h})^{W_{\mathbb{C}}}$ whose Fourier transform is

$$\hat{\alpha}(\nu) = P(\nu)\,\hat{\alpha}_1(\nu)^n .$$

It satisfies (i) and (ii) if n is large enough; it satisfies also (iii) since the quadratic form $Q_{M,\alpha,\nu}$ induced by $Q = 4Q_0 + nQ_1$ on $\mathfrak{a}_M^*$ is a positive definite quadratic form.

□

I.8.3. Lemma. – *Let ν_0 and α as in I.8.2. Let g be a continuous and integrable function on $\mathfrak{a}_M^*$. Fix $\nu \in \mathfrak{h}_u$ and let $N_M(\nu_0, \nu)$ be the set of elements $\Lambda \in \mathfrak{a}_M^*$ such that $\nu + i\Lambda$ belongs to the $W_{\mathbb{C}}$-orbit of ν_0 . Then*

$$\lim_{m\to\infty} \left(\sqrt{\frac{m}{\pi}}\right)^{\dim \mathfrak{a}_M} \int_{\mathfrak{a}_M^*} \hat{\alpha}^m(\nu + i\Lambda')\,g(\Lambda')\;d\Lambda' = \sum_{\Lambda\in N_M(\nu_0,\nu)} \frac{g(\Lambda)}{\sqrt{\det\, Q_{M,\alpha,\nu+i\Lambda}}} .$$

Proof. Recall that $\hat{\alpha}$ restricted to $\nu + i\mathfrak{a}_M^*$ is positive, bounded by 1 . Let $\nu' = \nu + i\Lambda$ with $\Lambda \in N_M(\nu_0, \nu)$. For Λ' sufficiently small

$$\hat{\alpha}(\nu' + i\Lambda') = \exp\left(-Q_{M,\alpha,\nu'}(\Lambda')\;(1 + \epsilon(\Lambda'))\right)$$

with $\epsilon(\Lambda') \to 0$ as $\Lambda' \to 0$. Denote by χ_U the characteristic function of U a sufficiently small open neighbourhood of 0 in $\mathfrak{a}_M^*$. If Λ' is outside of the union of the $\Lambda + U$ where Λ runs over $N_M(\nu_0, \nu)$ we have $\hat{\alpha}(\nu + i\Lambda') < c < 1$. Using changes of variable $\Lambda' \to \Lambda + \frac{\Lambda''}{\sqrt{m}}$ we see that

$$\int_{\mathfrak{a}_M^*} \hat{\alpha}^m(\nu + i\Lambda')\,g(\Lambda')\;d\Lambda'$$

equals the sum over $\Lambda \in N_M(\nu_0, \nu)$ of

$$\left(\frac{1}{\sqrt{m}}\right)^{\dim \mathfrak{a}_M} \int_{\mathfrak{a}_M^*} \exp\left(-Q_{M,\alpha,\nu+i\Lambda}(\Lambda'')\ (1+\epsilon(\frac{\Lambda''}{\sqrt{m}}))\right)\ g(\Lambda+\frac{\Lambda''}{\sqrt{m}})\,\chi_U(\frac{\Lambda''}{\sqrt{m}})\ d\Lambda''$$

up to an exponentially small error term $o(c^m)$. Since g is continuous and integrable, dominated convergence shows that

$$\lim_{m\to\infty} \int_{\mathfrak{a}_M^*} \exp\left(-Q_{M,\alpha,\nu+i\Lambda}(\Lambda'')\ (1+\epsilon(\frac{\Lambda''}{\sqrt{m}}))\right)\ g(\Lambda+\frac{\Lambda''}{\sqrt{m}})\,\chi_U(\frac{\Lambda''}{\sqrt{m}})\ d\Lambda''$$

equals

$$\frac{(\sqrt{\pi}\,)^{\dim \mathfrak{a}_M}}{\sqrt{\det\, Q_{M,\alpha,\nu+i\Lambda}}}\ g(\Lambda)\ .$$

□

II. – THE TRACE FORMULA

In this chapter we first review the basic results on the trace formula established in [A1] [A3] and [A4], that have been extended to the nonconnected case in [CLL]. Some reformulations are borrowed from [A8], but we shall neither use the fine geometric expansion nor the invariant form of the trace formula. The absolute convergence of the spectral expansion is then stated as conjecture A, that will be proved in some particular cases. We also recall Arthur's substitute for absolute convergence that can be used to separate infinitesimal characters at archimedean places via multipliers.

II.1 – Trace formula and geometric expansion for regular functions.

Let F be a global field. The right regular representation of $L^0(\mathbb{A}_F)^1$ in

$$\mathbf{L}^2(L^0(F)\backslash L^0(\mathbb{A}_F)^1)$$

may be extended to a representation, denoted by ρ^1 , of the group $L^+(F)L^0(\mathbb{A}_F)^1$ using that

$$L^0(F)\backslash L^0(\mathbb{A}_F)^1 \simeq L^+(F)\backslash L^+(F)L^0(\mathbb{A}_F)^1 \ .$$

The operator $\rho^1(f^1)$ defined by the restriction f^1 to $L(\mathbb{A}_F) \cap L^+(F)L^0(\mathbb{A}_F)^1$ of a function $f \in \mathcal{C}_c^\infty(L(\mathbb{A}_F))$ acting by ρ^1 in $\mathbf{L}^2(L^0(F)\backslash L^0(\mathbb{A}_F)^1)$ is not in general of trace class and the integral over the diagonal of the kernel associated to $\rho^1(f^1)$ is divergent. J. Arthur has shown how to construct a truncated version of the restriction to the diagonal of this kernel, whose integral is convergent and defines a noninvariant distribution $J^L(f)$ that can be used as a substitute to the trace (cf. [A1] and [CLL]). If Q is a parabolic subset with Levi subset L_Q we define $J^Q(f)$ by

$$J^Q(f) = J^{L_Q}(f_Q) \ .$$

The noninvariant trace formula is the equality of a geometric and a spectral expansions for $J^L(f)$. Let γ be a regular element in $L(F)$; let

$$a^M(\gamma) = \iota(\gamma)^{-1} \operatorname{vol} \ (M^0_\gamma(F)\backslash M^0_\gamma(\mathbb{A}_F)^1)$$

if γ is M-elliptic and $a^M(\gamma) = 0$ otherwise. Given $M \in \mathcal{L}$ denote by

$$\{M(F)\}_{L-\mathrm{reg}}$$

a set of representative of $M^0(F)$-conjugacy classes of L-regular elements in $M(F)$.

II.1.1. Definition. – *We shall say that a smooth compactly supported decomposable function*

$$f_S = \otimes_{v\in S} f_v$$

is regular if for (at least) one place $v \in S$ the support of f_v is contained in the open set of regular elements in $L_v = L(F_v)$.

Let w^Q denote the cardinal of the Weyl set W_0^Q of automorphism of $\mathfrak{a}^Q_{M_0}$ induced by elements in $Q(F)$.

II.1.2. Proposition. – *Let f be a regular function.*

$$J^Q(f) = \sum_{M\in\mathcal{L}} \frac{w^M}{w^Q} \sum_{\gamma\in\{M(F)\}_{L-\mathrm{reg}}} a^M(\gamma)\, J^Q_M(\gamma, f) \ .$$

Proof. The contribution of strongly regular elements to the geometric expansion is computed in [A1] for connected groups; this has been extended to all regular elements in the nonconnected case in [CLL] (see also [Lab1]). Since we work with regular functions we do not need the more advanced results on the fine geometric expansion to compute the contributions of conjugacy classes of nonregular elements : they vanish. □

II.1.3. Proposition. – *Let S be a finite set of places of a global field F . Consider a pair of functions f_S and ϕ_S such that for any parabolic subset $Q \subset L$:*

$$J^Q(\phi) = J^Q(f)$$

whenever $f = f_S \otimes f^S$ and $\phi = \phi_S \otimes f^S$; then f_S and ϕ_S have the same weighted orbital integrals for regular elements $\gamma \in L(F_S)$.

Proof. If at some place $v \notin S$ the support of f_v is small enough, only one conjugacy class may contribute non trivially to the geometric expansion. This, together with the splitting formula I.6.3, yields the proposition. For a more detailed account of the proof in a similar but more complicated situation we refer the reader to the proof of V.2.1 and its corollary; the norm map there has to be replaced here by the identity map.

□

II.2 – The spectral expansion.

The fine spectral expansion has been established in [A4] for connected groups, and was extended to the nonconnected case by Langlands in [CLL] Lecture 15. To state it, we use the notation of [A8]; in particular we define numbers $a^L_{\text{disc}}(\pi)$ following [A8] p. 516-517.

We say that a representation $\pi \in \Pi(L(\mathbb{A}_F))$ occurs in the discrete spectrum for L if its restriction to $L^0(\mathbb{A}_F)$ is an irreducible direct factor of $\mathbf{L}^2(L^0(F)\backslash L^0(\mathbb{A}_F)^1)$ and one denotes by $m^L_{\text{disc}}(\pi)$ its multiplicity. By Langlands theory of Eisenstein series, we know that any automorphic representation π in the discrete spectrum for L^0 comes from iterated residues of Eisenstein series attached to some pair (M,σ) where σ is a cuspidal representation for a Levi subgroup M of L^0 . The conjugacy class χ of the pair (M,σ) is called a cupidal datum.

Let $\mathcal{L}^0$ be the set of Levi subgroups in L^0 containing M_0^0 and let Q_0 be a parabolic subgroup of L^0 with Levi subgroup $L_0 \in \mathcal{L}^0$. We shall denote by $\mathfrak{s}$ the section of the map

$$H_{L_0} \ : \ L_0(\mathbb{A}_F) \to \mathfrak{a}_{L_0}$$

such that $\mathfrak{s}(\mathfrak{a}_{L_0})$ is the connected component of 1 of the group of the real points in the split component of $\mathrm{Res}_{F/\mathbb{Q}} L_0$, the group deduced from L_0 by restriction of scalars from F to $\mathbb{Q}$. In particular

$$L_0(\mathbb{A}_F) \simeq L_0(\mathbb{A}_F)^1 \times \mathfrak{s}(\mathfrak{a}_{L_0}) .$$

Denote by $\rho_\chi^{L_0}$ the representation of $L_0(\mathbb{A}_F)$ in

$$\mathbf{L}^2_{\mathrm{disc},\chi}(\mathfrak{s}(\mathfrak{a}_{L_0})\, L_0(F)\backslash L_0(\mathbb{A}_F))$$

the sum of representations in the discrete spectrum attached to some cuspidal datum χ . Denote by $\rho_{Q_0,\chi}(\Lambda)$ the representation of $L^0(\mathbb{A}_F)$ unitarily induced from the representation of $Q_0(\mathbb{A}_F)$ defined by $\rho_\chi^{L_0}$ shifted by the character defined by $\Lambda \in i\mathfrak{a}^*_{L_0}$ and extended trivially on the unipotent radical. Consider $w \in L(F)$ which normalizes L_0 and fixes Λ and denote by s the image of w in $W_0^L/W_0^{L_0}$. Let $x \in L(\mathbb{A}_F)$ act on the right and w^{-1} on the left. This defines an operator $\rho_{Q_0,\chi}(s,\Lambda,x)$, which depends only on s , from the space of $\rho_{Q_0,\chi}(\Lambda)$ to the space of $\rho_{sQ_0,\chi}(\Lambda)$. Let us denote by $M_{Q_0|sQ_0}(\Lambda)$ the intertwining operator between these two representation spaces (in the notation of [A4]). The discrete part of the trace formula, relative to χ , is the following expression :

$$J^L_{\mathrm{disc},\chi}(f) = \sum_{L_0\in\mathcal{L}^0} \frac{w^{L_0}}{w^L} \sum_s |\det\,(s-1)_{\mathfrak{a}^L_{L_0}}|^{-1}\ \mathrm{trace}\ (M_{Q_0|sQ_0}(0)\rho_{Q_0,\chi}(s,0,f))$$

where the second sum is over the subset of $s \in W^L(\mathfrak{a}_{L_0})$ such that :

$$\det\,(s-1)_{\mathfrak{a}^L_{L_0}} \neq 0 .$$

One denotes by $\Pi_{\mathrm{disc}}(L,\chi)$ the set of equivalence classes of representations π of $L(\mathbb{A}_F)^+$ that contribute non trivially to the spectral expansion of $J^L_{\mathrm{disc},\chi}(f)$. If π belongs to $\Pi_{\mathrm{disc}}(L,\chi)$, the number $a^L_{\mathrm{disc}}(\pi)$ is defined by the spectral expansion of $J^L_{\mathrm{disc},\chi}$:

$$J^L_{\mathrm{disc},\chi}\,(f) = \sum_{\pi\in\Pi_{\mathrm{disc}}(L,\chi)} a^L_{\mathrm{disc}}(\pi)\ \mathrm{trace}\ (\pi(f)) .$$

We say that a representation $\pi \in \Pi(L(\mathbb{A}_F))$ occurs discretely in the trace formula for L if $a^L_{\mathrm{disc}}(\pi) \neq 0$. Note that only representations π of $L(\mathbb{A}_F)^+$ whose restriction to

$L^0(\mathbb{A}_F)$ remain irreducible may contribute nontrivially to the spectral expansion of $J^L_{\mathrm{disc},\chi}(f)$ for f supported on $L(\mathbb{A}_F)$. By construction of the representations $\rho_{Q_0,\chi}(0)$ the restriction of π to $\mathfrak{s}(\mathfrak{a}_{L^0})$ must be a multiple of the trivial representation. Observe that the representations which contribute nontrivially to $J^L_{\mathrm{disc},\chi}$, need not occur in the discrete spectrum for L^0 – this is already the case for $L = GL(2)$ – and the numbers $a^L_{\mathrm{disc}}(\pi)$ need not be positive integers. We may now state the so-called fine χ-expansion.

II.2.1. Proposition. – *The distribution $J^Q(f)$ can be expressed as a series indexed by cuspidal data:*

$$J^Q(f) = \sum_\chi J^Q_\chi(f)$$

of finite sums over Levi subsets M :

$$J^Q_\chi(f) = \sum_{M\in\mathcal{L}^Q} \frac{w^M}{w^Q} J^Q_{M,\chi}(f)$$

of absolutely convergent expressions

$$J^Q_{M,\chi}(f) = \sum_{\pi\in\Pi_{\mathrm{disc}}(M,\chi)} \int_{i\mathfrak{a}^*_M} \mathcal{J}^Q_M(\pi_\Lambda, f)\, d\Lambda\ .$$

Proof. This is nothing but a reformulation of the main result of [A4] extended to the nonconnected case by Langlands in [CLL]. A variant of our formula for J^Q_χ occurs at the bottom of p. 521 in [A8] :

$$J^Q_\chi(f) = \sum_{M\in\mathcal{L}} \sum_{M'\in\mathcal{L}^Q(M)} \frac{w^M}{w^Q} \sum_{\pi\in\Pi_{\mathrm{disc}}(M,\chi)} \int_{i\mathfrak{a}^*_M} a^M_{\mathrm{disc}}(\pi)\ r^{M'}_M(\pi_\Lambda)\ J^Q_{M'}(\pi'_\Lambda, f)\, d\Lambda\ .$$

Here π'_Λ is the representation in $\Pi(M'(\mathbb{A}_F))$ obtained from π_Λ by parabolic induction. Our assertion is obtained using distributions $\mathcal{J}^Q_M(\pi_\Lambda, f)$ instead of their developped expression in term of logarithmic derivatives of normalized intertwining operators and normalizing factors:

$$\mathcal{J}^Q_M(\pi, f) = a^M_{\mathrm{disc}}(\pi) \sum_{M'\in\mathcal{L}^Q(M)} r^{M'}_M(\pi)\ J^Q_{M'}(\pi', f)\ .$$

□

Remark. – The special case $L = M$ is of particular importance; to simplify the notation let us assume that $\mathfrak{a}_L = \mathfrak{a}_{L^0}$ so that $L(\mathbb{A}_F)^1 = L^0(\mathbb{A}_F)^1$; denote by π^1 the restriction of π to $L(\mathbb{A}_F)^1$. We have

$$J^L_{L,\chi}(f) = \sum_{\pi\in\Pi_{\mathrm{disc}}(L,\chi)} a^L_{\mathrm{disc}}(\pi) \int_{i\mathfrak{a}^*_L} \text{trace } \pi_\Lambda(f)\ d\Lambda$$

$$= \sum_{\pi\in\Pi_{\mathrm{disc}}(L,\chi)} a^L_{\mathrm{disc}}(\pi) \text{ trace } \pi^1(f^1)\ .$$

II.2.2. Corollary. – *Assume that f_S and ϕ_S have the same weighted characters*

$$J^Q_M(\pi,\phi_S) = J^Q_M(\pi,f_S)$$

for all representations $\pi \in \Pi(M(F_S))$ all parabolic subset Q and all Levi subset M , then

$$J^Q(\phi_S \otimes f^S) = J^Q(f_S \otimes f^S)\ .$$

Proof. This is an immediate consequence of the spectral expansion II.2.1 and of the splitting formula I.6.3 for the weighted characters.

□

II.2.3. Proposition. – *Let S be a set of places, containing all archimedean ones, and outside of which (L_v, K_v) is unramified. Let $f = f_S \otimes h$ be a smooth compactly supported function where h is a K^S-bi-invariant function on $L(\mathbb{A}^S_F)$, then*

$$J^Q_{M,\chi}(f) = \sum_{L_1,L_2} d^Q_M(L_1,L_2)\ J^{L_1,Q_2}_{M,\chi}(f)$$

where

$$J^{L_1,Q_2}_{M,\chi}(f) = \sum_{\pi\in\Pi_{\mathrm{disc}}(M,\chi)} a^M_{\mathrm{disc}}(\pi) \int_{i\mathfrak{a}^*_M} r^{L_1}_M(\pi_\Lambda) J^{Q_2}_M(\pi_{\Lambda,S}, f_S) \text{ trace } \pi^S_\Lambda(h_M)\ d\Lambda\ .$$

Proof. This follows from the spectral expansion II.2.1, the descent formula I.6.2 and I.6.5.

□

II.2.4. Conjecture A. – *The triple sum and integral over χ , π and Λ , in the spectral expansion of*

$$J_M^{L_1,Q_2}(f)=\sum_{\chi} J_{M,\chi}^{L_1,Q_2}(f),$$

is absolutely convergent.

For $\mathbb{Q}$-rank one groups, conjecture A is an immediate consequence of a result of Langlands (Assertion D in [Lan2] p. 118). A first step toward a proof of conjecture A for arbitrary groups is the proposition II.4.1, due to W. Müller [Mü]. This will allow us to establish particular cases of the conjecture (proposition II.4.5). For the general case one would need moreover some control on the constants in the estimates of logarithmic derivatives of L-functions proved in [A4] lemma 8.4 p. 1330.

II.3 – Estimates and multipliers.

An estimate is proved in [A8] Corollary 6.5, for the invariant form of the trace formula. The proof there applies almost verbatim to the proof of II.3.1 below but of course, since we work with the noninvariant trace formula, we do not need the arguments – in the middle of p. 536 of [A8] – used to show that this estimate is also valid for the invariant distributions ! To separate the contribution, to the spectral expansion, of representations with infinitesimal character ν at archimedean places, one may use this estimate as a substitute for the conjectural absolute convergence of the spectral expansion of the trace formula. It is quite powerful when used with the multipliers constructed in I.8.2.

We may group together the contribution of the cuspidal data χ defined by the conjugacy classes of pairs (M,σ) whose archimedean component have an infinitesimal character $\nu_\chi := \nu_{\sigma_\infty}$ such that $||\mathfrak{Im}\,(\nu_\chi)|| = t$. We thus define distributions

$$J_t^L(f)=\sum_{||\mathfrak{Im}\,(\nu_\chi)||=t} J_\chi^L(f)\ .$$

Let $C_N^\infty(\mathfrak{h})^{W_\mathbb{C}}$ be the space of $W_\mathbb{C}$-invariant smooth functions with support in the ball of radius N .

II.3.1. Lemma. – *Given a K-finite function $f \in C_c^\infty(L(\mathbb{A}_F))$, there exist constants C, k, and r such that for any $T > 0$ any $\alpha \in C_N^\infty(\mathfrak{h})^{W_{\mathbb{C}}}$ one has*

$$\sum_{t \geq T} |J_t^L(f_\alpha)| < Ce^{kN} \, sup \, \{|\hat{\alpha}(\nu)| \, , \, \nu \in \mathfrak{h}_u(r,T)\} \, .$$

where $\mathfrak{h}_u(r,T)$ is the set of $\nu \in \mathfrak{h}_u$ such that $\mathrm{Re}(\nu) < r$ *and* $\mathfrak{Im}(\nu) > T$.

Proof. A first approximation of the estimate we need, for distributions $J_\chi^L(f_\alpha)$, is given in the course of the proof of lemma 6.3 of [A8] p. 535; it can then be refined using lemma 6.4 of [A8] (see I.8.1); repeating the proof of Corollary 6.5. of [A8] we get the final form of our estimate.

□

Let us denote by $\alpha^{*m} = \alpha * \ldots * \alpha$ the m-th convolution power of α .

II.3.2. Corollary. – *Given a K-finite function $f \in C_c^\infty(L(\mathbb{A}_F))$, $\alpha \in C_N^\infty(\mathfrak{h})^{W_{\mathbb{C}}}$, and T large enough, the trace formula for $f_{\alpha^{*m}}$ has the following asymptotic expansion when $m \to \infty$:*

$$J^L(f_{\alpha^{*m}}) = \sum_{t<T} J_t^L(f_{\alpha^{*m}}) + o(c^m)$$

for some $0 < c < 1$.

Proof. Remark that $\hat{\alpha}$ is rapidly decreasing at infinity in the vertical strip $|\mathrm{Re}(\nu)| < r$. We choose $T > 0$ big enough so that, with the notation of the previous lemma

$$e^{kN} \sup \{|\hat{\alpha}(\nu)| \, , \, \nu \in \mathfrak{h}_u(r,T)\} < c < 1 \, .$$

Since the support of α is contained in a ball of radius N the support of α^{*m} is contained in a ball of radius mN ; we now apply II.3.1 to

$$\sum_{t \geq T} |J_t^L(f_{\alpha^{*m}})| \, .$$

□

Remark. – Given a K-finite function f and $T \in \mathbb{R}$, there is only a finite set of cuspidal data χ such that

$$||\mathfrak{Im}\,(\nu_\chi)|| < T$$

which may give a nonvanishing contribution to the χ-expansion of $J^L(f)$ (cf [A8] lemma 4.2 p. 517) and hence the spectral expansion of

$$\sum_{t<T} J_t^L(f)$$

is absolutely convergent.

II.4 – More on absolute convergence.

As in [A2] section 5, consider Δ'_L a left invariant self-adjoint positive elliptic operator of order 2 that commutes with K_∞ , more precisely, a suitable linear combination of the Casimir operators Ω_L and Ω_K for the derived groups of L^0_∞ and K_∞ , and of positive self-adjoint elliptic operators of order 2 on their center. If π is an irreducible unitary representation of $L^0(F_\infty)$ we denote by $||\pi||$ the smallest eigenvalue of $\pi(\Delta'_L)$; similarly if τ is an irreducible unitary representation of K_∞ one defines $||\tau||$. Let $\Delta_L = 1 + \Delta'_L$. If A is an operator in a Banach space we denote by $||A||$ the operator norm.

Denote by $m^L_{\text{disc}}(\pi^1)$ the multiplicity of the representation π^1 of $L^0(\mathbb{A}_F)^1$ in the discrete spectrum $\mathbf{L}^2_{\text{disc}}(L^0(F)\backslash L^0(\mathbb{A}_F)^1)$.

II.4.1. Proposition. – *The operator, induced in the discrete spectrum by a smooth compactly supported function $f \in \mathcal{C}^\infty_c(L^0(\mathbb{A}_F))$, is of trace class :*

$$\sum m^L_{\text{disc}}(\pi^1) \mid \text{trace } (\pi^1(f^1))| < \infty .$$

Proof. If our function f_∞ is K_∞-finite this is Müller's theorem [Mü]. If we remove the K_∞-finiteness assumption, the result is stated as "conceivable" in [Mü] and an argument is outlined. The details have now been checked by Müller (private communication). We shall give an independent proof in the case $L = L^0 = GL(n)$ using

Mœglin-Waldspurger's description of the discrete spectrum for $GL(n)$ ([MW]). The discrete spectrum is the direct sum of the generalized Speh representations that are defined as follows. Let P_1 be a parabolic subgroup whose Levi subgroup M_1 is a product of r groups $M_2 = GL(d)$ with $n = dr$, and $\sigma' = \sigma \otimes \ldots \otimes \sigma$ where σ is a cuspidal representations of M_2 . The generalized Speh representation :

$$\pi = \mathrm{Speh}(\sigma)$$

is the Langlands quotient of the parabolically induced representation

$$\mathcal{I}^L_{P_1}(\sigma' \otimes \delta^{1/2}_{P_1}) \ .$$

This last representation is not unitary if $P_1 \neq L$. Let g^∞ be the caracteristic function of a small enough open compact subgroup, divided by its volume, over the finite adeles such that $f * g^\infty = f$. We have

$$|\text{ trace } \pi(f)| \leq ||\pi(f * \Delta^n_L)|| \text{ trace } \pi(\Delta^{-n}_L \otimes g^\infty) \leq ||f * \Delta^n_L||_1 \text{ trace } \pi(\Delta^{-n}_L \otimes g^\infty) \ .$$

Since π is unitary the minimal eigenvalue $||\pi_\infty||$ of $\pi_\infty(\Delta'_L)$ is positive; on the other hand the minimal eigenvalue of $\mathcal{I}^L_{P_1}(\sigma'_\infty \otimes \delta^{1/2}_{P_1}, \Delta'_L)$ occurs for the minimal K_∞-types; since the minimal K_∞-types occur in the Langlands quotient π_∞ , all eigenvalues are positive and hence

$$\text{trace } \pi(\Delta^{-n}_L \otimes g^\infty) \leq \text{ trace } \mathcal{I}^L_{P_1}(\sigma' \otimes \delta^{1/2}_{P_1}, \Delta^{-n}_L \otimes g^\infty) \ .$$

Given an integer r , if n is large enough, by a standard parametrix construction, there exist compactly supported functions $g_{1,\infty} \in \mathcal{C}^r_c(L^0_\infty)$ and $g_{2,\infty} \in \mathcal{C}^\infty_c(L^0_\infty)$ such that

$$\delta = \Delta^n_L * g_{1,\infty} + g_{2,\infty}$$

where δ is the Dirac measure. Let $g_i = g_{i,\infty} \otimes g^\infty$; since $||\mathcal{I}^L_{P_1}(\sigma'_\infty \otimes \delta^{1/2}_{P_1}, \Delta^{-n}_L)|| \leq 1$ we get

$$\text{trace } \mathcal{I}^L_{P_1}(\sigma' \otimes \delta^{1/2}_{P_1}, \Delta^{-n}_L \otimes g^\infty) \ \leq \sum_i |\text{ trace } \mathcal{I}^L_{P_1}(\sigma' \otimes \delta^{1/2}_{P_1}, g_i)|$$

Now we have

$$\text{trace } \mathcal{I}_{P_1}^L(\sigma' \otimes \delta_{P_1}^{1/2}, g_i) = \text{ trace } \sigma'(\delta_{P_1}^{1/2} g_{i,P_1})$$

where g_{i,P_1} is the constant term of g_i along P_1 . Summing up we have proved that

$$|\text{ trace } \pi(f)| \leq ||f * \Delta_L^n||_1 \sum_i |\text{ trace } \sigma'(\delta_{P_1}^{1/2} g_{i,P_1})| \ .$$

Hence we are reduced to show that convolution operators by sufficiently regular functions in the cuspidal spectrum yield trace class operators: this is well known.

□

II.4.2. Lemma. – *Given a smooth and compactly supported function f there is a positive definite smooth and compactly supported function g such that for any unitary representation π one has*

$$|\text{ trace } \pi(f)| \leq \text{ trace } \pi(g) \ .$$

By Dixmier-Malliavin's factorisation theorem [DM] we may write f as a finite sum of convolution products $f = \sum f_j * f_j'$; each product $f_j * f_j'$ is a linear combination of positive definite functions $g_k * \check{g}_k$ where $g_k = (f_j \pm f_j')$ or $(f_j \pm i f_j')$ and

$$\check{g}_k(x) = \overline{g_k(x^{-1})} \ ;$$

hence

$$f = \sum \lambda_k \ g_k * \check{g}_k \ .$$

The function

$$g = \sum |\lambda_k| \, g_k * \check{g}_k$$

is a solution.

□

II.4.3. Lemma. – *If n is large enough, the operator norm*

$$||\mathcal{R}_M^Q(\pi_\infty, P)\mathcal{I}_P^L(\pi_\infty, \Delta_L^{-n})||$$

has a bound independent of the unitary representation π .

Proof. The derivatives of the matrix coefficients of operators $R_{P'|P}(\pi_\infty)$ restricted to a K_∞-type τ are rational functions of Λ that have bounds of the form

$$C(1+||\tau||)^N(1+||\pi_\infty||)^N$$

(see [A2] (7.6) p. 42). Using the definition of numbers c_M^Q attached to (L,M) families in term of derivatives of the $c_P^Q(\Lambda)$ (lemma I.1.1), this yields similar estimates for the matrix coefficients of $\mathcal{R}_M^Q(\pi_\infty,P)$. Given r , the eigenvalue of $\mathcal{I}_P^L(\pi_\infty,\Delta_L^{-n})$ on the subspace defined by the K_∞-type τ is, for n large enough, bounded by a constant times

$$(1+||\tau||)^{-r}(1+||\pi_\infty||)^{-r} .$$

So, for n large enough

$$||\mathcal{R}_M^Q(\pi_\infty,P)\mathcal{I}_P^L(\pi_\infty,\Delta_L^{-n})||$$

is uniformly bounded.

□

Remark. – The above proof is reminiscent of the proof of proposition 9.1 in [A2].

II.4.4. Corollary. – *There exist a positive definite function* $g_\infty \in \mathcal{C}_c^\infty(L^0(F_\infty))$ *such that*

$$|J_M^Q(\pi_\infty,f_\infty)| \leq C \text{ trace } \mathcal{I}_P^L(\pi_\infty,g_\infty) = C \text{ trace } \pi_\infty(g_{\infty,P})$$

where M *is a Levi subset of* $P\subset Q$ *and* $g_{\infty,P}$ *is the constant term of* g_∞ *along* P^0 .

Proof. Since

$$|J_M^Q(\pi_\infty,f_\infty)| \leq ||\mathcal{R}_M^Q(\pi_\infty,P)\mathcal{I}_P^L(\pi_\infty,\Delta_L^{-n})||\,|\text{ trace } \mathcal{I}_P^L(\pi_\infty,f_\infty*\Delta_L^n)| .$$

By II.4.3, for n large enough, there is a constant C such that :

$$|J_M^Q(\pi_\infty,f_\infty)| \leq C\,|\text{ trace } \mathcal{I}_P^L(\pi_\infty,f_\infty*\Delta_L^n)| .$$

We conclude by applying II.4.2 to the function $f_\infty * \Delta_L^n$.

□

II.4.5. Proposition. – *Conjecture A holds for* $L_1 = M$:

$$\sum_\chi C^{M,Q}_{M,\chi}(f) = \sum_\chi \sum_{\pi\in\Pi_{\mathrm{disc}}(M,\chi)} |a^M_{\mathrm{disc}}(\pi)| \int_{i\mathfrak{a}^*_M} |J^Q_M(\pi_{\Lambda,S}, f_S) \text{ trace } \pi^S_\Lambda(h_M)| \, d\Lambda$$

is convergent.

Proof. The splitting formulas allow us to write the above expression as a sum of integrals of products of terms at finite and infinite places. The finite places are easily dealt with: they give uniformly bounded contributions since unitary representations with a nonzero fixed vector under some fixed open compact subgroup form a quasi-compact set. Hence, by II.4.4, there exist a smooth positive definite function g on $L^0(\mathbb{A}_F)$ compactly supported such that

$$|J^Q_M(\pi_{\Lambda,\infty}, f_\infty)| \le C_1 \text{ trace } \mathcal{I}^L_P(\pi_\Lambda, g) = C_1 \text{ trace } \pi_\Lambda(g_P)$$

where g_P is the constant term along P of g . This shows that

$$\begin{aligned}\sum_\chi C^{M,Q}_{M,\chi}(f) &= \sum_{\pi\in\Pi_{\mathrm{disc}}(M,\chi)} |a^M_{\mathrm{disc}}(\pi)| \int_{i\mathfrak{a}^*_M} |J^Q_M(\pi_{\Lambda,S}, f_S) \text{ trace } \pi^S_\Lambda(h_M)| \, d\Lambda \\ &\le C_1 \sum_\chi \sum_{\pi\in\Pi_{\mathrm{disc}}(M,\chi)} |a^M_{\mathrm{disc}}(\pi)| \int_{i\mathfrak{a}^*_M} \text{trace } \pi_\Lambda(g_P) \, d\Lambda \ .\end{aligned}$$

Moreover

$$\int_{i\mathfrak{a}^*_M} \text{trace } \pi_\Lambda(g_P)\, d\Lambda = \text{trace } \pi^1(g^1_P)$$

where π^1 and g^1_P are the restrictions of π and g_P to the kernel of the map

$$M^0(\mathbb{A}_F) \to \mathfrak{a}_M \ .$$

We get

$$\sum_\chi C^{M,Q}_{M,\chi}(f) \le C_1 \sum_\chi \sum_{\pi\in\Pi_{\mathrm{disc}}(M,\chi)} |a^M_{\mathrm{disc}}(\pi)| \text{ trace } \pi^1(g^1_P) \ .$$

The representations π that occur discretely in the trace formula for M are constituents of the extension to $M(\mathbb{A}_F)^+$ of representations $\pi_1^{M^0} = \mathcal{I}^{M^0}_{P_1}(\pi_1)$ parabolically induced from representations π_1 in the discrete spectrum of M_1^0 , a Levi subgroup of P_1 . Let $Q_1 \subset P^0$ be the parabolic subgroup of L^0 such that $P_1 = Q_1 \cap M$. We have

$$\sum_{\pi \subset \pi_1^{M^0}} \text{trace } \pi(g_P) = \text{trace } \mathcal{I}_{P_1}^{M^0}(\pi_1, g_P) = \text{trace } \pi_1(g_{Q_1}) \ .$$

Let g_1 denote the integral of g_{Q_1} on $\mathfrak{z}(\mathfrak{a}_{M_1^0}^{M^0})$, this is a function on $M_1^0(\mathbb{A}_F)$ such that

$$\sum_{\pi \subset \pi_1^{M^0}} |a_{\text{disc}}^M(\pi)| \text{ trace } \pi^1(g_P^1) \leq m_{\text{disc}}^{M_1}(\pi_1^1) \text{ trace } \pi_1^1(g_1^1) \ ,$$

where π_1^1 and g_1^1 denote the restriction to $M_1^0(\mathbb{A}_F)^1$. We conclude using II.4.1.

□

II.5 – Absolute convergence and measures on the unramified dual.

Let S be a finite set of places, containing all archimedean ones, and outside of which (L_v, K_v) is unramified. Let us denote by $\Pi_{nr,u}(L(\mathbb{A}_F^S))$ the space of equivalence classes of unitary unramified representations of $L^0(\mathbb{A}_F^S)$, restrictions of representations of $L(\mathbb{A}_F^S)^+$. This compact space is the product for $v \notin S$ of the $\Pi_{nr,u}(L_v)$. We denote by $\mathcal{H}^L(\mathbb{A}_F^S)$ the restricted product of unramified Hecke algebras outside S .

II.5.1. Proposition. – *Let h_0 be a smooth compactly supported function on $L(\mathbb{A}_F^S)$ bi-invariant under K^S . Assume that conjecture A holds for the pair (L_1, Q_2). The linear form on the space of Fourier transform $\hat{h}$ of functions h in the unramified Hecke algebra $\mathcal{H}^L(\mathbb{A}_F^S)$, defined by the composition of the inverse Fourier transform and the series of terms $J_{M,\chi}^{L_1,Q_2}$ in the spectral expansion of the trace formula*

$$\hat{h} \mapsto \sum_{\chi} J_{M,\chi}^{L_1,Q_2}(f_S \otimes (h_0 * h)),$$

has a unique extension to a Radon measure on the compact space $\Pi_{nr,u}(L(\mathbb{A}_F^S))$.

Proof. We have

$$\sum_{\chi} |J_{M,\chi}^{L_1,Q_2}(f_S \otimes (h_0 * h))| \leq \sum_{\chi} C_{M,\chi}^{L_1,Q_2}(f_S \otimes h_0) \ ||\hat{h}||_L$$

where $C_{M,\chi}^{L_1,Q_2}(f_S \otimes h_0)$ equals

$$\sum_{\pi \in \Pi_{\text{disc}}(M,\chi)} \int_{i\mathfrak{a}_M^*} |a_{\text{disc}}^M(\pi) r_M^{L_1}(\pi_\Lambda) J_M^{Q_2}(\pi_{\Lambda,S}, f_S) \text{ trace } \pi_\Lambda^S(h_{0,M})| \, d\Lambda \; .$$

Conjecture A for (L_1, Q_2) tells us that

$$\sum_\chi C_{M,\chi}^{L_1,Q_2}(f_S \otimes h_0) < +\infty \; .$$

On the other hand, the algebra of functions $\hat{h} \; : \; \pi^S \mapsto \text{ trace } \pi^S(h)$ is self-adjoint, contains the scalars and separates the points on the compact space $\Pi_{nr,u}(L(\mathbb{A}_F^S))$; by Stone-Weierstraß theorem, this is a dense subalgebra in the algebra of continuous functions on this space. The linear form extends uniquely by continuity.

□

III. – NORM MAP AND ENDOSCOPIC TRANSFER

In this chapter F is a global or a local field of characteristic zero. Let E be a cyclic Galois algebra over F of degree $\ell = \ell_1\ell_2$. The algebra E is a direct sum ℓ_2 copies of a cyclic field extension E_1 of F of degree ℓ_1 . Let θ be a generator of the Galois group; θ acts as follows :

$$\theta(x_1, \ldots, x_{\ell_2}) = (x_2, \ldots, x_{\ell_2}, \theta_1(x_1))$$

where θ_1 is a generator of the Galois group E_1/F . Let

$$H = \prod_i GL(n_i)$$

be a product of linear groups and let G be an inner form of H over F ; we denote by η the isomorphism

$$\eta : G \to H$$

over the algebraic closure. Denote by $\mathrm{Res}_{E/F}G$ the group scheme obtained by restriction of scalars from E et F . We want to compare the trace formula and the harmonic analysis for the component

$$L = \mathrm{Res}_{E/F}G \rtimes \theta$$

of the nonconnected reductive group

$$L^+ = \mathrm{Res}_{E/F}G \rtimes \mathrm{Gal}(E/F) ,$$

and on the reductive groups H which is the only endoscopic group for L. We refer to [KS1] and [KS2] for the definition and the properties of twisted endoscopy. Let us introduce a notation that will appear from time to time : if A is an F-algebra, and given a function f defined on $L^0(A)$, one defines a function f_θ on $L(A)$ by

$$f_\theta(x \rtimes \theta) = f(x) .$$

The comparison of the geometric expansions of two trace formulas, in the base change situation we are to study, will be quite simple thanks to two facts :

1 - The derived group of $GL(n)$ being simply connected, the centralizers in L^0 of semisimple elements are connected; in particular centralizers of regular elements are tori.

2 - Conjugacy and stable conjugacy coincide in $L(F)$.

III.1 – Stable conjugacy and the norm map.

The definition of stable conjugacy, and the basic study of the *norm map* is to be found in [Ko1]. The derived group of $GL(n)$ being simply connected we may use the following definition. Let $\bar{F}$ be the algebraic closure of F .

III.1.1. Definition. – *We say that two elements δ and δ' in $L(F)$ are stably conjugate if they are conjugate by an element x in $L^0(\bar{F})$.*

III.1.2. Lemma. – *Two elements δ and δ' in $L(F)$ are stably conjugate if and only if they are already conjugate by an element y in $L^0(F) = G(E)$.*

Proof. If $\delta = x^{-1}\delta' x$ then for any σ in the Galois group $\mathfrak{G} = \mathrm{Gal}(\bar{F}/F)$:

$$\delta = \sigma(x)^{-1}\delta'\sigma(x)$$

and hence $\sigma \mapsto x^{-1}\sigma(x)$ is a 1-cocycle with value in the centralizer L_δ . It suffices to show that there exist $u \in L_\delta(\bar{F})$ such that $y = ux \in L^0(F)$. But this follows from the triviality of $H^1(\mathfrak{G}, L_\delta)$ the first Galois cohomology group with value in L_δ ; this in turn uses that such a centralizer L_δ is the multiplicative group of a finite dimensional algebra over F ([S] Exercice 2 p. 160).

□

The *norm map* $\mathcal{N}_{E/F}$ between conjugacy classes in $L(F)$ and stable conjugacy classes in $H(F)$ is induced by the ℓ-th power in L^+ : the conjugacy class of $\delta \in L(F)$ under $L^0(F)$, is mapped to the intersection of $H(F)$ with the stable conjugacy class of δ^ℓ . In fact $\theta(\delta^\ell)$ is stably conjugate to δ^ℓ , and since H is split with a simply connected derived group, this intersection is non empty [Ko1]. It is a single $H(F)$-conjugacy class since, according to III.1.2, stable conjugacy and ordinary conjugacy coincide.

III.1.3. Lemma. – *Let F be a local or a global field. The norm map induces an injection from the set of conjugacy classes of regular elements in $L(F)$ into the set of conjugacy classes of regular elements in $H(F)$.*

Proof. The stable conjugacy class of δ^ℓ contains some $\gamma \in H(F)$, and the centralizer of δ in $L^0(F)$ is isomorphic to an inner form of the centralizer of γ in $H(F)$. In particular δ is regular if and only if γ is also regular, and in such a case their centralizers being tori are isomorphic : $L_\delta \simeq H_\gamma = T$ is a maximal F-torus in H ; the centralizer of δ^ℓ in $L^0(F)$ is isomorphic to $T(E)$ and conjugation by δ induces a Galois automorphism denoted θ_T on $T(E)$. Again the triviality of the first Galois cohomology group with value in the centralizers shows that the norm map induces an injection from the set of conjugacy classes under $L^0(F) = G(E)$ in $L(F)$ into the set of conjugacy classes in $H(F)$ (cf. [Lan1] lemma 4.2 p. 33).

□

III.1.4. Lemma. – *Let F be a local field. A regular element $\gamma \in H(F)$ which is elliptic is a norm from $L(F)$ if and only if $\xi(\gamma) \in N_{E/F}E^\times$ for all F-rational characters $\xi \in X(H)_F$.*

Proof. We refer to [A7] lemma 10.4 page 376.

□

The next lemma is an elaboration for inner forms of $GL(n)$ of the well known similar result for $GL(1)$: given a cyclic global field extension E/F , an element in $F^\times$ is a norm from $E^\times$ if and only if it is a norm locally everywhere.

III.1.5. Lemma. – *If F is global, a regular element in $H(F)$ is a norm from $L(F)$ if and only if it is a norm locally everywhere.*

Proof. We refer to [Lan1] lemma 4.9 p. 37 or [AC] chapter 1 lemma 1.2 p. 4 for the split case $G = H$. For the general case we refer to [KS2] lemma 6.3.A.

□

III.1.6. Lemma. – *Let F be a local field and asume that $G = H$. There is an open neighbourhood of the identity in $H(F)$ in which any element is a norm from $L(F)$.*

Proof. Since $G = H$ there is an injection

$$H(F) \rtimes \theta \hookrightarrow L(F)$$

and the norm map is induced by the ℓ-th power; but the map $\gamma \mapsto \gamma^\ell$ induces a diffeomorphism of a small enough neighbourhood of 1 in $H(F)$ onto its image.

□

III.1.7. Lemma. – *Let F be a local field and let $G = H$. An admissible irreducible representation π of $H(F)$ has a character distribution Θ_π that does not vanish identically on the set of regular elements that are norms of elements in $L(F)$.*

Proof. According to III.1.6 there is a neighbourhood U of the identity in $H(F)$ which contains only norms from $L(F)$. Let $f = f_1 * \check{f}_1$ be a function of positive type; here $\check{f}_1(x) = \overline{f_1(x^{-1})}$. If the support of f_1 is small enough trace $\pi(f)$ is strictly positive and the support of f is in U . It suffices now to recall that the character distribution is defined by a function in $\mathbf{L}^1_{loc}(H(F))$ and that the set of regular elements is open and dense.

□

III.1.8. Lemma. – *Let F be a local or a global field. If γ is stably conjugate to δ^ℓ*

$$\widetilde{D}^L(\delta) = D^H(\gamma) \ .$$

Proof. In fact since $\tilde{\mathfrak{l}}_\delta \simeq \mathfrak{l}_\gamma \otimes E$ this follows from the following elementary calculation. Let V be a finite dimensional vector space over F ; consider $a \in GL(V)$ and

$$b \in GL(V \otimes E) \rtimes \theta$$

such that $a = b^\ell$; we want to compare $\det(1 - a|V)$ and $\det(1 - b|V \otimes E)$. Over the algebraic closure $\bar{F}$ one has

$$V \otimes E \otimes_F \bar{F} \simeq (V \otimes \bar{F})^\ell$$

the factors being cyclicly permuted by θ and one can write b as a bloc-matrix

$$b = \begin{pmatrix} 0 & 0 & \dots & 0 & b_\ell \\ b_1 & 0 & \dots & 0 & 0 \\ \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & \dots & b_{\ell-1} & 0 \end{pmatrix}$$

with the $b_i \in GL(V \otimes \bar{F})$. An elementary computation shows that

$$\det(1 - b|V \otimes E) = \det(1 - b_\ell b_{\ell-1} \dots b_1 |V \otimes \bar{F}) \ .$$

Now $a = b^\ell$ is conjugate to $b_\ell b_{\ell-1} \dots b_1$ and hence $\det(1-a|V) = \det(1-b|V \otimes E)$. □

Remark. – A similar lemma holds for the normalizing factors used by Arthur and Clozel but powers of ℓ show up (cf. [AC] p. 80).

We have fixed once for all a minimal Levi subset M_0^L in L and a minimal Levi subgroup M_0^H in H . The isomorphism η induces an injection $M \mapsto \eta_{E/F}(M)$ of the set $\mathcal{L}^L(M_0^L)$ of semistandard F-Levi subsets in L into the set $\mathcal{L}^H(M_0^H)$ of semistandard Levi subgroups in H ; this is, a bijection if G is quasi-split. Given $M \in \mathcal{L}^H(M_0^H)$ we shall denote by M^L the preimage of M via $\eta_{E/F}$; this preimage may be empty. In particular $(M_0^H)^L$ is empty if G is not quasi-split. If the preimage of M is empty distributions $J_{M^L}^{Q^L}$ are understood to be zero. If M belongs to the range of $\eta_{E/F}$ the isomorphism η induces a bijection of the set $\mathcal{P}^L(M)$ of F-parabolic subsets in L with Levi subset M onto the set $\mathcal{P}^H(\eta_{E/F}(M))$ of parabolic subgroups Q in H with Levi subgroup $\eta_{E/F}(M)$. Again the preimage of Q is denoted by Q^L . The isomorphism η induces an isomorphism $\eta_H : \mathfrak{a}_H \to \mathfrak{a}_L$. More generally we have a canonical isomorphism

$$\eta_M^Q : \mathfrak{a}_M^Q \to \mathfrak{a}_{M^L}^{Q^L}$$

if M and Q are in the range of $\eta_{E/F}$. We shall use η_M^Q to transfer Haar measures. The covolumes that showed up in I.1 match : $a_M^Q = a_{M^L}^{Q^L}$. Let M^L be a Levi subset of L , we also have

$$d_M^Q(L_1, L_2) = d_{M^L}^{Q^L}(L_1^L, L_2^L) .$$

There is an other isomorphism induced by the norm map $\mathfrak{a}_{M^L}^{Q^L} \to \mathfrak{a}_M^Q$; the map η_M^Q followed by the norm map is the multiplication by ℓ on $\mathfrak{a}_M^Q$. This introduces factors $\ell^{\dim \mathfrak{a}_M^Q}$ in the transfer of Haar measures on the duals via the norm map.

III.2 – Endoscopic correspondence.

Let F be a local field. Dual to the concept of norm map for conjugacy classes is the concept of endoscopic correspondence (or base change) for representations; it is defined, at least for tempered representations, by a character identity which relates values of characters at points connected by the norm map.

III.2.1. Definition. – *Let π be an admissible irreducible representation of $H(F)$; denote by Θ_π its character distribution. An admissible irreducible representation π' of $L(F)^+$ with character distribution $\Theta_{\pi'}$ is said to be a strict base change lift of π if there is a nonzero constant e such that*

$$\Theta_{\pi'}(x) = e\ \Theta_\pi(y)$$

whenever $y \in H(F)$ is regular and stably conjugate to the ℓ-th power of $x \in L(F)$.

Remarks.

(i) It is usual to be more precise and to ask that the constant e is the Kottwitz constant, a sign which depends only on the local group $L(F)$ (cf. [AC] p. 78). In particular if L^0 is split the Kottwitz constant is 1 . If e is chosen to be the Kottwitz constant we say that π' is the canonical base change of π . Otherwise a base change π' of π , if it exists, is only defined up to a twist by a character of $L^0(F)\backslash L^+(F)$, and the constant e is the Kottwitz constant up to an ℓ-th root of unity. Nevertheless all base change lifts have the same restriction $\pi_{E/F}$ to $L^0(F)$.

(ii) While a character identity is the usual definition for tempered representations, the base change correspondence, in general, connects the Langlands quotients of standard representations (i.e. representations parabolically induced from quasi-tempered ones) that satisfy the character identity. Writing a representation as a linear combination of standard representations one can show that a strict base change is a base change; the converse need not be true for nontempered representations.

III.3 – Endoscopic transfer.

We now recall the definition of the endoscopic transfer, also called association in our setting.

III.3.1. Definition. – *A pair of smooth compactly supported functions f and ϕ on $H(F)$ and $L(F)$ respectively, are said to be associated, if for any regular element $\gamma \in H(F)$*

$$J_H(\gamma, f) = \sum_{\delta} \Delta_H^L(\gamma, \delta)\ J_L(\delta, \phi) .$$

where $\Delta_H^L(\gamma, \delta) = 1$ if γ is stably conjugate to δ^ℓ and equals 0 otherwise; the sum is over a set of representatives of $L^0(F)$-conjugacy classes in $L(F)$. We assume that the choice of Haar measures on the centralizers L_δ and H_γ , implicit in the definition of orbital integrals, is compatible with the natural isomorphism between centralizers if δ^ℓ is conjugate to γ .

Remarks.

(i) In the above definition it is equivalent to deal with orbital integrals instead of the normalized ones. This follows from lemma III.1.8.

(ii) We say that the endoscopic transfer holds if: given ϕ_v there exist an associated function f_v , and given f_v with vanishing orbital integrals for elements that are not norms there exists an associated function ϕ_v . It is shown in [AC] chapter 1 section 3 that the endoscopic transfer holds at least when either $G = H$ or $E = F$. We shall prove the particular cases we need.

We shall now introduce a noninvariant avatar of the concept of association.

III.3.2. Definition. – *Let F be a local field. Two functions f and ϕ on $H(F)$ and $L(F)$ respectively are called strongly associated if for all Levi subgroup $M \in \mathcal{L}^H$ and all parabolic subgroup Q of H , one has*

(i) If $\gamma \in M(F)$ is stably conjugate in M to δ^ℓ with $\delta \in M^L(F)$ the weighted orbital integrals match :

$$J^Q_M(\gamma, f) = J^{Q^L}_{M^L}(\delta, \phi)$$

(ii) If the value at γ of some F-rational character $\xi \in X(Q)_F$ is not a norm

$$\xi(\gamma) \notin N_{E/F}(E^\times)$$

or if Q does not belong to the image of $\eta_{E/F}$ then

$$J^Q_M(\gamma, f) = 0 \ .$$

Remarks.

(i) To check that two functions f and ϕ are strongly associated, it is enough, thanks to the descent formula (lemma I.6.3), to establish the matching statement

$$J^Q_M(\gamma, f) = J^{Q^L}_{M^L}(\delta, \phi)$$

when γ is the norm of δ , and the vanishing statement, for a Levi subgroup M in which γ is elliptic.

(ii) Given a pair (f, ϕ) of strongly associated functions, the same is true for pairs (f_P, ϕ_{P^L}) of constant terms.

III.3.3. Lemma. – *Strongly associated functions are associated.*

Proof. We have to check the vanishing property. Consider a regular element γ in $H(F)$ elliptic in M. It is a norm if and only if $\xi(\gamma)$ is a norm for all $\xi \in X(Q)_F$ where Q is a parabolic subgroup with M as Levi subgroup (lemma III.1.4). But

$$J_L(\gamma, f) = J^Q_M(\gamma, f)$$

if Q is parabolic subgroup with M as Levi subgroup. In particular, if (f, ϕ) are strongly associated $J_L(\gamma, f) = 0$ if γ is not a norm.

□

III.3.4. Lemma. – *Let F be a local field. Let M be a Levi subgroup in the image of $\eta_{E/F}$. Let $\gamma \in M_1(F) \subset M(F)$ such that $\xi(\gamma)$ is a norm for all $\xi \in X(M)_F$. Let f be in a pair (f, ϕ) of strongly associated functions. Then $J_M^Q(\gamma, f) = 0$ if $\xi(\gamma)$ is not a norm for some $\xi \in X(M_1)_F$.*

Proof. By the descent formula one has

$$J_M^Q(\gamma, f) = \sum d_{M_1}^Q(M, L_1) J_{M_1}^{Q_1}(\gamma, f)$$

Observe that the natural map

$$X(Q_1)_F \oplus X(M)_F \to X(M_1)_F$$

is surjective. Hence if $\xi(\gamma)$ is not a norm for some $\xi \in X(M_1)_F$ then $\xi_1(\gamma)$ is not a norm for some $\xi' \in X(Q_1)_F$ and hence $J_{M_1}^{Q_1}(\gamma, f) = 0$.

□

Remark. – In the proof of V.1.4 this lemma will play a role similar to proposition 10.2 page 373 of [A7] in the proof of proposition 8.1 p. 542 in [A8].

To separate infinitesimal characters it is convenient to deal with pairs compatible with multipliers.

III.3.5. Definition. – *Let F be an archimedean field and let (f, ϕ) be a pair of K-finite strongly associated smooth and compactly supported functions on $H(F)$ and $L(F)$ respectively. We say that the pair (f, ϕ) is compatible with multipliers if f_β and ϕ_α are strongly associated whenever α and β are compatible with base change i.e.*

$$\hat{\beta}(\nu) = \hat{\alpha}(\nu_{E/F})$$

where $\nu_{E/F}$ is the composition of ν with the map induced by the norm.

We hope that the endoscopic transfer can be supplemented in the following way to yield a noninvariant endoscopic transfer compatible with multipliers.

III.3.6. Conjecture B. – *Let F be a local field and let (f, ϕ) be a pair of associated smooth and compactly supported functions on $H(F)$ and $L(F)$ respectively.*

(B1) There exists a pair (f', ϕ') of strongly associated smooth compactly supported functions, with f' equivalent to f and ϕ' equivalent to ϕ .

(B2) If F is archimedean and if f and ϕ are K-finite the functions f' and ϕ' may be chosen K-finite and such that the pair (f', ϕ') is compatible with multipliers.

Remarks.

(i) One could formulate an analogue of B2 for nonarchimedean fields, with distributions in Bernstein's center playing the role of multipliers.

(ii) For functions with regular support, B1 follows from proposition III.4.1.

(iii) At split places, we shall prove B1 for K-finite functions, and B2 (III.5.5).

(iv) If $E = F$ and G is a division algebra conjecture B holds since associated functions are automatically strongly associated.

III.4 – Regular transfer and base change.

III.4.1. Proposition. – *Let F be a local field. Given ϕ a smooth function on $L(F)$, compactly supported supported inside the open set of regular elements, there exists a pair of smooth compactly supported function ϕ' on $L(F)$ and f' on $H(F)$ strongly associated with ϕ' equivalent to ϕ . Conversely if the orbital integrals of f vanish whenever $\gamma \in H(F)$ is not stably conjugate the ℓ-th power of an element $\delta \in L(F)$, and if f has a regular support there exist ϕ' and f' strongly associated with f' equivalent to f .*

Proof. Given ϕ lemma I.7.1 proves the existence of ϕ' equivalent but with vanishing weighted orbital integrals for nontrivial weights. Now any regular δ defines a torus $T(F)$ in $L(F)$ and its norm γ defines a torus $T_0(F)$ in $H(F)$ and $T^0(F) \simeq T_0(F)$. The image of the norm map is a finite covering from $T(F)_{reg}$ onto the open set of ℓ-th powers elements in $T_0(F)_{reg}$ and is a injection from the set of $W^{L^0}(T)$-orbits into the set of $W^H(T_0)$-orbits (lemma III.1.3) and hence ϕ' defines a smooth compactly supported $W^H(T_0)$-invariant function

$$\varphi(\gamma) = \sum_\delta \Delta_H^L(\gamma, \delta)\ J_L^L(\delta, \phi)$$

on $T_0(F)$. The lemma I.7.1 or rather its proof allows one to construct a function f' on $H(F)$ with vanishing weighted orbital integrals for nontrivial weights and with ordinary orbital integrals given by φ . The other half of the proposition is proven similarly.

□

III.4.2. Corollary. – *Let $\pi \in \Pi(H(F))$ and let $\pi' \in \Pi(L(F))$. Assume there is a nonzero constant c , such that for all pairs (f, ϕ) of strongly associated functions with regular support*

$$\text{trace}\ \pi'(\phi) = c\ \text{trace}\ \pi(f)\ .$$

Then π' is a strict base change lift of π .

Proof. We want to prove that

$$\Theta_{\pi'}(x) = e\ \Theta_\pi(y)$$

whenever $y \in H(F)$ is regular and stably conjugate to the ℓ-th power of $x \in L(F)$. This is an immediate consequence of Weyl's integration formula, using III.1.8 and III.4.1.

Remark. – The two constants c and e differ by a power of $|\ell|_F$.

III.5 – Strong association at split places.

Let F be a local field. If E/F splits and if $G = H$, the construction of strongly associated pairs can be made quite explicit thanks to the next lemma.

III.5.1. Lemma. – *Let f_i be smooth compactly supported functions on $H(F)$, and consider $\phi = (f_1 \otimes f_2 \otimes \ldots \otimes f_\ell)_\theta$ on $L(F)$. Let M be a Levi subgroup in H and let Q be a parabolic subgroup containing M . Consider $\delta \in M^L(F)$ regular in $L(F)$:*

$$\delta = (\gamma_1, \ldots, \gamma_\ell) \rtimes \theta \ ,$$

and let $\gamma = \gamma_1 \ldots \gamma_\ell$. The weighted orbital integral $\Phi^{Q^L}_{M^L}(\delta, \phi)$ equals

$$\int_{M(F)\backslash H(F)} \int_{H(F)} \cdots \int_{H(F)} f_1(x_\ell^{-1} \ldots x_1^{-1} \gamma x_1)\, f_2(x_2) \ldots f_\ell(x_\ell)$$
$$v^{Q^L}_{M^L}(x_1\ ,\ x_1 x_2\ , \ldots ,\ x_1 \ldots x_\ell)\ d\dot{x}_1\, dx_2 \ldots dx_\ell\ .$$

Proof. By definition

$$\Phi^{Q^L}_{M^L}(\delta, \phi) = \int_{M(F)\backslash L^0(F)} f_1(x_\ell^{-1} \gamma_1 x_1)\, f_2(x_1^{-1} \gamma_2 x_2) \ldots f_\ell(x_{\ell-1}^{-1} \gamma_\ell x_\ell)$$
$$v^{Q^L}_{M^L}(x_1, x_2, \ldots, x_\ell)\ \frac{dx_1\, dx_2 \ldots dx_\ell}{dm}\ .$$

The desired formula follows from simple changes of variables using the left invariance of the weight under $M(F) \times \ldots \times M(F)$.

□

The lemma shows that a function f on $H(F)$ is associated to ϕ if and only if f is equivalent to $f' = f_1 * \ldots * f_\ell$. Observe that any smooth compactly supported function on $H(F)$ can be written as a finite sum of convolution products. This follows from Dixmier-Malliavin's factorization theorem [DM] for the group $H(F)$. This shows that given f there exist ϕ such that (f, ϕ) are associated and conversely. To obtain strongly associated pairs of functions one has to specify the f_i .

The lemma also shows that, at split places, weighted orbital integrals still make sense if one replaces

$$f_2(x_2)\, dx_2 \otimes \ldots \otimes f_\ell(x_\ell)\, dx_\ell$$

by a tensor product of compactly supported Radon measures $\mu_2 \otimes \ldots \otimes \mu_\ell$.

Let τ be a finite dimensional irreducible representation of K^H , denote by e_τ the measure on $L(F)$, supported on K^H , defined by :

$$e_\tau(f) := \frac{\dim(\tau)}{\text{vol}\,(K^H)} \int_{K^H} f(k) \ \text{ trace } \tau(k)\, dk \ .$$

This is an idempotent in the convolution algebra. The sum over a finite set Γ of inequivalent irreducible τ is again an idempotent

$$e = \sum_{\tau \in \Gamma} e_\tau \ ,$$

that will be called an elementary idempotent. A function f is K-finite if and only if there is an elementary idempotent e such that $f * e = e * f = f$.

III.5.2. Lemma. – *Let f be a K-finite smooth compactly supported function on $H(F)$ and let e be an elementary idempotent such that*

$$e * f = f * e = f \ .$$

Then f and $(f \otimes e \otimes \ldots \otimes e)_\theta$ are strongly associated.

Proof. Since $v_{M^L}^{Q^L}(x_1, x_1, \ldots, x_1) = v_M^Q(x_1)$ this follows from III.5.1, using the right invariance of weights under

$$K^{L^0} = K^H \times \ldots \times K^H$$

□

This shows that B1 holds for split nonarchimedean places. Assume from now on that F is archimedean. For archimedean fields e is not a smooth function; we shall use multipliers to transform measures into functions. We need a lemma.

III.5.3. Lemma. – *Let $\phi = (f_1 \otimes f_2 \otimes \ldots \otimes f_\ell)_\theta$. Consider multipliers defined by $\alpha = (\alpha_1, \ldots, \alpha_\ell)$ and $\alpha' = (\alpha'_1, \ldots, \alpha'_\ell)$ such that*

$$\alpha_1 * \ldots * \alpha_\ell = \alpha'_1 * \ldots * \alpha'_\ell = \beta \ .$$

For any $\delta \in M^L(F)$ regular in $L(F)$

$$J^{Q^L}_{M^L}(\delta,\phi_\alpha) = J^{Q^L}_{M^L}(\delta,\phi_{\alpha'}) \; .$$

Proof. By proposition I.6.6 it suffices to show that the functions ϕ_α and $\phi_{\alpha'}$ have the same weighted characters. If the restriction of π' to $H(E) = H(F)^\ell$ equals $\pi\otimes\ldots\otimes\pi$, the weighted characters are of the form

$$J^L_M(\pi',\phi_\alpha) = \text{ trace } (R\,\pi'(\phi_\alpha))$$

where R is a sum of decomposed operator

$$R = \sum_i R^i_1 \otimes R^i_2 \otimes \ldots \otimes R^i_\ell \; .$$

Since $\pi'(\theta)$ permutes the factors cyclicly

$$\begin{aligned}\text{trace } (R\,\pi'(\phi_\alpha)) = \\ &\sum_i \hat\alpha_1(\nu_\pi)\ldots\hat\alpha_\ell(\nu_\pi) \text{ trace } (R^i_1\,\pi(f_1)R^i_2\,\pi(f_2)\ldots R^i_\ell\,\pi(f_\ell)) \\ &= \hat\beta(\nu_\pi) \text{ trace } (R\,\pi'(\phi))\end{aligned}$$

and hence

$$J^L_M(\pi',\phi_{\alpha'}) = J^L_M(\pi',\phi_\alpha) \; .$$

□

III.5.4. Proposition. – *Let $\alpha = (\alpha_1,\ldots,\alpha_\ell)$ be a family of smooth $W_{\mathbb{C}}$-invariant compactly supported functions on $\mathfrak{h}$, let $\beta = \alpha_1 * \ldots * \alpha_\ell$ and*

$$\phi_\alpha = (f_{\alpha_1}\otimes e_{\alpha_2}\otimes\ldots\otimes e_{\alpha_\ell})_\theta \; .$$

The functions ϕ_α and f_β are smooth K-finite compactly supported functions on $L(F)$ and $H(F)$ respectively. They are strongly associated.

Proof. Consider a sequence $g^{(n)}$ for $n \in \mathbb{N}$ of smooth functions on $H(F)$, with support in a fixed compact set such that the sequence of measures $g^{(n)}(x)\,dx$ is a sequence of

Radon measures with bounded norm that converge to the measure e as $n \to \infty$, in the weak topology. In particular $g^{(n)} \otimes \ldots \otimes g^{(n)}$ converges in the weak topology to $e \otimes \ldots \otimes e$. Consider the distribution α' such that $\hat{\alpha}' = (\hat{\beta}, 1, \ldots, 1)$ and let

$$\phi_{\alpha'}^{(n)} = (f_\beta \otimes g^{(n)} \otimes \ldots \otimes g^{(n)})_\theta \ .$$

Lemma III.5.2 shows that

$$J_M^Q(\gamma, f_\beta) = \lim_n J_{M^L}^{Q^L}(\delta, \phi_{\alpha'}^{(n)}) \ .$$

Now let

$$\phi_\alpha^{(n)} = (f_{\alpha_1} \otimes g_{\alpha_2}^{(n)} \otimes \ldots \otimes g_{\alpha_\ell}^{(n)})_\theta \ .$$

According to lemma III.5.3

$$J_{M^L}^{Q^L}(\delta, \phi_\alpha^{(n)}) = J_{M^L}^{Q^L}(\delta, \phi_{\alpha'}^{(n)}) \ .$$

To conclude we observe that $g_{\alpha_i}^{(n)}$ tends to e_{α_i} in $\mathbf{L}^1(H(F))$ when $g^{(n)}$ tend to e and hence

$$\lim_n J_{M^L}^{Q^L}(\delta, \phi_\alpha^{(n)}) = J_{M^L}^{Q^L}(\delta, \phi_\alpha) \ .$$

□

III.5.5. Proposition. – *Conjecture B2 holds for split archimedean places.*

Proof. Consider a pair of associated functions f and $\phi = (f_1 \otimes \ldots \otimes f_\ell)_\theta$; this means that f is equivalent to $f' = f_1 * \ldots * f_\ell$. Lemma I.8.1 and the factorization theorem [DM] for functions on $\mathfrak{h}$ shows that f is a finite sum of functions of the form g_β where β is a convolution products of ℓ factors $\alpha_1 * \ldots * \alpha_\ell$. Now proposition III.5.4 allows us to construct a K-finite function ϕ' equivalent to ϕ such that (f, ϕ') is a pair of strongly associated functions compatible with multipliers.

□

IV. – UNRAMIFIED PLACES

In this chapter F is a nonarchimedean local field, E is an unramified Galois algebra of degree ℓ over F , the group G is a product of groups $GL(n_i)$; in particular G is split

$$H(F) = G(F) = \prod_i GL(n_i, F)$$

and

$$L^0(F) = G(E) = \prod_i GL(n_i, E) .$$

Elementary functions were introduced in [Lab2] and used to prove the fundamental lemma for stable base change. Here we show that the various properties established there have noninvariant analogues. We show that pairs of elementary functions defined by semisimple elements connected by the norm map are in fact strongly associated. Moreover we show that weighted characters for elementary functions connected by the norm map are compatible with base change. This uses that elementary functions are closely related to functions bi-invariant under Iwahori subgroups. At the end of this chapter we state a noninvariant form of the "fundamental lemma".

IV.1 – Elementary functions and Iwahori subgroups double cosets.

Following Deligne [Deligne] and Casselman [Cas1], to any semisimple element y in $L(F)$ we associate a parabolic subset $P^{(L,y)}(F)$ in $L(F)$: the subset of $x \in L(F)$ such that $y^n x y^{-n}$ remains bounded for $n \in \mathbb{N}$. This parabolic subset has a Levi decomposition $P^{(L,y)} = M^{(L,y)}N^{(L,y)}$ with Levi subset $M^{(L,y)}(F)$, the subset of $x \in L(F)$ such that $y^n x y^{-n}$ remains bounded for $n \in \mathbb{Z}$. The nilpotent radical $N^{(L,y)}$ is often simply denoted $N^{(y)}$. We shall denote by $\bar{N}^{(y)}$ the opposite unipotent subgroup. The Levi subgroup of L^0 defined by y will be denoted $M^{(L^0,y)}$ or simply $M^{(y)}$. These Levi subsets need not be semistandard.

Denote by $\mathfrak{O}_F$ the ring of integers in F and let $\mathfrak{P}_F = \varpi\mathfrak{O}_F$ denote its maximal ideal where ϖ is an uniformizing parameter in $\mathfrak{O}_F$. Assume that L^0 is a quasi-split scheme over $\mathfrak{O}_F$ split over an unramified field extension F_1 of F so that the subgroup $K = L^0(\mathfrak{O}_F)$ is an hyperspecial maximal compact subgroup. We assume that some (and hence any) $\varepsilon \in M_0(\mathfrak{O}_F)$ normalizes K . In the terminology of section I.2 the pair (L, K) is uramified. This applies in particular to our situation: G is split, and $L = \mathrm{Res}_{E/F} G \rtimes \theta$ where E is an unramified Galois algebra of degree ℓ over F , whose Galois group is generated by θ and $\varepsilon = 1 \rtimes \theta$. To y is associated a parahoric subset $B^{(L,y)}$ and a parahoric subgroup $B^{(L^0,y)}$, also sometimes denoted $B^{(y)}$, with Iwahori decomposition

$$B^{(L,y)} = \bar{N}^{(y)}(\mathfrak{P}_F)M^{(L,y)}(\mathfrak{O}_F)N^{(y)}(\mathfrak{O}_F) .$$

Let A_0 be the split component of the minimal Levi subset M_0 . The image of the map

$$H_0 \ : \ A_0(F) \to \mathfrak{a}_0$$

is a lattice X_* isomorphic to the group of one parameter subgroups in A_0 . There is a natural section of the map $A_0(F) \to X_*$ up to the choice of a uniformizing parameter. Given $\tau \in X_0$ we denote by $t_0 = \varpi^\tau$ its image by the section in $A_0(F)$. Choose a minimal Levi subset $P_0 = M_0N_0$. We say that $t = t_0\varepsilon$ is antidominant if $|t^\alpha| \leq 1$ for all roots α of A_0 in N_0 . This is equivalent to say that τ is dominant : $< \tau, \alpha > \geq 0$ for all α . The Levi subset $M^{(L,t)}$ is the centralizer of t_0 in $L(F)$. We say that t is very regular if $|t^\alpha| \neq 1$ for all roots α . If t is antidominant and very regular $P^{(L,t)} = P_0$ the minimal parabolic subset and $B^{(t)} = B_0$ is an Iwahori subgroup.

Let B be an open compact subgroup of K such that $B \cap M^{(t)}(F)$ satisfies the Kottwitz' conditions ([Ko2] p. 240 or [Lab2] p. 520). Let us denote by Char (X) the characteristic function of X . The elementary function f_t^B is defined as follows :

$$f_t^B = \frac{1}{\text{vol }(B)} \text{ Char } \{k^{-1} m\, t\, k | k \in B\, , \, m \in B \cap M^{(t)}(F)\} \, .$$

Here we shall consider elementary functions defined by taking B to be a standard parahoric subgroup for $L(F)$ i.e. parahoric subgroups of $L^0(F)$ defined from a standard parabolic subset (i.e. containing P_0). If $B = K$ and we shall sometimes write f_t^L instead of f_t^K . As in [Lab2] only the two extreme cases will be used : either t is very regular and $M^{(t)} = M_0^0$, or $t = \varepsilon$ and $M^{(t)} = L^0$. We begin by exhibiting the close connection between elementary functions and function in the Hecke-Iwahori algebra i.e. compactly supported functions bi-invariant under an Iwahori subgroup.

IV.1.1. Proposition. – *(i) If B is a standard parahoric subgroup for $L(F)$ such that $B \subset B^{(t)}$, the elementary function f_t^B is the characteristic function of the double-coset $B\, t\, B$ divided by the volume of B .*

(ii) If $B \supset B^{(t)}$ the elementary function is the caracteristic function of a disjoint union of conjugates by a set of representatives of $B^{(t)} \backslash B$ of the double-coset $B^{(t)}\, t\, B^{(t)}$, divided by the volume of B .

$$f_t^B = \frac{1}{\text{vol }(B)} \sum_{b \in B^{(t)} \backslash B} \text{Char } (b^{-1} B^{(t)}\, t\, B^{(t)}\, b) \, .$$

Proof. Assume first that $B \subset B^{(t)}$. Let $A^{(t)} = M^{(t)}(F) \cap B \subset M^{(t)}(\mathfrak{O}_F)$. This is a parahoric subgroup in $M^{(t)}(F)$. Consider the map

$$(b, m) \mapsto b^{-1} m\, t\, b$$

from $B \times A^{(t)}$ to $B.t.B$. It induces a map

$$c_t \;:\; A^{(t)} \backslash (B \times A^{(t)}) \to B.t.B$$

where $A^{(t)}$ acts by left translation on B and by ε-twisted conjugacy on the second factor $A^{(t)}$. This is a bijection onto the double coset $B.t.B$. We prove first that c_t is an injection : let $b \in B$ such that

$$b^{-1} m\, t\, b = m_1 t$$

with m and m_1 in $A^{(t)}$; then $t^n b t^{-n}$ is bounded for all $n \in \mathbb{N}$; hence b belongs to the Levi subgroup $M^{(t)}(F)$ defined by t ; this implies $b \in A^{(t)}$ and proves the injectivity of c_t . To see that c_t is a surjection it suffices for example to show that the volume of the image of the map c_t (which is open and compact) equals the volume of $(B.t.B)$. But the volume of the image of c_t is vol $(B)\ |\widetilde{D}^L(t)|$ since the Jacobian is $|\widetilde{D}^L(t)|$. On the other hand

$$\text{vol}\ (B\,t\,B) = \text{vol}\ (B)\ \text{Card}\,(C^{(t)} \setminus B)$$

where $C^{(t)} = B \cap t\,B\,t^{-1}$. This is an open subgroup of B with an Iwahori decomposition

$$C^{(t)} = \bar{N}^{(t)}(\mathfrak{P}_F)\ A^{(t)}\ t\ N^{(t)}(\mathfrak{O}_F)\ t^{-1}\ .$$

The covolume of $C^{(t)}$ is $\delta_{P^{(t)}}(t)^{-1}$. The surjectivity of c_t follows from the equality

$$\delta_{P^{(t)}}(t)^{-1} = |\widetilde{D}^L(t)| = |D^H(t^\ell)|\ .$$

This proves (i). Assertion (ii) is an easy consequence of (i).

□

IV.1.2. Corollary. – *Let π be an irreducible admissible representation of $L^+(F)$ in a vector space V , let $\mathcal{R}$ be an endomorphism of V that commutes with the restriction of π to K . If $B \subset B^{(t)}$*

$$\text{trace}\ (\mathcal{R}\pi(f_t^B)) = \delta_{P^{(t)}}(t)^{-1}\ \text{trace}\ (\mathcal{R}^B \pi^B(t))$$

where $\mathcal{R}^B$ and π^B are the operators in the space V^B of B-invariant vectors in V , deduced from $\mathcal{R}$ and π by left and right multiplication by the projector on V^B . If $B \supset B^{(t)}$ one has

$$\text{trace}\ (\mathcal{R}\pi(f_t^B)) = \delta_{P^{(t)}}(t)^{-1}\ \text{trace}\ (\mathcal{R}^{B^{(t)}} \pi^{B^{(t)}}(t))\ .$$

□

From the IV.1.1 we deduce also a noninvariant version of proposition 5 of [Lab2].

IV.1.3. Proposition. – *Let π be an irreducible admissible representation of $L^+(F)$, then*

$$\pi(f_t) = 0$$

unless the restriction of π to $L^0(F)$ is a sum of subquotients of unramified principal series.

Proof. Observe that the only representations of $L^0(F)$, with nonzero vectors fixed under an Iwahori subgroup, are subquotients of unramified principal series.

□

Any $w \in W_0^L$ can be written $w = s_w \rtimes \theta$ with $s_w \in W_0^H$; the map $w \mapsto s_w$ is a bijection between W_0^L and the Weyl group W_0^H . For $w \in W_0^H$ let $t^w = w^{-1}tw$.

IV.1.4. Lemma. – *Let λ be an unramified character of $M_0^+(F)$. Let π be a subquotient of the principal series representation $\mathcal{I}_{P_0}^L(\lambda)$ of $M(F)^+$ defined by λ . There is a subset $W(\pi) \subset W_0^H$ such that, as a function of t , for t antidominant with respect to P_0 and very regular,*

$$J_M^Q(\pi, f_t) = \sum_{w \in W(\pi)} \lambda(t^w)\ P_w(log\, \nu_1(t), \dots, log\, \nu_r(t))$$

where P_w is a polynomial and the ν_i are positive real valued characters. If moreover λ is regular (i.e. $w(\lambda) = \lambda$ implies $w = 1$)

$$J_M^Q(\pi, f_t) = \sum_{w \in W(\pi)} c_M^Q(w, \lambda, P_0)\ \lambda(t^w)$$

where the $c_M^Q(w, \lambda)$ are analytic functions of λ in the regular set.

Proof. We assume t antidominant and very regular; and hence $B^{(t)} = B_0$. It follows from IV.1.2 that

$$J_M^Q(\pi, f_t^B) = J_M^Q(\pi, f_t^{B_0})$$

where B_0 is the Iwahori subgroup. Now the convolution subalgebra of the Hecke-Iwahori algebra generated by the $f_t^{B_0}$ with t antidominant is abelian :

$$f_{t_1}^{B_0} * f_{t_2}^{B_0} = f_{t_1 t_2}^{B_0} \ .$$

The set of operators $\pi(f_t)$ is a finitely generated commutative family of finite rank operators. There is a basis in which they are simultaneously upper triangular. This representation can be computed using the Jacquet module for M_0 whose semisimplification is the sum over $W(\pi)$ of characters $t \mapsto \lambda(t^w)$. Moreover if λ is regular the Jacquet module is semisimple.

□

IV.2 – Elementary functions and constant terms.

In this section we establish a compatibility between elementary functions and constant terms. The elementary functions f_t^B with $B = K^{L^0}$ the hyperspecial maximal compact subgroup will also be denoted f_t^L . We assume that $t = t_0\epsilon$ is very regular. Let M be a semistandard Levi subset of a parabolic subset $P = MN$.

IV.2.1. Lemma. – *Consider $n \in N(F)$ and m regular in $M(F)$. Then, $f_t^L(mn) \neq 0$ if and only if*

$$mn = p^{-1}\eta t_0^w p$$

with $\eta \in K^L \cap M_0(F)$ for some w in K normalizing M_0 and $p \in K \cap P^0(F)$.

Proof. Since t is very regular $M^{(L,t)} = M_0$. If $f_t^L(mn) \neq 0$ then, by definition of f_t^L ,

$$x = mn = k^{-1}\eta_0 t_0 k$$

for some $k \in K$ and $\eta_0 \in K^L \cap M_0(F)$. The Levi subset $M^{(L,x)}$ defined by x in L , contains the Levi subset defined by x in P . Up to conjugacy by an element p_0 of $P^0(F)$ we may assume that the latter is the minimal Levi subset M_0 , and hence

$$p_0 M^{(L,x)} p_0^{-1} = M_0 \ ,$$

and there exists $m_0 \in M_0(F)$ such that

$$x = mn = p_0^{-1} m_0 p_0 = k^{-1}\eta_0 t_0 k \ .$$

Hence $y = p_0 k^{-1}$ normalize M_0 ; we may write $y = m_1 w$ with $w \in K$ normalizing M_0 and $m_1 \in M_0(F)$. Let $p = m_1^{-1} p_0$; we get that $f_t^L(mn)$ is nonzero only if

$$mn = p^{-1}\eta t_0^w p$$

with $\eta = w^{-1}\eta_0 w = K^L \cap M_0(F)$ and $p = w^{-1}k \in K \cap P^0(F)$. The converse is clear.

□

Let W_M^L denote the "quotient" $W_0^M \backslash W_0^L$. The quotient is defined using the bijection between W_0^L and the Weyl group W_0^H . For m regular in $M(F)$ let

$$\Delta_M^L(m) = \frac{|\widetilde{D}^L(m)|}{|\widetilde{D}^M(m)|} .$$

IV.2.2. Proposition. – *We assume that t is very regular. The constant term along P of f_t^L is a linear combination of elementary functions on $M(F)$:*

$$f_{t,P}^L = \sum_{w \in W_M^L} \Delta_M^L(t^w)^{1/2} \, f_{t^w}^M .$$

In particular $f_{t,P}^L$ is independent of the parabolic subset P with Levi subset M .

Proof. Recall that the Haar measures at finite places are normalized so that

$$\text{vol}\ (K) \ = \ \text{vol}\ (K^{M^0}) = \text{vol}\ (N(F) \cap K) = 1 .$$

Consider $m \in M(F)$ regular semisimple, we want to compute the constant term

$$f_{t,P}^L(m) = \delta_P(m)^{1/2} \int_K \int_{N(F)} f_t^L(k^{-1}\, m\, n\, k)\ dn\, dk .$$

This can be rewritten :

$$f_{t,P}^L(m) = \ \Delta_M^L(m)^{1/2} \int_{N(F)} f_t^L(n_1^{-1} m n_1)\ dn_1 .$$

But IV.2.1 shows that $f_t^L(n_1^{-1}mn_1) \neq 0$ if and only if $n_1^{-1}mn_1 = mn = p^{-1}\eta t_0^w p$ with $\eta \in K^L \cap M_0(F)$ for some $w \in W_0^L$ and $p \in K \cap P(F)$. This implies $n_1 \in K \cap N(F)$ and

$$\Delta_M^L(m)^{1/2} = \Delta_M^L(t^w)^{1/2} .$$

We get

$$f_{t,P}^L(m) = \ \Delta_M^L(t^w)^{1/2}$$

if $m = k^{-1}\eta t_0^w k$ for some $k \in K \cap M^0(F)$, some $\eta \in K^L \cap M_0(F)$ and some $w \in K$ normalizing M_0 ; it vanishes otherwise.

□

IV.3 – Elementary functions and noninvariant endoscopic transfer.

In this section E is a field extension of F. As already suggested in [Lab2] p. 522, the proof of proposition 3 of [Lab2] can be generalized to weighted orbital integrals. If t is very regular the proof relies on the surjectivity of the norm map from $\mathfrak{O}_E^\times$ onto $\mathfrak{O}_F^\times$; in general, the basic ingredient in the proof is the non abelian analogue of this fact, due to Kottwitz [Ko2].

Let F^{nr} be the maximal unramified extension of F. Let M be a semistandard Levi subgroup of H. Let w be a function on $G(F^{nr})$ right invariant by $G(\mathfrak{O}_{F^{nr}})$ and left invariant by $M(F^{nr})$. Denote by B^L the translate by ε of a standard parahoric subgroup B. The elementary function f_ε^B is the characteristic function of B^L divided by its volume. Given $\delta \in M^L(F)$ regular in $L(F)$ we denote the weighted orbital integral of the function f_ε^B with the weight w by $\Psi(\delta, B^L, w)$:

$$\Psi(\delta, B^L, w) = \int_{L_\delta(F)\backslash L^0(F)} f_\varepsilon^B(x^{-1}\delta x)w(x)\, dx \ .$$

Similarly $f_1^{B^H}$ is the characteristic function of $B^H = B \cap H(F)$ divided by its volume.

IV.3.1. Lemma. – *Given $\delta \in M^L(F)$ and $\eta \in M(F)$ such that δ^ℓ is stably conjugate to γ, then*

$$\Psi(\delta, B^L, w) = \Psi(\gamma, B^H, w) \ .$$

Proof. This is stated and proved by Kottwitz in [Ko2] p. 248-249; recall that $M(\mathfrak{O}_E)\cap B$ satisfies Kottwitz' conditions and that conjugacy and stable conjugacy coincide for our groups.

□

IV.3.2. Lemma. – *Let $M \in \mathcal{L}^H(M_0^0)$. Given $\delta \in M^L(F)$ such that $x^{-1}\delta x \in K^L$ for some $x \in L^0(F)$ then there exist $m \in M^0(E)$ such that $m^{-1}\delta m = \nu \in M^L(\mathfrak{O}_F)$*

Proof. By Iwasawa decomposition one has $x = mnk$ with $k \in K^{L^0}$, $n \in N(E)$ and $m \in M^0(E)$. Hence

$$n^{-1}m^{-1}\delta mn = m^{-1}\delta m n_1 \in K$$

for some $n_1 \in N(E)$. This implies $m^{-1}\delta m \in K^L \cap M^L(F) = M^L(\mathfrak{O}_F)$.

□

IV.3.3. Proposition. – *The characteristic function of B^L divided by its volume and the characteristic function of B^H divided by its volume are strongly associated.*

Proof. According to Kottwitz [Ko2] the norm map from B^L into the set of conjugacy classes in B^H , is surjective, the proposition is an immediate consequence of the above two lemmas since the weights v_M^Q are the restriction to $L(F)$ of weights which satisfy the assumptions of lemma IV.3.1.

□

Remark. – This proposition proved in Kottwitz [Ko2] is also stated as lemma 4.3 in [AC] chapter 2 p. 103. A more general result is true; before proving it, we need some lemmas.

IV.3.4. Lemma. – *Assume that for some $x \in L^0(F)$, and some $\nu \in M^{(L,t)}(\mathfrak{O}_E)$ one has $x^{-1}\delta x = \nu t_0$. The weighted orbital integrals for f_t^L can be rewritten as a weighted orbital integral on $M^{(L,t)}$ the Levi subset defined by t in L*

$$\Phi_{M^L}^{Q^L}(\delta, f_t^B) = \Psi(\nu, M^{(L,t)}(F) \cap B^L, w^x)$$

with the weight $w^x(y) = v_M^Q(xy)$.

Proof. By hypothesis

$$\Phi_{M^L}^{Q^L}(\delta, f_t^B) = \int_{L_{\nu t_0}(F)\backslash L^0(F)} f_t^B(y^{-1}\nu t_0 y) v_M^Q(xy) dy \ .$$

By lemma 1 in [Lab2] $f_t^B(y^{-1}\nu t_0 y) \neq 0$ only if $y = mk$ with $k \in B$ and $m \in M^{(t)}(E)$. The weight being right invariant under K , we may reduce the above integral to a weighted orbital integral for the weight $w^x(y) = v_M^Q(xy)$ over the Levi subset $M^{(L,t)}$. In fact, integrating first over $M^{(t)}(E)$ and then over $M^{(t)}(E)\backslash L^0(F)$ the above integral equals :

$$\frac{\text{vol }(B)}{\text{vol }\left(M^{(t)}(E) \cap B\right)} \int_{L_{\nu t_0}(F)\backslash M^{(t)}(E)} f_t^B(m^{-1}\nu t_0 m) w^x(m)\, dm \ .$$

To conclude we remark that $L_{\nu t_0} = M_\nu^{(L,t)}$.

□

IV.3.5. Lemma. – *Let $\gamma \in M(F)$. Assume that $x \in H(F)$, and $\eta \in M^{(t)}(\mathfrak{O}_F)$ are such that $x^{-1}\gamma x = \eta t_0^\ell$. We may find $\gamma_1 \in M(F)$ conjugate to γ by an element of $M(F)$ and $\eta_1 \in M^{(t)}(\mathfrak{O}_F)$ conjugate to η by an element of $M^{(t)}(\mathfrak{O}_F)$ and such that $x_1^{-1}\gamma_1 x_1 = \eta_1 t_0^\ell$ for some $x_1 \in H(F)$ and such that moreover $M_1 = x_1^{-1}Mx_1 \cap M^{(t)}$ is a Levi subgroup containing M_0 .*

Proof. By conjugacy of γ by some $m \in M(F)$ we may assume that $\gamma_1 = m^{-1}\gamma m$ is such that $M(F) \cap M^{(\gamma_1)}$ is a Levi subgroup containing M_0 . There is $s \in H(F)$ representing an element in the Weyl group W_0^H such that $\gamma_2 = s^{-1}\gamma_1 s$ belongs to $M^{(t)}(F)$ and is conjugate to ηt_0^ℓ by some $m' \in M^{(t)}(F)$. We conclude using lemma IV.3.3.

□

IV.3.6. Proposition. – *The elementary functions f_t^B on $L(F)$ and $f_{t^\ell}^{B^H}$ on $H(F)$ are strongly associated.*

Proof. Let Q be a parabolic subgroup of H with Levi subgroup M . Consider $\delta \in M^L(F)$, regular in $L(F)$. The weighted orbital integral $\Phi_{M^L}^{Q^L}(\delta, f_t^B)$ is not zero only if for some $x \in L^0(F)$, and some $\nu \in M^{(L,t)}(\mathfrak{O}_E)$ one has $x^{-1}\delta x = \nu t_0$. In such a case by lemma IV.3.4

$$\Phi_{M^L}^{Q^L}(\delta, f_t^B) = \Psi(\nu, M^{(L,t)}(F) \cap B^L, w^x) .$$

Consider now $\gamma \in H(F)$, $x \in H(F)$ and $\eta \in M^{(t)}(F) \cap B^H$ such that $x^{-1}\gamma x = \eta t^\ell$ then by the same lemma

$$\Phi_M^Q(\gamma, f_{t^\ell}^{B^H}) = \Psi(\eta, M^{(t)}(F) \cap B^H, w^x) .$$

Since parahoric subgroups satisfy the Kottwitz' conditions ([Ko2] p. 240), one can find $\nu \in M^{(L,t)}(E) \cap B^L$ so that $(\nu)^\ell = \eta$; then $\delta = x\nu t_0 x^{-1}$ is such that $\delta^\ell = \gamma \in M(F)$; in particular γ is a norm. Note that the function $y \mapsto v_M^Q(xy)$ is the restriction to $M^{(t)}(E)$ of a function on $M^{(t)}(F^{nr})$ that is $M^{(t)}(\mathfrak{O}_{F^{nr}})$ invariant on the right and $M(F^{nr})^x$ invariant on the left. But according to lemma IV.3.5 γ , η , ν and x may be replaced by γ_1 etc..., without changing the value of the weighted integrals, and so that the subgroup $M_1 = M^{x_1} \cap M^{(t)}$ is a Levi subgroup containing M_0 . Since $\nu_1 \in M_1^L(\mathfrak{O}_F)$ and $\eta_1 \in M_1(\mathfrak{O}_F)$ we may apply lemma IV.3.1 and we get

$$\Psi(\nu_1, M^{(L,t)}(F) \cap B^L, w^{x_1}) = \Psi(\eta_1, M^{(t)}(F) \cap B^H, w^{x_1}) .$$

□

Remark. – For t very regular, one has $M^{(t)} = M_0$ and the above proof shows that in fact, thanks to I.3.1, the weighted orbital integral vanish if the weight v_M^Q is nontrivial (i.e. M is not the Levi subset of Q).

IV.4 – Elementary functions and base change for weighted characters.

In this section assumptions are the same as in IV.3. We assume moreover that t is very regular. Let M be a Levi subgroup of H ; we denote by $\Xi(M(F))_{nr}^{\ell}$ the group of unramified characters of $M(F)$ of order ℓ . Let λ be an unramified character of $M_0(F)$; denote by $\lambda_{E/F}$ the composition with the norm map $N_{E/F} \,:\, M_0(E) \to M_0(F)$. It has a canonical extension to $M_0^L(F)^+$: we let $\lambda_{E/F}(1 \rtimes \theta) = 1$.

IV.4.1. Lemma. – *For any unramified character* λ *of* $M_0(F)$

$$\ell^{\dim \mathfrak{a}_{M_0}} J^L_{M_0^L}(\lambda_{E/F}, f_t^L) = \sum_{\xi \in \Xi(M_0(F))^{\ell}_{nr}} J^H_{M_0}(\lambda \otimes \xi, f^H_{t^\ell}) .$$

Proof. Any unitary unramified character for a local field can be extended to an automorphic character and any unramified extension of local fields can be embedded into a cyclic extension of global fields that splits at archimedean places. Hence, the expected equality follows from V.5.1.

□

Remark. – For $GL(2)$ it is an exercise to prove the lemma by a direct computation using Casselman's explicit description of intertwining operators [Cas2].

Let M be a Levi subgroup of H and P_0 a minimal parabolic subgroup of M containing M_0 . Let $\mathcal{I}^M_{P_0}(\lambda, \cdot)$ be the unramified principal series representation of

$M(F)$ defined by λ , i.e. the representation induced from λ considered as a character of $P_0(F)$ a minimal parabolic subgroup in M . The unramified principal series representation of $M(E)$ defined by $\lambda_{E/F}$ has a canonical extensions to a representation $\mathcal{I}_{P_0^L}^{M^L}(\lambda_{E/F},\cdot)$ of $M^L(F)^+$ denoted simply by $\mathcal{I}_{\lambda_{E/F}}^{M^L}$

IV.4.2. Corollary. – *Consider Q some parabolic subgroup of H containing M . Then*

$$\ell^{\dim \mathfrak{a}_M} J_{M^L}^{Q^L}(\mathcal{I}_{\lambda_{E/F}}^{M^L}, f_t^L) = \sum_{\xi\in\Xi(M(F))_{nr}^\ell} J_M^Q(\mathcal{I}_{\lambda\otimes\xi}^M, f_{t^\ell}^H) \ .$$

Proof. The formation of constant terms IV.2.2 of elementary functions is compatible with the norm map thanks to III.1.8. The compatibility of distributions J_M^L with constant terms (I.6.4) on one hand, the descent formulas (I.6.2) on the other hand, reduce the proof to the particular case $Q = H$ and $M = M_0$ the minimal Levi subgroup; our assertion follows from IV.4.1.

□

We say that a character λ of $M_0(F)$ is M-antidominant, with respect to some Weyl chamber, if the linear form $\mu(\lambda) \in \mathfrak{a}_{M_0}^*$ defined by

$$|\lambda(m)| = e^{<\mu(\lambda)\,,\,H_{M_0}(m)>}$$

is negative i.e. $< \mu(\lambda), \check{\alpha} > \leq 0$ for any positive coroot $\check{\alpha}$ of M_0 in M .

IV.4.3. Proposition. – *Let λ be a character of M_0 and let t be very regular. Assume that λ and t are both M-antidominant with respect to the same Weyl chamber. Let π_λ be the unramified representation of M defined by λ . The coefficient of $\lambda_{E/F}(t)$ in :*

$$t \mapsto \ell^{\dim \mathfrak{a}_M} J_{M^L}^{Q^L}(\pi_{\lambda_{E/F}}, f_t^L)$$

considered as a function of t , equals the coefficient of $\lambda(t^\ell) = \lambda_{E/F}(t)$ in

$$t \mapsto \sum_{\varepsilon\in\Xi(M(F)_{nr}^\ell} J_M^Q(\pi_{\lambda\otimes\varepsilon}, f_{t^\ell}^H).$$

Proof. Observe that since $\lambda_{E/F}$ is M-antidominant with respect to some Weyl chamber the unramified subquotient $\pi_{\lambda_{E/F}}$ of $\mathcal{I}_{\lambda_{E/F}}^{M^L}$ the character $\lambda_{E/F}$ occurs in the semisimplification of the Jacquet module attached to the same Weyl chamber of $\pi_{\lambda_{E/F}}$ and for no other subquotient of the principal series representation defined by $\lambda_{E/F}$ ([Lab2] proposition 8). Our proposition now follows from the previous corollary since the function $t \mapsto \lambda_{E/F}(t)$ is linearly independent from the other terms in the expression for $t \mapsto J_{M^L}^{Q^L}(\pi_{\lambda_{E/F}}, f_t^L)$ given by IV.1.4.

□

IV.5 – A noninvariant fundamental lemma for base change.

In this section we state a noninvariant form of the fundamental lemma for base change: pairs of functions in the unramified Hecke algebra connected by base change are strongly associated. The vanishing result IV.5.1 has a simple local proof. To prove the *matching* result we use in V.6.3 a local-global argument similar to those of [Clo] and [Lab2].

For any unramified character λ of $M_0(F)$ let $\lambda_{E/F}$ be the composition of λ and the norm map. Denote by $b_{E/F}$ the base change homomorphism between the unramified Hecke algebras $\mathcal{H}_E := \mathcal{H}_F^L$ on $L^0(F) = H(E)$ and $\mathcal{H}_F := \mathcal{H}_F^H$ on $H(F)$

$$b_{E/F} : \mathcal{H}_E \longrightarrow \mathcal{H}_F \ .$$

The map $b_{E/F}$ is such that, for any $h \in \mathcal{H}_E$

$$\text{trace } \mathcal{I}_{P_0^L}^L(\lambda_{E/F}, h) = \text{ trace } \mathcal{I}_{P_0}^H(\lambda, b_{E/F}(h)) \ .$$

We observe that the base change map is clearly compatible with constant terms:

$$b_{E/F}(h_{Q^L}) = b_{E/F}(h)_Q \ .$$

IV.5.1. Lemma. – *The normalized weighted integral of functions in the unramified Hecke algebra obtained by base change from E vanish :*

$$J_M^Q(\gamma, b_{E/F}(h)) = 0$$

if $\xi(\gamma) \notin N_{E/F}(E^\times)$ *for some F-rational character* $\xi \in X(H)_F$.

Proof. We first show that

$$J_M^H(\gamma, b_{E/F}(h)) = 0$$

if $\xi(\gamma) \notin N_{E/F}(E^\times)$ for some F-rational character $\xi \in X(H)_F$. It suffices to remark that

$$h_{E/F}(x) := b_{E/F}(h)(x) = 0$$

if $\xi(x)$ is not a norm for some ξ . It is equivalent to show that

$$h_{E/F}(x) = \chi(x) h_{E/F}(x)$$

if χ is any complex valued character of $H(F)$ trival on norms from $L(F)$. The scalar Fourier transform is enough to characterize functions in the unramified Hecke algebra; hence it is enough to show that $h_{E/F}$ and $\chi\, h_{E/F}$ have the same scalar Fourier transform. If λ is an unramified character of the minimal Levi subgroup and π_λ is the unramified representation defined by λ one has

$$\text{trace } \pi_\lambda(\chi\, h_{E/F}) = \text{ trace } (\pi_\lambda \otimes \chi)(h_{E/F}) = \text{ trace } \pi_{(\lambda\otimes\chi)}(h_{E/F}) \ .$$

Since χ is trivial on the norms $(\lambda \otimes \chi)_{E/F} = \lambda_{E/F}$ and hence

$$\text{trace } \pi_{\lambda\otimes\chi}(h_{E/F}) = \text{ trace } \pi_{\lambda_{E/F}}(h) = \text{ trace } \pi_\lambda(h_{E/F}) \ .$$

The general case where $Q \neq H$ follows from the compatibility with constant terms (lemma I.6.4).

□

Remark. – This is the kind of proof suggested by Clozel in a footnote in [Clo] §6 p. 294, and used in [AC] lemma I.4.11 p. 46.

IV.5.2. Theorem. – *Given $h \in \mathcal{H}_E$, then $b_{E/F}(h)$ and h_θ are strongly associated.*

Proof. The vanishing statement – assertion (ii) in the definition III.3.2 of strong association – is our lemma IV.5.1. The matching when E is a field will be established as proposition V.6.3. Let us show how the case of an unramified cyclic Galois algebra E of degree $\ell = \ell_1\ell_2$ algebra over a local field F is reduced to the case where E is a field. The algebra E is a direct sum ℓ_2 copies of a cyclic field extension E_1 of F of degree ℓ_1 . Let θ be a generator of the Galois group; θ acts as follows :

$$\theta(x_1, \ldots, x_{\ell_2}) = (x_2, \ldots, x_{\ell_2}, \theta_1(x_1))$$

where θ_1 is a generator of the Galois group E_1/F . One defines $b_{E/F}$ as follows : given $h_i \in \mathcal{H}_{E_1}$ for $i = 1, \dots, \ell_2$

$$b_{E/F}(h_1 \otimes \dots \otimes h_{\ell_2}) = b_{E_1/F}(h_1) * \dots * b_{E_1/F}(h_{\ell_2}) = b_{E_1/F}(h_1 * \dots * h_{\ell_2}) \ .$$

Now if $h_2 = \dots = h_\ell = h_0$ the unit in $\mathcal{H}_{E_1}$, then $h_\theta = (h_1 \otimes h_0 \otimes \dots \otimes h_0)_\theta$ and $b_{E/F}(h) = b_{E_1/F}(h_1)$ are strongly associated. This follows immediately from the theorem assumed to hold for E_1/F , a cyclic field extension, and of III.5.2. We conclude using the following lemma.

□

IV.5.3. Lemma. – *The functions $h_\theta = (h_1 \otimes h_2 \otimes \dots \otimes h_\ell)_\theta$ and $h'_\theta = (h_1 * h_2 * \dots * h_\ell \otimes h_0 \otimes \dots \otimes h_0)_\theta$ have the same weighted orbital integrals.*

Proof. This follows from proposition I.6.6 using that h_θ and h'_θ have the same weighted characters.

□

V. – THE BASE CHANGE IDENTITY: FIRST APPLICATIONS

In this chapter F is a global field. We state the base change identity V.1.2: the equality of the trace formula for L and H when applied to pairs (ϕ, f) of rationally strongly associated functions. Then we show how to use conjecture B2 to refine this identity. This is used to give global proofs for local results stated in the previous chapters.

V.1 – The base change identity.

We first define a global noninvariant endoscopic transfer.

V.1.1. Definition. – *A pairs of functions (f, ϕ) is said to be rationally strongly associated if, for any Levi subgroup $M \in \mathcal{L}^H(M_0^H)$, any parabolic subgroups Q containing M and any semisimple elements $\gamma \in M(F)$, regular in $H(F)$ one has :*

$$J_M^Q(\gamma, f) = \sum_{\delta} \Delta_M^{M^L}(\gamma, \delta)\ J_{M^L}^{Q^L}(\delta, \phi)$$

the sum being over the $\delta \in M^L$ modulo conjugacy under $M^{L^0}(F)$ and where M^L and Q^L are preimage of M and Q via $\eta_{E/F}$; the subsets M^L and Q^L exist if γ is a norm.

V.1.2. Proposition. – *If ϕ on $L(\mathbb{A}_F)$ and f on $H(\mathbb{A}_F)$ are rationally strongly associated regular functions one has*

$$J^{Q^L}(\phi) = J^Q(f) \ .$$

Proof. If δ^ℓ is stably conjugate to γ regular, the centralizers L_δ and H_γ are inner forms of the same torus and hence are isomorphic. The choice for Haar measures implicit in the definition of rational strong association is compatible with this isomorphism. We get

$$a^{M^L}(\delta) = a^M(\gamma) .$$

On the other hand, the functions being rationally strongly associated we have

$$J_M^Q(\gamma, f) = \sum_\delta \Delta_M^{M^L}(\gamma,\delta)\; J_{M^L}^{Q^L}(\delta,\phi) .$$

We conclude the proof using the proposition II.1.2 and the lemma III.1.3.

□

V.1.3. Definition. – *Let F be a global field and let S be a set of places of F . Two decomposable functions $f_S = \otimes_{v\in S} f_v$ and $\phi_S = \otimes_{v\in S}\phi_v$ are said to be S-strongly associated, or even simply – strongly associated – if (f_v, ϕ_v) are strongly associated for all $v \in S$.*

V.1.4. Proposition. – *If $f = \otimes f_v$ and $\phi = \otimes\phi_v$ are strongly associated locally everywhere, then f and ϕ are rationally strongly associated.*

Proof. The splitting formula I.6.3 can be generalized to products over all places and can be written

$$J_M^Q(\gamma, f) = \sum d_M^Q(\{L_v\}) \prod_v J_M^{Q_v}(\gamma, f_v)$$

the sum is over collections $\{Q_v\}$ of parabolic subgroups with Levi subgroups L_v indexed by places; the numbers $d_M^Q(\{L_v\})$ are non zero only when

$$\mathfrak{a}_M^Q \simeq \bigoplus_v \mathfrak{a}_M^{Q_v} .$$

Similarly

$$J_{M^L}^{Q^L}(\delta,\phi) = \sum d_M^Q(\{L_v\}) \prod_v J_{M^L}^{Q_v^L}(\delta, \phi_v)$$

The global matching

$$J_{M^L}^{Q^L}(\delta,\phi) = J_M^Q(\gamma, f)$$

if γ is the norm of δ follows from the local ones via these splitting formulas. We also have to prove that $J_M^Q(\gamma, f) = 0$ if γ is not a norm. Consider $\xi \in X(M)_F$. If $\xi(\gamma)$ is not a global norm then locally it is not a norm at some place say v. We have

$$J_M^Q(\gamma, f) = \sum d_M^Q(L_1, L_2)\, J_M^{Q_1}(\gamma, f_v)\, J_M^{Q_2}(\gamma, f^v)\ .$$

Observe that the natural map

$$X(Q_1)_F \oplus X(Q_2)_F \to X(M)_F$$

is surjective. We may write $\xi = \xi_1\xi_2$ with $\xi_i \in X(Q_i)_F$. Then either $\xi_1(\gamma)_v$ is not a norm and then $J_M^{Q_1}(\gamma, f_v) = 0$ or $\xi_2(\gamma)_v$ is not a norm but then $\xi_2(\gamma)_w$ is not a norm for some place $w \neq v$ and hence $J_M^{Q_2}(\gamma, f^v) = 0$. Hence $J_M^Q(\gamma, f) = 0$ unless $\xi(\gamma)$ is a norm for all $\xi \in X(M)_F$. Using III.3.4 and III.1.4 we see that $J_M^Q(\gamma, f) = 0$ unless γ is norm locally everywhere but, by III.1.5, this equivalent to say that γ is a norm.

□

Remarks.

(i) The proof of the vanishing statement is similar to the proof of proposition 8.1 p. 542 in [A8].

(ii) There are other ways to construct pairs of functions that are rationally strongly associated. For example, if for all places v the functions f_v and ϕ_v are (simply) associated and if for v in a finite set S of places, of cardinal greater than 2, the functions f_v and ϕ_v are *very cuspidal* – i.e. their constant terms along all proper parabolic subgroups vanish – then (f, ϕ) are rationally strongly associated.

V.2 – A twisted noninvariant version of Kazdan's density theorem.

In the proof of V.6.3 we need a partial converse to proposition V.1.2.

V.2.1. Proposition. – *Let S be a finite set of places of a global field F . Consider a pair of functions f_S and ϕ_S such that, for any parabolic subgroup $Q \subset H$:*

$$J^{Q^L}(\phi) = J^Q(f) \ ,$$

whenever $f = f_S \otimes f^S$ and $\phi = \phi_S \otimes \phi^S$ with f_v and ϕ_v strongly associated for all $v \notin S$ and regular at some place $v \notin S$. Then

$$J_M^Q(\gamma, f_S) = \sum_\delta \Delta_M^{M^L}(\gamma,\delta)\ J_{M^L}^{Q^L}(\delta,\phi_S)$$

for all $\gamma \in M(F)$ regular in $H(F)$ and such that γ_v is a norm for all $v \notin S$.

Proof. Consider $\gamma \in M(F)$ semisimple, H regular. By descent (I.6.2) it is enough to prove the assertion when γ is M-elliptic and $Q = H$. Let Σ be a set of places which contains S and all the ramified places. The set Σ is taken big enough so that $\gamma_v \in K_v^H$ for all $v \notin \Sigma$. Let f^Σ (resp. ϕ^Σ) be the product of the characteristic function of K_v^H (resp. K_v^L) divided by its volume, for v outside of Σ ; these functions are strongly associated according to a result of Kottwitz recalled in lemma IV.3.3. Let S_1 be the complement of S in Σ . Take f_{S_1} and ϕ_{S_1} to be a pair of strongly associated functions with regular support, the existence of such pairs follows from lemma III.4.1. We may assume that the function f^S is such that

$$J_H(\gamma, f^S) = J_M^Q(\gamma, f^S) \neq 0$$

if M is the Levi subgroup of Q . Assume that at one place $v \in S_1$ the function f_v has its support in a small enough neighbourhood of γ so that, the geometric expansion of the trace formula reduces to the contribution of the conjugacy class of γ which, using the invariance under the Weyl group of weighted orbital integrals (I.6.1), can be written :

$$J^H(f) = \ a^M(\gamma) J_M^H(\gamma, f_S \otimes f^S),$$

and similarly at most one conjugacy class contributes, the conjugacy class of some δ if γ is a global norm to the geometric expansion of $J^L(\phi)$:

$$J^L(\phi) = \sum_{\delta} \Delta_M^{M^L}(\gamma,\delta)\, a^{M^L}(\delta) J_{M^L}^L(\delta, \phi_S \otimes \phi^S) \ .$$

Since $a^{M^L}(\delta) = a^M(\gamma)$ if γ is the norm of δ , the equality of the two trace formulas can be written, using the splitting formula (I.6.3) :

$$\sum_{L_1,L_2} d_M^H(L_1,L_2) \left(J_M^{Q_1}(\gamma, f_S) - \sum_{\delta} \Delta_M^{M^L}(\gamma,\delta) J_{M^L}^{Q_1^L}(\delta,\phi_S) \right) J_M^{Q_2}(\gamma, f^S) = 0 \ .$$

If we assume the proposition to hold for any Levi subgroups $L_1 \subsetneq L$, and we get

$$\left(J_M^H(\gamma, f_S) - \sum_{\delta} \Delta_M^{M^L}(\gamma,\delta) J_{M^L}^L(\delta,\phi_S) \right) \ J_M^Q(\gamma, f^S) = 0 \ .$$

We have chosen the function f^S so that $J_M^Q(\gamma, f^S) \neq 0$ if M is the Levi subgroup of Q and hence

$$J_M^H(\gamma, f_S) = \sum_{\delta} \Delta_M^{M^L}(\gamma,\delta) J_{M^L}^L(\delta,\phi_S) \ .$$

Since we have assumed that the trace formula identity holds for all constant terms, the proposition follows by induction on L .

□

Remark. – The proof extends immediately to the case where the pair of functions (f^S, ϕ^S) may only vary in a subset of strongly associated regular pairs provided that, given any regular semisimple point γ , this subset allows one to take functions f_v , at some place $v \notin S$, with support in arbitrary small neighbourhoods of γ and such that the ordinary orbital integrals $J_H(\gamma, f^S)$ does not vanish.

V.2.2. Corollary. – *If S is a finite set of places of a global field F . Consider a pair of functions f_S and ϕ_S are such that for any parabolic subgroup $Q \subset H$:*

$$J^{Q^L}(\phi_S \otimes \phi^S) = J^Q(f_S \otimes f^S)$$

whenever f^S *and* ϕ^S *are strongly associated regular outside of* S . *Then for all* $\delta \in M(F_S)$ *regular in* $L(F_S)$ *with norm* $\gamma \in H(F_S)$:

$$J_M^Q(\gamma, f_S) = J_{M^L}^{Q^L}(\delta, \phi_S) \ .$$

Proof. The proposition V.2.1 shows that f_S and ϕ_S have matching weighted orbital integrals for pairs (γ, δ) if γ is the norm of a regular element $\delta \in L(F)$, but these elements are dense in the set of regular elements in $L(F_S)$.

□

V.3 – Separation of infinitesimal characters via multipliers.

In this section we deal with pairs of rationally strongly associated regular functions, that are K-finite and compatible with multipliers at archimedean places. This is useful if, in particular, conjecture B2 holds at archimedean primes; for example, when $L^0(F_\infty) \simeq H(F_\infty)^\ell$. Let $\mathfrak{h} = \mathfrak{h}(L)$ and consider $\mu \in \mathfrak{h}_{\mathbb{C}}^*$, we denote by $\Pi_{\rm disc}(M^L, L, \mu)$ (or simply $\Pi_{\rm disc}(L, \mu)$ if $M^L = L$) the set of representations π that occur discretely in the trace formula for M^L and such that the representation of $L^0(F_\infty)$ obtained from π_∞ by parabolic induction has an infinitesimal character given by the orbit of μ under the complex Weyl group of $L^0(F_\infty)$. We denote by $\mu_{E/F} \in \mathfrak{h}(H)^* \otimes \mathbb{C}$ the composition of μ with the map induced by the norm.

V.3.1. Proposition. – *Let* (f, ϕ) *be a pair of* K*-finite, rationally strongly associated, regular functions compatible with multipliers at archimedean places. Given* $\mu \in \mathfrak{h}(L)^* \otimes \mathbb{C}$ *and* M , *for any* $\Lambda \in \mathfrak{a}_M$ *one has*

$$\ell^{\dim \mathfrak{a}_M} \sum_{\pi' \in \Pi_{\rm disc}(M^L, L, \mu)} \mathcal{J}_{M^L}^{Q^L}({\pi'}_{\Lambda_{E/F}}, \phi) = \sum_{\{\nu \mid \nu_{E/F} = \mu\}} \sum_{\pi \in \Pi_{\rm disc}(M, H, \nu)} \mathcal{J}_M^Q(\pi_\Lambda, f) \ .$$

Proof. Let α and β be a pair of multipliers compatible with the base change. By hypothesis $(\phi_{\alpha * m}, f_{\beta * m})$ are rationally strongly associated regular functions for all m . By proposition V.1.2 we have,

$$J^{Q^L}(\phi_{\alpha * m}) - J^Q(f_{\beta * m}) = 0 \ .$$

By lemma II.3.2, for T large enough

$$\sum_{t<T}\sum_{M\in\mathcal{L}} \frac{w^M}{w^Q}\left(J^{Q^L}_{M^L,t}(f_{\alpha*m}) - J^Q_{M,t}(f_{\beta*m})\right) = o(c^m)$$

with $c<1$. Let $\mathcal{J}^Q_{M,\mu}(f,\phi,\Lambda)$ be defined by

$$\ell^{\dim \mathfrak{a}_M} \sum_{\pi'\in\Pi_{\rm disc}(M^L,L,\mu)} \mathcal{J}^{Q^L}_{M^L}(\pi'_{\Lambda_{E/F}},\phi) - \sum_{\{\nu\mid \nu_{E/F}=\mu\}}\ \sum_{\pi\in\Pi_{\rm disc}(M,H,\nu)} \mathcal{J}^Q_M(\pi_\Lambda,f)\ ,$$

where $\Lambda_{E/F}$ is the composition of Λ with the map induced by the norm. These functions are analytic on $\mathfrak{a}_M$. For each t , taking into account the factor $\ell^{\dim \mathfrak{a}_M}$ due to the transfer of measures, we have

$$\sum_{t<T}\left(J^{Q^L}_{M^L,t}(f_{\alpha*m}) - J^Q_{M,t}(f_{\beta*m})\right) = \sum_{\|\Im\mathfrak{m}\,\mu\|<T}\int_{\mathfrak{a}^*_M} \hat\alpha^m(\mu+i\Lambda_{E/F})\mathcal{J}^Q_{M,\mu}(f,\phi,i\Lambda)\ d\Lambda$$

so that

$$\sum_{M\in\mathcal{L}}\frac{w^M}{w^Q}\sum_{\|\Im\mathfrak{m}\,\mu\|<T}\int_{\mathfrak{a}^*_M} \hat\alpha^m(\mu+i\Lambda_{E/F})\mathcal{J}^Q_{M,\mu}(f,\phi,i\Lambda)\ d\Lambda\ = o(c^m)\ .$$

Let $\mu_0\in\mathfrak{h}_u$ and let α be chosen as in I.8.2. We have by I.8.3 an estimate as $m\to\infty$:

$$\sum_{\|\Im\mathfrak{m}\,\mu\|<T}\int_{\mathfrak{a}^*_M} \hat\alpha^m(\mu+i\Lambda_{E/F})\mathcal{J}^Q_{M,\mu}(f,\phi,i\Lambda)\ d\Lambda\ =$$

$$\left(\sqrt{\frac{\pi}{m}}\right)^{dim\mathfrak{a}_M}\sum_{\|\Im\mathfrak{m}\,\mu\|<T}\ \sum_{\Lambda\in N_M(\mu_0,\mu)}\frac{1}{\sqrt{\det\,Q_{M,\alpha,\mu+i\Lambda_{E/F}}}}\left(\mathcal{J}^Q_{M,\mu}(f,\phi,i\Lambda)+o(1)\right)$$

but of course the integral vanishes if $\mathcal{J}^Q_{M,\mu}(f,\phi,i\Lambda)$ is identically 0 . Assume for a while that some term does not vanish and let

$$d=\inf_{M\in\mathcal{L}^Q}\{\dim\mathfrak{a}_M\ ,\quad \text{such that}\quad \Lambda\mapsto\mathcal{J}^Q_{M,\mu}(f,\phi,i\Lambda)\quad\text{is not identically } 0\}\ .$$

Denote by $\mathcal{L}_d$ the set of Levi subgroups containing M_0 such that $\dim\mathfrak{a}_M=d$. Let

$$C_d(f,\phi,\mu_0)=\sum_{\|\Im\mathfrak{m}\,\mu\|<T}\ \sum_{M\in\mathcal{L}_d}\frac{w^M}{w^Q}\sum_{\Lambda\in N_M(\mu_0,\mu)}\frac{1}{\sqrt{\det\,Q_{M,\alpha,\mu+i\Lambda_{E/F}}}}\mathcal{J}^Q_{M,\mu}(f,\phi,i\Lambda)$$

then

$$(C_d(f,\phi,\mu_0)+o(1))\left(\frac{1}{\sqrt{m}}\right)^d = o(c^m)\ .$$

This implies $C_d(f,\phi,\mu_0) = 0$ for all choices of μ_0 (and α). Take $\mu_0 = \mu + i\Lambda_{E/F}$ with $\Lambda \in \mathfrak{a}_M$ for some $\mu \in \mathfrak{h}_u$ and $M \in \mathcal{L}_d$ and such that $\mathcal{J}^Q_{M,\mu}(f,\phi,i\Lambda) \neq 0$. There is an open set of Λ so that, for no other couple (μ',M') there exist $\Lambda' \in \mathfrak{a}^Q_{M'}$ with $w\mu_0 = \mu' + \Lambda'$ for some $w \in w^Q_{\mathbb{C}}$ and $\mathcal{J}^Q_{M',\mu'}(f,\phi,i\Lambda') \neq 0$ (recall that the set of such couples is finite), unless this couple is deduced form (μ,M) by some $w' \in W^L_{\mathbb{C}}$. Modulo $W^M_{\mathbb{C}}$ this can be achieved by a $w'' \in W^Q$. For such a choice of $\mu_0 = \mu + i\Lambda_{E/F}$, taking into account the invariance of the family of distributions $\mathcal{J}^Q_{M,\mu}(f,\phi,i\Lambda)$ under the Weyl group (I.6.1) we see that $C_d(f,\phi,\mu_0)$ is proportional, by a non zero constant, to $\mathcal{J}^Q_{M,\mu}(f,\phi,i\Lambda)$; this is a contradiction.

□

V.4 – Some auxiliary results.

In this section E/F is a cyclic extension of global fields and $G = H$. Let M be a Levi subgroup of H . Let us denote by $\Xi(M)_{E/F}$ the group of characters of $M(\mathbb{A}_F)$ trivial on $M(F)$ and the norms of elements in $M^L(\mathbb{A}_E)$. Recall that if v is a place such that $E \otimes F_v$ is an unramified field extension of F_v we denote by $\Xi(M_v)^\ell_{nr}$ the group of unramified characters of M_v of order ℓ . Let L_1 and L_2 be two proper Levi subgroups of H containing M_0 and let Q_1 and Q_2 the corresponding parabolic subgroups by the sections defined by some generic $\zeta \in \mathfrak{a}^H_{M_0}$.

V.4.1. Lemma. – *Let E/F be a cyclic extension of global fields, let v be a place where $E \otimes F_v$ is an unramified field extension of F_v .*

(i) The cardinal of the set of $\Xi(M)_{E/F}$ equals $\ell^{dim\,\mathfrak{a}_M}$.

(ii) The restriction map

$$\Xi(M)_{E/F} \to \Xi(M_v)^\ell_{nr}$$

is bijective.

(iii) Assume that $\mathfrak{a}^{L_1}_M \oplus \mathfrak{a}^{L_2}_M = \mathfrak{a}^H_M$. The map

$$\Xi(M)_{E/F}/\Xi(H)_{E/F} \to (\Xi(M)_{E/F}/\Xi(L_1)_{E/F}) \oplus (\Xi(M)_{E/F}/\Xi(L_2)_{E/F})$$

induced by the diagonal map

$$\Xi(M)_{E/F} \to \Xi(M)_{E/F} \oplus \Xi(M)_{E/F}$$

is bijective.

Proof. Statements (i) and (ii) follow from class field theory since E/F and E_v/F_v are cyclic of order ℓ . Statement (iii) follows from [A7] lemma 10.1 together with (i) (see also [AC] p. 126).

□

We shall now study distributions attached to representations of the minimal Levi subgroup M_0 ; this is a split torus and irreducible representations are one dimensional. The base change of a character π of a split torus is simply the character $\pi_{E/F}$ obtained by composition of π with the norm map; this character has a canonical extension denoted again $\pi_{E/F}$ to the semidirect product with the Galois group. We first draw some further consequences of IV.4.1.

V.4.2. Lemma. – *Let (f,ϕ) be a pair of K-finite strongly associated regular functions, compatible with multipliers at archimedean places. Assume that IV.4.1 hold; then if $\pi_{E/F}$ is the base change of an automorphic character of π of $M_0(\mathbb{A}_F)$ and if Q is a parabolic subgroup of H containing M_0 we have*

$$\ell^{\dim \mathfrak{a}_{M_0}} \mathcal{J}_{M_0^L}^{Q^L}(\pi_{E/F},\phi) = \sum_{\xi\in\Xi(M_0)_{E/F}} \mathcal{J}_{M_0}^{Q}(\pi\otimes\xi,f) \ .$$

Proof. The descent formula I.6.2 and the compatibility with constant terms I.6.4 shows that it is enough to prove the lemma when $Q = H$. Proposition V.3.1 shows that

$$\ell^{\dim \mathfrak{a}_{M_0}} \sum_{\pi'\in\Pi_{\mathrm{disc}}(M_0^L,L,\nu_{E/F})} \mathcal{J}_{M_0^L}^{L}(\pi'{}_{\Lambda_{E/F}},\phi) = \sum_{\pi\in\Pi_{\mathrm{disc}}(M_0,H,\nu)} \mathcal{J}_{M_0}^{H}(\pi_\Lambda,f) \ .$$

For w outside a finite set S of places $\phi_w = h_0^L$ the translate by θ of the unit in the unramified Hecke algebra and

$$\mathcal{J}_{M_0^L}^{L}(\pi'{}_{\Lambda_{E/F}},\phi) = \mathcal{J}_{M_0^L}^{L}(\pi'{}_{\Lambda_{E/F}},\phi_S)$$

if π' is unramified outside of S . Replace the pair (f^S, ϕ^S) by a strongly associated pair of elementary functions: this is possible by IV.3.6. Assume inductively that the lemma is proved for parabolic subgroups $Q \subsetneq H$. Using the splitting formula I.6.3 for $f = f_S \otimes f^S$, the inductive assumption and IV.4.1, we may use lemma V.4.3 to cancel all terms but those where $Q_1 = H$ or $Q_2 = M_0$ and we are left with :

$$\ell^{\dim \mathfrak{a}_{M_0}} \sum_{\pi' \in \Pi_{\mathrm{disc}}(M_0^L, L, \nu_{E/F})} \mathcal{J}_{M_0^L}^L(\pi'_{\Lambda_{E/F}}, \phi_S) \ \text{trace}\ \pi'_{\Lambda_{E/F}}(\phi_{M_0}^S)$$
$$= \sum_{\pi \in \Pi_{\mathrm{disc}}(M_0, H, \nu)} \mathcal{J}_{M_0}^H(\pi_\Lambda, f_S) \ \text{trace}\ \pi_\Lambda(f_{M_0}^S).$$

Now we may separate W^H-orbits of characters by varying the functions outside S among pairs of associated elementary functions (IV.3.6 and IV.1.4). By I.6.1 our distributions are W^H-invariant. We have

$$\text{trace}\ \pi'_{\Lambda_{E/F}}(\phi_{M_0}^S) = \text{trace}\ \pi_\Lambda(f_{M_0}^S) ,$$

and the lemma now follows from the properties of base change for split tori.

□

V.4.3. Lemma. – *Let v be a place where $E \otimes F_v$ is an unramified field extension of F_v . Assume that lemma IV.4.1 hold for all Levi subgroups $L \subsetneq G$. Assume that $\mathfrak{a}_{M_0}^{L_1} \oplus \mathfrak{a}_{M_0}^{L_1} = \mathfrak{a}_{M_0}^H$. Let (f^v, ϕ^v) be a pair of K-finite strongly associated regular functions outside v , compatible with multipliers at archimedean places. At the place v consider a pair of elementary functions f_t^L and $f_{t'}^H$. Let π be an automorphic character of $M_0(\mathbb{A}_F)$ with base change $\pi_{E/F}$; let $\lambda = \pi_v$. If $L_i \neq G$ we have*

$$\ell^{\dim \mathfrak{a}_{M_0}} \mathcal{J}_{M_0^L}^{Q_1^L}(\pi_{E/F}, \phi^v)\ J_{M_0^L}^{Q_2^L}(\lambda_{E_v/F_v}, f_t^L)$$
$$= \sum_{\xi \in \Xi(M_0)_{E/F}} \mathcal{J}_{M_0}^{Q_1}(\pi \otimes \xi, f^v)\ J_{M_0}^{Q_2}(\lambda \otimes \xi_v, f_{t'}^H) .$$

Proof. Using the compatibility with constant terms (I.6.4), lemma V.4.2 for L_1 , and proposition IV.4.3 for L_2 , we see that if π lifts to $\pi_{E/F}$

$$\ell^{2 \dim \mathfrak{a}_{M_0}} \mathcal{J}_{M_0^L}^{Q_1^L}(\pi_{E/F}, \phi^v)\ J_{M_0^L}^{Q_2^L}(\lambda_{E_v/F_v}, f_t^L)$$

equals

$$\sum_{\xi\in\Xi(M_0)_{E/F}} \mathcal{J}^{Q_1}_{M_0}(\pi\otimes\xi, f^v) \sum_{\varepsilon\in\Xi(M_{0,v})^\ell_{nr}} J^{Q_2}_{M_0}(\lambda\otimes\varepsilon, f^H_{t^\ell}) \ .$$

Note that characters $\xi \in \Xi(L_1)_{E/F}$ act trivially in the first factor and that characters $\varepsilon \in \Xi(L_2)^\ell_{nr}$ act trivially in the second. The assertion follows from lemma V.4.1.

□

V.5 – Proof of a spectral matching.

In this section we establish the spectral matching result, for weighted characters and elementary functions, used in IV.4.1.

V.5.1. Proposition. – *Let E/F is a cyclic extension split at archimedean places and unramified at v. Let λ be an unramified character of $M_0(F_v)$ which is the component at v of an automorphic character. Then*

$$\ell^{\dim \mathfrak{a}_{M_0}} J^L_{M^L_0}(\lambda_{E_v/F_v}, f^L_t) = \sum_{\varepsilon\in\Xi(M_0(F_v))^\ell_{nr}} J^H_{M_0}(\lambda\otimes\varepsilon, f^H_{t^\ell}) \ .$$

Proof. Let (f,ϕ) be a pair of K-finite, strongly associated, regular functions, compatible with multipliers at archimedean places. Since the hypothesis of V.3.1 are satisfied we have

$$\ell^{\dim \mathfrak{a}_{M_0}} \sum_{\pi'\in\Pi_{\mathrm{disc}}(M^L_0,L,\nu_{E/F})} \mathcal{J}^L_{M^L_0}(\pi'_{\Lambda_{E/F}}, \phi) = \sum_{\pi\in\Pi_{\mathrm{disc}}(M_0,H,\nu)} \mathcal{J}^H_{M_0}(\pi_\Lambda, f) \ .$$

Consider functions f and ϕ bi-invariant under a fixed open compact subgroup at finite places; since at archimedean places the representations have a fixed infinitesimal character, only a finite number of terms in the sum above do not vanish. As already observed, M_0 is abelian and irreducible automorphic representations are simply characters. Let S be a finite set of places outside of which E_w/F_w is unramified and fix $v \notin S$. By lemma IV.1.4 we may, at any place $v' \notin S\cup\{v\}$, separate a finite set of W^H-orbits of liftings of unramified characters of $M_0(F_v)$ by varying $(\phi_{v'}, f_{v'})$ among pairs of elementary functions attached to very regular elements $t_{v'}$ and $t^\ell_{v'}$. By rigidity we may choose a finite set of places S' disjoint from $S\cup\{v\}$, large enough so that, in the above identity considered as an identity between linear forms on regular elementary functions over S' , we may separate the contributions of the various W^H-orbits of liftings $\pi_{E/F}$ of automorphic characters π . Recall that our distributions are W^H-invariant (I.6.1). Consider an automorphic character π of $M_0(\mathbb{A}_F)$, unramified at v . Given an open compact subgroup U_w at each finite places $w \in S \cup \{v\}$ we may find S' such that

$$\ell^{\dim \mathfrak{a}_{M_0}} \mathcal{J}^L_{M_0^L}(\pi_{E/F}, \phi_S \otimes \phi_{S'} \otimes \phi_v) = \sum_{\xi \in \Xi(M_0)_{E/F}} \mathcal{J}^H_{M_0}(\pi \otimes \xi, f_S \otimes f_{S'} \otimes f_v)$$

where $\phi_S \otimes \phi_v$ and $f_S \otimes f_v$ may vary among pairs of strongly associated functions bi-invariant under the fixed open compact subgroup at finite places; the functions $\phi_{S'}$ and $f_{S'}$ may vary among tensor products over $w \in S'$ of elementary functions $f^L_{t_w}$ and $f^H_{t^\ell_w}$ with t_w very regular. Assume that U_v is the Iwahori subgroup. In particular

$$(1) \quad \ell^{\dim \mathfrak{a}_{M_0}} \mathcal{J}^L_{M_0^L}(\pi_{E/F}, \phi_S \otimes \phi_{S'} \otimes f^L_t) = \sum_{\xi \in \Xi(M_0)_{E/F}} \mathcal{J}^H_{M_0}(\pi \otimes \xi, f_S \otimes f_{S'} \otimes f^H_{t^\ell})$$

Denote by h^H_0 the characteristic function, of the maximal compact K^H_v divided by its volume and by h^L_0 the characteristic function of K^L_v divided by its volume, we also have

$$(2) \quad \ell^{\dim \mathfrak{a}_{M_0}} \mathcal{J}^L_{M_0^L}(\pi_{E/F}, \phi_S \otimes \phi_{S'} \otimes h^L_0) = \sum_{\xi \in \Xi(M_0)_{E/F}} \mathcal{J}^H_{M_0}(\pi \otimes \xi, f_S \otimes f_{S'} \otimes h^H_0)$$

Let P_0 be a parabolic subgroup; if π is an automorphic character we have

$$\mathcal{J}^{P_0}_{M_0}(\pi, f) = \text{ trace } \pi(f_{M_0}) \ .$$

We shall assume that all archimedean primes of F split completely in E . Proposition III.5.5 shows that we may find a pair of functions $(f_S \otimes f_{S'}, \phi_S \otimes \phi_{S'})$ such that the hypothesis of V.3.1 are satisfied and such that moreover

$$(3) \qquad \mathcal{J}^{P_0^L}_{M_0^L}(\pi_{E/F}, \phi_S \otimes \phi_{S'}) = \mathcal{J}^{P_0}_{M_0}(\pi \otimes \xi, f_S \otimes f_{S'}) \neq 0 \ .$$

Applying the splitting formula I.6.3 we get

$$\ell^{\dim \mathfrak{a}_{M_0}} \sum_{L_1, L_2} d^H_{M_0}(L_1, L_2)\ \mathcal{J}^{Q_1^L}_{M_0^L}(\pi_{E/F}, \phi_S \otimes \phi_{S'})\ J^{Q_2^L}_{M_0^L}(\lambda_{E_v/F_v}, f^L_t)$$
$$= \sum_{L_1, L_2} d^H_{M_0}(L_1, L_2) \sum_{\xi \in \Xi(M_0)_{E/F}} \mathcal{J}^{Q_1}_{M_0}(\pi \otimes \xi, f_S \otimes f_{S'})\, J^{Q_2}_{M_0}(\lambda \otimes \xi_v, f^H_{t^\ell}) \ .$$

Assume by induction that lemmas IV.4.1 (and hence also V.4.2) holds for Levi subset $L_i \subsetneq H$. This implies that in the above equation all terms cancel except maybe those with $L_i = M$ or H . The equation (2) above can also be written

$$(2') \qquad \ell^{\dim \mathfrak{a}_{M_0}} \mathcal{J}^{L}_{M_0^L}(\pi_{E/F}, \phi_S \otimes \phi_{S'}) = \sum_{\xi \in \Xi(M_0)_{E/F}} \mathcal{J}^{H}_{M_0}(\pi \otimes \xi, f_S \otimes f_{S'}) \ .$$

Taking into account the compatibility with contant terms I.6.4 and IV.2.2, we have

$$(4) \qquad J^{P_0^L}_{M_0^L}(\lambda_{E_v/F_v}, f_t) = J^{P_0}_{M_0}(\lambda \otimes \varepsilon, f_{t^\ell}) \ ,$$

for any $\varepsilon \in \Xi(M_0(F_v))^{\ell}_{nr}$. Multiplying equations (2') and (4) we see that terms with $L_1 = G$ and $L_2 = M$ also match. We are left with

$$\ell^{\dim \mathfrak{a}_{M_0}} \ \mathcal{J}^{P_0^L}_{M_0^L}(\pi^v_{E/F}, \phi_S \otimes \phi_{S'}) \ J^{L}_{M_0{}^L}(\lambda_{E_v/F_v}, f^L_t) = \sum_{\xi \in \Xi(M_0)_{E/F}} \mathcal{J}^{P_0}_{M_0}(\pi \otimes \xi, f_S \otimes f_{S'}) \ J^{H}_{M_0}(\lambda \otimes \xi_v, f^H_{t^\ell}) \ .$$

Using the nonvanishing condition (3) above and V.4.1 (ii) we get the expected equality.
□

V.6 – Proof of a geometric matching.

In this section E/F is a cyclic extension of global fields and v is a place where $E_v = E \otimes F_v$ is an unramified field extension of F_v . Moreover assume that all archimedean primes of F split completely in E . In particular conjecture B2 holds.

Given a pair of representations $\pi_v \in \Pi(H_v)$ and $\pi'{}_v \in \Pi(L_v)$ and if ψ_v is a character of the unramified Hecke algebra $\mathcal{H}^L_v$, we define $\delta^L_H(\pi_v, \psi_v)$ (resp. $\delta^L_L(\pi'{}_v, \psi_v)$) to be 1 if :

$$\text{trace } \pi_v(b_{E/F}(h)) = \hat{h}(\psi_v)$$

(resp. trace $\pi'{}_v(h) = \hat{h}(\psi_v)$). Let $\delta^L_H(\pi_v, \psi_v) = 0$ otherwise. Given $\Lambda \in \mathfrak{a}_M$, we denote by $\Pi_{\text{disc}}(M, H, \nu, \Lambda)$ the set of representations π_Λ with $\pi \in \Pi_{\text{disc}}(M, H, \nu)$.

V.6.1. Lemma. – *Assume that theorem IV.5.2 holds. Let S be a finite set of places, containing ramified places for E/F and archimedean places. Let (f_S, ϕ_S) be a pair of K-finite strongly associated regular functions compatible with multipliers. For any character ψ_v of the unramified Hecke algebra $\mathcal{H}_v^{M^L}$ we have*

$$\ell^{\dim \mathfrak{a}_M} \sum_{\pi' \in \Pi_{\rm disc}(M^L, L, \mu, \Lambda_{E/F})} \delta_{M^L}^{M^L}(\pi'_v, \psi_v)\ \mathcal{J}_{M^L}^{Q^L}(\pi', \phi_S)$$
$$= \sum_{\{\nu \mid \nu_{E/F} = \mu\}} \sum_{\pi \in \Pi_{\rm disc}(M, H, \nu, \Lambda)} \delta_M^{M^L}(\pi_v, \psi_v)\ \mathcal{J}_M^Q(\pi, f_S)\ .$$

Proof. Proposition V.3.1 allows us to separate infinitesimal characters at archimedean places and yields identities where only finite sets of representations may contribute if we work with pairs of strongly associated functions invariant by some fixed open compact subgroup at the finite places. Since we assume that theorem IV.5.2 holds we may construct strongly associated pairs of functions using at unramified places, for example at v , pairs of functions in the unramified Hecke algebra connected by the base change and hence the characters of the Hecke algebra $\mathcal{H}_v^L$ can then be separated; this allows us to separate W^H-orbits of characters of the unramified Hecke algebra $\mathcal{H}_v^{M^L}$; we conclude using I.6.1.

□

Let L_1 and L_2 be two Levi subgroups in $\mathcal{L}^Q(M)$ and let Q_1 and Q_2 be the corresponding parabolic subgroups via the sections defined by some generic $\zeta \in \mathfrak{a}_M^Q$. Assume that $\mathfrak{a}_M^{L_1} \oplus \mathfrak{a}_M^{L_2} = \mathfrak{a}_M^Q$. The next lemma is a generalization of V.4.3.

V.6.2. Lemma. – *Assume that theorem IV.5.2 hold for all proper Levi subgroups of H . Consider (f_S, ϕ_S) a pair of strongly associated regular functions as above. Choose a Weyl chamber in $\mathfrak{a}_{M_0}^M$. For $v \notin S$ consider associated elementary functions f_t and f_{t^ℓ} with t very regular M-antidominant. Consider $Q_1 \neq H$ and λ the M-antidominant character of $M_0(F_v)$ defined by ψ_v . The coefficient of $\lambda_{E/F}(t)$ in :*

$$t \mapsto \ell^{\dim \mathfrak{a}_M} \sum_{\pi' \in \Pi_{\rm disc}(M^L, L, \mu, \Lambda_{E/F})} \mathcal{J}_{M^L}^{Q_1^L}(\pi', \phi_S)\ J_{M^L}^{Q_2^L}(\pi'_v, f_t)$$

equals the coefficient of $\lambda(t^\ell) = \lambda_{E/F}(t)$ *in*

$$t \mapsto \sum_{\{\nu \mid \nu_{E/F}=\mu\}} \sum_{\pi\in\Pi_{\rm disc}(M,H,\nu,\Lambda)} \mathcal{J}_M^{Q_1}(\pi, f_S)\ J_M^{Q_2}(\pi_v, f_{t^\ell}).$$

Proof. Let ψ_v be a character of the unramified Hecke algebra $\mathcal{H}_v^{M^L}$. Using the compatibility with constant terms I.6.4 and lemma V.6.1 for L_1 , and using IV.4.3 for L_2 , we get that the coefficient of $\lambda(t^\ell) = \lambda_{E/F}(t)$ in

$$\ell^{2\dim \mathfrak{a}_M} \sum_{\pi'\in\Pi_{\rm disc}(M^L,L,\mu,\Lambda_{E/F})} \delta_{M^L}^{M^L}(\pi'_v, \psi_v)\ \mathcal{J}_{M^L}^{Q_1^L}(\pi', \phi_S)\ J_{M^L}^{Q_2^L}(\pi'_v, f_t)$$

equals the coefficient of $\lambda(t^\ell)$ in

$$\sum_{\{\nu \mid \nu_{E/F}=\mu\}} \sum_{\pi\in\Pi_{\rm disc}(M,H,\nu,\Lambda)} \delta_M^{M^L}(\pi_v, \psi_v)\ \mathcal{J}_M^{Q_1}(\pi, f_S) \sum_{\varepsilon\in\Xi(M_v)^\ell_{nr}} J_M^{Q_2}(\pi_v \otimes \varepsilon, f_{t^\ell})\ .$$

We conclude using lemma V.4.1.

□

We are now in position to give a proof of the matching of weighted orbital integrals for functions in the unramified Hecke algebra connected by the base change map :

V.6.3. Proposition. – *Let* $h \in \mathcal{H}_v^L$. *If* $\gamma \in M(F)$ *is the norm of* δ *then for any* $Q \in \mathcal{L}^H(M)$

$$J_M^Q(\gamma, b(h)) = J_{M^L}^{Q^L}(\delta, h_\theta)\ .$$

Proof. By induction we assume that theorem IV.5.2 holds for Levi subgroups $L_1 \neq H$. Consider a pair of strongly associated regular functions ϕ^v and f^v outside v compatible with multipliers, and at v elementary regular associated functions; we have

$$\ell^{\dim \mathfrak{a}_M} \sum d_M^Q(L_1, L_2) \sum_{\pi'\in\Pi_{\rm disc}(M^L,L,\mu,\Lambda_{E/F})} \mathcal{J}_{M^L}^{Q_1^L}(\pi', \phi^v)\ J_{M^L}^{Q_2^L}(\pi'_v, f_t)$$
$$= \sum d_M^Q(L_1, L_2) \sum_{\{\nu \mid \nu_{E/F}=\mu\}} \sum_{\pi\in\Pi_{\rm disc}(M,H,\nu,\Lambda)} \mathcal{J}_M^{Q_1}(\pi, f^v)\ J_M^{Q_2}(\pi_v, f_{t^\ell})\ .$$

Since infinitesimal characters are fixed we are left with finite sums and, according to IV.1.4, we may decompose each term into a finite sum of characters evaluated at t , times polynomials of linear forms of t , when t varies among very regular antidominant elements. By inductive assumption, using V.6.2, and the compatibility with constant terms, we see that if λ is antidominant the coefficients of $\lambda_{E_v/F_v}(t)$ and of $\lambda(t^\ell)$ are already known to match in all terms unless maybe if $Q_1 = L_1 = H$ and hence the matching also holds for these remaining terms. This shows that the coefficients of $\lambda_{E_v/F_v}(t)$ in

$$\ell^{\dim \mathfrak{a}_M} \sum_{\pi' \in \Pi_{\mathrm{disc}}(M^L, L, \mu, \Lambda_{E/F})} \mathcal{J}_{M^L}^{Q^L}(\pi', \phi^v) \text{ trace } (\pi'_v(f_{t,M^L}))$$

equals the coefficients of $\lambda(t^\ell)$ in

$$\sum_{\{\nu \mid \nu_{E/F}=\mu\}} \sum_{\pi \in \Pi_{\mathrm{disc}}(M,H,\nu,\Lambda)} \mathcal{J}_M^Q(\pi, f^v) \text{ trace } (\pi_v(f_{t^\ell,M})) .$$

Proposition 8 of [Lab2] allows us to substitute pairs $(h_\theta, b(h))$ where h is a function in the unramified Hecke algebra for L_v^0 , to our pair of associated elementary functions in this identity and, taking into account lemma I.5.1, we get :

$$\ell^{\dim \mathfrak{a}_M} \sum_{\pi' \in \Pi_{\mathrm{disc}}(M^L, L, \mu, \Lambda_{E/F})} \mathcal{J}_{M^L}^{Q^L}(\pi', \phi^v \otimes h_\theta)$$
$$= \sum_{\{\nu \mid \nu_{E/F}=\mu\}} \sum_{\pi \in \Pi_{\mathrm{disc}}(M,H,\nu,\Lambda)} \mathcal{J}_M^Q(\pi, f^v \otimes b(h)) .$$

Using the spectral expansion II.2.1 this shows that

$$J^Q(f^v \otimes b(h)) = J^{Q^L}(\phi^v \otimes h_\theta)$$

whenever, outside v , the functions f^v and ϕ^v are strongly associated regular and K-finite. We conclude the proof using corollary V.2.2 (or rather, its variant for K-finite functions which holds thanks to III.5.5).

□

Remarks.

(i) For $GL(2)$ the matching of weighted orbital integrals V.6.3 has been proved by a direct local computation in [Lan1] (lemma 5.11 p. 74).

(ii) One can deduce V.6.3 from the main theorems in [AC]. Recall that h_θ and $b_{E/F}(h)$ are already known to be associated. Observe also that normalized weighted orbital integrals J_{M^L} of functions in the unramified Hecke algebra, coincide with invariant distributions I_{M^L} :

$$J_{M^L}(\delta, h_\theta) = I_{M^L}(\delta, h_\theta) \ .$$

This is an immediate consequence of I.5.1 (cf. [A7] lemma 2.1). To prove the matching – assertion (i) in the definition III.3.2 of strong association – we embbed our local situation in a global one : our local field is now the completion at a place v of a global field F . Theorem A of [AC], chap. 2 p. 108, tells us that for pairs of associated functions (f, ϕ) over $H(F_S)$ and $L(F_S)$, for a large enough finite set of places S , the invariant distributions I_M match i.e.

$$I_{M^L}(\delta, \phi) = I_M(\gamma, f)$$

if γ is stably conjugated to δ^ℓ . We may apply this statement also to pairs

$$(f, \phi)$$

and

$$(f \otimes b_{E_v/F_v}(h), \phi \otimes h_\theta) \ .$$

Assuming by induction the result proved for all smaller Levi subgroups, and using the descent and splitting formulas we see that the invariant distributions $I_M(\gamma, b_{E_v/F_v}(h))$ and $I_{M^L}(\delta, h_\theta)$ also match.

(iii) The local-global argument that has been used to prove the matching V.6.3 does not yield, right away, the vanishing IV.5.1. In fact such an argument based on the equality of two trace formulas for pairs of functions (f, ϕ) only yields the vanishing of distributions $J_M^Q(\gamma, f_S)$ where S contains the set of all places where γ is a not a norm. In particular S has at least two elements. G. Henniart has shown me how to build a global situation in which $J_H(\gamma, f_S)$ is a product over $v \in S$ of local distributions $J_H(\gamma, f_v)$ independent of $v \in S$. This would allow one to establish the vanishing for ordinary orbital integrals, but this is not enough for our needs.

VI. – BASE CHANGE FOR AUTOMORPHIC REPRESENTATIONS

In this chapter F is a global field and E is a cyclic algebra over F. An automorphic representation π' is said to be a base change of π if ${\pi'}_v$ is a base change of π_v for all places v . We would like to prove that if π' occurs discretely in the trace fromula for L then there exist π that occurs discretely in the trace formula for H and such that π' is the base change of π. Conversely, if the character of π does not vanish on norms, there should exist π' that occurs discretely in the trace formula for L and such that π' is the base change of π.

In the first section we prove our main technical result: a refined base change identity VI.1.4 in which the unramified infinitesimal characters are separated. We give a first proof using the noninvariant fundamental lemma IV.5.2. If the full conjecture A was known to hold this would be easy. We only have at hand the particular case II.4.5 of conjecture A. We shall in fact prove a stronger result VI.1.3, which is tantamount to a compatibility of weighted characters with base change, provided that the normalizing factors are compatible with the weak base change. A simpler proof is also given using a preliminary separation of infinitesimal characters at archimedean places and then elementary functions at unramified places but, for the first step, we need conjecture B2.

If G is split i.e. $G = H$, using rigidity properties, we may extract the spectral informations coded in the identity VI.1.4; this yields our main theorem VI.4.1. Unfortunately such rigidity properties are not a priori available for inner forms and we are not able to complete the proof of the analogue of VI.4.1 in general.

VI.1 – Separation of unramified infinitesimal characters.

Given a pair of representations $\pi \in \Pi_{\text{disc}}(H)$ and $\pi' \in \Pi_{\text{disc}}(L)$ let

$$\delta_H^L(\pi,\pi') = 1\ ,$$

if π'_v is the base change of π_v for almost all v and let

$$\delta_H^L(\pi,\pi') = 0$$

otherwise.

VI.1.1. Definition. – *If $\delta_H^L(\pi,\pi') = 1$ we say that π' is a weak base change of π.*

Similarly, given ψ a character of the unramified Hecke algebra $\mathcal{H}^L(\mathbb{A}_F^S)$ outside of S , a finite set of places containing all ramified places, we define $\delta_H^L(\pi,\psi)$ (resp. $\delta_L^L(\pi',\psi)$) to be 1 if :

$$\text{trace } \pi^S(b_{E/F}(h)) = \hat{h}(\psi)$$

(resp. trace ${\pi'}^S(h) = \hat{h}(\psi)$), and $\delta_H^L(\pi,\psi) = 0$ otherwise.

VI.1.2. Definition. – *We shall say that the normalizing factors are compatible with the weak base change if*

$$\ell^{\dim \mathfrak{a}_M^H}\ r_{M^L}^L(\pi') = \sum_{\xi \in \Xi(M)_{E/F}/\Xi(H)_{E/F}} r_M^H(\pi \otimes \xi)$$

whenever $\delta_M^{M^L}(\pi,\pi') = 1$.

VI.1.3. Proposition. – *Assume that the normalizing factors are chosen to be compatible with the weak base change. Let S be a finite set of places outside of which E/F is unramified. Given a Levi subgroup M consider a character ψ of the unramified Hecke algebra $\mathcal{H}^{M^L}(\mathbb{A}_F^S)$. If (f_S,ϕ_S) is a pair of strongly associated regular functions one has*

$$\ell^{\dim \mathfrak{a}_M} \sum_{\pi' \in \Pi_{\text{disc}}(M^L)} \delta_{M^L}^{M^L}(\pi',\psi)\, a_{\text{disc}}^{M^L}(\pi')\ J_{M^L}^{Q^L}(\pi',\phi_S)$$

$$= \sum_{\pi \in \Pi_{\text{disc}}(M)} \delta_M^{M^L}(\pi,\psi)\ a_{\text{disc}}^M(\pi)\ J_M^Q(\pi,f_S)\ .$$

Proof. By proposition V.1.2 we have

$$J^{Q^L}(\phi) = J^Q(f)$$

for pairs (f, ϕ) of strongly associated regular functions. Recall that according to II.2.3

$$J^{Q^L}(\phi) = \sum_{\chi'} \sum_{M \in \mathcal{L}} \frac{w^M}{w^Q} \sum_{L_1, L_2} d_M^Q(L_1, L_2)\ J_{M^L,\chi'}^{L_1^L, Q_2^L}(\phi)$$

where

$$J_{M,\chi'}^{L_1^L,Q_2^L}(\phi) =$$

$$\sum_{\pi' \in \Pi_{\mathrm{disc}}(M^L,\chi')} a_{\mathrm{disc}}^{M^L}(\pi') \int_{i\mathfrak{a}_M^*} r_{M^L}^{L_1^L}(\pi'_{\Lambda_{E/F}})\ J_{M^L}^{Q_2^L}(\pi'_{\Lambda_{E/F},S}, \phi_S) \text{ trace } \pi'^S_{\Lambda_{E/F}}(h_{M^L})\ d\Lambda\ .$$

Assume that $\mathfrak{a}_M^{L_1} \oplus \mathfrak{a}_M^{Q_2} = \mathfrak{a}_M^Q$ and assume inductively that VI.1.3 holds for parabolic subgroups $Q_2 \subsetneq Q$. Using V.4.1 and the compatibility of normalizing factors with weak base change, we see that

$$\ell^{\dim \mathfrak{a}_M} \sum_{\pi' \in \Pi_{\mathrm{disc}}(M^L)} \delta_{M^L}^{M^L}(\pi', \psi)\, a_{\mathrm{disc}}^{M^L}(\pi')\, r_{M^L}^{L_1^L}(\pi')\ J_{M^L}^{Q_2^L}(\pi', \phi)$$

equals

$$\sum_{\pi \in \Pi_{\mathrm{disc}}(M)} \delta_M^{M^L}(\pi, \psi)\ a_{\mathrm{disc}}^M(\pi) r_M^{L_1}(\pi)\ J_M^{Q_2}(\pi, f)\ .$$

if $Q_2 \neq Q$. Hence we may cancel all terms attached to pairs (L_1, Q_2) in the difference of spectral expansions provided $Q_2 \neq Q$ and we are left with the identity

$$\sum_{M \in \mathcal{L}^H} \frac{w^M}{w^Q} \left(\sum_{\chi'} J_{M^L,\chi'}^{M^L,Q^L}(\phi) - \sum_{\chi} J_{M,\chi}^{M,Q}(f) \right) = 0\ .$$

Consider h in the unramified Hecke algebra $\mathcal{H}^L(\mathbb{A}_F^S)$ and let $m^S(\phi_S, f_S; \hat{h})$ be defined by

$$\sum_{M \in \mathcal{L}^H} \frac{w^M}{w^Q} \left(\sum_{\chi'} J_{M^L,\chi'}^{M^L,Q^L}(\phi_S \otimes h_\theta) - \sum_{\chi} J_{M,\chi}^{M,Q}(f_S \otimes b_{E/F}(h)) \right)\ .$$

By proposition II.4.5 the spectral expansion recalled above, for the right hand side of this equation is absolutely convergent and thanks to II.5.1 it defines a Radon measure on the unramified unitary dual outside S : the compact space $\Pi_{nr,u}(L(\mathbb{A}_F^S))$. We have an explicit expression for this measure : $m^S(\phi_S, f_S; \hat{h})$ can be written

$$\sum_{M\in\mathcal{L}^H} \frac{w^M}{w^Q}\int_{i\mathfrak{a}_M^*}\sum_{\psi}\left(\ell^{\dim\mathfrak{a}_M} c_M^{Q^L}(\psi,\Lambda_{E/F};\phi_S) - c_M^Q(\psi,\Lambda;f_S)\right)\ \hat{h}_{M^L}(\psi_{\Lambda_{E/F}})\ d\Lambda$$

where ψ runs over character of the unramified Hecke algebra $\mathcal{H}^{M^L}(\mathbb{A}_F^S)$ and

$$c_M^{Q^L}(\psi,\Lambda;\phi_S) = \sum_{\pi'\in\Pi_{\mathrm{disc}}(M^L)} \delta_{M^L}^{M^L}(\pi',\psi)\ a_{\mathrm{disc}}^{M^L}(\pi')\ J_{M^L}^{Q^L}(\pi'_\Lambda,\phi_S)$$

$$c_M^Q(\psi,\Lambda;f_S) = \sum_{\pi\in\Pi_{\mathrm{disc}}(M)} \delta_M^{M^L}(\pi,\psi)\ a_{\mathrm{disc}}^M(\pi)\ J_M^Q(\pi_\Lambda,f_S)\ .$$

We have used that

$$\hat{h}_{M^L}(\psi_{\Lambda_{E/F}}) = \text{ trace } {\pi'}^S_{\Lambda_{E/F}}(h_{M^L}) = \text{ trace } \pi_\Lambda^S(b_{E/F}(h_{M^L}))$$

if

$$\delta_{M^L}^{M^L}(\pi',\psi) = \delta_M^{M^L}(\pi,\psi) = 1\ .$$

But, thanks to theorem IV.5.2 pairs of functions $(h_\theta, b_{E/F}(h))$ are strongly associated and hence, if f_S and ϕ_S are strongly associated and regular

$$m^S(\phi_S,f_S;\hat{h}) = 0$$

for all $h\in\mathcal{H}^L(\mathbb{A}_F^S)$. The associated Radon measure also vanishes. As in the proof of V.3.1 we may first separate the various contributions according to the dimension d of $\dim\mathfrak{a}_M$ by varying the functions at one place. Continuous functions on $\Pi_{nr,u}(L(\mathbb{A}_F^S))$ separate only the Weyl group orbits of characters of unramified Hecke algebra attached to the various M . Hence the sum over Weyl group orbits of coefficients of $\hat{h}_{M^L}(\psi_{\Lambda_{E/F}})$, considered as functions on $i\mathfrak{a}_M^*$, must vanish almost everywhere; they are analytic and hence vanish everywhere. But, taking into account the invariance under the Weyl group of distribution J_M^H (lemma I.6.1) we get

$$\ell^{\dim\mathfrak{a}_M} c_M^{Q^L}(\psi,\Lambda_{E/F};\phi_S) - c_M^Q(\psi,\Lambda;f_S) = 0$$

for any character ψ of the unramified Hecke algebra $\mathcal{H}^{M^L}(\mathbb{A}_F^S)$.

□

Remark. – A variant of the above argument is used in Langlands' original proof of the base change for $GL(2)$ [Lan1]. A similar argument is also used in [Rog].

VI.1.4. Proposition. – *Let S be a finite set of places containing the archimedean ones. Assume that E/F is unramified outside S and consider a pair (f_S, ϕ_S) of strongly associated regular functions. Assume that either*

(a) the normalizing factors may be chosen to be compatible with the weak base change,

or

(b) the functions are K-finite and the pair is compatible with multipliers at archimedean places.

Let ψ be a character of the unramified Hecke algebra $\mathcal{H}^L(\mathbb{A}_F^S)$, then

$$\ell^{\dim \mathfrak{a}_H} \sum_{\pi' \in \Pi_{\rm disc}(L)} \delta_L^L(\pi', \psi)\; a^L_{\rm disc}(\pi') \text{ trace } \pi'_S(\phi_S)$$

$$= \sum_{\pi \in \Pi_{\rm disc}(H)} \delta_H^L(\pi, \psi)\; a^H_{\rm disc}(\pi) \text{ trace } \pi_S(f_S)\ .$$

Proof. In case (a) the assertion is the particular case $M = H$ of VI.1.3. Consider now the case (b). If at archimedean places the functions are K-finite and that the pair is compatible with multipliers, we may use V.3.1 to separate infinitesimal characters at archimedean places and we get

$$\ell^{\dim \mathfrak{a}_H} \sum_{\pi' \in \Pi_{\rm disc}(L, \nu_{E/F})} a^L_{\rm disc}(\pi') \text{ trace } \pi'(\phi_S \otimes \phi^S)$$

$$= \sum_{\pi \in \Pi_{\rm disc}(H, \nu)} a^H_{\rm disc}(\pi) \text{ trace } \pi(f_S \otimes f^S)\ ,$$

where $(\nu_{E/F}, \nu)$ is a pair of infinitesimal character associated by base change at archimedean places. The infinitesimal characters $\nu_{E/F}$ and ν being prescribed, only a finite set of representations may contribute nontrivially to both side if we consider pairs of functions left and right invariant at finite places under some fixed open compact subgroup. We may use associated elementary functions to separate unramified infinitesimal characters ([Lab2] proposition 8).

□

Remarks.

(i) A similar result, with a proof as in case (b), is true for pairs of functions (f_S, ϕ_S), that give rise to rationally strongly associated pairs (f, ϕ) compatible with multipliers at archimedean places, whenever (f^S, ϕ^S) are associated. It suffices for example that at two places $v \in S$ the functions in the pairs (f_v, ϕ_v) are very cuspidal. Cuspidal functions at two places is even enough but the proof uses the invariant form of the trace formula.

(ii) If we assume that conjecture A holds, an argument similar to the proof of V.3.1, but using the noninvariant fundamental lemma IV.5.2 instead of multipliers at archimedean places, yields a simple proof of the separation of unramified infinitesimal characters for the discrete part of the base change identity for pairs of strongly associated regular functions.

In some cases one may relax the regularity assumption in the base change identities.

VI.1.5. Proposition. – *Let (f, ϕ) be a pair of strongly associated functions. Assume that either*

(a) $G = H$,

or

(b) the functions are K-finite and the pair is compatible with multipliers at archimedean places.

Then

$$J^L(\phi) = J^H(f) \ .$$

Under the same assumptions, the regularity assumption can be removed from V.3.1 or VI.1.4.

Proof. Consider a pair of strongly associated function and let v be an unramified place for E/F such that f_v and ϕ_v are the characteristic functions of K_v^H and K_v^L respectively. Substitute at the place v associated regular elementary functions; then by V.1.2 the identity between trace formulas hold

$$J^L(\phi^v \otimes f_t^L) = J^H(f^v \otimes f_{t'}^H) \ .$$

By VI.1.3 in case (a) using moreover the rigidity VI.2.2 or by V.3.1 in case (b), this identity can be refined into an identity for each Levi subgroup M and with infinitesimal character at archimedean places in a finite set. This being done, the conductor being fixed, we deal with a finite set of representations and we may separate unramified infinitesimal characters at the place v ([Lab2] proposition 8); as in the proof of IV.5.2 we may now reverse the process and substitute back the original functions in the unramified Hecke algebra to our elementary functions; then using II.2.3 we recover an identity between trace formulas.

□

VI.2 – L-functions and rigidity.

In this section $G = H$. Recall that a representation $\pi' \in \Pi(L(\mathbb{A}_F))$ is said to occurs in the discrete spectrum for L if its restriction to $L^0(\mathbb{A}_F)$ is an irreducible direct factor of $\mathbf{L}^2(L^0(F)\backslash L^0(\mathbb{A}_F)^1)$ and one denotes by $m^L_{\text{disc}}(\pi')$ its multiplicity. The multiplicity one theorem for the cuspidal spectrum of $GL(n)$ [Shal], which readily extends to the discrete spectrum by [MW] , tells us that $m^L_{\text{disc}}(\pi') \in \{0,1\}$. As a supplement to the multiplicity one theorem one has rigidity properties VI.2.2 that follow from the next theorem.

VI.2.1. Theorem. – *Let π_1 and π_2 be two cuspidal unitary automorphic representations for $GL(n)$; denote by $\widetilde{\pi}$ the contragredient of π . Let S be a finite set of places containing the archimedean ones and outside of which the representations are unramified. The partial L-function of pairs $L^S(s, \pi_1 \times \widetilde{\pi}_2)$ is regular nonzero for* $\mathrm{Re}(s) > 1$ *; it has a simple pole at $s = 1$ if $\pi_1 \simeq \pi_2$; otherwise it is regular and nonzero at $s = 1$.*

Proof. This follows from theorems of Jacquet and Shalika [JS] and Shahidi [Shah1] (see [AC] chapter 3 p. 200).

□

The numbers $a^L_{\text{disc}}(\pi')$ have been defined in II.2. We may be more explicit in the case $G = GL(n)$. We say that a representation $\pi' \in \Pi(L(\mathbb{A}_F))$ occurs discretely in the trace formula for L if $a^L_{\text{disc}}(\pi') \neq 0$. This is the case only if the restriction π of π' to $L^0(\mathbb{A}_F)$ is a constituent of a representation parabolically induced from a unitary representation π_0 of a Levi subgroup L^0_0 that occurs discretely in $\mathbf{L}^2(L^0_0(F)\backslash L^0_0(\mathbb{A}_F)^1)$ and if there is an element s in the Weyl group that normalizes L^0_0 which is such that :

(i) $\det\,(s-1)_{\mathfrak{a}^L_{L_0}} \neq 0$,

(ii) let $w = s \rtimes \theta$, then w stabilizes π_0 .

Since we are working with $GL(n)$ unitary parabolic induction preserves irreducibility. Condition (i) is equivalent to say that

$$L^0_0 = L^0_1 \times \ldots \times L^0_1$$

and that w permutes transitively the m factors $L^0_1 = GL(n_1)$ with $n = mn_1$. We know by [MW] that π_0 is a tensor product of Speh representations

$$\pi_1 = \text{Speh}(\sigma_1) \otimes \ldots \otimes \text{Speh}(\sigma_m)$$

where the σ_i are unitary cuspidal representations of $M^0_2 = GL(d)$ with $n = mdr$. Condition (ii) tells us that the representation π_0 can be extended to a representation π'_0 of the semidirect product $L_0(\mathbb{A}_F)^+$ of $L^0_0(\mathbb{A}_F)$ by the cyclic group generated by s . This is possible if and only if

$$\sigma = \sigma_1 \otimes \ldots \otimes \sigma_m$$

is extendable to a representation σ' of $M_1(\mathbb{A}_F)^+$ the semidirect product of

$$M^0_1 = M^0_2 \times \ldots \times M^0_2$$

by the cyclic group generated by s . Observe that in the case $L = H$ this implies that the σ_i are all equal; moreover the intertwining operators that show up in the defintion of numbers $a^H_{\text{disc}}(\pi)$ in II.2 are scalars since unitary parabolic induction preserves irreducibility; these scalars are roots of unity.

VI.2.2. Proposition. – *(i) Consider two automorphic representations* π_1 *and* π_2 *of* $H(\mathbb{A}_F)$ *that occur discretely in the trace formula for* H *and let* ψ *be a character of* $\mathcal{H}^L(\mathbb{A}^S_F)$ *then*

$$\delta^L_H(\pi_1, \psi) = \delta^L_H(\pi_2, \psi) = 1$$

if and only if there exist a character $\xi \in \Xi(H)_{E/F}$ such that $\pi_2 = \pi_1 \otimes \xi$.
(ii) Consider two automorphic representations π'_1 and π'_2 of $L^0(\mathbb{A}_F)$ that occur discretely in the trace formula for L and let ψ be a character of $\mathcal{H}^L(\mathbb{A}_F^S)$ then

$$\delta_L^L(\pi'_1, \psi) = \delta_L^L(\pi'_2, \psi) = 1$$

if and only if $\pi'_1 = \pi'_2$.

Proof. It suffices to consider the case $H = GL(n)$. If the two representations π_i with $i = 1, 2$ occur discretely in the trace formula for H , they are parabolically induced from two automorphic representations that are tensor products of m_i copies of Speh representations Speh(σ_i) constructed from two unitary cuspidal representations σ_i for some $M_i = GL(d_i)$. We have $n = m_i d_i r_i$. The relation $\delta_H^L(\pi_1, \psi) = \delta_H^L(\pi_2, \psi)$ implies the following identity between products of partial L-functions:

$$\prod_\xi L^S(s, \sigma_1 \times \widetilde{\pi}_1 \times \xi) = \prod_\xi L^S(s, \sigma_1 \times \widetilde{\pi}_2 \times \xi) ;$$

where ξ runs over Größencharactere of F trivial on the norms from E . By VI.2.1 the left hand side has, a pole of order m_1 at $s = 1 + \frac{r_1 - 1}{2}$ and is analytic for

$$\mathrm{Re}(s) > 1 + \frac{r_1 - 1}{2} .$$

The same must be true for the right hand side. This implies $m_1 = m_2$, $d_1 = d_2$ and $\sigma_1 = \sigma_2 \otimes \xi$ for some ξ and hence $\pi_2 = \pi_1 \otimes \xi$. The proof of (ii) is similar except that the two inducing representations need not be isotypic products; using VI.2.1 one shows that, up to a permutation of the factors, the inducing representations are equal; the induced representations are equal.

□

Observe that tensorisation by a unitary automorphic character ξ trivial on $\sigma(\mathfrak{a}_H)$ preserves the discrete part of the trace formula:

$$a_{\mathrm{disc}}^H(\pi) = a_{\mathrm{disc}}^H(\pi \otimes \xi) .$$

Given $\pi \in \Pi_{\mathrm{disc}}(H)$, let $c(\pi, E/F)$ be cardinal of the subgroup of characters $\xi \in \Xi(H)_{E/F}$ such that $\pi \otimes \xi = \pi$. We shall use a variant of numbers a_{disc}^H :

$$a_{\mathrm{disc}}^H(\pi, E/F) = \ell^{-\dim \mathfrak{a}_H} c(\pi, E/F)^{-1} \sum_{\xi \in \Xi(H)_{E/F}} a_{\mathrm{disc}}^H(\pi \otimes \xi) = c(\pi, E/F)^{-1} a_{\mathrm{disc}}^H(\pi) .$$

VI.3 – Normalizing factors and base change.

If $G = H$ one can use the canonical normalization of global intertwining operators by global L-functions. This is possible thanks to Shahidi's results ([Shah2] and [Shah3]). Moreover the L-functions are compatible with base change since this is true locally everywhere (VI.5.2) and hence the canonical normalizing factors are compatible with the base change.

VI.3.1. Proposition. – *If $G = H$ the canonical normalizing factors are compatible with the base change:*

$$\ell^{\dim \mathfrak{a}_M^H}\ r_{M^L}^L(\pi') = \sum_{\xi\in\Xi(M)_{E/F}/\Xi(H)_{E/F}} r_M^H(\pi\otimes\xi)$$

if π' is a base change of π.

Proof. If P and Q have Levi M , the canonical factors $r_{P|Q}(\pi_\Lambda)$ are products of terms indexed by a set, depending on P and Q , of roots α of M of functions $r_\alpha(\pi_\Lambda)$ which in turn can be expressed in term of L-functions of pairs. Hence if we use canonical global normalizing factors $r_{P|Q}(\pi_\Lambda)$, the factor $r_M^H(\pi)$ is the product of a constant times logarithmic derivatives of L-functions of pairs (this is proposition 7.5 of [A4] p. 1323). The constant is a_M^H the covolume of the coroot lattice in $\mathfrak{a}_M^H$ introduced in I.1. One has similar expressions for L . The lemma follows from the compatibility of L-functions of pairs with base change (VI.5.2) up to the power of ℓ: the factor $\ell^{\dim \mathfrak{a}_M^H}$ shows up since we use the norm map to transfer linear forms Λ to compute the logarithmic derivatives; note that characters in $\Xi(H)_{E/F}$ act trivially.

□

Remarks.

(i) This proposition is nothing but lemma 11.1 of [AC] chapter 2 p. 147.

(ii) It is proved in [AC], using lemma 2.2.1 page 88, that one can also define for inner forms normalizing factors compatible with the base change. The definition relies on the local correspondence for which Arthur and Clozel refer to [DKV].

VI.4 – Cyclic base change for $GL(n)$.

We may now state and prove our main theorem. In [AC] a similar theorem is proved only for automorphic representations induced from cuspidal ones ([AC] chapter 3, theorem 4.2 and 5.1). This restriction can be lifted thanks to [MW].

VI.4.1. Theorem. – *Let F be a global field and let E be a cyclic algebra over F . Assume that $G = H$.*

(i) Given $\pi' \in \Pi_{\text{disc}}(L)$ there exist $\pi \in \Pi_{\text{disc}}(H)$ such that $\delta_H^L(\pi,\pi') = 1$. Moreover such a representation π is unique up to twists by characters $\xi \in \Xi(H)_{E/F}$.

(ii) Given $\pi \in \Pi_{\text{disc}}(H)$ there exist a unique $\pi' \in \Pi_{\text{disc}}(L)$ such that $\delta_H^L(\pi,\pi') = 1$.

(iii) Let $\pi' \in \Pi_{\text{disc}}(L)$ and $\pi \in \Pi_{\text{disc}}(H)$; if $\delta_H^L(\pi,\pi') = 1$ the representation π' is a strict base change of π .

Proof. Assume inductively that VI.4.1 (iii) holds for proper Levi subgroups; hence weak base change and base change coincide for them. Then VI.3.1 shows that the canonical normalizing factors are compatible with the weak base change and hence we may use VI.1.4. This inductive step is not necessary if conjecture B2 holds. Now VI.1.4 and proposition VI.2.2, show that given a representation π' such that

$$a^L_{\text{disc}}(\pi') \neq 0$$

then, for S finite large enough and for any pair (f_S, ϕ_S) of strongly associated functions, we have

$$a^L_{\text{disc}}(\pi') \text{ trace } \pi'_S(\phi_S) = \sum_{\pi\in\Pi_{\text{disc}}(H)} \delta^L_H(\pi',\pi)\, a^H_{\text{disc}}(\pi) \text{ trace } \pi_S(f_S) \ .$$

We want to show the existence of π such that $\delta_H^L(\pi,\pi') = 1$ and $a^H_{\text{disc}}(\pi) \neq 0$. It suffices to exhibit a strongly associated pair (f_S,ϕ_S) such that trace $\pi'_S(\phi_S) \neq 0$. Let $\delta_0 \in L_v$ be a regular semisimple point for which the character of π'_v does not vanish. Proposition III.4.1, allows one to construct a pair of strongly associated functions (f_v,ϕ_v) with trace $\pi'_v(\phi_v) \neq 0$ by taking ϕ_v with support in a small enough neighbourhood of δ_0 with positive ordinary orbital integrals $J_L(\delta,\phi_v)$ nonvanishing at δ_0 . The uniqueness of π up to twists follows from VI.2.2. This proves (i). Assume now that $\pi \in \Pi_{\rm disc}(H)$. We get using VI.1.4 and VI.2.2 :

$$\sum_{\pi' \in \Pi_{\rm disc}(L)} \delta_H^L(\pi',\pi)\, a_{\rm disc}^L(\pi') \ \text{trace}\ \pi'_S(\phi_S) = a_{\rm disc}^H(\pi, E/H)\ \text{trace}\ \pi_S(f_S)\ .$$

We have to exhibit a pair (f_S, ϕ_S) such that the right hand side of does not vanish. Lemma III.1.7 shows that any representation π_v has a character distribution that does not vanish identically on the set of regular elements that are norms of elements in L_v . Let γ_0 be a regular norm where the character of π_v does not vanish. Proposition III.4.1, allows one to construct a pair of strongly associated functions (f_v, ϕ_v) with trace $\pi_v(f_v) \neq 0$ by taking f_v with support in a small enough neighbourhood of γ_0 with positive ordinary orbital integrals $J_H(\gamma, f_v)$ nonvanishing at γ_0 . This proves (ii). Consider π such that $a_{\rm disc}^H(\pi) \neq 0$ and π' such that $a_{\rm disc}^L(\pi') \neq 0$. Assume moreover that $\delta_H^L(\pi, \pi') = 1$. Let $c(\pi,\pi') \in \mathbb{C}^\times$ such that

$$c(\pi,\pi')\, a_{\rm disc}^L(\pi') = a_{\rm disc}^H(\pi, E/F)\ .$$

For any large enough finite set S of places, VI.1.4 and VI.2.2 show that

$$\text{trace}\ \pi'_S(\phi_S) = c(\pi,\pi')\ \text{trace}\ \pi_S(f_S)$$

for all pairs of strongly associated functions with regular support. Assertion (iii) now follows from III.4.2.

□

VI.4.2. Proposition. – *Assume that π' and π are both in the discrete spectrum for L and H respectively. If $\delta_H^L(\pi,\pi') = 1$ then*

$$c(\pi,\pi') = 1\ .$$

Proof. Recall that

$$c(\pi,\pi')\, a_{\rm disc}^L(\pi') = a_{\rm disc}^H(\pi, E/F)\ .$$

Since π' and π are both in the discrete spectrum, we know by VI.2.2 that

$$a_{\rm disc}^L(\pi') = m_{\rm disc}^L(\pi') = 1 = m_{\rm disc}^H(\pi) = a_{\rm disc}^H(\pi)\ .$$

But if π' is in the discrete spectrum, by [MW] it is a Speh representation $\pi' = \mathrm{Speh}(\sigma')$ where σ' is cuspidal on some other group G_1 . We may assume that $G = GL(n)$ and that $G_1 = GL(d_1)$ with $n = r_1 d_1$. The partial L-function $L^S(s, \pi' \times \widetilde{\pi'})$ is the product of $L^S(s-k, \sigma' \times \widetilde{\sigma}')$ where k is an integer with $|k| \leq r_1 - 1$ and the L-function $L^S(s, \pi' \times \widetilde{\pi'})$ has a simple pole at $s = r_1$. One has also $\pi = \mathrm{Speh}(\sigma)$ where σ is cuspidal on some other group $G_2 = GL(d_2)$ and one has a similar expression for $L^S(s, \pi \times \tilde{\pi})$. Moreover

$$L^S(s, \pi' \times \widetilde{\pi'}) = \prod_{\xi} L^S(s, \pi \times \tilde{\pi} \times \xi)$$

is regular nonzero for $\mathrm{Re}(s) > r_1$ and has a simple pole at $s = r_1$. This implies in particular that $d_1 = d_2$ and that $\pi \neq \pi \otimes \xi$ unless $\xi = 1$ i.e. $c(\pi, E/F) = 1$.

□

Remark. – The local components of π' are not canonically defined a priori. The previous proposition allows one to show that ${\pi'}_v$ can be taken to be *the* canonical base change of π_v at all places. We refer the reader to [AC] section 1.6.3 p. 56 for a proof.

VI.4.3. Corollary. – *Assume that $G = H$. Let S be a finite set of places outside of which E/F is unramified. Given a Levi subgroup M , consider π' and π that occur in the discrete spectrum for M^L and M respectively. Assume that π' and π are unramified outside S and such that $\delta_M^{M^L}(\pi, \pi') = 1$. If (f_S, ϕ_S) is a pair of strongly associated functions one has*

$$\ell^{\dim \mathfrak{a}_M^H} J_{M^L}^L(\pi', \phi_S) = \sum_{\xi \in \Xi(M)_{E/F}/\Xi(H)_{E/F}} J_M^H(\pi \otimes \xi, f_S)$$

Proof. It follows from VI.4.1 (iii) and VI.3.1 that the canonical normalizing factors are compatible with the weak base change. Hence we may use VI.1.3 when $G = H$. The corollary now follows from VI.1.3, VI.2.2 and VI.4.2.

□

VI.5 – The local base change.

The base change theorem for automorphic representations yields the local base change. Let F be a local field and E a cyclic algebra over F.

VI.5.1. Theorem. – *Assume that $G = H$. Any $\pi \in \Pi(H(F))$ has a base change $\pi' \in \Pi(L(F))$ and conversely any $\pi' \in \Pi(L(F))$ is a base change of some π.*

Proof. Using the Langlands classification and since, for our groups, representations unitarily induced from tempered ones are irreducible, we are reduced to consider discrete series (resp. θ-discrete series.) We refer the reader to [AC] section 1.6.2 for a detailed account of this reduction step. One may now embed the local situation in a global one and one observes that any discrete series (resp. θ-discrete series) representation occurs as the local component of a cuspidal automorphic representation. This is classical and is an easy consequence of the existence of pseudo-coefficients. For local components of cuspidal representations the theorem follows immediately from our main theorem VI.4.1.

□

VI.5.2. Proposition. – *Let σ and τ be irreducible admissible representations of $GL(n_1, F)$ and $GL(n_2, F)$ with base change $\sigma_{E/F}$ and $\tau_{E/F}$. Then*

$$L(s, \sigma_{E/F} \times \tau_{E/F}) = \prod_{\xi} L(s, \sigma \times \tau \times \xi) \ ;$$

where ξ runs over characters of $F^\times$ trivial on the norms from $E^\times$.

Proof. For unramified situations this is clear. Using this and a local-global argument, which relies on the functional equation of L-functions, the assertion can be shown to be true for any non archimedean field using the properties of the local base change. We refer the reader to the proof of proposition 1.6.9 of [AC] p. 60. For archimedean fields this can be checked directly.

□

This compatibility has been used in the proof of VI.4.1, inductively for proper Levi subgroups via VI.3.1.

VI.6 – Inner forms.

We return to the global field case. We believe that our main theorem VI.4.1 should hold in general, when G is an inner form of H, except that (ii) should read:

(ii') Given $\pi \in \Pi_{\text{disc}}(H)$ there exist a unique $\pi' \in \Pi_{\text{disc}}(L)$ such that $\delta_H^L(\pi, \pi') = 1$ if and only if for any place v the character of π_v does not vanish almost everywhere on regular norms from $L(F_v)$.

But the proof given for VI.4.1 does not extend readily: there we use the rigidity. We do not know an a priori proof of the rigidity for inner forms, and in fact we want to deduce it from the endoscopic correspondence. The result of [MW] would also have to be extended. When G is a non split inner form of H we have this partial results.

VI.6.1. Proposition. – *Assume that $E = F$ and that $L = G$ is an inner form of H. Assume moreover that conjecture B2 holds or that one can use normalizing factors compatible with the weak base change. Then, given $\pi \in \Pi_{\text{disc}}(H)$ there exist $\pi' \in \Pi_{\text{disc}}(L)$ such that $\delta_H^L(\pi, \pi') = 1$ if and only if for any place v the character of π_v does not vanish almost everywhere on regular norms from $L(F_v)$.*

Proof. For S large enough, proposition VI.1.4 shows that

$$\sum_{\pi' \in \Pi_{\text{disc}}(L)} \delta_L^L(\pi', \psi)\ a_{\text{disc}}^L(\pi')\ \text{trace}\ \pi'_S(\phi_S)$$

$$= \sum_{\pi \in \Pi_{\text{disc}}(H)} \delta_H^L(\pi, \psi)\ a_{\text{disc}}^H(\pi)\ \text{trace}\ \pi_S(f_S)\ .$$

If $a_{\text{disc}}^H(\pi) \neq 0$ then VI.2.2 tells us that

$$\sum_{\pi' \in \Pi_{\text{disc}}(L)} \delta_H^L(\pi, \pi')\, a_{\text{disc}}^L(\pi')\ \text{trace}\ (\pi'_S(\phi_S)) = a_{\text{disc}}^H(\pi)\ \text{trace}\ (\pi_S(f_S))\ .$$

We see that, if there exists a pair (f_S, ϕ_S) of strongly associated functions such that

$$\text{trace}\ \pi_S(f_S) \neq 0\ ,$$

then there exist π' with nonzero multiplicity in the discrete part of the trace formula such that $\delta_H^L(\pi, \pi') = 1$. Such a pair of functions will exist if and only if the character of π_S does not vanish almost everywhere on norms from $L(F_S)_{\text{reg}}$.

□

Remark. – We observed that one can use normalization factors compatible with base change. One would need moreover the rigidity for proper Levi subgroups to show they are compatible with the weak base change. In the particular case where G is the multiplicative group of a division algebra there is nothing to prove.

One can think of different approaches to prove in general the analogue of VI.4.1. Starting again from VI.1.4 one may try to use the linear independence of characters of inequivalent representations in a finite set against functions with regular support. But, to do this, we need some a priori finiteness of the number of non trivial terms in the sum. This in turn would follow from a fairly general conjecture, which is a weak form of the rigidity:

VI.6.2. Conjecture C – *Given ψ an unramified infinitesimal character outside some finite set S of places, then there is a finite set of representations π' unramified outside S such that $\delta_L^L(\pi', \psi) = 1$.*

An other way would be to establish the noninvariant endoscopic transfer (without support restrictions). For the groups we study, the ordinary endoscopic transfer can be established thanks to results of Shelstad and Vigneras. This is used in [DKV] and in [AC]. To be used in our setting we need moreover conjecture B1. If for example, $H = GL(2)$ and if $L = G$ is a quaternion algebra, conjecture B1 holds trivially: pairs of associated functions are automatically strongly associated. This kind of argument is used by Jacquet and Langlands in [JL] chapter 16. But in general a proof of conjecture B1 will require more work.

BIBLIOGRAPHY

[A1] J. ARTHUR, A trace formula for reductive groups I, Duke Math. J. **45** (1978), 911-952.

[A2] J. ARTHUR, The trace formula in invariant form, Ann. of Math. **114** (1981), 1-74.

[A3] J. ARTHUR, On a family of distributions obtained from Eisenstein series I, Am. J. Math. **104** No. 6 (1982), 1243-1288.

[A4] J. ARTHUR, On a family of distributions obtained from Eisenstein series II, Am. J. Math. **104** No. 6 (1982), 1289-1336.

[A5] J. ARTHUR, A Paley-Wiener theorem for real reductive groups, Acta Math. **150** (1983), 1-89.

[A6] J. ARTHUR, The local behaviour of weighted orbital integrals, Duke Math. J. **56** (1988), 223-293.

[A7] J. ARTHUR, The invariant trace formula I, J. Am. Math. Soc. **1** (1988), 323-383.

[A8] J. ARTHUR, The invariant trace formula II, J. Am. Math. Soc. **1** (1988), 501-554.

[A9] J. ARTHUR, Intertwining Operators and Residues I. Weighted Characters, J. Fonct. Anal. **84** (1989), 19-84.

[A10] J. ARTHUR, On the Fourier transform of weighted orbital integrals, J. für reine und angewandete Math. **452** (1994), 163-217

[AC] J. ARTHUR L. CLOZEL, *Simple algebras, base change, and the advanced theory of the trace formula*, Annals of Math. Studies **120** Princeton Univ. Press (1989).

[Cas1] W. CASSELMAN, Characters and Jacquet Modules, Math. Annalen **230** (1977), 101-105.

[Cas2] W. CASSELMAN, The unramified principal series of $\mathfrak{p}$-adic groups I. The spherical function, Compositio Math. **40** (1980), 387-406.

[Clo] L. CLOZEL, The fundamental lemma for stable base change, Duke Math. J. **61** No. 1 (1990), 255-302.

[CLL] L. CLOZEL J.-P. LABESSE R.P. LANGLANDS, Morning seminar on the trace formula, Lecture Notes, Institute for Avanced Study, Princeton (1984).

[Deligne] P. DELIGNE, Le support du caractère d'une représentation supercuspidale, C. R. Acad. Sci. **283** Série A (1976), 155-157.

[DKV] P. DELIGNE D. KAZDAN M.-F. VIGNERAS, Représentations des algèbres centrales simples p-adiques, in *Représentations des groupes réductifs sur un corps local*, Hermann (1984), 33-117.

[Delorme] P. DELORME, Multipliers for the convolution algebra of left and right K-finite compactly supported smooth functions on a semi-simple Lie group, Invent. Math. **75** (1984), 9-23.

[DM] J. DIXMIER P. MALLIAVIN, Factorisations de fonctions et de vecteurs indéfiniment différentiables, Bull. Sci. Math. **102** (1978), 307-330.

[JL] H. JACQUET R.P. Langlands, *Automorphic forms on* $GL(2)$, Lecture Notes in Math. **114** Springer Verlag (1970).

[JS] H. JACQUET J.A. SHALIKA, On Euler products and the classification of automorphic representations I and II, Amer. J. Math. **103** (1981), 499-558 and 777-815.

[Ko1] R. KOTTWITZ, Rational conjugacy classes in reductive groups, Duke Math. J. **49** (1982), 785-806.

[Ko2] R. KOTTWITZ, Base change for units of Hecke algebras, Compositio Math. **60** (1986), 237-250.

[KS1] R. KOTTWITZ D. SHELSTAD, Twisted Endoscopy I: Definitions, Norm mappings and Transfer factors, preprint (1993).

[KS2] R. KOTTWITZ D. SHELSTAD, Twisted Endoscopy II: Basic global theory, preprint (1993).

[Lab1] J.-P. LABESSE, La formule des traces d'Arthur–Selberg, Séminaire Bourbaki, exposé **636** Novembre 1984.

[Lab2] J.-P. LABESSE, Le lemme fondamental pour le changement de base stable, Duke Math. J. **61** No.2 (1990), 519-530.

[Lab3] J.-P. LABESSE, Noninvariant trace formula comparison for the cyclic base change, preprint (1992).

[Lan1] R.P. LANGLANDS, *Base Change for* $GL(2)$, Annals of Math. Studies **96** Princeton Univ. Press (1980).

[Lan2] R.P. LANGLANDS, Rank one residues of Eisenstein series, Israel Mathematical Conference Proceedings **3** (1990), 111-125.

[MW] C. MŒGLIN J.-L. WALDSPURGER, Le spectre résiduel de $GL(n)$, Ann. Sci. E.N.S. 4^e série **22** (1989), 605-674.

[Mü] W. MÜLLER, The trace class conjecture in the theory of automorphic forms, Annals of Math. **130** (1989), 473-529.

[Rog] J. ROGAWSKI, *Automorphic Representations of Unitary Groups in Three Variables*, Annals of Math. Studies **123** Princeton Univ. Press (1990).

[S] J.-P. SERRE, *Corps locaux*, Actualités scientifiques et industrielles **1296** Hermann Paris (1962).

[Shah1] F. SHAHIDI, On nonvanishing of L-functions, Bull. A.M.S. (new series) **2** (1980), 442-464.

[Shah2] F. SHAHIDI, On certain L-functions, Amer. J. Math. **103** (1981), 297-356.

[Shah3] F. SHAHIDI, Local coefficients and normalization of intertwining opertors for $GL(n)$, Composition Math. **39** (1983), 271-295.

[Shal] J. SHALIKA, The multiplicity one theorem for $GL(n)$, Annals of Math. **100** (1974), 171-193.

[T] J. TITS, Reductive groups over local fields, Proceedings of symposia in pure Math. **33** Amer. Math. Soc. (1979), Part 1, 29-69.