B. HELFFER
J. SJÖSTRAND

Semi-classical analysis for Harper's equation. III : Cantor structure of the spectrum

Mémoires de la S. M. F. 2e série, tome 39 (1989)
<http://www.numdam.org/item?id=MSMF_1989_2_39__1_0>

Mémoire de la Société Mathématique de France, n° 39
Supplément au Bulletin de la S.M.F.
Tome 117, 1989, fascicule 4

SEMICLASSICAL ANALYSIS FOR HARPER'S EQUATION III
CANTOR STRUCTURE OF THE SPECTRUM

by

B. Helffer and **J. Sjöstrand**

Dépt. de Mathématiques,
Université de Nantes,
2, Chemin de la Houssinière,
F-44072 Nantes, CEDEX 3,
France

Dépt. de Mathématiques, Bât.425
Université de Paris Sud,
F-91405 Orsay,
France

RESUME. Dans ce travail nous continuons notre étude de l'opérateur de Harper, $\cos hD + \cos x$ dans $L^2(\mathbb{R})$, par des méthodes d'analyse microlocale et de renormalisation. On obtient une description assez complète du spectre dans le cas où $h/2\pi$ est irrationnel avec un développement en fraction continue : $h/2\pi = 1/(q_0 + 1/(q_1 + \dots))$, si $q_j \in \mathbb{Z}$, $|q_j| \geq C_0$ et $C_0 > 0$ est assez grand. En particulier le spectre est un ensemble de Cantor de mesure 0. Nos résultats sont aussi valables pour certaines perturbations de l'opérateur de Harper et on donne une application à l'opérateur de Schrödinger magnétique périodique sur $\mathbb{R}^2$.

ABSTRACT. In this paper we continue our study of Harper's operator $\cos hD + \cos x$ in $L^2(\mathbb{R})$, by means of microlocal analysis and renormalization. A rather complete description of the spectrum is obtained in the case when $h/2\pi$ is irrationnel and has a continued function expansion : $h/2\pi = 1/(q_0 + 1/(q_1 + \dots))$ with $q_j \in \mathbb{Z}$, $|q_j| \geq C_0$, provided that $C_0 > 0$ is sufficiently large. In particular, the spectrum is a Cantor set of measure 0. Our results are also valid for certain perturbations of Harper's operator and an application to the periodic magnetic Schrödinger operator on $\mathbb{R}^2$ is given.

Texte reçu le 14 avril 1988, révisé le 6 janvier 1989

Contents

0. Introduction.

This work is a continuation of our study, started in [HS1,2], of the spectrum of Harper's operator, by the use of semi-classical methods. If $h \in \mathbb{R}$, $h \neq 0$, then the problem is to study the union of the spectra, when θ varies in $\mathbb{R}$, of the operators in $\mathcal{L}(l^2(\mathbb{Z}), l^2(\mathbb{Z}))$, given by,

(0.1) $$H_\theta u(n) = \tfrac{1}{2}(u(n+1) + u(n-1)) + \cos(hn+\theta)u(n).$$

As a set, this union of spectra coincides with the spectrum of,

(0.2) $P_0 = \cos(hD_x) + \cos(x)$

in $\mathcal{L}(L^2(\mathbb{R}), L^2(\mathbb{R}))$, where $D_x = i^{-1}\partial/\partial x$, so that $\cos(hD_x) = \frac{1}{2}(\tau_h + \tau_{-h})$, where $\tau_h u(x) = u(x-h)$. Inspired by ideas of Wilkinson [W1], we obtained a partial Cantor structure result for the spectrum, $Sp(P_0)$ of P_0 under the assumption that $h/2\pi$ is irrational and,

(0.3) $h/2\pi = 1/(q_1 + 1/(q_2 + \ldots))$, $q_j \in \mathbb{Z}$, $1 \leq j < \infty$,

and

(0.4) $|q_j| \geq C_0$,

for some sufficiently large constant C_0. (See Théorème 1 in [HS1].) Roughly, our result was that if $\varepsilon_0>0$ and if $C_0>0$ is sufficiently large (as a function of ε_0), then outside $[-\varepsilon_0,\varepsilon_0]$ the spectrum of P_0 is contained in a union of intervals, J_j of width $\exp(-\sim|q_1|)$ and such that the separation between neighboring intervals (on the same side of $[-\varepsilon_0,\varepsilon_0]$) is $\sim 1/|q_1|$. If κ_j is the increasing affine function that maps J_j onto $[-2,2]$, then outside $[-\varepsilon_0,\varepsilon_0]$, the set $\kappa_j(J_j\cap Sp(P_0))$ can again be localized into a finite union of closed intervals, having widths and separations of the same order of magnitude as for the J_j , but in terms of q_2 instead of q_1. This procedure can then be continued indefinitely.

The proof of this result was obtained by applying first microlocal analysis near a "potential well", i.e. a component of $\cos(\xi)+\cos(x)=\mu$, where $\mu\in[-2,2]\backslash[-\varepsilon_0,\varepsilon_0]$, in order to obtain certain discrete eigenvalues, well defined up to $\mathcal{O}(e^{-1/Ch})$. It then followed that $Sp(P_0)$ is localized to certain intervals, J_j, exponentially close to these eigenvalues. After that we analyzed the tunnel effect between the potential wells, and this permitted us to describe $Sp(P_0)\cap J_j$ as the spectrum of a certain infinite "interaction" matrix. Exploiting certain translation invariance properties of the resulting matrix, we could then reduce the study of it's spectrum to that of $P(x,h'D_x)$, the Weyl quantization of $P(x,h'\xi)$, (and by definition, the h'-Weyl quantization of $P(x,\xi)$,) where $P=P_{j,h}$ is a small perturbation of $P_0=\cos(\xi)+\cos(x)$. Here $h'/2\pi=1/(q_2+1/(q_3+\ldots))$. For P we could then start over again ....

In this paper, we shall be able to eliminate the intervals, $[-\varepsilon_0,\varepsilon_0]$, and obtain a fairly complete description of $Sp(P_0)$, under the assumption (0.3),(0.4) with $C_0>0$ sufficiently large. When trying to make this improvement at the first level of the iteration scheme, an obvious difficulty is that for $\mu\approx 0$, $P_0^{-1}(\mu)$ is close to the union of the lines $\xi=\pm x+(2k+1)\pi$, $k\in\mathbb{Z}$, and there is no more obvious localization into potential wells.

As before, we can however study microlocal solutions of the homogeneous equation, $(P_0-\mu)u=0$, and as a matter of fact, this was done heuristically already by Azbel [Az]. Away from the saddle points, $(k\pi, l\pi)$, $k+l\in 2\mathbb{Z}+1$, the characteristic set, $P_0^{-1}(\mu)$ is a smooth analytic curve, and near a point in this part of the set, the microlocal kernel of $(P_0-\mu)$ is a one dimensional space, generated by a standard WKB solution. Near a saddle point the space of microlocal solutions is two dimensional, and can be computed more or less explicitly. If we choose for instance the point $(0,\pi)$, then an element of the microlocal kernel near that point, is determined by its behaviour near the open segments $](-\pi,2\pi),(0,\pi)[$ and $](0,\pi),(\pi,0)[$. Using a microlocal study of $P_0-\mu$, we can then obtain a globally defined, well posed "Grushin" problem,

(0.5) $(P_0-\mu)u+R_-u^-=v$, $R_+u=v^+$,

where, $u,v\in L^2(\mathbb{R})$, $u^-,v^+\in l^2(\mathbb{Z}^2;\mathbb{C}^2)$. Roughly (thinking of the case, $v=0$), the condition $R_+u=v^+$ fixes the microlocal behaviour of u near all segments of the form $]((k-1)\pi,(l+1)\pi),(k\pi,l\pi)[$, $k+l\in 2\mathbb{Z}+1$, and R_-u^- provides a one-dimensional inhomogeneity near each segment of the form, $](k\pi,l\pi),((k+1)\pi,(l+1)\pi)[$. Denoting the solution by,

(0.6) $u=Ev+E_+v^+$, $u^-=E_-v+E_{-+}v_+$,

where all operators depend on μ, it is easy to show that μ belongs to the spectrum of P_0 if and only if 0 belongs to the spectrum of E_{-+}. Now E_{-+} may be viewed as a block matrix, $(E_{-+}(\alpha,\beta))_{\alpha,\beta\in\mathbb{Z}^2}$, where each entry is a 2×2 matrix. By the same procedure as for the matrix W above, we then see that $0\in Sp(E_{-+})$ iff $0\in Sp(P)$, where P is a 2×2 matrix of h'-pseudodifferential operators. After rescaling, we see that (in the most interesting spectral region) P falls into a certain class of "strong type 2 operators". We also define strong type 1 operators, as scalar h-pseudodifferential operators, satisfying certain commutation relations and which are close to $P_0(x,hD)$. Fortunately, the study of strong type 2 operators is often very close to the study of s-type 1 operators, and we can again divide the problem into certain potential well cases and a branching case.

An interesting feature is that we loose the linear dependence of the spectral parameter, already after considering the first branching problem, so we shall systematically work with operators $P=P_\mu$, and define μ-$Sp(P)$ as the set of μ such that 0 belongs to the spectrum of P_μ. Theorem 6.2 below shows that the study of the μ-spectrum of a strong type 1 h-pseudodifferential operator sufficiently close to P_0 can, when h is sufficiently small, be localized into into a union of closed disjoint intervals, such that the further study of the μ-spectrum in each of these intervals leads to an operator either of s-type 1 or 2. Theorem 9.2 gives the corresponding result for s-type 2 operators. Theorem 9.3 is a combination of the Theorems

6.2 and 9.2 and says that if we start with a strong type 1 operator sufficiently close to P_0 and if (0.3),(0.4) hold with C_0 sufficiently large, then we get a complete description of the μ-Spectrum by means of an infinite sequence of localizations into finite disjoint unions of closed subintervals and rescalings. From the additional quantitative informations stated after the Theorems 6.2 and 9.2 about the lengths and separations of the various intervals appearing in those theorems, combined with Theorem 9.3, we obtain the following result (which expresses only a small part of the very precise information that our methods produce).

<u>Theorem 0.1.</u> Let $P=P(x,hD_x)$ be a self-adjoint h-pseudodifferential operator such that the corresponding Weyl symbol, $P(x,\xi)$ extends holomorphically to the "band" $|\mathrm{Im}(x,\xi)|<1/\varepsilon$, and satisfies:

(0.7) $P((x,\xi)+2\pi\alpha)=P(x,\xi)$, for all $\alpha\in\mathbb{Z}^2$,

(0.8) $P(\xi,-x)=P(\xi,x)$,

(0.9) $|P(x,\xi)-(\cos(\xi)+\cos(x))|\leq\varepsilon$, when $|\mathrm{Im}(x,\xi)|<1/\varepsilon$.

If (0.3),(0.4) hold with $C_0>0$ sufficiently large, and if $0<\varepsilon<\varepsilon_1$ with $\varepsilon_1>0$ sufficienly small, then Sp(P) is of Lebesgue measure 0, has no isolated points and is nowhere dense. (The last statement means that Sp(P) is dense in no non-trivial open interval.)

As already mentioned, the method produces a much more precise description of the spectrum, which is unfortunately rather lengthy to formulate in terms of a theorem, but the interested reader will be able to extract that information from the proofs. This refined description will no doubt be useful when studying the Hausdorff dimension of the spectrum as a function of the sequence (q_j). From the point of view of applications, it is important that our results apply also to small perturbations of Harper's operator. In appendix e we show that under suitable assumptions, the spectrum of a periodic magnetic Schrödinger operator is near the bottom a Cantor set of measure 0.

In [HS2], the results of [HS1] where extended to the case when for some N: $|q_j|\geq C_N(q_1,..,q_N,\varepsilon_0)$ for $j\geq N+1$, but still with the same incompleteness as in [HS1]. We believe that the techniques of the present paper rather automatically lead to a more complete Cantor structure result also in that case. One could probably generalize the result even to the case when there is a sequence $1\leq j_1<j_2<j_3<\ldots$ of integers such that

$|q_{j_k}|\geq C_{j_k-j_{k-1}}(q_{j_{k-1}+1},\ldots,q_{j_k-1})$, for suitable functions, C_N.

The plan of the paper is the following:

<u>Section 1</u>. Here we introduce and study certain auxiliary operators.

<u>Section 2</u> contains a formal study of the iteration steps that we will encounter, and we show that certain crucial symmetries are conserved.

<u>Section 3.</u> Here we treat the potential well problem for s-type 1 operators by suitable modifications of the methods in [HS1].

Section 4 treats the branching problem for strong type 1 operators and we obtain a "renormalized" 2×2 system of h'-pseudodifferential operators.
Sections 5 and 6 contain some preliminary results for the renormalized operator in section 4. It is showed that after rescaling and depending on the spectral region, the renormalized operator is either of s-type 1 or 2. Theorem 6.2 gives the main iteration step for s-type 1 operators.
Section 7 treats the totally degenerate potential well case, which is the only case genuinely non-scalar case for s-type 2 operators. Here we use some ideas from [HS2].
Section 8 treats the non-degenerate potential well case for s-type 2 operators.
Section 9 is devoted to the branching case for s-type 2 operators. Theorem 9.2 gives the main iteration step for s-type 2 operators.

Various results are collected into 5 appendices:
Appendix a contains various general results in microlocal analysis. The paragraph a.1 recollects the approach of [S1] to analytic microlocal analysis via FBI-transforms. We refer to that book for a more thorough treatment. In paragraph a.2 we developp a simple functional calculus for analytic pseudodifferential operators. Paragraph a.3 may be of independent interest. It gives a refined correspondence between unitary Fourier integral operators and canonical transformations.
Appendix b. Here we give local normal forms for self-adjoint pseudodifferential operators when the symbol has a saddle point or a minimum. We only allow unitary conjugations and taking functions of the operator. We believe that the results of this appendix will be useful in other contexts.
Appendix c. Here we show that certain 2×2 systems of pseudodifferential operators can be reduced to the case when the diagonal terms are scalars. This is of use in section 7. See also [HS2].
Appendix d contains some justifications of the arguments in section 4.
Appendix e gives an application to magnetic Schrödinger operators. This is a modification of the corresponding arguments in [HS1]. Since the symmetry (0.8) was never assumed in [HS1], we have to add an extra symmetry to the magnetic and electric fields and check that this leads to (0.8).

Some of the results of the present paper have been announced in [HS3]

We would finally like to thank A.Grigis for a large number of interesting and stimulating dicussions with the authors during the preparation of this long work.

1. Various operators with commutation relations.

In this section, we introduce various auxiliary operators, that will play an important role later, and we study their commutation relations. Some of this was already done in [HS1,2], but we think it is convenient to have all at the same place. Let $h \in \mathbb{R}$, $h \neq 0$. All operators will act on $L^2(\mathbb{R})$. The first operators we study are natural h-quantizations of the translations: $(x,\xi) \to (x,\xi)+2\pi\alpha$, $\alpha \in \mathbb{Z}^2$ (and sometimes even in $\mathbb{R}^2$). Let $\tau=\tau_{2\pi}$ denote the operator of translation by 2π; $\tau u(x)=u(x-2\pi)$, let τ^* denote the operator of multiplication by $e^{2\pi ix/h}$, and put

(1.1) $T_\alpha=\tau^{\alpha_1}\circ\tau^{*\alpha_2}$, for $\alpha \in \mathbb{Z}^2$.

Sometimes, we shall also use that there is a natural extension of the definition of T_α to the case when $\alpha \in \mathbb{R}^2$, since there is an obvious definition of real powers of τ and τ^*. In a way, the crucial phenomenon that causes all the interesting phenomena for Harper's operator, is that τ and τ^* do not commute in general. In fact, $\tau\circ\tau^*=\exp(-i(2\pi)^2/h)\tau^*\circ\tau$. Let $h' \in \mathbb{R}$, be a number such that,

(1.2) $2\pi/h=k+h'/2\pi$,

for some integer k. Then $\tau\circ\tau^*=\exp(-ih')\,\tau^*\circ\tau$, and more generally we get,

(1.3) $T_\alpha T_\beta= e^{ih'\alpha_2\beta_1}\, T_{\alpha+\beta}$,

(1.4) $T_\alpha T_\beta= e^{ih'\sigma(\alpha,\beta)}T_\beta T_\alpha$,

for $\alpha,\beta \in \mathbb{Z}^2$, where σ denotes the standard symplectic form on $\mathbb{R}^2$, given by $\sigma(x,\xi;y,\eta)=\xi y-x\eta$. (1.3) and (1.4) remain valid for $\alpha,\beta \in \mathbb{R}^2$, provided that we replace h' by $(2\pi)^2/h$. The next operator we introduce is the unitary Fourier transform $\mathcal{F}_h=\mathcal{F}$, which can be viewed as an h-quantization of the map κ: $\mathbb{R}^2 \to \mathbb{R}^2$, given by,

(1.5) $\kappa(x,\xi)=(\xi,-x)$.

Later on we shall also need the maps κ_t: $\mathbb{R}^2 \to \mathbb{R}^2$, defined as rotation by the angle t, so that $\kappa=\kappa_{-\pi/2}$. By definition,

(1.6) $\mathcal{F}u(\xi)=(2\pi h)^{-1/2}\int e^{-ix\xi/h}\, u(x)\, dx$, $h>0$,

and as already mentionned, $\mathcal{F}$: $L^2(\mathbb{R}) \to L^2(\mathbb{R})$ is unitary, $\mathcal{F}^{-1}=\mathcal{F}^*$, where $\mathcal{F}^*$ denotes the complex adjoint in the L^2-sense. It is easy to check, (starting with the operators τ and τ^*,) that,

(1.7) $\mathcal{F}\circ T_\alpha=e^{-ih'\alpha_1\alpha_2}\, T_{\kappa(\alpha)}\circ\mathcal{F}$,

for $\alpha \in \mathbb{Z}^2$, and the same relation with h' replaced by $(2\pi)^2/h$, when $\alpha \in \mathbb{R}^2$.

It will be useful in the following, to recall the relation between $\mathcal{F}$ and the unitary group associated to the harmonic oscillator, $R=\frac{1}{2}(h^2D^2+x^2-h)$, $D=i^{-1}(d/dx)$. Let $U_t=e^{itR/h}$. Since $u_0=e^{-x^2/2h}$ is in the kernel of R, we have $U_tu_0=u_0$. On the other hand, we know (Leray [L]), that $U_{-\pi/2}$ and $\mathcal{F}$ are metaplectic (unitary) operators with the same canonical transformation; $\kappa_{-\pi/2}$, and hence that $U_{-\pi/2}=\omega\mathcal{F}$, for some ω of modulus 1. Since

$U_{-\pi/2}u_0=u_0$ and $\mathcal{F}u_0=u_0$, it follows that $\omega=1$;

(1.8) $U_{-\pi/2}=\mathcal{F}=\mathcal{F}_h$, $h>0$.

For later reference, it will also be convenient to know $U_{\pi/4}$ explicitely. Using that the phase $\varphi(x,y)=-x^2/2+(2)^{1/2}xy-y^2/2$ generates the correct canonical transformation, namely $\kappa_{\pi/4}$, we first see that

(1.9) $U_{\pi/4}u(x)=Ch^{-1/2}\int e^{i\varphi(x,y)/h}u(y)\,dy$,

for some constant C. In order to determine this constant, we again use that $U_{\pi/4}u_0=u_0$ and that the corresponding integral in (1.8) can be evaluated exactly, to obtain that, $C=2^{-1/4}\pi^{-1/2}e^{i\pi/8}$.

Let Γ denote the antilinear operator of complex conjugation; $\Gamma u=\bar{u}$. To this operator we associate the transformation of phase-space; $(x,\xi)\to(x,-\xi)$. We notice that this transformation is anti-canonical, in the sense that the Jacobian is equal to -1. As a general rule we shall associate anti-canonical transformations to antilinear operators. The present association is justified by the following fact. Let $A=A(x,hD)$ be the h-Weyl quantization of the symbol $A(x,\xi)\in S^0(\mathbb{R}^2)=\{a\in C^\infty(\mathbb{R}^2);$ for all $j,k\in\mathbb{N}$, there exists $C_{j,k}$ such that $|\partial_x^j\partial_\xi^k a(x,\xi)|\le C_{j,k}$, for all $(x,\xi)\in\mathbb{R}^2\}$, defined by,

(1.10) $Au(x)=(2\pi h)^{-1}\iint e^{i(x-y)\theta/h}A((x+y)/2,\theta)u(y)\,dyd\theta$, $h>0$,

so that A is $\mathcal{O}(1)$ as a bounded operator on $L^2(\mathbb{R})$ by standard theorems. (See for instance [HS1] for a non standard proof.) When we want to distinguish more clearly between the operator and its symbol, we shall sometimes write $Op_h(A)$ or simply $Op(A)$ for the operator. The justification of the association is then given by,

(1.11) $\Gamma Op(A)=Op(B)\Gamma$,

where $B(x,\xi)=\bar{A}(x,-\xi)$. Notice that $\Gamma^2=id$, so that (1.11) may take many equivalent forms.

Thus in a way, Γ is a natural quantization of reflection in the x-axis. We shall also need quantizations of other reflections, such as in some of the lines $l_\theta=\{t(\cos\theta,\sin\theta);\ t\in\mathbb{R}\}$. To define such reflections, it is natural to rotate l_θ by $\kappa_{-\theta}$ to the x-axis (l_0), then reflect in the x-axis, and finally rotate back again. More precisely, the quantization of reflection γ_θ in l_θ, is defined by,

(1.12) $\Gamma_\theta= U_\theta\Gamma U_{-\theta}$,

so that $\Gamma_0=\Gamma$. This corresponds to $\gamma_\theta=\kappa_\theta\gamma_0\kappa_{-\theta}$. From the definition of U_θ, it is easy to verify that,

(1.13) $\Gamma U_\theta=U_{-\theta}\Gamma$ (,and classically, $\gamma_0\kappa_\theta=\kappa_{-\theta}\gamma_0$),

which gives rise to several obvious equivalent forms of (1.12). We get the general relations,

(1.14) $\Gamma_b U_a=U_\alpha\Gamma_\beta$, $\gamma_b\kappa_a=\kappa_\alpha\gamma_\beta$, if $2b-a=\alpha+2\beta$.

Now it is a general fact, that

(1.15) $U_{-\theta}Op(a)U_\theta=Op(a\circ\kappa_\theta)$, and combining this with (1.11), we get,

(1.16) $A\circ\Gamma_\theta=\Gamma_\theta\circ B$, if on the symbol level, $A\circ\gamma_\theta=\bar{B}$.

Again, we notice that $\Gamma_\theta{}^2=\mathrm{id}$, so (1.16) can take many equivalent forms. Later, we shall make a particular use of $V=_{\mathrm{def.}}\Gamma_{\pi/4}$, whose associated transformation is $\gamma_{\pi/4}=_{\mathrm{def.}}\delta:(x,\xi)\rightarrow(\xi,x)$. Using that $V=U_{\pi/2}\circ\Gamma=\Gamma\circ U_{-\pi/2}$, it is easy to check, using (1.7),(1.6) that,

(1.17) $VT_\alpha=e^{ih'\alpha_1\alpha_2}T_{\delta(\alpha)}V$.

We finally notice that,

(1.18) $(\Gamma_\theta u|v)=\overline{(u|\Gamma_\theta v)}$, $u,v\in L^2$.

2. Formal study of the iteration process.

In the process that we are going to study, there will appear infinitely often one of the following two types of operators, namely:

Type 1. $P=P(x,hD_x)$ is a scalar pseudodifferential operator with the following properties:

(2.1) P commutes with the operators T_α, $\alpha\in\mathbb{Z}^2$.

This means that the Weyl symbol; $P(x,\xi)$ of P is 2π-periodic both in x and in ξ.

(2.2) $P\mathfrak{F}=\mathfrak{F}P^*$, $\mathfrak{F}P=P^*\mathfrak{F}$.

This means for the Weyl symbol, that $P\circ\kappa=\overline{P}$, where $\kappa=\kappa_{-\pi/2}$ is the map of rotation of $\mathbb{R}^2$ by the angle $-\pi/2$, i.e. the canonical transformation associated to $\mathfrak{F}$. We also assume a symmetry under reflection in $x=\xi$:

(2.3) $PV=VP$.

Here we recall that V quantizes the reflection map $(x,\xi)\rightarrow(\xi,x)$, and is an anti-linear operator. At the very first iteration step, P will be selfadjoint (for real values of the spectral parameter μ), but that property will be lost later on in the iteration and will be replaced by:

(2.4) P_1^*P and PP_2^* are selfadjoint,

where P_1 and P_2 are bounded pseudo-differential operators, satisfying (2.1) and :

(2.5) $P_1\mathfrak{F}=\mathfrak{F}P_2^*$, $P_2\mathfrak{F}=\mathfrak{F}P_1^*$.

(2.6) $P_jV=VP_j$, $j=1,2$.

In the beginning of the iteration, we take P_1 and P_2 to be the identity operator, and later, they will be elliptic near the characteristics of P.

Type 2. P is a 2×2 system of pseudodifferential operators : $L^2(\mathbb{R},\mathbb{C}^2_{\{1,3\}})\rightarrow L^2(\mathbb{R},\mathbb{C}^2_{\{2,4\}})$, where the subscripts indicate the coordinate indices that we use for the two different copies of $\mathbb{C}^2$. We shall always think of these indices, as defined modulo 4, and in order to simplify the notation we write from now on $\mathbb{C}^2_{\mathrm{odd}}=\mathbb{C}^2_{\{1,3\}}$, $\mathbb{C}^2_{\mathrm{even}}=\mathbb{C}^2_{\{2,4\}}$. Let T denote one of the two operators $T_1:\mathbb{C}^2_{\mathrm{odd}}\rightarrow\mathbb{C}^2_{\mathrm{even}}$ or $T_2:\mathbb{C}^2_{\mathrm{even}}\rightarrow\mathbb{C}^2_{\mathrm{odd}}$, where each one is defined by the general formula $(Tx)_j=x_{j-1}$. Then the operator T^2, defined either as T_1T_2 or T_2T_1 is given in the standard basis of $\mathbb{C}^2$ by the matrix : $\begin{pmatrix}0&1\\1&0\end{pmatrix}$. We then assume that P satisfies (2.1) and

(2.7) $P\mathfrak{F}T=\mathfrak{F}TP^*$, $P^*\mathfrak{F}T=\mathfrak{F}TP$,

(2.8) $VP=PVT^2$.

Again, we assume that there are operators P_1 and P_2 satisfying (2.1), such that (2.4) holds and we replace (2.5) and (2.6) by:

(2.9) $P_1\mathfrak{F}T=\mathfrak{F}TP_2^*$, $P_2\mathfrak{F}T=\mathfrak{F}TP_1^*$,

(2.10) $VP_j=P_jVT^2$, $j=1,2$.

It will be important later on,to observe that these two relations and (2.4) remain valid, if we replace P_1, P_2, by P_1+tP, P_2+tP for any real number t.

In this section, we study formally certain Grushin type problems for the operator P either of type 1 or of type 2, and we shall see that we obtain in each case a new "renormalized" operator which is again of one of the two types. For each type there are two types of Grushin problems; either a "potential well" problem or a "branching" problem. As we shall see, the potential well problem leads always to a renormalized operator of type 1, (regardless of whether we start from an operator of type 1 or of type 2), and a branching problem always leads to a renormalized operator of type two. Hence there are 4 cases to check. The richest one is the branching problem for an operator of type 2 (leading to a new operator of type 2), and the other 3 cases are simple adaptations of this one. For an explanation of the terminology "branching" and "potential well", we refer to later sections dealing with more substantial (microlocal) analysis.

Case 1. The branching problem for a type 2 operator. Let $f_{0,1}\in L^2(\mathbb{R};\mathbb{C}^2_{\{1,3\}})$ be a function, whose form will be specified later, and in some sense (that we do not need to specify yet) concentrated near the segment $](0,\pi),(\pi,0)[$. We put $f_{0,j}=(\mathcal{T}T)^{1-j}f_{0,1}$, for $j\in\mathbb{Z}_4=\mathbb{Z}/(4)$, and then $f_{\alpha,j}=T_\alpha f_{0,j}$, for $\alpha\in\mathbb{Z}^2$. We put,

(2.11) $R_+u(\alpha,j)=(u|f_{\alpha,j})$, $u\in L^2$, $(\alpha,j)\in\mathbb{Z}^2\times\{1,3\}$,

From the choice of the $f_{\alpha,j}$, it will be clear that R_+: $L^2(\mathbb{R};\mathbb{C}^2_{odd})\rightarrow l^2(\mathbb{Z}^2\times\{1,3\})$ is a bounded operator. Similarly, we define a bounded operator

R_-: $l^2(\mathbb{Z}^2\times\{2,4\})\rightarrow L^2(\mathbb{R};\mathbb{C}^2_{even})$, by

$$(2.12)\quad R_-u^- = \sum_{\mathbb{Z}^2\times\{2,4\}} u^-_{\alpha,j}\, f_{\alpha,j}\,.$$

From the definition of the $f_{\alpha,j}$ and from the commutation relations, that we have studied earlier, it follows that,

(2.13) $T_\gamma f_{\alpha,j}=\exp(i\gamma_2\alpha_1 h')\, f_{\alpha+\gamma,j}$.

We conclude that,

(2.14) $T_\gamma R_-=R_-\mathcal{T}_\gamma$, $\mathcal{T}_\gamma R_+=R_+T_\gamma$,

where $\mathcal{T}_\gamma$: $l^2(\mathbb{Z}^2\times\{1,3\}(\{2,4\}))\rightarrow l^2(\mathbb{Z}^2\times\{1,3\}(\{2,4\}))$ is defined by,

(2.15) $\mathcal{T}_\gamma w(\alpha,j)=\exp(ih'\gamma_2(\alpha_1-\gamma_1))\, w(\alpha-\gamma,j)$.

Using this together with (2.1), we obtain,

$$(2.16)\qquad \mathcal{P}\begin{pmatrix} T_\gamma & 0 \\ 0 & \mathcal{T}_\gamma \end{pmatrix} = \begin{pmatrix} T_\gamma & 0 \\ 0 & \mathcal{T}_\gamma \end{pmatrix}\mathcal{P},$$

where,

$$(2.17) \qquad \mathcal{P} = \begin{pmatrix} P & R_- \\ R_+ & 0 \end{pmatrix} .$$

Throughout this section, we shall assume that $\mathcal{P}$ is bijective $L^2\times l^2 \to L^2\times l^2$, and we denote the inverse by,

$$(2.18) \qquad \mathcal{E} = \begin{pmatrix} E & E_+ \\ E_- & E_{-+} \end{pmatrix}.$$

Then, (2.16) remains valid, if we replace $\mathcal{P}$ by $\mathcal{E}$. As a general fact, we notice that if E_{-+} is bijective with a bounded inverse, then $P: L^2 \to L^2$ has the same property, and the inverse of P is given by $E-E_+(E_{-+})^{-1}E_-$. Conversely, if P is bijective with a bounded inverse, then the same holds for E_{-+}, and the inverse is given by $-R_+P^{-1}R_-$. In particular, we know that 0 belongs to the spectrum of P if it belongs to the spectrum of E_{-+}.

The fact that $\mathcal{E}$ satisfies (2.16) implies in particular that

$$(2.19) \qquad \mathcal{T}_\gamma E_{-+} = E_{-+}\mathcal{T}_\gamma,\ \gamma\in\mathbb{Z}^2.$$

Now E_{-+} has a matrix $E(\alpha,j;\beta,k)$, $\alpha,\beta\in\mathbb{Z}^2, j=2,4, k=1,3$, and if we identify $l^2(\mathbb{Z}^2\times\{1,3\}(\{2,4\}))\cong l^2(\mathbb{Z}^2;\mathbb{C}^2_{odd(even)})$, then we can also think of the matrix of E_{-+} as an infinite matrix of 2×2 blocks ; $E_{-+}(\alpha,\beta)= (E_{-+}(\alpha,j;\beta,k))_{j=2,4,k=1,3}$. Analyzing the relations (2.19), we then get the equivalent statement,

$$(2.20) \qquad E_{-+}(\alpha,\beta)=\exp(ih'\beta_2(\alpha_1-\beta_1))\, f(\alpha-\beta) ,$$

where f is a matrix valued function on $\mathbb{Z}^2$. We will always be in the situation when f is exponentially decreasing; $\| f(\alpha)\| \le C\exp(-|\alpha|/C)$, for some $C>0$, so we assume this property from now on. Noticing as in [HS1],section 6, that E_{-+} is a convolution in the variables α_1 and that after a suitable conjugation, E_{-+} may be viewed as a convolution in the variables α_2, we can show that $0\in \mathrm{Spec}(E_{-+})$ iff $0\in\mathrm{Spec}(Q)$, where Q: $L^2(\mathbb{R};\mathbb{C}^2_{odd})\to L^2(\mathbb{R};\mathbb{C}^2_{even})$, is the h'-Weyl quantization of the symbol,

$$(2.21) \qquad Q(x,\xi)=\Sigma\, f(\alpha)\exp(-i\alpha_1\alpha_2h'/2)\, e^{-i\langle \delta(\alpha),(x,\xi)\rangle},$$

where $\delta(x,\xi)=(\xi,x)$. Here we assume that $0<h'<2\pi$. (When $h'=0$, E_{-+} is a convolution operator and the condition that 0 belongs to the spectrum is equivalent to : $0\in Q(\mathbb{R}^2)$.) In view of the exponential decrease of f, we see that Q is a well defined analytic matrix-valued function, 2π-periodic both in x and in ξ, so Q satisfies (2.1), if the T_α (as always on the renormalized operator level) are defined with h replaced by h'. What we shall verify (in each of the four cases), is that Q inherits all the invariance properties of P. Before that, it will be convenient to make some general remarks about the renormalization map $:E_{-+}\to Q$, namely that this map respects composition, and passage to the adjoint.

Proposition 2.1. Let $e:\mathbb{Z}^2 \to \mathrm{Mat}(n\times 1)$, and $f:\mathbb{Z}^2 \to \mathrm{Mat}(1\times m)$ be exponentially decreasing functions, so that we can define the composition $E\circ F$, of the two matrices,

(2.22) $E(\alpha,\gamma)=\exp(ih'\gamma_2(\alpha_1-\gamma_1))\, e(\alpha-\gamma)$,

(2.23) $F(\gamma,\beta)=\exp(ih'\beta_2(\gamma_1-\beta_1))\, f(\gamma-\beta)$.

Then,

(2.24) $(E\circ F)(\alpha,\beta)= \exp(ih'\beta_2(\alpha_1-\beta_1))(e*f)(\alpha-\beta)$,

where by definition,

(2.25) $e*f(\alpha-\beta)= \Sigma_\gamma \exp(ih'(\gamma_2-\beta_2)(\alpha_1-\gamma_1))\, e(\alpha-\gamma)f(\gamma-\beta)$.

Moreover, if we denote by Q_e, Q_f, Q_{e*f}, the renormalized symbols, defined as in (2.21), then for the corresponding h'-quantizations, we have,

(2.26) $Q_e\circ Q_f=Q_{e*f}$.

Proof. The proof is by straight forward calculation: We have,

$$E\circ F(\alpha,\beta)= \Sigma_\gamma \exp(ih'(\gamma_2(\alpha_1-\gamma_1)+\beta_2(\gamma_1-\beta_1))\, e(\alpha-\gamma)f(\gamma-\beta),$$

and in order to obtain (2.24),(2.25), it is enough to check that,

$$\gamma_2(\alpha_1-\gamma_1)+\beta_2(\gamma_1-\beta_1)-\beta_2(\alpha_1-\beta_1)=(\gamma_2-\beta_2)(\alpha_1-\gamma_1).$$

If we compose the corresponding h'-Weyl quantizations, we find that

$$Q_e\circ Q_f= \Sigma\Sigma\, e(\alpha)\, f(\beta) \exp(-i(\alpha_1\alpha_2+\beta_1\beta_2)h'/2)\, A_{\alpha,\beta},$$

$$A_{\alpha,\beta}= e^{-i\alpha_2 x/2}\circ e^{-i\alpha_1 h'D}\circ e^{-i(\alpha_2+\beta_2)x/2}\circ e^{-i\beta_1 h'D}\circ e^{-i\beta_2 x/2}.$$

Using the relations:

$$e^{-i\alpha_1 h'D}\circ e^{-i\beta_2 x/2} = e^{i\beta_2\alpha_1 h'/2}\circ e^{-i\beta_2 x/2}\circ e^{-i\alpha_1 h'D},$$
$$e^{-i\alpha_2 x/2}\circ e^{-i\beta_1 h'D}=e^{-i\alpha_2\beta_1 h'/2}\circ e^{-i\beta_1 h'D}\circ e^{-i\alpha_2 x/2},$$

we see that,

$$A_{\alpha,\beta}=$$
$$e^{i(\beta_2\alpha_1-\alpha_2\beta_1)h'/2}\, e^{-i(\alpha_2+\beta_2)x/2}\, e^{-i(\alpha_1+\beta_1)h'D}\, e^{-i(\alpha_2+\beta_2)x/2} =$$
$$e^{i(\beta_2\alpha_1-\alpha_2\beta_1)h'/2}\, \mathrm{Op}_{h'}(e^{-i((\alpha_2+\beta_2)x+(\alpha_1+\beta_1)\xi}),$$

where $\mathrm{Op}_{h'}(\ldots)$ denotes the h'-Weyl quantization of ... . Hence,

$$Q_e\circ Q_f= \Sigma_\gamma\, g(\gamma)\, e^{-i\gamma_1\gamma_2 h'/2}\, \mathrm{Op}_{h'}(e^{-\gamma_2 x-\gamma_1\xi}),$$

where,

$$g(\gamma)= e^{i\gamma_1\gamma_2 h'/2}\, \Sigma_{\alpha+\beta=\gamma} e(\alpha)f(\beta)e^{-i(\alpha_1\alpha_2+\beta_1\beta_2-\beta_2\alpha_1+\alpha_2\beta_1)h'/2}.$$

Rewriting (2.25) on the form,

$$e*f(\gamma)= \Sigma_{\alpha+\beta=\gamma}\, e^{ih'\alpha_1\beta_2}\, e(\alpha)f(\beta),$$

we see that $g=e*f$, which gives (2.26). ■

For the passage to the adjoint, we have,

Proposition 2.2. Let $F(\alpha,\beta)= e^{ih'\beta_2(\alpha_1-\beta_1)}f(\alpha-\beta)$, where f is an exponentially decreasing function with values in the space of $n\times m$

matrices. If F^* denotes the complex adjoint of F in the sense of infinite matrices , then,

(2.27) $F^*(\alpha,\beta)=e^{ih'\beta_2(\alpha_1-\beta_1)}g(\alpha-\beta)$, $g(\gamma)=e^{ih'\gamma_1\gamma_2}f(-\gamma)^*$,

where $f(\alpha)^*$ denotes the complex adjoint of $f(\alpha)$ for each fixed α. If Q_f and Q_g denote the corresponding renormalized symbols, then we have

(2.28) $Q_g(x,\xi)=Q_f(x,\xi)^*$,

and hence the corresponding relation for the h'-quantizations,

(2.29) $Q_g=Q_f^*$.

In other words, the map $F\longrightarrow Q_f$ respects passage to the complex adjoints.

The proof of this result is just a routine calculation, starting from the fact that $F^*(\alpha,\beta)=(F(\beta,\alpha))^*$, and we omit the details.

We now continue the study of the case 1, and we shall next look at the Fourier invariance. From the commutation relations between the T_γ and $\mathcal{F}$, and the definition of $f_{\alpha,j}$, we obtain,

(2.30) $(\mathcal{F}T)f_{\alpha,j}=e^{-ih'\alpha_1\alpha_2}f_{\kappa(\alpha),j-1}$,

where we recall that $\kappa=\kappa_{-\pi/2}$, denotes rotation by the angle $\pi/2$. From this and the definition of $R_\pm$, it is easy to show that

(2.31) $R_+\mathcal{F}T=\mathcal{G}R_-^*$ (implying $\mathcal{F}TR_-=R_+^*\mathcal{G}$),

(2.32) $\mathcal{F}TR_+^*=R_-\mathcal{G}$(implying $\mathcal{G}R_+=R_-^*\mathcal{F}T$).

Here $\mathcal{G}$denotes (the only possible) one of the two operators :
$l^2(\mathbb{Z}^2\times\{1,3\})\longrightarrow l^2(\mathbb{Z}^2\times\{2,4\})$, $l^2(\mathbb{Z}^2\times\{2,4\})\longrightarrow l^2(\mathbb{Z}^2\times\{1,3\})$,given in both cases by,

(2.33) $\mathcal{G}w(\alpha,j)=e^{ih'\alpha_1\alpha_2}w(\kappa^{-1}(\alpha),j+1)$.

Combining (2.7) with (2.31),(2.32), we get,

$$\begin{pmatrix}\mathcal{F}T & 0\\ 0 & \mathcal{G}\end{pmatrix}\mathcal{P}=\mathcal{P}^*\begin{pmatrix}\mathcal{F}T & 0\\ 0 & \mathcal{G}\end{pmatrix},\ \mathcal{P}\begin{pmatrix}\mathcal{F}T & 0\\ 0 & \mathcal{G}\end{pmatrix}=\begin{pmatrix}\mathcal{F}T & 0\\ 0 & \mathcal{G}\end{pmatrix}\mathcal{P}^*\quad ,$$

which implies the corresponding relations for $\mathcal{E}$:

$$(2.34)\quad \begin{pmatrix}\mathcal{F}T & 0\\ 0 & \mathcal{G}\end{pmatrix}\mathcal{E}=\mathcal{E}^*\begin{pmatrix}\mathcal{F}T & 0\\ 0 & \mathcal{G}\end{pmatrix},\quad \mathcal{E}\begin{pmatrix}\mathcal{F}T & 0\\ 0 & \mathcal{G}\end{pmatrix}=\begin{pmatrix}\mathcal{F}T & 0\\ 0 & \mathcal{G}\end{pmatrix}\mathcal{E}^*\quad .$$

In particular, we obtain,

(2.35) $\mathcal{G}E_{-+}=E_{-+}^*\mathcal{G}$, $\quad\mathcal{G}E_{-+}^*=E_{-+}\mathcal{G}$.

More explicitly, the last relation means that,

(2.35) $E(\alpha,\beta)=e^{ih'(\alpha_1\alpha_2-\beta_1\beta_2)}\,T\circ E(\kappa(\beta),\kappa(\alpha))^*\circ T^{-1}$,

or in view of (2.20),

(2.36) $f(\kappa(\alpha))=T\circ f(\alpha)^*\circ T^{-1}$.

Using the relations, ${}^t\kappa=-\kappa$, ${}^t\delta=\delta$, $\kappa\delta=-\delta\kappa$, we get for the corresponding renormalized operator,

$$Q\circ\kappa(x,\xi)= \Sigma\, f(\alpha)\, e^{-i\alpha_1\alpha_2 h'/2}\, e^{-i\langle\delta(\alpha),\kappa(x,\xi)\rangle} =$$
$$\Sigma\, f(\alpha)\, e^{-i\alpha_1\alpha_2 h'/2}\, e^{-i\langle\delta\kappa(\alpha),(x,\xi)\rangle} =$$
$$\Sigma\, f(\kappa^{-1}(\alpha))\, e^{i\alpha_1\alpha_2 h'/2}\, e^{-i\langle\delta(\alpha),(x,\xi)\rangle} =$$
$$\Sigma\, f(\kappa(\alpha))\, e^{i\alpha_1\alpha_2 h'/2}\, e^{i\langle\delta(\alpha),(x,\xi)\rangle} =$$
$$T(\,\Sigma\, f(\alpha)\, e^{-i\alpha_1\alpha_2 h'/2}\, e^{-i\langle\delta(\alpha),(x,\xi)\rangle})^* T^{-1} = TQ^*T^{-1},$$

(2.37) $Q\circ\kappa= TQ^*T^{-1}$.

For the h'-quantizations, this means that $\mathcal{F}^{-1}Q\mathcal{F}= TQ^*T^{-1}$, or rather,

(2.38) $Q(\mathcal{F}T)=(\mathcal{F}T)Q^*$.

This is precisely the relation (2.7) for the renormalized operator,(with the only difference, that now $\mathcal{F}$ denotes the h'-quantization).

We shall next study the invariance under reflection in the line $x=\xi$. In view of (2.8), we shall assume that it is possible (subject to verification later), to choose $f_{0,1}$ such that,

(2.39) $VT^2 f_{0,1}=f_{0,1}$.

Using the various commutation relations between the operators $\mathcal{F}$, T_α, and V, it is easy to check first that $Vf_{0,2}=f_{0,4}$, $VT^2f_{0,3}=f_{0,3}$ and then more generally that,

(2.40) $VT^2 f_{\alpha,j}=e^{ih'\alpha_1\alpha_2}\, f_{v(\alpha,j)}$, for j odd,

$Vf_{\alpha,j}=e^{ih'\alpha_1\alpha_2}\, f_{v(\alpha,j)}$, for j even,

where $v(\alpha,j)=(\delta(\alpha),2-j)$. From this it follows that,

(2.41) $VR_-=R_-V'$, $R_+VT^2=V'R_+$, where the antilinear operator V' is defined by,

(2.42) $V'w(\alpha,j)= e^{ih'\alpha_1\alpha_2}\bar{w}(v(\alpha,j))$.

We notice that $V'^2=\mathrm{id}$. Combining (2.41),(2.42),(2.8), we get,

$$\begin{pmatrix} V & 0\\ 0 & V'\end{pmatrix}\mathcal{P} = \mathcal{P}\begin{pmatrix} VT^2 & 0\\ 0 & V'\end{pmatrix},$$

which implies,

(2.43) $$\mathcal{E}\begin{pmatrix} V & 0\\ 0 & V'\end{pmatrix} = \begin{pmatrix} VT^2 & 0\\ 0 & V'\end{pmatrix}\mathcal{E} \quad .$$

In particular, we have,

(2.44) $V'E_{-+} = E_{-+}V'$,

or more explicitely,

(2.45) $E(v(\alpha,j),(v(\beta,k))= e^{ih'(\alpha_1\alpha_2-\beta_1\beta_2)}\,\bar{E}(\alpha,j;\beta,k)$.

For the corresponding matrix $f(\alpha)=(f(\alpha;j,k))$, we then get,

(2.46) $f(\delta(\alpha);2-j,2-k)= e^{ih'\alpha_1\alpha_2}\,\bar{f}(\alpha;j,k)$,

which implies for the corresponding renormalized symbol, that

(2.47) $Q(-\xi,-x;2-j,2-k)= \bar{Q}(x,\xi;j,k)$.

This does not use quite the reflection that we want, but iterating (2.37), we get $Q\circ\kappa^2= T^2QT^{-2}$, and this relation can be written explicitely as, $Q(-x,-\xi;j,k)=Q(x,\xi;j+2,k+2)$, and combining this with (2.46), we get

$Q(\xi,x;4-j,4-k)=\bar{Q}(x,\xi;j,k)$, which can be written in matrix form as,
(2.48) $\bar{Q}(\xi,x)=Q\circ T^2$.
Here the left hand side is the symbol of the operator $V\circ Q\circ V$, where we now use the h'-Weyl quantization, so on the renormalized operator-level, (2.48) can be written as:
(2.49) $VQ=QVT^2$.
This is the same relation (for the h'-quantizations) as (2.8).

To finish completely the case 1, we have to find natural operators Q_1 and Q_2, satisfying the obvious analogues of (2.4),(2.9),(2.10). Under the hypotheses above, we notice that the adjoint of $\mathcal{P}$ gives the Grushin problem,
(2.50) $P^*u+R_+^*u_+=v, \quad R_-^*u=v_-$,
which is well posed with inverse given by $\mathcal{E}^*$;
(2.51) $u=E^*v+E_-^*v_-$
$u_+=E_+^*v+E_{-+}^*v_-$.
Let us take $v=0$ in these relations, and multiply the first equation of (2.50), by P_2. Using that $P_2P^*=PP_2^*$, we get $PP_2^*u=-P_2R_+^*u_+$. We can view this as the first equation of a Grushin problem for P_2^*u, and hence,
(2.52) $P_2^*u=-EP_2R_+^*u_++E_+R_+P_2^*u$,
$0=-E_-P_2R_+^*u_++E_{-+}R_+P_2^*u$.
Substituting (2.51) with $v=0$ into the last equation of (2.52), we obtain, $E_-P_2R_+^*E_{-+}^*v_-= E_{-+}R_+P_2^*E_-^*v_-$, for all v_-, and this means that $E_{-+}(E_-P_2R_+^*)^*$ is self-adjoint. If we replace $(\mathcal{P},\mathcal{E},P_2)$ in these arguments by $(\mathcal{P}^*,\mathcal{E}^*,P_1^*)$, we also get that $E_{-+}^*(E_+^*P_1^*R_-)^*$ is self-adjoint, or equivalently, that $(R_-^*P_1E_+)^*E_{-+}$ is self-adjoint. Using the various translation-invariance properties, it is easy to check that $E_-P_2R_+^*$ and $R_-^*P_1E_+$ commute with $\mathcal{T}_\gamma$, so we can define the corresponding renormalized operators (by the same procedure as $E_{-+}\rightarrow Q$): Q_2, and Q_1. From the propositions 2.1 and 2.2, it is then clear that Q_1^*Q and QQ_2^* are self-adjoint and that Q_1, Q_2 commute with the operators T_γ (where we now use the h'-quantizations). Moreover, if we repeat the arguments leading to the other invariance properties for Q, we obtain the natural analogues of (2.9) and (2.10), namely that,
(2.53) $Q_1\mathcal{F}T=\mathcal{F}TQ_2^*$, $Q_2\mathcal{F}T=\mathcal{F}TQ_1^*$,
(2.54) $VQ_j=Q_jVT^2$, $j=1,2$.
In fact, with $A_1=R_-^*P_1E_+$, $A_2=E_-P_2R_+^*$, we first find $A_1\mathcal{G}=\mathcal{G}A_2^*$, $\mathcal{G}A_1=A_2^*\mathcal{G}$. In general, if M and M′ commute with the $\mathcal{T}_\gamma$ and $M\mathcal{G}=\mathcal{G}M'$, then for the corresponding renormalized operators, we find,

$Q_m(\mathcal{T}T)^{-1}=(\mathcal{T}T)^{-1}Q_{m'}$, and this implies (2.53). To see that we have (2.54), we first notice that $(V'u|v)_{l^2}=\overline{(u|V'v)}_{l^2}$, and that the analogous relation for V has been obtained in section 1. The commutation relations for $R_\pm$ and V, VT^2, V', then give, $R_-^*V=V'R_-^*$, $VT^2R_+^*=R_+^*V'$. Using this and (2.43), we see that A_j commutes with V', j=1,2. From this we get (2.54) by the argument leading to (2.49).
This completes the study of the case 1.

<u>Case 2</u>. <u>The potential-well problem for a 2×2 system</u>. Assume as in the preceding case, that P is a type 2-operator. There are two subcases, corresponding to whether the potential wells are associated naturally to $2\pi\mathbb{Z}^2$ or to $2\pi\mathbb{Z}^2+(\pi,\pi)$. Let us first reduce completely the second case to the first one. Given P, P_j, we put $P'=S^{-1}PS$, $P_j'=S^{-1}P_jS$, where $S=T_{(\frac{1}{2},\frac{1}{2})}$.Using the commutation relations between T_γ, $\mathcal{T}$, V, it is easy to check that (P',P'_1,P'_2) satisfies the relations (2.1),(2.4),(2.7)-(2.10). Since S is a natural quantization of the translation by (π,π), we get a complete reduction to the subcase n°1. To describe the kind of Grushin-problem that we shall study in this case, let $f_{0,1}$ be an L^2-function suitably concentrated to the component of the characteristic set of P in $\mathbb{R}^2$, which is "close to" (0,0) (in a sense that will be made precise later). As we shall see later, it will be possible to choose $f_{0,1}$ in such a way that,

(2.55) $(\mathcal{T}T)^2f_{0,1}=\pm f_{0,1}$,

which we here take as an assumption. We now let the index j vary in $\mathbb{Z}_2=\mathbb{Z}/(2)$, and put $f_{0,2}=(\mathcal{T}T')^{-1}f_{0,1}$, where for convenience T' is defined as T when the +sign is valid in (2.55) and defined as iT in the other case. Then in general, $f_{0,j}=(\mathcal{T}T')^{k-j}f_{0,k}$, and since $T^4=\mathrm{id}$, we also have $(T')^4=\mathrm{id}$. Following closely the case 1, we define $f_{\alpha,j}=T_\alpha f_{0,j}$. and the operators $R_+:L^2(\mathbb{R};\mathbb{C}^2)\to l^2(\mathbb{Z}^2)$, $R_-: l^2(\mathbb{Z}^2)\to L^2(\mathbb{R};\mathbb{C}^2)$, by:

(2.56) $R_+u(\alpha)=(u|f_{\alpha,1})$,

(2.57) $R_-w=\Sigma w(\alpha)f_{\alpha,2}$.

As in the earlier case, we shall assume that the corresponding operator $\mathcal{P}$ is bijective and has bounded inverse $\mathcal{E}=\begin{pmatrix}E & E_+\\ E_- & E_{-+}\end{pmatrix}$.

Repeating the argument of the case 1, we then obtain,

(2.58) $T_\gamma R_-=R_-\mathcal{T}_\gamma$, $R_+T_\gamma=\mathcal{T}_\gamma R_+$,

(2.59) $\mathcal{T}_\gamma E_{-+}=E_{-+}\mathcal{T}_\gamma$,

where $\mathcal{T}_\gamma : l^2(\mathbb{Z}^2)\to l^2(\mathbb{Z}^2)$ is given by the obvious modification of (2.15), and where we now notice that E_{-+} is given by an infinite matrix

$(E_{-+}(\alpha,\beta))$, where $E_{-+}(\alpha,\beta)$ are scalars rather than 2×2 blocks. We still have (2.20) and we can define the renormalized operator as before, having the crucial property that 0 belongs to the spectrum of P iff it belongs to the spectrum of Q. Now Q is a scalar pseudodifferential operator instead of a 2×2 system. Repeating the earlier arguments, we also find,

(2.60) $\mathcal{F}T'R_- = R_+^*\mathcal{G},\ R_-^*\mathcal{F}T' = \mathcal{G}R_+$,

(2.61) $\mathcal{G}E_{-+} = E_{-+}^*\mathcal{G}$,

where $\mathcal{G}: l^2(\mathbb{Z}^2) \to l^2(\mathbb{Z}^2)$ is defined by the obvious simplification of (2.33). As before this leads to the relation $Q\circ\kappa = \bar{Q}$ for the renormalized symbol, and hence for the corresponding h'-quantizations,

(2.62) $Q\mathcal{F} = \mathcal{F}Q^*$.

In order to treat the invariance under reflection in the line $x=\xi$, we add the assumption that

(2.63) $VT^2 f_{0,1} = f_{0,1}$.

Then using the commutation relations and (2.63),(2.55), we get after a simple computation,

(2.64) $Vf_{0,2} = f_{0,2}$.

Using again the commutation relations, we obtain from (2.63),(2.64),

(2.65) $VT^2 f_{\alpha,1} = e^{ih'\alpha_1\alpha_2} f_{\delta(\alpha),1}$, $Vf_{\alpha,2} = e^{ih'\alpha_1\alpha_2} f_{\delta(\alpha),2}$,

and as in the case 1, this leads to

(2.66) $VR_- = R_-V',\ R_+VT^2 = V'R_+$,

where the antilinear operator V' is defined by,

(2.67) $V'w(\alpha) = e^{ih'\alpha_1\alpha_2}\bar{w}(\delta(\alpha))$.

Continuing as in case 1, we then obtain (2.44), which now means more explicitely that,

(2.68) $E(\delta(\alpha),\delta(\beta)) = e^{ih'(\alpha_1\alpha_2-\beta_1\beta_2)}\,\bar{E}(\alpha,\beta)$.

For the corresponding matrix $f(\alpha)$, we then get,

(2.69) $f(\delta(\alpha)) = e^{ih'\alpha_1\alpha_2}\,\bar{f}(\alpha)$,

which implies for the corresponding renormalized symbol, that

(2.70) $Q(-\xi,-x) = \bar{Q}(x,\xi)$.

This does not use quite the reflection that we want, but iterating the relation $Q\circ\kappa = \bar{Q}$, we get $Q\circ\kappa^2 = Q$, and this relation can be written explicitely as,

(2.71) $Q(-x,-\xi) = Q(x,\xi)$,

and combining this with (2.70), we get,

(2.72) $\bar{Q}(\xi,x) = Q(x,\xi)$.

Here the left hand side is the symbol of the operator $V\circ Q\circ V$, where we now use the h'-Weyl quantization, so on the renormalized operator-level, (2.72) can be written as:

(2.73) $QV = VQ$.

This is the same relation (for the h'-quantizations) as (2.3).

To finish completely the case 2, we have to find natural operators Q_1 and Q_2, satisfying the natural analogues of (2.4)-(2.6). The first part of the corresponding argument in case 1 is completely general, and as there, we see that $E_{-+}(E_-P_2R_+^*)^*$ and $(R_-^*P_1E_+)^*E_{-+}$ are self-adjoint. Again $E_-P_2R^*$ and $R_-^*P_1E_+$ commute with $\mathcal{T}_\gamma$, so we can define the corresponding renormalized operators Q_2 and Q_1. It is then clear that Q_1^*Q and QQ_2^* are self-adjoint and that Q_1, Q_2 commute with the operators T_γ (where we now use the h'-quantizations). Moreover, if we repeat the arguments leading to the other invariance properties for Q, we obtain the natural analogues of (2.5) and (2.6), namely that,

(2.74) $Q_1\mathcal{T}=\mathcal{T}Q_2^*$, $Q_2\mathcal{T}=\mathcal{T}Q_1^*$,

(2.75) $VQ_j=Q_jV$, $j=1,2$.

This completes the study of the case 2.

Case 3. The branching problem for a type 1-operator. This case is a complete repetition of the case 1, with the obvious simplification that all the invariance relations before renormalization, do not involve any operators "T". As in case 1, the renormalized operator is of type 2.

Case 4. The potential well problem for a type 1-operator. This case is a complete repetition of the case 2, with the obvious simplification that all the invariance relations before renormalization, do not involve any operators "T". As in case 2, the renormalized operator is of type 1.

3. The potential-well problem for a type 1 operator.

Let P be of type 1, as defined in the section on the formal study of the iteration process, and let P_1,P_2 be corresponding operators such that P_1^*P, PP_2^* are self adjoint et c . We shall assume that P, P_j depend analytically on a real parameter $\mu\in]-4,4[$, and more precisely that

(3.1) $P=Op_h(P(\mu,x,\xi))$, $P_j=Op_h(P_j(\mu,x,\xi))$,

where P,P_j are holomorphic in $D(\varepsilon)=\{\mu;|\mu|<4\}\times\{(x,\xi)\in\mathbb{C}^2;|\mathrm{Im}(x,\xi)|<1/\varepsilon\}$, for some sufficiently small $\varepsilon>0$. In this domain and with the same ε, we also assume that,

(3.2) $|P-(\cos\xi+\cos x-\mu)|\le\varepsilon$, $|P_j-1|\le\varepsilon$, for $(\mu,x,\xi)\in D(\varepsilon)$.

We formalize this by introducing the following definition.

Definition 3.1. We say that (P,P_1,P_2) (or simply P) is of strong type 1, if it is of type 1 in the sense of section 2, and has the properties above. If (P,P_1,P_2) is of strong type 1, we let $\varepsilon(P)$ (or rather $\varepsilon(P,P_1,P_2)$) be the infimum of all $\varepsilon>0$, such that (3.2) holds.

In this definition, $P(x,\xi)$ and $P_j(x,\xi)$ may depend on many more parameters than μ . Then $\varepsilon(P)$ will depend on these parameters also. The goal of this section (and the following ones) is to obtain results which are uniform with respect to all these additional parameters, valid when $\varepsilon(P)$ is sufficiently small. Among the additional parameters we may also have h, but the most important h-dependence, and the only one that we explicitly take into account in our notation, is the one resulting from the fact that we work with h-quantizations.

Definition 3.2. If the operator P is bounded in some Hilbert space, and depend on a parameter $\mu\in M$, then we define μ-Sp(P) as the set of μ in M, such that $0\in \mathrm{Sp}(P)$.

In this section we shall assume that (P,P_1,P_2) is of strong type 1, and that $0<h\le2\pi$. Notice that if $\varepsilon(P)$ is sufficiently small, the operators P_j are bijective on L^2, so 0 belongs to the spectrum of P iff it belongs to the spectrum of $H_1=P_1^*P$ (or equivalently, if it belongs to the spectrum of $H_2=PP_2^*$). Recall that H_1 and H_2 are selfadjoint, and intertwined by $\mathcal{F}$. In particular,

(3.3) $H_1\mathcal{F}^2=\mathcal{F}^2H_1$,

where we recall that $\mathcal{F}^2$ is a quantization of κ^2 which is the reflection in (0,0). Also recall that,

(3.4) $VH_1=H_1V$,

where V is a quantization of reflection in the line $x=\xi$. V is antilinear and $V^2=I$. From (3.2) it follows that H_j also satisfy (3.2) with a new $\varepsilon>0$, in a region defined as before with this new ε. The new ε tends to 0, when $\varepsilon(P)$ tends to zero. The Weyl symbol H_1 is real valued for real μ and (3.3) and (3.4) imply that $H_1(\mu,x,\xi)$ is invariant under reflections in (0,0) and in the line

$x=\xi$; $H_1(-x,-\xi)=H_1(x,\xi)$, $H_1(x,\xi)=H_1(\xi,x)=H_1\circ\delta(x,\xi)$. Moreover H_1 is 2π-periodic in x and in ξ, and $H_1(x,\xi)+\mu$ is on the real domain an arbitrarily small perturbation in C^2 of the symbol $\cos\xi+\cos x$. We start by discussing the symbol $H_1+\mu$ on the real domain, following closely the argument of section 7 in [HS1]. Modulo $2\pi\mathbb{Z}^2$ there are precisely 4 critical points of this symbol, namely a non-degenerate maximum close to (0,0), a non-degenerate minimum close to (π,π) and two non-degenerate saddle points close to $(0,\pi)$ and $(\pi,0)$ respectively. From the invariance under the map $(x,\xi)\to(-x,-\xi)$, it follows that the point of maximum is equal to (0,0). Using also the translation invariance, we see that the symbol is invariant under the maps $(x,\xi)\to(2\pi-x,2\pi-\xi)$, $(x,\xi)\to(-x,2\pi-\xi)$, $(x,\xi)\to(2\pi-x,-\xi)$, which are respectively the reflections in (π,π), $(0,\pi)$ and $(\pi,0)$. From this we conclude that the point of minimum is (π,π) and that the two saddle points are $(0,\pi)$ and $(\pi,0)$. The map δ leaves H_1 invariant and exchanges the two saddle points, hence we have the same critical value at the two saddle points. Summing up, we have,

<u>Proposition 3.3.</u> For real μ's, the symbol H_1 of $H_1=P_1^*P$ is invariant under reflection in any of the points $(k\pi,l\pi)$, $k,l\in\mathbb{Z}$, under translation by 2π in x or in ξ, and under reflection in the line $x=\xi$ (and more generally in any of the lines $\xi=\pm x+k2\pi$; $k\in\mathbb{Z}$). Modulo $2\pi\mathbb{Z}^2$ there are precisely 4 critical points, all non-degenerate; a maximum at (0,0) with $H_1(0,0)+\mu=2+\mathcal{O}(\varepsilon)$, a minimum at (π,π) with $H_1(\pi,\pi)+\mu=-2+\mathcal{O}(\varepsilon)$, and saddle points at $(0,\pi)$ and $(\pi,0)$ with $H_1(0,\pi)+\mu=H_1(\pi,0)+\mu=\mathcal{O}(\varepsilon)$.

Let $c(\mu)=H_1(\pi,0)+\mu=\mathcal{O}(\varepsilon)$. The discussion below will be uniformly valid, provided that $\varepsilon(P)>0$ is sufficiently small, that $\mu\geq c(\mu)+\varepsilon_0$ for some arbitrary but fixed $\varepsilon_0\geq 0$, and finally that $0<h\leq h_0(\varepsilon_0)$ for some $h_0(\varepsilon_0)>0$. The assumption that $\mu\geq c(\mu)+\varepsilon_0$ can also be written; $H_1(\pi,0)\leq-\varepsilon_0$, so the real characteristic set of H_1 avoids the saddle points $(k\pi,l\pi)$, $k+l\in 2\mathbb{Z}+1$. For $\lambda>c(\mu)$, let $U_\alpha=U_\alpha(\lambda)$ be the component of $(H_1+\mu-\lambda)^{-1}(0)$ in $\mathbb{R}^2$, naturally associated to $\alpha\in\mathbb{Z}^2$, and let $\tilde{H}_1+\mu$ be a modification of $H_1+\mu$ obtained by filling all the potential wells $U_\alpha(c(\mu)+\varepsilon_0)$ for $\alpha\neq(0,0)$ but leaving H_1 unchanged near U_0 and in the compact domain to which U_0 is the boundary. For the moment, we can do this by standard C^∞-theory as in section 2 of [HS1]. We know that the eigenvalues of $\tilde{H}_1+\mu$ in the interval $[c(\mu)+\varepsilon_0,\infty[$ are simple, bounded from above by $H_1(0,0)+\mu+\mathcal{O}(h)$, and mutually separated by a distance $\geq h/C_0$ for some constant C_0. Moreover, if we use (3.2) and ordinary perturbation theory, we see that each such eigenvalue $\lambda(\mu)$ will depend analytically on μ with a holomorphic extension (as well as for the eigenfunction) to the strip $|\mathrm{Im}\mu|<h/C_0$. Moreover, in this strip, we have $d\lambda(\mu)/d\mu=\mathcal{O}(\varepsilon)$, so if μ is close to $\lambda(\mu)$ to start with, then there will be a

simple real zero $\mu=\mu_1$ of $\mu-\lambda(\mu)$, and from now on we restrict the attention to an interval $]\mu_1-h/C_0,\mu_1+h/C_0[$, where we may choose C_0 as large as we like. It is then clear that $\tilde{H}_1=\tilde{H}_1+\mu-\mu$ has an inverse of norm $\mathcal{O}(1/|\mu-\mu_1|)$, for μ complex with $0<|\mu-\mu_1|<h/C_0$. If we do the "filling" with some care, using quantizations of the modifying operators based on superpositions of Gaussians, then we also know that the spectrum of $H_1+\mu$ in $]\lambda(\mu)-h/C_0,\lambda(\mu)+h/C_0[$ for C_0 sufficiently large, is concentrated to an exponentially short intervall centered at $\lambda(\mu)$, when μ is real. (More details about gaussian quantizations will be given in section 4.) Let π_μ be the associated spectral projection. Writing π_μ as a contour integral with the resolvent of $H_1+\mu$, we see that the definition of π_μ extends holomorphically to $|\mathrm{Im}\mu|<h/C_0$, and we have $\| d\pi_\mu/d\mu \| =\mathcal{O}(\varepsilon/h)$.

We shall now fix $\mu=\mu_1$ for a while, and we start by recalling the definition of certain distances as in [HS1]. With $U_\alpha=U_\alpha(\mu_1)$, we now put $U_j=\pi_x(U_{(j,k)})=$ the x-space projection of $U_{(j,k)}$. Then for x real between U_0 and U_1, the complex zeros of $H_1(x,\xi)$ are of the form $\xi(x)+2\pi k\pm i\Phi_1'(x)$, $k\in\mathbb{Z}$, where $\xi(x)$ is real and $(x,\xi(x))$ tends to a point of $U_{(0,0)}$, when x tends to the right boundary of U_0, and $\Phi_1'(x)\geq 0$. We then extend Φ_1 to a C^1 function on $\mathbb{R}$, such that $\Phi_1'(x)$ is 2π periodic and vanishes on U_0. Finally, we put $D_1(x,y)=|\Phi_1(x)-\Phi_1(y)|$. Since $H_1(-x,-\xi)=H_1(x,\xi)$, we obtain that Φ_1 is odd, and hence that $D_1(-x,-y)=D_1(x,y)$. Let us define D_2 associated to H_2 in the same way. As we shall see (or rather recall) solutions to the equation $H_ju=0$ will often have weighted estimates in terms of D_j. Since $\mathcal{F}$ exchanges H_1 and H_2, the Fourier transforms of solutions of $H_1u=0$ will satisfy estimates in terms of D_2 and vice versa. Since our weighted estimates will allow for losses $=\mathcal{O}(e^{\varepsilon/h})$ for every $\varepsilon>0$, we can often replace D_1 and D_2 by one single distance function by means of the following observation: We have that $H_1(x,\xi)=0$ implies $P(x,\xi)$, $\bar{P}(\bar{x},\bar{\xi})$, $H_2(x,\xi)=\mathcal{O}(h)$, simply because on the operator level, $P=(P_1^*)^{-1}H_1$ and so on, and the composition formula for the full symbols is reduced to multiplication, up to an error $\mathcal{O}(h)$. If we add the assumption that $d_\xi H_1(x,\xi)\neq 0$, then there is a unique $\xi'=\xi+\mathcal{O}(h)$ such that $H_2(x,\xi')=0$. Completing this argument with a simple discussion of what happens in a small neighborhood of the x-projected wells, we see that $\Phi_2(x)-\Phi_1(x)=o(1)(1+|x|)$ uniformly, when $h\to 0$. Thanks to this we will usually be able to replace D_1 or D_2 by $D=_{\mathrm{def.}}\frac{1}{2}(D_1+D_2)$.

We choose $\varphi=\varphi_0$ (where $0=(0,0)$) the same way as the function g_0 in [HS1],section 4, to be a suitable Gaussian of L^2-norm $\mathcal{O}(h^{-N_0})$, such that

(3.5) $f_0=\pi_{\mu_1}\varphi_0$ has norm 1,

(3.6) f_0, $\mathcal{F}f_0=\hat{\mathcal{O}}(e^{-f/h})$,

in the sense that for every $\varepsilon>0$, there is an $h_\varepsilon>0$, such that $\|e^{(1-\varepsilon)f/h}f_0\|=\mathcal{O}(e^{\varepsilon/h})$ for $0<h<h_\varepsilon$, and similarly for $\mathcal{F}f_0$. Here $f(x)=\min_{k\in\mathbb{Z}} D(U_k,x)+v_0|k|$, where $0<v_0\leq S_0=_{def.}D(0,2\pi)$. Without changing the growth estimates for f_0, we shall next modify this function in order to fulfill some symmetry relations, required in our formal iteration scheme. Microlocally, we know that f_0 is concentrated to a neighborhood of U_0 and by construction, we also know that H_1f_0 is exponentially small in L^2, and in particular, it makes sense to say that f_0 belongs to the kernel of H_1 microlocally near U_0. (See appendix a.) Now this kernel is one-dimensional (in a sense that we leave to the reader to define, possibly after reading [HS1]), and invariant under the operator $\mathcal{F}^2$. Hence, $f_0=\pm\mathcal{F}^2f_0$ up to an exponentially small error in L^2, and without changing any of the essential properties of f_0, we can then replace this function by $\frac{1}{2}(f_0\pm\mathcal{F}^2f_0)=\pi_{\mu_1}(\frac{1}{2}(\varphi_0\pm\mathcal{F}^2\varphi_0))$. Since $\mathcal{F}^4=I$, the new function f_0 will satisfy:

(3.7) $\mathcal{F}^2f_0=\pm f_0$.

We can also add symmetry under V. Microlocally near $U_{0,0}$, we have $Vf_0\sim\omega f_0$ for some complex ω of modulus 1, since V commutes with H_1. Writing this as $V\omega^{\frac{1}{2}}f_0\sim\omega^{\frac{1}{2}}f_0$, (since V is anti linear,) we get approximate V-invariance by replacing f_0 by $\omega^{\frac{1}{2}}f_0$, and this does not destroy any of the earlier properties. If f_0 denotes this new function, we make a last modification by replacing f_0 by $\frac{1}{2}(f_0+Vf_0)$. This will not alter (3.7), since V and $\mathcal{F}^2$ commute, and we have gained the property,

(3.8) $Vf_0=f_0$.

Recall now that the subscript 0 stands for 0 in $\mathbb{Z}^2$, and in order to use the notation of section 2, we put $f_{0,1}=f_{(0,0),1}=f_0$, and from this function, we define $f_{\alpha,j}$ for $\alpha\in\mathbb{Z}^2$, $j=1,2$, as in case 4 of that section, (which is an obvious modification of the case 2 in the same section,) as well as the corresponding operators R_+ and R_-. The corresponding Grushin problem is then,

(3.9) $Pu+R_-u^-=v,\ \ R_+u=v_+$,

and we shall first study this problem microlocally near the potential well U_0 for $\mu=\mu_1$. We then get the problem,

(3.10) $$Pu+u^-(0)f_{0,2}=v,$$ $$(u|f_{0,1})=v_+(0).$$

Since $\mu=\mu_1$, we know that $f_{0,1}$ is microlocally in Ker P_1^*P=Ker P, and

$f_{0,2} \in \mathrm{Ker}\, PP_2^* = (\mathrm{Im}\, PP_2^*)^{\perp} = (\mathrm{Im}\, P)^{\perp}$. Here we proceed formally and we leave to the reader to give a precise sense to the arguments by introducing the convenient error-estimates and pseudodifferential cutoffs. (See also appendix d, where a more complicated Grushin problem is treated.) In order to solve (3.10), we write $u=u'+xf_{0,1}$, $u' \in (f_{0,1})^{\perp}$, $v=v'+yf_{0,2}$, $v' \in (f_{0,2})^{\perp} = \mathrm{Im}(P)$, $x=(u|f_{0,1})$, $y=(v|f_{0,2})$. We notice that $P:(f_{0,1})^{\perp} \to (f_{0,2})^{\perp}$ is microlocally a bijection, and that the inverse E_0 is of $\mathcal{L}(L^2,L^2)$-norm $\mathcal{O}(h^{-1})$. We then get the microlocal solution,

(3.11) $$u=E_0 v'+v_+(0)f_{0,1}, \quad u^-(0)=(v|f_{0,2}),$$

so the inverse of,

$$\mathcal{P}=\begin{pmatrix} P & R_- \\ R_+ & 0 \end{pmatrix}$$

is microlocally of the form,

(3.12) $$\mathcal{E}_0 = \begin{pmatrix} E_0 & E_{0,+} \\ E_{0,-} & E_{0,-+} \end{pmatrix} = \begin{pmatrix} E_0 & R_+^* \\ R^*_- & 0 \end{pmatrix} = \begin{pmatrix} \mathcal{O}(h^{-1}) & \mathcal{O}(1) \\ \mathcal{O}(1) & 0 \end{pmatrix},$$

and by standard microlocal cutoffs et c, (or by microlocal apriori estimates as in appendix d,) we see that $\mathcal{P}$ also globally has an inverse of the form (3.12) up to an error $\mathcal{O}(h^{\infty})$, and where the description of E_0 is now only microlocal near the wells U_α. By ordinary perturbation theory, we see that if we let μ vary in the complex disc $D(\mu_1,\varepsilon_0 h)$ of center μ_1 and radius $\varepsilon_0 h$, then $\mathcal{P}$ defined above is still bijective, with an inverse,

(3.13) $$\mathcal{E}=\begin{pmatrix} E & E_+ \\ E_- & E_{-+} \end{pmatrix} = \mathcal{O}(h^{-1}) .$$

Representing $\mathcal{E}$ by a perturbation series in the usual way, we see that,

(3.14) $$E_{-+} = ((P-P_0)f_{0,1}|f_{0,2})I+\mathcal{O}(v^2/h^3)+\mathcal{O}(h^{\infty}),$$

where we put $v=\mu-\mu_1$, $P=P(\mu,x,hD)$, $P_0=P(\mu_1,x,hD)$. Let us study the first term to the right in (3.14). In view of (3.2), we have :

(3.15) $$f_{0,1}=P_1 f_{0,1}+\mathcal{O}(\varepsilon).$$

Using the selfadjointness of $P_1^*P_0$ and $P_0P_2^*$ and the fact that $P_0 f_{0,1}=\mathcal{O}(h^{\infty})$ (and even exponentially small if we are careful), we see that $P_0P_2^*P_1 f_{0,1}=\mathcal{O}(h^{\infty})$, so $P_1 f_{0,1}=\omega f_{0,2}+\mathcal{O}(h^{\infty})$. From this and (3.15), we conclude that,

(3.16) $$f_{0,1}=\omega f_{0,2}+\mathcal{O}(\varepsilon)+\mathcal{O}(h^{\infty}).$$

Since $f_{0,2}=(i)\mathcal{F} f_{0,1}$, and $((i)\mathcal{F})^2 f_{0,1}=f_{0,1}$, we can permute $f_{0,1}$ and $f_{0,2}$ in (3.16). It follows that we may assume that $\omega^2=1$, so finally,

(3.17) $$f_{0,1}=\pm f_{0,2}+\mathcal{O}(\varepsilon)+\mathcal{O}(h^{\infty}).$$

(3.2) and Cauchy's inequality give,

(3.18) $$P-P_0=-vI+\mathcal{O}(\varepsilon v),$$

and from this and (3.17) we obtain:

(3.19) $((P-P_0)f_{0,1}|f_{0,2})=\mp v(1+\mathcal{O}(\varepsilon)+\mathcal{O}(h^\infty))$.

Substitution in (3.14) gives,

(3.20) $E_{-+}=(\mp v+\mathcal{O}(\varepsilon v))I+\mathcal{O}(v^2/h^3)+\mathcal{O}(h^\infty)$.

This gives interesting information about the behaviour of the diagonal part of E_{-+} when $v<<h^3$, and this region contains the possible values of v in $[-h/C_0,h/C_0]$ such that 0 is in the spectrum of P. We restrict the attention to this region from now on. After a translation by $\mathcal{O}(h^\infty)$ (and even by an exponentially small quantity), we may assume also that the diagonal part of E_{-+} vanishes for v=0. This is possible since the diagonal part of E_{-+} is constant by translation invariance, real by reflection invariance with respect to $x=\xi$ (,see section 2, case 4, which is the analogue of case 2), and given by (3.20). We recall (3.6) and the subsequent definition of f and of v_0. Thus, if z(v)I is the diagonal part of E_{-+}, we know that z(v) is real for real v, (since z(v)=f(0) and we have (2.69),) and we may assume that

(3.21) $z(v)=(\mp v+\mathcal{O}(\varepsilon v))I+\mathcal{O}(v^2/h^3)$.

The next problem is to study the off diagonal part of E_{-+}. Using the techniques of section 7 in [HS1], it is clear that we have the following result: Let F be a real C^∞-function with all derivatives in L^∞, such that

(3.22) $|F'(x)|\le(|D'(x)|-\delta)_+$,

(3.23) $|F(U_k)-F(U_j)|\le(1-\delta)v_0|k-j|$,

for some $\delta>0$. Then if $F^*((j,k))=F(U_j)$, $\mathcal{P}$ is bijective from $L^2(\mathbb{R};e^{2F(x)/h})\times l^2(e^{2F^*/h})$ onto itself for sufficiently small h, and the inverse $\mathcal{E}$ is $\mathcal{O}(h^{-1})$ in the associated operator-norm. (Here $l^2(e^{2F^*/h})=\{w;\ we^{F^*/h}\in l^2\}$. We also have an analogous result with weighted spaces on the Fourier transform side, and using both results, we obtain,

(3.24) $E_{-+}(\alpha,\beta)=\mathcal{O}(e^{-(1-\delta)v_0|\alpha-\beta|_\infty/h})$,

for every $\delta>0$. Here $|\alpha|_\infty=\max(|\alpha_1|,|\alpha_2|)$. From this it follows that 0 can belong to the spectrum only if $|v|$ is exponentially small.

Recall that $(R_-^*P_1E_+)^*E_{-+}$ and $E_{-+}(E_-P_2R_+^*)^*$ are self adjoint. Put $A_1=R_-^*P_1E_+$, $A_2=E_-P_2R_+^*$. The matrix of A_1 is given by $A_1(\alpha,\beta)=(P_1E_+\delta_\beta|f_{\alpha,2})=\mathcal{O}(e^{-(1-\delta)v_0|\alpha-\beta|/h})$ for all $\delta>0$, and by translation invariance, $A_1(\alpha,\alpha)=A_1(0,0)$. For v=0, we get, $A_1(0,0)=(P_1f_{0,1}|f_{0,2})$ ignoring errors which are $\mathcal{O}(h^\infty)$, so $|A_1(0,0)|\ge$const.>0. The reflection symmetry in $x=\xi$, implies that, $A_j(\delta(\alpha),\delta(\beta))=e^{ih'(\alpha_1\alpha_2-\beta_1\beta_2)}\bar{\bar{A}}_j(\alpha,\beta)$, which implies that $A_j(\alpha,\alpha)$ is real valued. For the corresponding renormalized operators Q_j, it follows that $Q_j=A_j(0,0)+\mathcal{O}(e^{-1/C_0h})$ in a complex strip $|\mathrm{Im}(x,\xi)|\le 1/C_0h$, and replacing Q_j

by $A_j(0,0)^{-1}Q_j$ this will imply after rescaling in the μ-variable, that the part of (3.2) concerning Q_j is satisfied with ε replaced by C_0h.

Following section 4 in [HS1], we shall now improve the estimates on $E_{-+}(\alpha,\beta)$ when $v=0$, so that $z(v)=E_{-+}(\alpha,\alpha)=0$. Define $\delta_\alpha\in l^2(\mathbb{Z}^2)$ by $\delta_\alpha(\beta)=1$ when $\beta=\alpha$ and $=0$ otherwise. Put $g_{\alpha,1}=E_+(\delta_\alpha)$, $g_{\alpha,2}=E_-^*(\delta_\alpha)$, and notice that $g_{\alpha,j}$ and their Fourier transforms satisfy the same growth estimates as $f_{\alpha,j}$. Since $R_+E_+=I$, we get,
$(g_{\alpha,1}|f_{\beta,1})=(E_+\delta_\alpha|f_{\beta,1})=R_+E_+\delta_\alpha(\beta)=\delta_{\alpha,\beta}$ (Kronecker's delta):

$$(3.25)\qquad (g_{\alpha,1}|f_{\beta,1})=\delta_{\alpha,\beta}.$$

Similarly, using that $R_-^*E_-^*=I$:

$$(3.26)\qquad (g_{\alpha,2}|f_{\beta,2})=\delta_{\alpha,\beta}.$$

Applying, $PE_++R_-E_{-+}=0$ to δ_β, we get

$$(3.27)\qquad Pg_{\beta,1}+\Sigma_{\alpha\neq\beta}E_{-+}(\alpha,\beta)f_{\alpha,2}=0,$$

which with (3.26) implies that,

$$(3.28)\qquad (Pg_{\beta,1}|g_{\alpha,2})=-E_{-+}(\alpha,\beta).$$

Taking the adjoint of the identity $E_-P+E_{-+}R_+=0$, and applying it to δ_β we get,

$$(3.29)\qquad P^*g_{\beta,2}+\Sigma_{\alpha\neq\beta}\bar{E}_{-+}(\beta,\alpha)f_{\alpha,1}=0,$$

which also implies (3.28).

We are looking for improved estimates for small values of α and β, so it will be natural to adopt the following terminology; we say that $A(\alpha,\beta)=\hat{\mathcal{O}}(e^{-a(\alpha,\beta)/h})$, where a is a real-valued function, if for every $\delta>0$, and for every bounded set K in $\mathbb{Z}^2$, there is a constant $C>0$, such that $|A(\alpha,\beta)|\le Ce^{(\delta+a(\alpha,\beta))/h}$, we shall also use the analogous terminology for functions of one variable α, and for functions of a real variable x we change the earlier convention, into: $u=\hat{\mathcal{O}}(e^{-f(x)/h})$, if for every $\delta>0$ and every compact set K in $\mathbb{R}$, the L^2-norm of $ue^{f(x)/h}$ on K is bounded by a constant times $e^{\delta/h}$, for sufficiently small h. In all that follows, the bounds near infinity in x or in α,β will be unchanged, but play an important role in the iteration process. Let us assume by induction that,

(H.1) $\quad E_{-+}(\alpha,\beta)=\hat{\mathcal{O}}(e^{-a(|\alpha-\beta|_\infty)/h})$, where $a:\mathbb{N}^*\to\mathbb{R}_+$ satisfies,

(H.2) $\quad 0\le a(k+1)-a(k)\le v_0$,

(H.3) $\quad |a(1)|\le S_0=_{\text{def.}}D(0,2\pi)$.

(H.4) $\quad |a(k)|\ge v_0k$.

The assumption, (H) is satisfied with $a(k)=v_0k$, and we also recall (3.24), which gives additional information about E_{-+} near infinity. Our object is to increase the function a. By translation invariance it is enough to assume (H.1) for $\alpha=0$ or for $\beta=0$, and also to prove (H.1) for a new function a for such values of α,β. Let us first study $Pg_{0,1}$ and $P^*g_{0,2}$ with the help of (3.27) and (3.29). The two cases are completely parallel, so we concentrate on $Pg_{0,1}$.

We recall that $g_{\alpha,j}$, $f_{\alpha,j}= \hat{\mathcal{O}}(e^{-f(x-2\pi\alpha_1)/h})$. It follows from (3.27) and (H.1) that,

$$Pg_{0,1}=\hat{\mathcal{O}}(e^{-(a(1)+f(x))/h})+\Sigma_{\beta_1\neq 0}\hat{\mathcal{O}}(e^{-(a(|\beta_1|)+f(x-2\pi\beta_1))/h}),$$

so

$$Pg_{0,1}=\hat{\mathcal{O}}(e^{-F(x)/h}),$$

where F is the even function given by,

$$F(x)=\min(a(1)+f(x),\min_{\beta_1\neq 0}(a(|\beta_1|)+f(x-2\pi\beta_1)).$$

Thanks to (H.2)-(H.4), we get for $2\pi k\leq |x|\leq 2\pi(k+1)$:

$$F(x)=\min(a(k)+D(2\pi k,|x|),a(k+1)+D(2\pi(k+1),|x|)).$$

Here we use the convention $a(0)=a(1)$. We also have the same estimate for $P^*g_{0,2}$. Now recall (3.28) for $\beta=0$:

(3.30) $\quad E_{-+}(\alpha,0)=-(Pg_{0,1}|g_{\alpha,2})$.

Assume $\alpha_1\neq 0$ and for instance that $\alpha_1>0$. We write,

(3.31) $\quad -E_{-+}(\alpha,0)=$

$$(\chi Pg_{0,1}|g_{\alpha,2})+(g_{0,1}|(1-\chi)P^*g_{\alpha,2})-(g_{0,1}|[P^*,\chi]g_{\alpha,2})=I+II+III.$$

Here $\chi=1_{]-\infty,\lambda]}$ is the characteristic function of $]-\infty,\lambda]$, where λ is to be chosen conveniently in $]0,2\pi\alpha_1[$.

<u>Case 1.</u> $\alpha_1=1$. We choose $\lambda=\pi$, which gives the same estimate for I and II: $I+II=\hat{\mathcal{O}}(1)e^{-(a(1)+v_0)/h}$, where now $\hat{\mathcal{O}}(1)$ simply means $\mathcal{O}(e^{\delta/h})$ for every fixed $\delta>0$. To estimate III, we also follow [HS1], but with the following slight simplification, based on the observation that $[P^*,\chi]=(1-\chi)P^*\chi-\chi P^*(1-\chi)$. This means that we can rely on the boundedness of P in weighted L^2-spaces, developed in [HS1],section7, and we obtain, $[P^*,\chi]g_{\alpha,2}=\hat{\mathcal{O}}(e^{-D(x,2\pi)/h})$ for $x\leq\pi$, and $=\hat{\mathcal{O}}(e^{-D(x,0)/h})$ for $x\geq\pi$. From this we see that $III=\hat{\mathcal{O}}(e^{-S_0/h})$, and hence, $E_{-+}(\alpha,0)=\hat{\mathcal{O}}(1)e^{-\min((a(1)+v_0,S_0)/h}$.

<u>Case 2.</u> α_1 is even ≥ 2. If we choose $\lambda/2\pi=\alpha_1/2$, then we will get the same bound for I and II, but for III we only get $\hat{\mathcal{O}}(e^{-v_0\alpha_1/h})$, which is no improvement compared to the initial function a. Instead, we take $\lambda/2\pi=(\alpha_1+1)/2$, which will produce a worse bound for I and a better bound for II, so it is enough to estimate I and III: We find I (and II) = $\hat{\mathcal{O}}(\exp(-(a(\alpha_1/2)+v_0\alpha_1/2)/h))$, and $III=\hat{\mathcal{O}}(\exp(-(v_0(\alpha_1-1)+S_0)/h))$, so

$$E_{-+}(\alpha,0)=\hat{\mathcal{O}}(1)e^{-\min(S_0+v_0(\alpha_1-1),a(\alpha_1/2)+v_0\alpha_1/2)/h}.$$

<u>Case 3.</u> α_1 is odd ≥ 3. We choose $\lambda/2\pi=\alpha_1/2=[\alpha_1/2]+1/2$. Then

$$I,II=\hat{\mathcal{O}}(1)e^{-(a([\alpha_1/2])+v_0(\alpha_1-[\alpha_1/2]))/h},$$

$$III=\hat{\mathcal{O}}(1)e^{-((\alpha_1-1)v_0+S_0)/h}, \text{ so}$$

$$E_{-+}(\alpha,0)=\hat{\mathcal{O}}(1)e^{-\min(((\alpha_1-1)v_0+S_0),(a([\alpha_1/2])+v_0(\alpha_1-[\alpha_1/2])))/h}.$$

Notice that this estimate also covers the case II.

Working with Fourier transforms, we get the same estimates with α_1 replaced by α_2, so finally,

$$E_{-+}(\alpha,0)= \hat{\mathcal{O}}(1)e^{-b(|\alpha|_\infty)/h},$$

where,

$$b(1)=\min(a(1)+v_0,S_0),$$
$$b(j)=\min(S_0+(j-1)v_0,a([j/2])+v_0(j-[j/2]))/h,\ j\geq 2.$$

As in [HS1], we check that (H) is verified with a replaced by b, and that after a finite number of iterations of this procedure, we get a function a(j) with $a(1)=S_0$, $a(j)>S_0$ for $j>1$. (An infinite iteration leads to $a(j)=S_0+(j-1)v_0$.) In other words, we have achieved that,

(3.32) $$E_{-+}(\alpha,\beta)=\hat{\mathcal{O}}(e^{-S_0/h}) \text{ for } |\alpha-\beta|_\infty=1, \text{ and } =\hat{\mathcal{O}}(e^{-(S_0+\varepsilon_0)/h}) \text{ for } |\alpha-\beta|_\infty\geq 2,$$

for some $\varepsilon_0>0$. Moreover, for $\alpha_1=1$, we have,

(3.33) $$E_{-+}(\alpha,0)=(g_{0,1}|[P^*,\chi]g_{\alpha,2})+\mathcal{O}(1)e^{-(S_0+\varepsilon_0)/h}= -([P,\chi]g_{0,1}|g_{\alpha,2})+\mathcal{O}(1)e^{-(S_0+\varepsilon_0)/h}$$

with $\chi=1_{]-\infty,t]}$, with t close to π. In order to study the last scalar product, we shall slightly modify the function χ (as in section 7 of [HS1]. Let $\pi_s(x)=C_1h^{-\frac{1}{2}}e^{-\varphi_s(x)/h}$, where $\varphi_s(x)=\varphi_0(x-s)$, $\varphi_0(x)=iC_0x^2$, with $C_0>0$ sufficiently large but fixed, and $C_1=C_1(C_0)$ the constant such that $\int\pi_s(x)\,ds=1$. Putting $K=\int\chi(s)\pi_s ds$, it is easy to check, using the growth estimates on $g_{0,1}, g_{\alpha,2}, Pg_{0,1}, P^*g_{\alpha,2}$, that we can replace $[P,\chi]$ in (3.33) by $\psi[P,K]\psi$, provided that C_0 is sufficiently large. Here $\psi\in C_0^\infty$ has its support near π and is equal to 1 near that point. (We now take $t=\pi$.) As in section 7 of [HS1], we obtain on the other hand,

(3.34) $$[P,\pi_s]=h\partial Q_s/\partial s+R_s, \text{ for } |x-s|+|y-s|\leq 1,\ x,y\in\mathbb{C},$$

where,

$$Q_s=Ch^{-3/2}\int e^{i((x-y)\theta+\frac{1}{2}(\varphi_s(x)+\varphi_s(y)))/h}\,Q(\tfrac{1}{2}(x+y),\theta,\tfrac{1}{2}(x+y)-s,h)d\theta,$$
$$R_s=h^{-3/2}\int e^{i((x-y)\theta+\frac{1}{2}(\varphi_s(x)+\varphi_s(y)))/h}\,R(\tfrac{1}{2}(x+y),\theta,\tfrac{1}{2}(x+y)-s,h)d\theta,$$

where Q is a realization of a classical analytic symbol of order 0, defined for x,y in the region in (3.34), and for $|\mathrm{Im}\,\theta|\leq C(\varepsilon)$, where ε is the parameter in (3.2) and $C(\varepsilon)$ tend to ∞ when ε tends to 0. R is holomorphic and $=\mathcal{O}(e^{-\varepsilon_0/h})$, where ε_0 depends on C_0. We have,

(3.35) $$\psi[P,K]\psi=h\psi Q_\pi\psi+\int_{-\infty}^{\pi}\psi R_s\psi ds.$$

In view of the properties of the symbol of R_s, we see that the contributions from the integral in (3.35) to $\psi[P,K]\psi$ (replacing $[P,\chi]$ in (3.33)) is negligible, and we obtain,

(3.36) $$E_{-+}(\alpha,0)=-h(\psi Q_\pi\psi g_{0,1}|g_{\alpha,2})+\mathcal{O}(e^{-(S_0+\varepsilon_0)/h}).$$

Now recall from [HS1] that Q_t is the solution of a division problem, and that the leading part q of the symbol Q, is obtained by:

(3.37) $$P(x,\theta+i\sigma)-P(x,\theta-i\sigma)=C_0\sigma q(x,\theta,\sigma).$$

As in [HS1], section 5, one can show that $g_{0,1}=c_0(h)a(x,h)e^{i\varphi(x)/h}$ on $[x_+ +\delta,\pi+\delta_0]$, $g_{(1,0),2}=c_1(h)b(x,h)e^{i\psi(x)/h}$ on $[\pi-\delta_0,2\pi-x_- -\delta]$, for every $\delta>0$ and for some fixed $\delta_0>0$, where a,b are realizations of elliptic analytic symbols of order 0 , $c_0,c_1,c_0^{-1},c_1^{-1}=\hat{\mathcal{O}}(1)$, $\pi_x(U_{(0,0)})=[x_-,x_+]$, and φ,ψ are the solutions of the eiconal equations, $H_1(x,\varphi')=0, H_1(x,\psi')=0$, satisfying $\mathrm{Im}\varphi'>0$, $\mathrm{Im}\psi'<0$, $\varphi(x)\to 0$, $\varphi'(x)\to\xi_+$, when $x\searrow x_+$, $\psi(x)\to 0$, $\psi'(x)\to\xi_-$, when $x\nearrow 2\pi+x_-$. Here $\xi_\pm$ are the unique values, such that $(x_\pm,\xi_\pm)\in U_{(0,0)}$. By symmetry considerations, we have

(3.38) $\mathrm{Im}(\varphi(x)+\psi(x))=S_0$, $\mathrm{Re}\varphi(x)=\mathrm{Re}\psi(x)+\text{const.}$.

Since, $\partial_\xi H_1(x,\varphi')\neq 0$, we obtain from (3.37) that $q(x,\varphi')\neq 0$, and by analytic stationary phase (see [S1]), we obtain

(3.39) $\psi Q_\pi \psi g_{0,1}=c_2(h)d(x,h)e^{i(\varphi_\pi(x)+\varphi(x))/h}$, near $x=\pi$,

where c_2 has the same properties as c_0 and c_1, and d is a realization of an elliptic analytic symbol of order 0. By the continuity of Q_π in weighted L^2-spaces, we also have

(3.40) $\psi Q_\pi \psi g_{0,1}=\hat{\mathcal{O}}(e^{i(\varphi_s(x)+\varphi(x))/h})$.

Combining this with the WKB-form of $g_{(1,0),2}$, and analytic stationary phase, we get,

(3.41) $(\psi Q_\pi \psi g_{0,1}|g_{(1,0),2})=c_3(h)e^{-S_0/h}$, $c_3,c_3^{-1}=\hat{\mathcal{O}}(1)$.

When $\alpha_2\neq 0$, we use that $g_{\alpha,2}=e^{-ih'\alpha_2}e^{2\pi i\alpha_2 x/2}g_{(1,0),2}$, and by contour deformation, we obtain,

(3.42) $(\psi Q_\pi \psi g_{0,1}|g_{\alpha,2})=\mathcal{O}(e^{-(S_0+\varepsilon_0)/h})$,

for some $\varepsilon_0>0$.

Summing up the discussion so far, we have for $\mu=\mu_1$ in addition to (3.24), that

(3.43) $E_{-+}(\alpha,0)=\mathcal{O}(e^{-(S_0+\varepsilon_0)/h})$, for $|\alpha|>1$,

(3.44) $E_{-+}(\alpha,0)=a(\alpha)e^{-S_0/h}$, for $|\alpha|=1$, where $a(\alpha),1/a(\alpha)=\hat{\mathcal{O}}(1)$.

Here $|\alpha|=|\alpha_1|+|\alpha_2|$, and $\varepsilon_0>0$. The corresponding renormalized operator, Q which is invariant under $\mathfrak{F}^2$ and V, is then of the form

(3.45) $Q=Q_0+R$,

where on the symbol level,

$Q_0=e^{-S_0/h}(a(1,0)e^{-i\xi}+a(-1,0)e^{i\xi}+a(0,1)e^{-ix}+a(-1,0)e^{ix})$, and $R=\mathcal{O}(e^{-(S_0+\varepsilon_0)/h})$ in a complex strip $|\mathrm{Im}(x,\xi)|\leq\varepsilon_0/h$. From the invariance under $\mathfrak{F}^2$, we deduce that $a(-\alpha)=a(\alpha)$, and from the V-invariance, that $a(1,0)=\bar{a}(-1,0)$, so with $a_0=a(1,0)$, we get,

$Q_0=2e^{-S_0/h}(a_0\cos\xi+\bar{a}_0\cos x)$. We now recall that we have already found Q_j, $j=1,2$, such that Q_1^*Q and QQ_2^* are self-adjoint and on the symbol-level,

$Q_j=1+\mathcal{O}(e^{-\varepsilon_0/h})$ for $|\mathrm{Im}(x,\xi)|\le\varepsilon_0/h$. This implies that $\mathrm{Im}(a_0)=\mathcal{O}(e^{-\varepsilon_0/h})$, so after multiplication of Q, by the real non-vanishing factor, $e^{S_0/h}(2\mathrm{Re}a_0)^{-1}$, we are reduced to the case when

(3.46) $\quad Q=\cos\xi+\cos x+\mathcal{O}(e^{-\varepsilon_0/h})$.

So far μ has been fixed$=\mu_1$, and it only remains to study E_{-+} for neighboring values. Combining (3.24) with (3.43),(3.44) and the Cauchy inequalities, we get for $|\mu-\mu_1|\le\varepsilon_0 h$,

(3.47) $\quad E_{-+}(\alpha,0)=\mathcal{O}(e^{-(S_0+\varepsilon_0)/h})+\mathcal{O}(|\mu-\mu_1|/h)\hat{\mathcal{O}}(e^{-V_0|\alpha|_\infty/h})$, $|\alpha|\ge2$,

(3.48) $\quad E_{-+}(\alpha,0)=a(\alpha)e^{-S_0/h}+\mathcal{O}(|\mu-\mu_1|/h)\hat{\mathcal{O}}(e^{-V_0|\alpha|_\infty/h})$, for $|\alpha|=1$.

Also recall that $z(v)=E_{-+}(0,0)$ has a simple zero at $v=0$, i.e. for $\mu=\mu_1$, and that the behaviour near $v=0$ is given by (3.21). From these three facts, we see that $0\notin\mathrm{Spec}(E_{-+})$ if $|\mu-\mu_1|\ge e^{(\delta-S_0)/h}$ and $h\le h(\delta)$, where $\delta>0$ may be arbitrarily small. We then restrict the attention to the values of μ such that $|\mu-\mu_1|<e^{(\delta-S_0)/h}$. As a new rescaled spectral parameter, we take, $\mu'=e^{S_0/h}(2\mathrm{Re}a_0)^{-1}z(v)$. Then we get for the corresponding renormalized operator, (given by (3.46) when $\mu=\mu_1$,)

(3.49) $\quad Q=(\mu'+\cos\xi+\cos x)+\mathcal{O}(e^{-\varepsilon_0/h})$, for $|\mathrm{Im}(x,\xi)|\le\varepsilon_0/h$.

This together with the information already obtained about the Q_j shows that (Q,Q_1,Q_2), is of strong type 1 with $\varepsilon(Q)=h/\varepsilon_0$.

Let us sum up the results of this section in the following

Proposition 3.4. Let (P,P_1,P_2) be of strong type 1 and $0<h\le2\pi$. Then for $0\le\varepsilon(P)\le\varepsilon_0<0$ sufficiently small, the symbol H_1 of P_1^*P has non-degenerate saddle points, all with the same critical value, $c(\mu)-\mu$, where $c(\mu)=\mathcal{O}(\varepsilon(P))$ is holomorphic in $D(0,4)=\{\mu\in\mathbb{C};\ |\mu|<4\}$. There is a unique real value $\mu_0=\mu_0(P)=\mathcal{O}(\varepsilon(P))$, such that $\mu_0-c(\mu_0)=0$. For every $\varepsilon_1>0$, there exists $C_1>0$, such the following holds when $0\le\varepsilon(P)\le\varepsilon_0$, $0<h\le1/C_1$: $\mu-\mathrm{Sp}(P)\subset\bigcup_{-N_-\le j\le N_+}J_j$, where J_j are closed disjoint intervals labelled in increasing order (so that J_{j+1} is the neighbor to the right of J_j), with the following properties:

1^0 If $a=\inf J_{-N_-}$, $b=\sup J_{N_+}$, then $a=-2+\mathcal{O}(\varepsilon(P))+\mathcal{O}(h)$, $b=2+\mathcal{O}(\varepsilon(P))+\mathcal{O}(h)$.

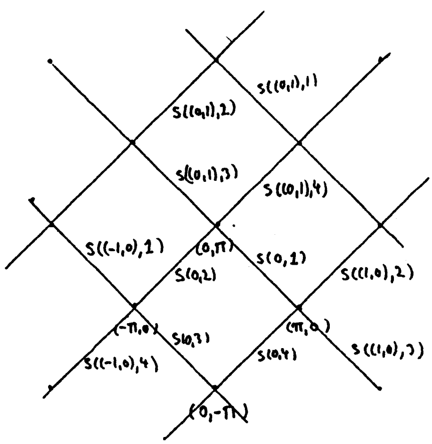
2^0 The distance between J_j and J_{j+1} is of the order of magnitude h.

3^0 $J_0=[\mu_0-\varepsilon_1+\mathcal{O}(h),\mu_0+\varepsilon_1+\mathcal{O}(h)]$, and $\partial J_0\subset\mu-\mathrm{Sp}(P)$.

4^0 For $j \neq 0$, the length of J_j is $e^{-l_j/h}$, where $l_j \sim 1$ (that is of the same order of magnitude as 1), and if $\mu \longrightarrow \kappa_j(\mu) = \mu'$ is a suitable increasing affine map, then $\kappa_j(\mu - Sp(P) \cap J_j) = \mu' - Sp(Q)$, where $Q = Q_j$ is the h'-quantization of a strong type 1 symbol, with $\varepsilon(Q) \longrightarrow 0$ as $h \longrightarrow 0$. Here $2\pi/h \equiv h'/2\pi \mod \mathbb{Z}$, $0 < h' \leq 2\pi$. (When $h' = 0$, the h'-quantization is defined as the multiplication by Q on $L^2(\mathbb{R}^2)$.)

4. The branching problem for a type 1 operator.

Let (P,P_1,P_2) be of strong type 1 with $\varepsilon(P)\leq\varepsilon>0$ sufficiently small , $0<h\leq 2\pi$. We then recall from Proposition 3.3, that the Weyl symbol H_1 of $H_1=P_1^*P$ has saddle points at $(\pi,0),(0,\pi)$ with the same critical value $c(\mu)-\mu$, where $c(\mu)=\mathcal{O}(\varepsilon)$. In section 3, we studied potential well Grushin problems for P when $|c(\mu)-\mu|\geq\varepsilon_0$, for any fixed $\varepsilon_0>0$, provided that $\varepsilon>0$, and $h>0$ are small enough. In this section, we study the case when $|c(\mu)-\mu|\leq\varepsilon_0$, and $\varepsilon_0>0$, $\varepsilon>0$, $h>0$ are sufficiently small. The real characteristics of H_1 are then included in a thin neighborhood of the union of all segments $s(\alpha,j)$, $\alpha\in\mathbb{Z}^2$, $j\in\mathbb{Z}_4$, where $s(0,1)$ is the segment $[(0,\pi),(\pi,0)]$, $s(0,j)=\kappa^{1-j}(s(0,1))$, $s(\alpha,j)=\{2\pi\alpha\}+s(0,j)$, where as before κ denotes rotation around $(0,0)$ by the angle $-\pi/2$.



The new difficulty is then to make a microlocal study of the operators P, H_1 and H_2 near the saddle points $(k\pi,l\pi)$, $k,l\in\mathbb{Z}$, $k+l$ odd. Because of the invariance properties of our operators, it will be essentially enough to concentrate on what happens near the point $(0,\pi)$. Let us start with H_1. We recall from Proposition 3.3 that the symbol H_1 is invariant under reflection in the point $(0,\pi)$. On the operator level, we have, $CH_1=H_1C$, where $C=T_{0,1}C_0$ is a quantization of reflection in $(0,\pi)$, and $C_0=\mathcal{F}^2$ quantizes the reflection in $(0,0)$. The real symbol $H_1+\mu-c(\mu)$ has critical value 0 at $(0,\pi)$ and applying the results of the section b of the appendix, we obtain a real valued analytic symbol $f(\mu,t,h)=f_0+f_1h+..$ of order 0, defined for μ,t near $(0,0)$, with $f_0(\mu,0)=0$, $\partial f_0(\mu,0)/\partial t>0$, and a unitary analytic Fourier integral operator U, whose associated canonical transformation κ_U maps a neighborhood of

$(0,0)$ onto a neighborhood of $(0,\pi)$, such that,

(4.1) $U^{-1}f(\mu,H_1+\mu-c(\mu),h)U=P_0=\frac{1}{2}(xhD+hDx)$.

Moreover, we can arrange so that,

(4.2) $CU=UC_0$,

and so that κ_U maps the part of the negative ξ-axis in the domain of definition, into a small neighborhood of $s(0,1)$. Formally, we observe that if $u=Uv$ and u is a microlocal solution of $H_1u=0$ (which is equivalent to $Pu=0$), then $P_0v=U^{-1}f(\mu,\mu-c(\mu),h)Uv=f(\mu,\mu-c(\mu),h)v$ microlocally near $(0,0)$. The map

(4.3) $\mu\mapsto\mu'=f(\mu,\mu-c(\mu),h)$

is invertible and its inverse is given by,

(4.4) $\mu=g(\mu',h)$,

where g is a classical analytic symbol of order 0.

The relation (4.1) will also allow us to treat the inhomogeneous equation $H_1u=w$ microlocally near $(0,\pi)$. Indeed, let $t\mapsto k(\mu,t,h)$ be the inverse map (where k is an analytic symbol) of the map $s\mapsto f(\mu,s+\mu-c(\mu),h)$. Then from (4.1), we get,

(4.5) $U^{-1}H_1U=k(\mu,P_0,h)$.

Now $k(\mu,\mu',h)=0$, where $\mu'(\mu,h)$ is given by (4.3), and we can factorize: $k(\mu,t,h)=l(\mu,t,h)(t-\mu'(\mu,h))$, where $l(\mu,t,h)$ is an elliptic analytic symbol of order 0. Using this in (4.5), we get,

(4.6) $U^{-1}H_1U=l(\mu,P_0,h)(P_0-\mu')$.

Here $l(\mu,P_0,h)$ is an elliptic operator, so we see that the microlocal inversion problem for H_1 can be reduced to the corresponding one for $P_0-\mu'$.

We now write $U=U_1$ and we shall see how to obtain an operator U_2 which reduces H_2. Let V be the antilinear operator, introduced in section 1, and put $A=V\mathcal{F}$, $B=T_{0,1}\mathcal{F}^2V\mathcal{F}$, which are antilinear realizations of the reflections in the ξ-axis and in the line $\xi=\pi$ respectively. Using the invariance properties for type 1 operators and the appropriate commutation relations of section 1, it is easy to check that

(4.7) $A^2=I,\ B^2=I,\ B=CA=AC$,

(4.8) $AH_1=H_2A,\quad BH_1=H_2B$.

Put $A_0=V=\Gamma\mathcal{F}$, and let $B_0=U_{3\pi/2}\Gamma$ be a quantization of reflection in $x+\xi=0$. Then we have,

(4.9) $A_0^2=I,\ B_0^2=I,\ B_0=C_0A_0$,

(4.10) $[A_0,P_0]=[B_0,P_0]=0$.

Let U_2 be the unitary operator defined by,

(4.11) $U_1A_0=AU_2$.

It is then easy to verify that we also have,

(4.12) $U_1B_0=BU_2$,

(4.13) $CU_2 = U_2C_0$,

(4.14) $U_2^{-1} f(\mu, H_2 + \mu - c(\mu), h) U_2 = P_0$.

The next step will be to study $P_0 - \mu'$, and we start by exhibiting suitable solutions to the homogeneous equation. The equation $(\frac{1}{2}(xD+Dx)-\alpha)u=0$ has four solutions u_+, u_-, v_+, v_-, given by:

(4.15) $u_\pm(x) = H(\pm x)|x|^{-\frac{1}{2}+i\alpha}$, $\mathcal{F}_1 v_\pm(\xi) = H(\pm\xi)|\xi|^{-\frac{1}{2}-i\alpha} = \bar{u}_\pm(\xi)$,

where $\mathcal{F}_1$ denotes the unitary Fourier transform defined in section 1 for h=1. The last relation can also be written $v_\pm = \mathcal{F}_1^{-1}\Gamma u_\pm$. The general solution in $\mathcal{S}'$ is of the form $u = \alpha_+ u_+ + \alpha_- u_- = \beta_+ v_+ + \beta_- v_-$, where the coefficients are related by,

(4.16) $$\begin{pmatrix}\beta_+\\ \beta_-\end{pmatrix} = A_\alpha \begin{pmatrix}\alpha_+\\ \alpha_-\end{pmatrix} ,$$

(4.17) $$A_\alpha = \Gamma(\tfrac{1}{2}+i\alpha)(2\pi)^{-\frac{1}{2}}([\exp(\pi\alpha/2-\pi i/4)]_{1,1}+[\exp(-\pi\alpha/2+\pi i/4)]_{1,2}+ [\exp(-\pi\alpha/2+\pi i/4)]_{2,1}+ [\exp(\pi\alpha/2-\pi i/4)]_{2,2}),$$

where $[a]_{j,k}$ denotes the 2×2 matrix whose only non-vanishing element is a in the j:th row and k:th column. Using the reflection identity $\Gamma(\frac{1}{2}+i\alpha)\Gamma(\frac{1}{2}-i\alpha) = \pi/\mathrm{ch}(\pi\alpha)$, we see that A_α is unitary (when α is real).

In the case of solutions to $(P_0-\mu')u=0$, we can apply the above with $\alpha = \mu'/h$, but it will be convenient to make also two renormalizations. The first one is due to the fact, that we prefer to work with $\mathcal{F} = \mathcal{F}_h$ instead of $\mathcal{F}_1$, and the second renormalization is due the fact that we wish u_+ to enjoy additional approximate symmetry under reflections in the line $x=\xi$, when $\mu'>0$. Assuming $\mu'>0$ (which is no essential restriction, as we shall see below), this leads to the choice,

(4.18) $$u_\pm^0(x,\mu') = e^{i\mu'(1-\log\mu')/2h + i\pi/8} H(\pm x)|x|^{-\frac{1}{2}+i\mu'/h},$$
$$w_\pm^0 = \mathcal{F}^{-1}\Gamma u_\pm^0 = V u_\pm^0.$$

Then using the method of stationary phase, (or (4.20) combined with the complex version of Stirling's formula, that we shall recall later in this section,) we check that,

(4.19) $$Vu_\pm^0 = (1+\mathcal{O}(h))u_\pm^0 + \mathcal{O}(h)u_\mp^0,$$

uniformly for μ' in any compact interval in $]0,\infty[$. From (4.16),(4.17), we obtain that the general (temperate) solution u of $(P_0-\mu')u=0$ can be written $u = \alpha_+ u_+^0 + \alpha_- u_-^0 = \gamma_+ w_+^0 + \gamma_- w_-^0$, where,

(4.20) $$\begin{pmatrix}\gamma_+\\ \gamma_-\end{pmatrix} = (h/\mu')^{i\mu'/h} e^{i(\mu'/h+\pi/4)} A_{\mu'/h} \begin{pmatrix}\alpha_+\\ \alpha_-\end{pmatrix}.$$

As a second preparation concerning $P_0-\mu'$, we study the inhomogeneous equation. We look for an inverse of this operator which propagates singularities in the forward Hamilton field direction. Following a well known procedure, we put,

(4.21) $E=(i/h)\int_0^\infty e^{-it(P_0-\mu')/h}dt=(i/h)\int_0^\infty U_t dt,$

where U_t can be constructed by the standard WKB-procedure for the strictly hyperbolic Cauchy problem. We get,

(4.22) $U_t u(x)=(2\pi h)^{-\frac{1}{2}}\int e^{ie^{-t}x\eta/h}\, e^{t(i\mu'/h-\frac{1}{2})}\, \mathcal{F}u(\eta)\, d\eta = e^{t(i\mu'/h-\frac{1}{2})}u(e^{-t}x).$

When μ' is real, there is no hope that E should be a globally L^2-bounded operator, but if we put $H_s(\mathbb{R})=\{u\in\mathcal{S}';\ \langle hD\rangle^s u\in L^2\}$, $\langle\xi\rangle=(1+\xi^2)^{\frac{1}{2}}$, $L^2_s(\mathbb{R})=\{u\in\mathcal{S}';\langle x\rangle^s u\in L^2\}$, then we shall see that E is bounded: $H_{\frac{1}{2}+\delta}\to L^2_{-\frac{1}{2}-\delta}$, for every $\delta>0$. The $\mathcal{L}(H_{\frac{1}{2}+\delta},L^2_{-\frac{1}{2}-\delta})$ norm of U_t is equal to the $\mathcal{L}(L^2,L^2)$-norm of the integral operator with kernel,

(4.23) $k_t(x,\eta)= (2\pi h)^{-\frac{1}{2}}e^{ie^{-t}x\eta/h}\, e^{-t(\frac{1}{2}+\mathrm{Im}\mu'/h)}\langle x\rangle^{-\frac{1}{2}-\delta}\langle\eta\rangle^{-\frac{1}{2}-\delta},$

and if we estimate this L^2-norm (brutally) with the corresponding Hilbert-Schmidt norm, we get,

(4.24) $\|U_t\|_{\mathcal{L}(H_{1/2+\delta},L^2_{-1/2-\delta})}\le C_\delta h^{-\frac{1}{2}}\, e^{-t(\frac{1}{2}+\mathrm{Im}\mu'/h)}.$

Using this in (4.21), we obtain:

(4.25) $\|E\|_{\mathcal{L}(H_{1/2+\delta},L^2_{-1/2-\delta})}\le C_\delta h^{-3/2}$, if $\mathrm{Im}\mu'/h+\frac{1}{2}\ge\delta>0.$

In appendix d we have defined wavefront sets for microlocally defined operators, and computed WF'(E).

Let $u_{0,1}$ be a WKB solution of $Pu=0$, defined microlocally near the interior, Int(s(0,1)) of the segment s(0,1) defined above. Since P_1^*P is of real principal type in this region, and since we work microlocally, and hence neglecting exponentially small errors, we know that $u_{0,1}$ (in this region) is unique up to a constant factor, and can therefore be expressed as a multiple of $U_1 w^0_-$. Let $\chi_{0,1}\in C_0^\infty(\mathbb{R}^2)$ be equal to 1 near $(0,\pi)=$ the end point of the segment s(0,1), equipped with the orientation of the Hamilton field of $\cos\xi+\cos x$, and with support in the disc of radius $(3/4)(2)^{\frac{1}{2}}\pi$, centered at $(0,\pi)$. (Following a general terminology, $(\pi,0)$ will be called the starting point of the segment s(0,1).) Define $\chi_{0,j}$ and more generally $\chi_{\alpha,j}$, by the relations,

(4.26) $\chi_{0,j}\circ\kappa^{1-j}=\chi_{0,1},\ \chi_{\alpha,j}(x,\xi)=\chi_{0,j}((x,\xi)-2\pi\alpha).$

We also choose $\chi_{0,1}$ invariant under reflection in the point $(0,\pi)$, which implies that $\chi_{\alpha,j}=\chi_{\beta,k}$ if the segments $s(\alpha,j)$ and $s(\beta,k)$ have the same last end point. Taking a suitable quantization of $\chi_{\alpha,j}$ using superpositions of gaussians,(see below for more details,) we can also arrange so that on the operator level,

(4.27) $\chi_{0,j}\circ\mathcal{F}^{1-j}=\mathcal{F}^{1-j}\circ\chi_{0,1},\ \chi_{\alpha,j}\circ T_\alpha=T_\alpha\circ\chi_{0,j},$

and again $\chi_{\alpha,j}=\chi_{\beta,k}$ if the segments $s(\alpha,j)$ and $s(\beta,k)$ have the same end point. The scalar product $(i[H_1,\chi_{0,1}]u_{0,1}|u_{0,1})$ is well defined modulo an exponentially small term, since we see that the essential contribution to this expression must come from the intersection between the cut-off region, $\chi_{0,1}(x,\xi)\notin\{0,1\}$ and the real characteristics of H_1, close to $s(0,1)$. It is clear that this scalar product can be evaluated by stationary phase, and we get $(i[H_1,\chi_{0,1}]u_{0,1}|u_{0,1})>0$. After multiplication of $u_{0,1}$ by an elliptic analytic symbol, depending on h and μ' only, we may assume that,

(4.28) $\quad (i[H_1,\chi_{0,1}]u_{0,1}|u_{0,1})=1.$

We shall then say that $u_{0,1}$ is a normalized microlocal solution of $H_1u=0$, defined near $\mathrm{int}(s(0,1))$. We notice that (4.28) is essentially independent of the choice of $\chi_{0,1}$, because if $\chi\in C_0^\infty$ vanishes near both endpoints of $s(0,1)$, and if χ is a corresponding gaussian quantization of χ, $(i[H_1,\chi]u_{0,1}|u_{0,1})=i(\chi u_{0,1}|H_1u_{0,1})-i(\chi H_1u_{0,1}|u_{0,1})=0$, since $H_1u_{0,1}=0$ microlocally near $\mathrm{int}(s(0,1))$.

We now put $f_{0,1}=i[H_1,\chi_{0,1}]u_{0,1}$, which makes sense (modulo an exponentially small uncertainty) as a globally defined function on $\mathbb{R}$, exponentially decreasing outside the intervall $[\delta,\pi-\delta]$, for some $\delta>0$, and generate $f_{\alpha,j}$ and $u_{\alpha,j}$ as in section 2,

(4.29) $\quad f_{0,j}=\mathfrak{F}^{1-j}f_{0,1},\ f_{\alpha,j}=T_\alpha f_{0,j},$

(4.30) $\quad u_{0,j}=\mathfrak{F}^{1-j}u_{0,1},\ u_{\alpha,j}=T_\alpha u_{0,j}\ ,$

Here $u_{\alpha,j}$ is defined microlocally near $\mathrm{int}(s(\alpha,j))$. Using the appropriate invariance and commutation relations of sections 1,2, it is easy to see that $u_{\alpha,j}$ is a normalized microlocal solution of $H_1u=0$ if j is odd and of $H_2u=0$, if j is even. (More precisely, for j even we have $(i[H_2,\chi_{\alpha,j}]u_{\alpha,j}|u_{\alpha,j})=1$, and for j odd we have the same relation with H_2 replaced by H_1.) Moreover, we have,

(4.31) $f_{\alpha,j}=i[H_1,\chi_{\alpha,j}]u_{\alpha,j}\ ,\ f_{\alpha,j}=i[H_2,\chi_{\alpha,j}]u_{\alpha,j}\ ,$

for j odd and even, respectively.

With these functions $f_{\alpha,j}$, we define R_+,R_- as in the case 3 of section 2. Microlocally, $f_{\alpha,j}$ is non-orthogonal to $\mathrm{Ker}\,P=\mathrm{Ker}\,H_1$, when j is odd, and non-orthogonal to $\mathrm{Ker}\,P^*=\mathrm{Ker}\,H_2$, and hence not in $\mathrm{Im}\,P$ when j is even. This makes it plausible that the corresponding operator $\mathcal{P}$ is bijective. In order to be completely in the case 3 of section 2, we have also to arrange so that (cf. (2.39)),

(4.32) $\quad f_{0,1}=Vf_{0,1}.$

To have this we choose $\chi_{0,1}$ real-valued and such that $\chi_{0,1}\circ\delta=1-\chi_{0,1}$ near $s(0,1)$. We also need to be more explicit about the choice of Gaussian quantization. Let $I=\int\pi_\alpha d\alpha$ be a resolution of the identity, where $\pi_\alpha=\pi_{\alpha,h}$ has the kernel,

(4.33) $$Ch^{-3/2}e^{i((x-y)\alpha_\xi+i(x-\alpha_x)^2/2+i(y-\alpha_x)^2/2)/h}= Ch^{-3/2}v_\alpha(x)\bar{v}_\alpha(y).$$

Here $C>0$. By definition, the Gauss quantization of $\chi_{0,1}$ is then,

(4.34) $\quad \chi_{0,1}=\int\chi_{0,1}(\alpha)\pi_\alpha d\alpha.$

Using that v_0 is invariant under $\mathcal{F},V,\Gamma$, and that $v_\alpha=T_{\alpha/2\pi}v_0$, we check that $T_{\gamma/2\pi}\pi_\alpha T^{-1}_{\gamma/2\pi}=\pi_{\alpha+\gamma}$, $\mathcal{F}\pi_\alpha\mathcal{F}^{-1}=\pi_{\kappa(\alpha)}$, $V\pi_\alpha V=\pi_{\delta(\alpha)}$. This shows that the Gauss quantization behaves as the Weyl quantization under conjugation by $T_\gamma,\mathcal{F},V$, and in particular, by the choice of $\chi_{0,1}$, we obtain,

(4.35) $\quad V\chi_{0,1}V=I-\chi_{0,1}$, near $s(0,1)$.

Using also that V commutes with H_1, we obtain,

$$Vf_{0,1}=-iV[H_1,\chi_{0,1}]u_{0,1}=i[H_1,\chi_{0,1}]Vu_{0,1}.$$

After multiplication of $u_{0,1}$ by a suitable scalar, we may assume that $Vu_{0,1}=u_{0,1}$, and then we get (4.32).

So far, (4.32) is only a microlocal relation, but we can make it global and exact by replacing $f_{0,1}$ by $\frac{1}{2}(f_{0,1}+Vf_{0,1})$, and then we have modified $f_{0,1}$ only by an exponentially small quantity.

We shall next see how certain WKB-considerations near the branching points, imply the wellposedness of our Grushin problem, and give the possibility to compute the leading contributions to E_{-+} appearing in the inverse $\mathcal{E}$ of $\mathcal{P}$. First we notice that microlocally, P_2^*: $\mathrm{Ker}(H_2)\longrightarrow\mathrm{Ker}(H_1)$, P_1: $\mathrm{Ker}(H_1)\longrightarrow\mathrm{Ker}(H_2)$, and that these maps are bijective. Microlocally, near $\mathrm{Int}(s(0,1))$, we have the function $u_{0,1}$, defined as a multiple of $U_1w_-^0$, and since the function w_-^0 is defined globally, we can extend the definition of $u_{0,1}$ in $\mathrm{Ker}(P)$ to a full neighborhood of the branching point $(0,\pi)$, and then to neighborhoods of $\mathrm{Int}(s(0,2))$ and $\mathrm{Int}(s((0,1),4))$, by standard WKB constructions. We then have a microlocal solution $u_{0,1}$, defined in a neighborhood of $\mathrm{Int}(s(0,1))\cup\{(0,\pi)\}\cup\mathrm{Int}(s(0,2))\cup\mathrm{Int}(s((0,1),4))$, which has its wavefront set (defined in appendix d) concentrated to a much smaller neighborhood of this set. Outside the point $(0,\pi)$, $u_{0,1}$ is of simple WKB-form, and there are constants $\alpha,\beta,\gamma,\delta$, such that,

(4.36) $\quad u_{0,1}=\beta P_2^*u_{0,2}$ near $\mathrm{Int}(s(0,2))$,

$\quad u_{0,1}=\alpha P_2^*u_{(0,1),4}$ near $\mathrm{Int}(s((0,1),4))$,

$\quad u_{(0,1),3}=\gamma P_2^*u_{0,2}$ near $\mathrm{Int}(s(0,2))$,

$\quad u_{(0,1),3}=\delta P_2^*u_{(0,1),4}$ near $\mathrm{Int}(s((0,1),4))$.

Here, we have extended $u_{(0,1),3}$ to a neighborhood of $(0,\pi)$, in the same way. When μ is real, it is clear from the reduction to $P_0-\mu'$, that $\alpha,\beta,\gamma,\delta$ exist and are $\mathcal{O}(1)$, and we shall compute these coefficients later.

Now consider the Grushin problem,

(4.37) $\quad Pu+R_-u^-=0,\ R_+u=\delta_{0,1},$

where $\delta_{\alpha,j}(\beta,k)=1$ if $(\beta,k)=(\alpha,j)$, and $=0$ otherwise. Microlocally, near $\mathrm{Int}(s(0,1))$, an approximate solution is given by $u=u_{0,1}$, still with $u^-=0$. In order to get a global approximate solution, we extend $u_{0,1}$ across the branching points $(0,\pi)$ and $(\pi,0)$ and then we truncate $u_{0,1}$ suitably on the segments $s(0,2),s((0,1),4),s(0,4),s((1,0),2)$. Let us discuss the details of this only near $(0,\pi)$. Let $\tilde{\chi}_{0,2}$ be a cutoff operator, $=I$ microlocally near $(-\pi,0)$, and $=0$ near $(0,\pi)$, such that $\tilde{\chi}_{0,2}P_2^*=P_2^*\chi_{0,2}$. Near $s(0,2)$, we put $u=(1-\tilde{\chi}_{0,2})u_{0,1}$. Then near the interior of that segment,

$$Pu=\beta P(1-\tilde{\chi}_{0,2})P_2^*u_{0,2}=-\beta P\tilde{\chi}_{0,2}P_2^*u_{0,2}=$$
$$-\beta PP_2^*\chi_{0,2}u_{0,2}=-\beta[H_2,\chi_{0,2}]u_{0,2}=i\beta f_{0,2}.$$

Similarly, let $\tilde{\chi}_{(0,1),4}$ have its support near $(\pi,2\pi)$, such that $\tilde{\chi}_{(0,1),4}P_2^*=P_2^*\chi_{(0,1),4}$. Then near $s((0,1),4)$, we put, $u=(1-\tilde{\chi}_{(0,1),4})u_{0,1}$, and essentially the same calculation gives $Pu=i\alpha f_{(0,1),4}$ near that segment. Similarly, we can extend u to a neighborhood of $(\pi,0)$, such that $Pu=\text{const.}f_{0,4}$, and $Pu=\text{const.}f_{(1,0),2}$ respectively near the corresponding segments. Summing up the discussion so far, we have solved the problem (4.37) with exponentially small errors, and the corresponding u^- has only 4 non-vanishing components, out of which, we have computed 2. In appendix d we show how to obtain from this discussion via some apriori estimates the following result.

Proposition 4.1. For real μ, $\mathcal{P}$ is bijective with an inverse $\mathcal{E}=[E]_{1,1}+[E_+]_{1,2}+[E_-]_{2,1}+[E_{-+}]_{2,2}=\mathcal{O}(h^{-3/2})$. Moreover, $E_{-+}=\mathcal{O}(1)$, and there is an $\varepsilon_0>0$, such that,

(4.38) $E_{-+}(\alpha,j;\beta,k)=\mathcal{O}(e^{-(1+|\alpha-\beta|)\varepsilon_0/h})$ uniformly for all $(\alpha,j),(\beta,k)$ such that $s(\alpha,j)\cap s(\beta,k)=\emptyset$.

(4.39) $E_{-+}((0,0),2;(0,0),1)=-i\beta$, $E_{-+}((0,1),4;(0,0),1)=-i\alpha$,
$E_{-+}((0,0),2;(0,1),3)=-i\gamma$, $E_{-+}((0,1),4;(0,1),3)=-i\delta$.

Here the last two relations of (4.39) are proved as above by considering the problem (4.37) with $\delta_{0,1}$ replaced by $\delta_{(0,1),3}$.

In block matrix form we can write,

(4.40) $E_{-+}(\alpha,\beta)=e^{ih'\beta_2(\alpha_1-\beta_1)}f(\alpha-\beta)$,

where $f(\alpha)$ is exponentially small for $|\alpha|>1$. Using the invariance properties (2.36),(2.46) (valid also in the case 3), we get,

(4.41) $f(0)=[b]_{2,1}+[\bar{b}]_{2,3}+[\bar{b}]_{4,1}+[b]_{4,3}$, and modulo $\mathcal{O}(e^{-\varepsilon_0/h})$:
$f((1,0))=[\bar{a}]_{2,1}, f((-1,0))=[\bar{a}]_{4,3}, f((0,1))=[a]_{4,1}, f((0,-1))=[a]_{2,3}$.

Here, we see that,

$$a=E((0,1),4;(0,0),1)=E((0,-1),2;(0,0),3),$$
$$\bar{a}=E((1,0),2;(0,0),1)=E((-1,0),4;(0,0),3),$$
$$b=E((0,0),2;(0,0),1))=E((0,0),4;(0,0),3),$$
$$\bar{b}=E((0,0),2;(0,0),3)=E((0,0),4;(0,0),1).$$

Comparing this with (4.39), we get, $-i\alpha=a$, $-i\beta=b$. Using also (4.40), we get, $-i\gamma=f((0,-1);2,3)=a$, $-i\delta=f((0,0);4,3)=b$, and hence, (4.42) $\alpha=\gamma,\ \beta=\delta,\ a=-i\alpha,\ b=-i\beta.$

Before the actual computation, we shall see to what extent the matrix,

$$\begin{pmatrix}\beta & \gamma\\ \alpha & \delta\end{pmatrix}=\begin{pmatrix}\beta & \alpha\\ \alpha & \beta\end{pmatrix}=i\begin{pmatrix}b & a\\ a & b\end{pmatrix},$$

is unitary. Let $v_{0,1}$ be a normalized WKB solution of $H_2v=0$ in the sense of (4.28), defined near $\mathrm{Int}(s(0,1))$, and generate $v_{\alpha,j}$, the same way as we generated the $u_{\alpha,j}$. Then, in view of (2.2),(2.5), $v_{\alpha,j}$ is a normalized solution of $H_2v=0$ when j is odd, and of $H_1v=0$, when j is even. Near $\mathrm{Int}(s(0,1))$, we can write $P_1u_{0,1}=\lambda v_{0,1}$, where $\lambda, 1/\lambda=\mathcal{O}(1)$. Using (2.5), we get $P_1u_{\alpha,j}=\lambda v_{\alpha,j}$, when j is odd, and $P_2^*u_{\alpha,j}=\lambda v_{\alpha,j}$, when j is even, so from (4.36), we get,

(4.43) $u_{0,1}=\lambda\beta v_{0,2}$ near $\mathrm{Int}(s(0,2))$,

$u_{0,1}=\lambda\alpha v_{(0,1),4}$ near $\mathrm{Int}(s((0,1),4))$,

$u_{(0,1),3}=\lambda\gamma v_{0,2}$ near $\mathrm{Int}(s(0,2))$,

$u_{(0,1),3}=\lambda\delta v_{(0,1),4}$ near $\mathrm{Int}(s((0,1),4))$.

Consider the general solution $u=x_1u_{0,1}+x_3u_{(0,1),3}=y_2v_{0,2}+y_4v_{(0,1),4}$, of $H_1u=0$, defined near $(0,\pi)$. Here we extend $u_{0,1}$ near $(\pi,0)$ and $v_{0,1}$ near $(0,\pi)$ and $(\pi,0)$ the same way as we did with $u_{0,1}$ near $(0,\pi)$, by using microlocal models. Then the coefficients x and y are related by

$$y=\lambda\begin{pmatrix}\beta & \gamma\\ \alpha & \delta\end{pmatrix}x\quad .$$

Let χ be a pseudodifferential cutoff, $=I$ near $(0,\pi)$ and with support close to that point. From the trivial identity $0=(i[H_1,\chi]u\,|\,u)$, we get after summing the non-exponentially small contributions from each of the four segments, meeting at $(0,\pi)$: $0=|x_1|^2+|x_3|^3-|y_2|^2-|y_4|^4+\mathcal{O}(e^{-\varepsilon_0/h})$.

This means that,

(4.44) $\lambda\begin{pmatrix}\beta & \gamma\\ \alpha & \delta\end{pmatrix}=i\lambda\begin{pmatrix}b & a\\ a & b\end{pmatrix}$ is unitary up to an exponentially small error,

in the sense that the adjoint of this matrix differs from the inverse by an exponentially small term. It is then an easy exercise to see that there are a', b' with $a-a'$ and $b-b'$ exponentially small, such that $\lambda\begin{pmatrix}b' & a'\\ a' & b'\end{pmatrix}$ is unitary.

We now attack the WKB-problem. We first notice that the problem (4.36) is independent of the normalization of the function $u_{0,1}$ (and of condition (4.32)), so we may assume from now on that,

(4.45) $u_{0,1}=U_1\,w_-^0$ near $\mathrm{Int}(s(0,1))$.

We choose,

(4.46) $v_{0,1}=U_2w_-^0$.

This may affect λ in (4.43) by a factor of modulus 1, since $U_2U_1^{-1}$ maps normalized solutions of $H_1u=0$ onto normalized solutions of $H_2v=0$. Our main problem, will then be to determine $\lambda\alpha$, $\lambda\beta$ in (4.43). We recall the definition of the operators A,B,C,A_0,B_0,C_0, earlier in this section and the symmetry relations between these operators and H_1,H_2,P_0,U_1,U_2. Put,

(4.47) $\tilde{u}_{0,2}=Au_{0,1}\in \mathrm{Ker}(H_2)$,

(4.48) $\tilde{v}_{0,2}=Av_{0,1}\in \mathrm{Ker}(H_1)$.

The problem of determining $\lambda\beta$, can then be decomposed into two problems;

(4.49) Find λ_1 such that $u_{0,1}=\lambda_1 Av_{0,1}=\lambda_1\tilde{v}_{0,2}$ near $\mathrm{Int}(s(0,2))$,

(4.50) find λ_2 such that $Av_{0,1}=\lambda_2 v_{0,2}$ near $\mathrm{Int}(s(0,2))$.

Then we will have $\lambda\beta=\lambda_1\lambda_2$. Notice that $|\lambda_2|=1$ since $Av_{0,1}$ and $v_{0,2}$ have the same normalization. In view of all our symmetry relations, the first problem reduces to a similar one on the model, namely to find λ_1 such that,

(4.51) $w_{0,-}=\lambda_1 u_{0,-}$ near Γ_2,

where Γ_1 is the open negative ξ-axis, and $\Gamma_j=\kappa^{j-1}(\Gamma_1)$. Now the solution of (4.51) is $\lambda_1=\alpha_-$ given by (4.20), with $\gamma_-=1$, $\gamma_+=0$, and we get,

(4.52) $\lambda_1=(\mu'/h)^{i\mu'/h}(2\pi)^{-\frac{1}{2}}\Gamma(\frac{1}{2}-i\mu'/h)e^{-i\mu'/h+\pi\mu'/2h}$.

For $x>0$, we can write,

(4.53) $u^0_+(x,\mu')=a(x,\mu',h)e^{i\varphi_0(x,\mu')/h}$,

where a is an analytic elliptic symbol, defined for $x>0$, $|\mu'|\le$const., of the form,

$$a=e^{i\pi/8}|x|^{-\frac{1}{2}}c(\mu,h),$$

where $c(\mu,h)>0$ is a normalization factor, and

$$\varphi_0=\tfrac{1}{2}\mu'(1-\log(\mu'))+\mu'\log|x|.$$

We have analogous representations for u^0_-, $\mathcal{F}w^0_\pm$. In particular,

$$\mathcal{F}w^0_-(\xi)=\bar{a}(-\xi,\mu',h)e^{-i\varphi_0(-\xi,\mu')/h},\ \xi<0, \text{ and hence}$$

$$\mathcal{F}^{-1}w^0_-=\bar{a}e^{-i\varphi_0/h},\ \xi>0, \text{ so } v_0=U_2\mathcal{F}(\bar{a}e^{-i\varphi_0/h}) \text{ near } s(0,1).$$

By stationary phase, we then obtain the representation,

(4.54) $v_{0,1}=a(x,\mu,h)\, e^{i\varphi(x,\mu)/h}$,

microlocally near $\mathrm{Int}(s(0,1))\setminus$(a neighborhood of the starting point of this interval). Here the new analytic symbol, a is defined for $|\mu'|<$const., and we have the following geometric description of φ: Recall (4.14), which implies,

$$U_2^{-1}g(H_2)U_2=P_0-\mu',$$

where, $g(t)=g(\mu',t,h)=f(\mu,t+\mu-c(\mu),h)-f(\mu,\mu-c(\mu),h)$, so that $g'(0)>0$, $g(0)=0$. Let κ_{U_2} be an associated canonical transformation in the precise sense of section a in the appendix. We may choose κ_{U_2} satisfying the natural intertwining relation resulting from (4.13), and so that the corresponding phase in U_2 (generating κ_{U_2},) is a classical analytic symbol. It is only after these choices that the symbol and the phase in (4.54) are completely defined by the stationary phase method, and if $\Lambda_\varphi=\{(x,\varphi'_x(x))\}$, then

$\Lambda_\varphi = \kappa_{U_2} \circ \kappa(\Lambda_{-\varphi_0})$. Since the symbol of $g(H_2)$ is $g \circ H_2 + \mathcal{O}(h^2)$, and κ_{U_2} is a well adapted canonical transformation, we get $g \circ H_2 \circ \kappa_{U_2} = x\xi - \mu' + \mathcal{O}(h^2)$, $g \circ H_2 \circ \kappa_{U_2} \circ \kappa = -(x\xi + \mu') + \mathcal{O}(h^2)$. It follows that φ satisfies an eiconal equation: $\tilde{H}_2(x, \varphi'_x) = 0$, where $\tilde{H}_2 = H_2 + \mathcal{O}(h^2)$ is defined by $g \circ \tilde{H}_2 \circ \kappa_{U_2} = x\xi - \mu'$. Since $\mu' > 0$, the real characteristics of $\tilde{H}_2$ near $(0,\pi)$ is the union of two disjoint curves, one which is close to $s(0,1) \cup s(0,2)$ and the other which is close to $s((0,1),3) \cup s((0,1),4)$. It is then clear that we can extend the definition of φ to negative x as a solution of the eiconal equation. Let $\Phi(x,y) = \Phi(x,y,\mu',h)$ be a generating function for the transformation $\kappa_U \circ \kappa$. Since $\kappa_U(0,0) = (0,\pi)$, we have $\partial_x \Phi(0,0) = \pi$, $\partial_y \Phi(0,0) = 0$, and we may assume that $\Phi(0,0) = 0$. Then $\varphi(x) = \mathrm{vc}_y(\Phi(x,y) - \varphi_0(y))$, and for $x = 0$, the critical point $y(\mu')$ is the projection of $(y(\mu'), \eta(\mu'))$; the intersection in the 4:th quadrant of the curves $y\eta + \mu' = 0$ and $(\kappa_U \circ \kappa)^{-1}(\xi\text{-axis})$. The last curve is of the form, $\eta = -z(y,\mu')$, with z analytic, $z(0,\mu') = 0$, $\partial_y z(0,\mu')$ close to 1. Using the reflection symmetries with respect to the points $(0,\pi)$ and $(0,0)$, we also see that z is an odd function of y. It follows that $y(\mu') = \mu'^{\frac{1}{2}} g(\mu')$, where g is analytic with $g(0) > 0$, and a simple calculation gives,

(4.55) $\qquad \varphi(0) = \mathcal{O}(\mu')$,

and this is again an analytic function of μ'.

The transport equation for a_0 will conserve the argument of a_0, thanks to the fact that the eiconal equation for the phase makes use of the full Weyl symbol $H_2 + \mathcal{O}(h^2)$. (See appendix a.) Examining the geometry, we also see that $\partial_y^2 \Phi > 0$, and with the representation of $\mathcal{F}^{-1} w_-^0$ above, this leads to: $\arg(a_0) = \pi/8$, if a_0 denotes the leading part of a in (4.54). Here we assume that the leading amplitude in $U_2 \circ \mathcal{F}^{-1}$ at $(0,0)$ is >0. (Substracting $\varphi(0) = \mathcal{O}(\mu')$ from the generating function of this operator, we may even assume that the right hand side in (4.55) is 0.)

Let (x_μ, x_μ) be the unique point of this form on the characteristics of H_2, close to $(\pi/2, \pi/2)$. Let us study (4.50) at the point $-x_\mu$. Since $Au(x) = \bar{u}(-x)$, we get,

(4.56) $\qquad Av_{0,1}(-x_\mu) = \bar{a}(x_\mu, \mu, h) e^{-i\varphi(x_\mu, \mu)/h}$.

On the other hand, by the method of stationary phase, we find,

(4.57) $\qquad v_{0,2}(-x_\mu) = \mathcal{F}^{-1} v_{0,1}(-x_\mu) =$
$$J(\mu,h)(a(x_\mu,\mu,h) + \mathcal{O}(h)) e^{i\varphi(x_\mu,\mu)/h - \pi i/4 - i x_\mu^2/h},$$

where $J > 0$ is an elliptic symbol of order 0, defined for $|\mu'| <$ const.. We already know that $|\lambda_2| = 1$, and if we compare (4.56),(4.57),(4.50), we get,

(4.58) $\qquad \arg(\lambda_2) = x_\mu^2/h - 2\varphi(x_\mu, \mu)/h + \mathcal{O}(1)$,

where "$\mathcal{O}(1)$" hides a real valued analytic symbol of order 0, defined for $|\mu'| \leq$ const.>0. (With the additional normalizations indicated after (4.55), we may reduce the $\mathcal{O}(1)$ term to $\mathcal{O}(h)$.)

Here we can make the following geometric interpretation: Let $A(\mu)$ be the area of the domain limited by the ξ-axis, $H_2^{-1}(0)$ and the line $\xi=x$. Let $B(\mu)$ be the area of the additional region, which is required to fill up the triangle with corners, $(0,0),(0,\pi),(\pi/2,\pi/2)$. (We think of the appropriate component of $H_2^{-1}(0)$ as lying below $s(0,1)$, otherwise just modify what we just said by introducing domains with negative area!). Then $B(\mu)+A(\mu)=\pi^2/4$. Since $\varphi(0,\mu)=\mathcal{O}(\mu')$, we have $\varphi(x_\mu)=\int_0^{x_\mu}\varphi' dx+\mathcal{O}(\mu')=A(\mu)+x_\mu^2/2+\mathcal{O}(\mu')$. Hence,

(4.59) $\quad \arg(\lambda_2)=-(2A(\mu)+\mathcal{O}(\mu'))/h+\mathcal{O}(1).$

Combining this with (4.52), we get,

(4.60) $\quad \lambda\beta=\lambda_1\lambda_2=$

$$(\mu'/h)^{i\mu'/h}(2\pi)^{-\frac{1}{2}}\Gamma(\tfrac{1}{2}-i\mu'/h)e^{(-i\mu'+\pi\mu'/2-i2A(\mu)+i\mathcal{O}(\mu'))/h+i\mathcal{O}(1)}.$$

(With additional normalizations indicated above, we may replace $\mathcal{O}(1)$ by $\mathcal{O}(h)$, and suppress completely the $\mathcal{O}(\mu')$ term.)Here, we write,

$-2A(\mu)/h=-\pi^2/2h+2B(\mu)/h$, $\pi^2/2h=h^*/8\equiv h'/8+k\pi/2 \bmod (2\pi\mathbb{Z})$, $k\in\mathbb{Z}_4$, (where h^* and $h'\in]0,2\pi]$ are defined by $2\pi/h=h^*/2\pi\equiv h'/2\pi \bmod (2\pi\mathbb{Z})$,) and we obtain,

(4.61) $\quad \lambda\beta=(2\pi)^{-\frac{1}{2}}\Gamma(\tfrac{1}{2}-i\mu'/h)\times$

$$e^{i(\mu'\log 1/h+2B(\mu)+\mu'\log\mu')/h+\pi\mu'/2h-ih^*/8+i\mathcal{O}(\mu')/h+i\mathcal{O}(1)},$$

where $\mathcal{O}(1)$ indicates a real valued analytic symbol of order 0, defined for μ' in a neighborhood of 0, and $\mathcal{O}(\mu')$ indicates a real valued analytic function defined near 0. In order to evaluate the singularity of $B(\mu)$ at $\mu'=0$, we let $\gamma=\kappa_{U_2}$(negative ξ-axis), and let $B'(\mu)$ be the area obtained the same way as $B(\mu)$, but with $s(0,1)$ replaced by γ. Then,

(4.62) $\quad B'(\mu)-B(\mu)=\mathcal{O}(\varepsilon),$

is an analytic function of μ, and ε is the parameter in the condition (3.2). By canonical transformation, we see that $B'(\mu)$ is equal to the sum of an analytic function $=\mathcal{O}(\mu')$, and the area of $0\le y\le 1$, $0\le\eta\le\min(z(y,\mu),\mu'/y)$, which gives,

(4.63) $\quad B'(\mu)=-\tfrac{1}{2}\mu'\log\mu'+\mu' f(\mu'),$

where f is analytic. We conclude that,

(4.64) $\quad 2B(\mu)+\mu'\log\mu'=2f(\mu')\mu'+\mathcal{O}(\varepsilon),$

so (4.61) gives,

(4.65) $\quad \lambda\beta=\ (2\pi)^{-\frac{1}{2}}\Gamma(\tfrac{1}{2}-i\mu'/h)\times$

$$e^{i\mu'\log(1/h)/h+\pi\mu'/2h+i(\mathcal{O}(\mu')+\mathcal{O}(\varepsilon))/h-ih^*/8+i\mathcal{O}(1)}$$

where $\mathcal{O}(\mu'),\mathcal{O}(\varepsilon)$ are analytic functions of μ', and $\mathcal{O}(1)$ is a real valued analytic symbol of order 0.

We now recall Stirling's formula in the complex (see [0]):

(4.66) $\quad (2\pi)^{-\frac{1}{2}}\Gamma(z)=e^{-z}z^{z-\frac{1}{2}}(1+1/12z+1/288z^2+\dots),$

valid uniformly asymptotically when $|z|\to\infty$, $\arg(z)\le\pi-\delta$, for any $\delta>0$. Writing the asymptotic sum in the parenthesis as $e^{k(z)/z}$, where k is holomorphic and bounded in any domain $|\arg z|\le\pi-\delta$, $|z|\ge$const., and real-valued on the real axis, we get,

$$(2\pi)^{-\frac{1}{2}}\Gamma(\tfrac{1}{2}-i\mu'/h)= e^{(-\frac{1}{2}+i\mu'/h-i(\mu'/h)\log(\frac{1}{2}-i\mu'/h)+k(\frac{1}{2}-i\mu'/h)/(\frac{1}{2}-i\mu'/h)}.$$

Here, $\log(\frac{1}{2}-i\mu'/h)=\log(\mu'/h)-i\pi/2+ih/2\mu'+(h/\mu')^2 l(ih/\mu')$, where l is holomorphic near 0, and real-valued on the real. We then get,

(4.67) $$(2\pi)^{-\frac{1}{2}}\Gamma(\tfrac{1}{2}-i\mu'/h)= e^{(i\mu'/h-i(\mu'/h)\log(\mu'/h)-\pi\mu'/2h+i(h/\mu')F(\mu'/h))},$$

where F is a bounded holomorphic function, defined in a domain, $|\arg z|<\text{const.}>0$, $|z|>\text{const.}$ Substitution into (4.65), gives for $\mu'/h>\text{const.}>0$:

(4.68) $$\lambda\beta= e^{-i(\mu'/h)\log(\mu')+i\mathcal{O}(\mu')/h+i\mathcal{O}(\varepsilon)/h+i\mathcal{O}(1)-ih^*/8+ i(h/\mu')F(\mu'/h))}.$$

Here we recall that the $\mathcal{O}$-terms indicate real-valued analytic symbols of order 0, defined for μ' in a small neighborhood of 0.

The reflection identity, $\Gamma(z)\Gamma(1-z)=\pi/\sin(\pi z)$, gives $\Gamma(\frac{1}{2}-i\mu'/h)\Gamma(\frac{1}{2}+i\mu'/h)=\pi/\mathrm{ch}(\pi\mu'/h)$. Combining this with (4.67) and its complex conjugate, we get for $\mu'/h\geq\text{const.}>0$,

$$e^{\pi\mu'/h}/2\mathrm{ch}(\pi\mu'/h)=e^{i(h/\mu')(F(\mu'/h)-\bar{F}(\mu'/h))},$$

from which we conclude that,

(4.69) $$\mathrm{Im}(F(\mu'/h))=\mathcal{O}(\mu'/h)e^{-2\pi\mu'/h}$$

$\mu'/h\geq\text{const.}>0$. Reinsertion in (4.68), then shows that,

(4.70) $$|\lambda\beta|=1+\mathcal{O}((e^{-2\pi\mu'/h}).$$

Naturally, all our calculations, so far are valid only up to some exponentially small error, $\mathcal{O}(e^{-1/Ch})$. After modification of $\lambda\beta,\lambda\alpha$ by such exponentially small terms, we know that the matrix $\begin{pmatrix}\lambda\beta & \lambda\alpha\\ \lambda\alpha & \lambda\beta\end{pmatrix}$ is unitary.

This, means that

(4.71) $$|\lambda\beta|^2+|\lambda\alpha|^2=1,\ \arg(\lambda\beta)-\arg(\lambda\alpha)=\pm\pi/2.$$

Combining (4.65) and the reflection relation, we know on the other hand, that,

(4.72) $$|\lambda\beta|^2=\tfrac{1}{2}(\mathrm{ch}(\pi\mu'/h))^{-1}e^{\pi\mu'/h}$$ (up to an exponentially small error),

so from the first part of (4.71), we conclude that,

(4.73) $$|\lambda\alpha|^2=\tfrac{1}{2}(\mathrm{ch}(\pi\mu'/h))^{-1}e^{-\pi\mu'/h}.$$

Combining this with (4.65) and the second part of (4.71), we get,

(4.74) $$\lambda\alpha= (2\pi)^{-\frac{1}{2}}\Gamma(\tfrac{1}{2}-i\mu'/h)\times e^{i\mu'\log(1/h)/h-\pi\mu'/2h+i(\mathcal{O}(\mu')+\mathcal{O}(\varepsilon))/h-ih^*/8+i\mathcal{O}(1)\mp i\pi/2},$$

where the $\mathcal{O}$-terms are the same as in (4.65).

We shall next extend our asymptotic results to the case $\mu'<0$, by symmetry arguments. For $\alpha\in\mathbb{R}^2$, let $\mathcal{F}_\alpha=T_\alpha\mathcal{F}T_\alpha^{-1}$, so that $\mathcal{F}_\alpha$ is a quantization of rotation around $2\pi\alpha$ by the angle $-\pi/2$. Put $\tilde{P}=\mathcal{F}_{0,\frac{1}{2}}P(\mathcal{F}_{0,\frac{1}{2}})^{-1}$ and define $\tilde{P}_j$, $j=1,2$, similarly. Define $\tilde{H}_j$ as before. Then $\tilde{H}_j$ can also be obtained from H_j by conjugation by $\mathcal{F}_{0,\frac{1}{2}}$. It is easy to check that, $T_\gamma\mathcal{F}_\alpha=e^{ih^*\theta(\gamma,\alpha)}\mathcal{F}_\alpha T_{\kappa^{-1}(\gamma)}$, where the function θ does not need to be explicited. From this relation we immediately obtain,

$T_{\kappa^{-1}(\gamma)}\mathcal{F}_{\bar\alpha}^{-1}=e^{-ih^*\theta(\gamma,\alpha)}\mathcal{F}_{\bar\alpha}^{-1}T_\gamma$, and it is then easy to check that,

(4.75) $\quad \tilde{P},\ \tilde{P}_j$ commute with T_γ for all γ.

In order to investigate the Fourier invariance, we first find that $\mathcal{F}\mathcal{F}_{(0,\frac{1}{2})}=e^{-ih^*/4}\mathcal{F}_{0,\frac{1}{2}}T_{0,1}\mathcal{F}$, which implies, $e^{ih^*/4}(\mathcal{F}_{0,\frac{1}{2}})^{-1}\mathcal{F}=T_{0,1}\mathcal{F}(\mathcal{F}_{0,\frac{1}{2}})^{-1}$. Using (2.5) it is then easy to check that, $\mathcal{F}\tilde{P}_1=\tilde{P}_2^*\mathcal{F}$ as well as the other parts of (2.2),(2.5) for $\tilde{P}$, $\tilde{P}_j$.

In order to check the reflection invariance, we first see that, $V\mathcal{F}_{0,\frac{1}{2}}=T_{\frac{1}{2},0}\mathcal{F}^{-1}T_{-\frac{1}{2},0}V$, which implies that, $V(\mathcal{F}_{0,\frac{1}{2}})^{-1}=T_{\frac{1}{2},0}\mathcal{F}T_{-\frac{1}{2},0}V$. Then we get,
$V\tilde{P}=T_{\frac{1}{2},0}\mathcal{F}^{-1}T_{-\frac{1}{2},0}PT_{\frac{1}{2},0}\mathcal{F}T_{-\frac{1}{2},0}V$. Using that $\mathcal{F}^2$ commutes with P, we get $V\tilde{P}=KPK^{-1}V$, where $K=T_{\frac{1}{2},0}\mathcal{F}^{-1}T_{-\frac{1}{2},0}\mathcal{F}^2=e^{iW}\mathcal{F}_{0,\frac{1}{2}}T_{1,1}$. Using the translation invariance of P, we get $KPK^{-1}=\tilde{P}$, and hence, $V\tilde{P}=\tilde{P}V$. This is the analogue of (2.3) and the analogue of (2.6) is obtained the same way.

In conclusion, we have checked that $\tilde{P}$ together with $\tilde{P}_j$ is of type 1. Since the symbols of these operators are obtained by rotation by $-\pi/2$ around $(0,\pi)$, we also see that $-\tilde{P}$, $\tilde{P}_j$ satisfy the more precise assumption (3.2) with μ replaced by $-\mu$. To make this even more precise, we also introduce $\tilde{U}_1=\mathcal{F}_{0,\frac{1}{2}}U_1\mathcal{F}^{-1}$, which is a unitary Fourier integral operator such that $C\tilde{U}_1=\tilde{U}_1C_0$, and whose associated canonical transformation is close to that of U_1. Since P_0 and $\mathcal{F}$ anticommute, we get,

(4.76) $\quad \tilde{U}_1^{-1}\tilde{H}_1\tilde{U}_1=-l(\mu,-P_0,h)(P_0+\mu')$,

where l is the same function as in (4.6). This shows that all the results, obtained above for P when $\mu'>0$, are valid also for $\tilde{P}$ when $\mu'<0$.

Recall that the general microlocal solution of $H_1u=0$ near $(0,\pi)$, is of the form $u=x_1u_{0,1}+x_3u_{(0,1),3}=y_2v_{0,2}+y_4v_{(0,1),4}$, where ${}^t(y_2,y_4)=([\lambda\beta]_{1,1}+[\lambda\alpha]_{1,2}+[\lambda\alpha]_{2,1}+[\lambda\beta]_{2,2})^t(x_1,x_3)$. Using that the matrix here is unitary, we get $x_1=\bar\lambda\bar\alpha$, $x_3=\bar\lambda\bar\beta$, if $y_2=0$, $y_4=1$. In other words,

(4.77) $\quad v_{(0,1),4}=\bar\lambda\bar\alpha u_{0,1}$ near $\mathrm{Int}(s(0,1))$,

and $v_{(0,1),4}=\bar\lambda\bar\beta u_{(0,1),3}$ near $\mathrm{Int}(s((0,1),3)$.

The functions $\tilde{u}_{0,1}=\mathcal{F}_{0,\frac{1}{2}}v_{(0,1),4}$, $\tilde{v}_{0,1}=\mathcal{F}_{0,\frac{1}{2}}u_{(0,1),4}$ are microlocally in the kernel of $\tilde{H}_2$ and $\tilde{H}_1$ respectively. (We get here a permutation of the indices 1,2, that will not cause any essential difficulty in the following.) From these functions, we generate $\tilde{u}_{\alpha,j}$, $\tilde{v}_{\alpha,j}$ as before. The analogue of problem (4.43) is then to determine $\tilde\beta$, $\tilde\alpha$ such that,

(4.78) $\quad \tilde{u}_{0,1}=\tilde\beta\,\tilde{v}_{0,2}$ near $\mathrm{Int}(s(0,2))$,

$\tilde{u}_{0,1}=\tilde\alpha\tilde{v}_{(0,1),4}$ near $\mathrm{Int}(s((0,1),4)$,

$\tilde{u}_{(0,1),3}=\tilde\alpha\,\tilde{v}_{0,2}$ near $\mathrm{Int}(s(0,2))$,

$$\tilde{u}_{(0,1),3}=\tilde{\beta}\,\tilde{v}_{(0,1),4} \text{ near Int}(s((0,1),4)).$$

Since μ' has been exchanged with $-\mu'$, we can apply the earlier results in the case when $\mu'<0$, and obtain,

(4.79) $\tilde{\beta}=(2\pi)^{-\frac{1}{2}}\Gamma(\frac{1}{2}+i\mu'/h)\times$
$e^{-i\mu'\log(1/h)/h-\pi\mu'/2h-ih^*/8+i\mathcal{O}(\mu')/h+i\mathcal{O}(\varepsilon)/h+i\mathcal{O}(1)}$,

where as before the $\mathcal{O}$-terms are real valued analytic symbols, of order 0, defined for μ' in a neighborhood of 0.

Let us now relate $\tilde{\beta}$ and $\bar{\alpha}$. Applying $\mathcal{F}_{0,\frac{1}{2}}$ to the first relation of (4.77), we see that we need to compare $\tilde{v}_{0,2}$ and $\mathcal{F}_{0,\frac{1}{2}}u_{0,1}$. By definition, $\tilde{v}_{0,2}=\mathcal{F}^{-1}\mathcal{F}_{0,\frac{1}{2}}T_{0,1}\mathcal{F}u_{0,1}=T_{-\frac{1}{2},\frac{1}{2}}\mathcal{F}u_{0,1}$, while $\mathcal{F}_{0,\frac{1}{2}}=e^{-ih^*/4}T_{-\frac{1}{2},\frac{1}{2}}\mathcal{F}$, so we get,

(4.80) $\mathcal{F}_{0,\frac{1}{2}}u_{0,1}=e^{-ih^*/4}\tilde{v}_{0,2}$.

Comparing this with (4.77) and (4.78), we get,

(4.81) $\tilde{\beta}=\bar{\lambda}\bar{\alpha}e^{-ih^*/4}$,

which together with (4.79) gives for $\mu'<0$,

(4.82) $\lambda\alpha=(2\pi)^{-\frac{1}{2}}\Gamma(\frac{1}{2}-i\mu'/h)\times$
$e^{i\mu'\log(1/h)/h-\pi\mu'/2h-ih^*/8+i\mathcal{O}(\mu')/h+i\mathcal{O}(\varepsilon)/h+i\mathcal{O}(1)\mp i\pi/2}$.

This is the same type of expression as (4.74), but it is not immediately clear that the analytic symbols hiding in the $\mathcal{O}$-terms are the same. Using again the unitarity, we obtain from (4.81) that a formula of type (4.65) is valid also for $\mu'<0$, although we can not be absolutely certain that the $\mathcal{O}$-terms contain the same analytic symbols.

After deformation to the case when $P=H_1=H_2=\cos(hD)+\cos(x)-\mu$, $P_j=1$, and some more detailed computations following the same ideas, it is possible to show in the model case and hence in general, that

(4.83) the + sign is valid in (4.71), hence the − sign is valid in (4.74).

In this model case, we can take $u_{0,1}=v_{0,1}$, $\lambda=1$. For $\mu=\mu'=0$, we have

$\tilde{P}=-P$, so $\tilde{\alpha}=\alpha$, $\tilde{\beta}=\beta$, and (4.81) gives, $|\alpha|=|\beta|$, $\arg\alpha+\arg\beta+h^*/4=2\pi k$, $k\in\mathbb{Z}$. Since $\arg\beta=\arg\alpha+\pi/2$, we get, $\arg\alpha=\pi k-\pi/4-h^*/8$, $\arg\beta=\pi k+\pi/4-h^*/8$ in this special case.

We shall next extend our results to the case when $|\mathrm{Im}\mu'|\le\varepsilon_1 h$ for some $\varepsilon_1<\frac{1}{2}$. Recall that P, P_j depend holomorphically on μ. For complex μ, we put, $H_1(\mu)=P_1^*(\bar{\mu})P(\mu)$, $H_2(\mu)=P(\mu)P_2^*(\bar{\mu})$ so that H_j depend holomorphically on μ. Restricting the attention to a square $|\mathrm{Im}\mu|<\varepsilon_0$, $|\mathrm{Re}\mu|<\varepsilon_0$, (where $\varepsilon_0>0$ is small but fixed and independent of "ε" in (3.2)), we first notice that we can choose the WKB-solution $u_{0,1}$, so that the phase depends analytically on μ, and so that the symbol is a classical analytic symbol with μ and x as base variables. Here x is restricted to some neighborhood of $\pi/2$. (Earlier in this section, the normalization of $u_{0,1}$ was choosen in such a way that the phase had a logarithmic singularity at $\mu'=0$, but this was for computational reasons only and did not affect the final results concerning the problem (4.43).) We may also arrange so that $u_{0,1}$ is normalized. We now restrict the attention to a "strip" $|\mathrm{Im}\mu'|<Ch$, $|\mathrm{Re}\mu'|<\varepsilon_0$. We have,

(4.84) $1=(u_{0,1}(\mu)|i[H_1(\bar{\mu}),\chi_{0,1}]u_{0,1}(\bar{\mu}))$.

Define $f_{\alpha,j}$ and $u_{\alpha,j}$ as for real values of μ. Again $f_{\alpha,j}$ is no more a microlocal function, but we may arrange so that it depends holomorphically on μ, and so that we have the same growth estimates for $f_{\alpha,j}$ and $\mathcal{F}f_{\alpha,j}$ as in the case of real μ. Then (4.31) remains valid, and $u_{\alpha,j}$ is a microlocal solution of $H_1u=0$ when j is odd and of $H_2u=0$, when j is even. We extend the definition of R_- in the obvious way, and we extend R_+ holomorphically, by putting $(R_+u)(\alpha,j)=(u|u_{\alpha,j}(\bar{\mu}))$. Restricting μ further, by imposing $|\mu'|<\varepsilon_1 h$, with $0<\varepsilon_1<\frac{1}{2}$, we see that the existence of an inverse for the corresponding Grushin problem can be proved the same way as in the case of real μ, and the inverse $\mathcal{E}$ is $\mathcal{O}(h^{-3/2})$ uniformly. Moreover, we see as before that the matrix elements $E_{-+}(\alpha,j;\beta,k)$ satisfy (4.38). The coefficients a, b defined in (4.41) are now holomorphic functions of μ, and if we choose $v_{0,1}$ by (4.45),(4.46), then $\lambda=e^{if(\mu,h)/h}$, where f is an analytic symbol of order 0, such that Im f is of order ≤ -1. This implies that $\lambda\beta$ and $\lambda\alpha$ are holomorphic functions of μ of at most temperate growth when $h\to 0$, for $|\mu'|<\varepsilon_1 h$, $|\mathrm{Re}\mu'|<\varepsilon_0$. Put $u=\lambda\beta$. According to (4.65) there is a holomorphic function u_+, defined on the same rectangle and of temperate growth there, such that $u-u_+$ is uniformly of exponential decrease on the intersection Γ_+ of our rectangle and the positive real axis. Similarly, as indicated after (4.82), there is a function u_- with the same properties, such that $u-u_-$ is exponentially small on Γ_-, which is the intersection of the negative real axis with our rectangle. From the maximum principle, it follows that if we decrease slightly our rectangle by decreasing ε_0 and ε_1, then $u-u_\pm$ is exponentially decreasing in $U_\pm=\{\mu'\in\mathbb{C},\mathrm{dist}(\mu',\Gamma_\pm)\leq\varepsilon_1 h\}$. For instance to get such an estimate for $u-u_+$ in a disc $D(0,\varepsilon_2 h)$, with ε_2 slightly smaller than the original ε_1, we introduce the subharmonic function $g=\log|u-u_+|$, and after some changes of scales, we get a subharmonic function $\tilde{g}$ with only finitely many log-singularities, defined in a neighborhood of the closure of the domain $\Omega=\{z\in\mathbb{C};\ |z|<1,\ 0<\arg(z)<2\pi\}$, satisfying $\tilde{g}(x)\leq -1$ for $0\leq x<1$, $\tilde{g}(z)\leq 0$ for $|z|=1$. By the maximum principle, (comparing with the harmonic solution of the corresponding Dirichlet problem,) we then obtain, $\tilde{g}\leq\alpha<0$ for $|z|\leq\varepsilon_2/\varepsilon_1$, and after rescaling back, we get the required exponential decrease.

We also know that $u_+/u_-=e^{ia(\mu',h)/h}$, where a is a real valued analytic symbol of order 1. From the exponential decrease of $u-u_\pm$ on the interval $[-\varepsilon_1 h,\varepsilon_1 h]$, it follows that a is exponentially small on the same interval (after adding an integer multiple of 2π). We conclude that a is a realization of the 0 symbol, hence a is exponentially small on the whole rectangle. The conclusion is then that we have extended the validity of (4.65) to the whole rectangle. The same then holds for $\lambda\alpha$, and we have proved,

Proposition 4.2. There is $\lambda=e^{if(\mu,h)/h}$, where f is an analytic symbol of order ≤ 0 such that Im f is of order ≤ -1, such that the following holds for $\alpha=a/i$, $\beta=b/i$: Let $\varepsilon_0>0$ be sufficiently small, and let $0<\varepsilon_1<\frac{1}{2}$. Then (4.65) and (4.82) hold uniformly in the rectangle $|\mathrm{Im}\mu'|<\varepsilon_1 h$, $|\mathrm{Re}\mu'|<\varepsilon_0$, where as before, the $\mathcal{O}$-terms indicate realizations (possibly with an exponentially small imaginary part) of real valued classical analytic symbols of order 0, which are the same in both expressions. Moreover up to exponentially small errors, we have (4.71), where the + sign is valid.

We shall next study the new self-adjoining operators. Let $A_1=R^*_-P_1E_+$, $A_2=E_-P_2R^*_+$, so that the operators Q_1,Q_2, defined in section 2, are the renormalizations of A_1, A_2. Repeating the arguments of that section, we get the following results (forming intermediate steps in the verification of the properties (2.7),(2.8) for Q,Q_1,Q_2.)

(4.85) $\mathcal{G}A_1=A_2^*\mathcal{G}$, $A_1\mathcal{G}=\mathcal{G}A_2^*$,

(4.86) $V'A_j=A_jV'$.

The fact that A_j commute with the $\mathcal{T}_\gamma$, implies on the block-matrix level, that,

(4.87) $A_j(\alpha,\beta)=e^{ih'\beta_2(\alpha_1-\beta_1)}a_j(\alpha-\beta)$,

and expliciting the properties (4.85),(4.86), we get,

(4.88) $a_1(\kappa(\alpha))=T\circ a_2(\alpha)^*\circ T^{-1}$, $a_1(\kappa^{-1}(\alpha))=T^{-1}\circ a_2(\alpha)^*\circ T$,

(4.89) $a_\nu(\delta(\alpha);2-j,2-k)=e^{ih'\alpha_1\alpha_2}\bar{a}_\nu(\alpha;j,k)$.

As with E_{-+}, we see that $a_\nu(\alpha)$ is exponentially small, except for $|\alpha|\leq 1$, and for $|\alpha|=1$, the only non-exponentially small entries are $a_\nu((1,0);2,1)$, $a_\nu((0,1);4,1)$, $a_\nu((-1,0);4,3)$, $a_\nu((0,-1);2,3)$. Using (4.88), (4.89), it is easy to check that we have,

(4.90) $a_1(0)=a_2(0)=[x]_{2,1}+[\bar{x}]_{2,3}+[\bar{x}]_{4,1}+[x]_{4,3}$.

Similarly, we get for $|\alpha|=1$,

(4.91) $a_\nu((1,0))\equiv[\bar{z}]_{2,1}$, $a_\nu((0,-1))\equiv[z]_{2,3}$, $a_\nu((-1,0))\equiv[\bar{z}]_{4,3}$,
$a_\nu((0,1))\equiv[z]_{4,1}$, modulo $\mathcal{O}(e^{-\varepsilon_0/h})$ for some $\varepsilon_0>0$.

(These relations could also have been obtained directly, by using (2.7),(2.8) for Q_j and studying the trigonometric polynomial of degree 1 in the Fourier series expansion of the symbols Q_j. Looking at higher order contributions, we also see that there is no reason to suspect that $Q_1=Q_2$.)

The next problem is then to study $x=A_\nu(0,2;0,1)$, $z=A_\nu((0,1),4;0,1)$. Applying the definitions of the various operators, we get,

$x=(P_1E_+(\delta_{0,1})|f_{0,2})\equiv\lambda\beta(P_1(1-\tilde{\chi}_{0,2})v_{0,2}|f_{0,2})$,

$z=(P_1E_+(\delta_{0,1})|f_{(0,1),4})\equiv\lambda\alpha(P_1(1-\tilde{\chi}_{(0,1),4})v_{(0,1),4}|f_{(0,1),4})$.

Recall that the corresponding coefficients b,a for E_{-+}, are given by $b=i^{-1}\beta$, $a=i^{-1}\alpha$, so we get the same coefficients, multiplied by a common factor,

$$\theta=i\lambda(P_1(1-\tilde{\chi}_{0,2})v_{0,2}|f_{0,2})=i\lambda(P_1(1-\tilde{\chi}_{(0,1),4})v_{(0,1),4}|f_{(0,1),4}).$$

Recalling that $\lambda v_{0,2}=P_2^*u_{0,2}$, we can also write

$$\theta=i(P_1(1-\tilde{\chi}_{0,2})P_2^*u_{0,2}|f_{0,2})=i(P_2^*(1-\tilde{\chi}_{0,1})P_1u_{0,1}|f_{0,1})= i(P_2^*P_1(1-\chi_{0,1})u_{0,1}|f_{0,1}),$$

where $\tilde{\chi}_{0,1}P_1=P_1\chi_{0,1}$. Here $P_2^*P_1$ maps Ker(H_1) to itself, so $P_2^*P_1u_{0,1}=\mu u_{0,1}$ for some complex number μ. Since V commutes with P_j and P_j^*, and since $Vu_{0,1}=u_{0,1}$, we find $P_2^*P_1u_{0,1}=\bar{\mu}u_{0,1}$, and hence μ is real. Using (1.18) and the fact that $V^2=I$,

$$(P_2^*P_1(1-\chi_{0,1})u_{0,1}|f_{0,1})=(V^2P_2^*P_1(1-\chi_{0,1})u_{0,1}|f_{0,1})$$
$$=\overline{(VP_2^*P_1(1-\chi_{0,1})u_{0,1}|Vf_{0,1})} = \overline{(P_2^*P_1\chi_{0,1}u_{0,1}|f_{0,1})}.$$

Here we also used that $V\chi_{0,1}V^{-1}=1-\chi_{0,1}$ and that $Vu_{0,1}=u_{0,1}$, $Vf_{0,1}=f_{0,1}$. Putting $\zeta=(P_2^*P_1(1-\chi_{0,1})u_{0,1}|f_{0,1})$, it follows that

$$\zeta+\bar{\zeta}=(P_2^*P_1u_{0,1}|f_{0,1})=\mu(u_{0,1}|f_{0,1})=\mu,$$

(proving again that μ is real). Since $\mu\neq 0$ by the ellipticity of P_j, we deduce that, $\mathrm{Re}\zeta\neq 0$, and hence that $\mathrm{Im}\theta\neq 0$. More precisely, we have showed that in the same rectangle as in Proposition 4.2, we have, $x=b\theta+\mathcal{O}(e^{-1/C_0h})$, $z=a\theta+\mathcal{O}(e^{-1/C_0h})$, where θ is an analytic symbol of order 0, such that $\mathrm{Im}\theta$ is elliptic. Here $C_0>0$. Recall that the renormalization Q of E_{-+} is the h'-quantization of the symbol Q, given by (2.21), and that the renormalizations Q_j are obtained from A_j by the formula,

(4.92) $\quad Q_j(x,\xi)=\Sigma_\alpha\, a_j(\alpha)e^{-i\alpha_1\alpha_2h'/2}\, e^{-i\langle\delta(\alpha),(x,\xi)\rangle}.$

We then get,

<u>Proposition 4.3.</u> There is a constant $C_0>0$, such that for μ' in the rectangle of Proposition 4.2 and for $|\mathrm{Im}(x,\xi)|\le 1/C_0h$, we have,

(4.93) $$Q(x,\xi)=[b+\bar{a}e^{-i\xi}]_{2,1}+[\bar{b}+ae^{ix}]_{2,3}+[\bar{b}+ae^{-ix}]_{4,1}+[b+\bar{a}e^{i\xi}]_{4,3}+\mathcal{O}(e^{-1/C_0h}),$$

(4.94) $$Q_j(x,\xi)= [\theta b+\bar{\theta}\bar{a}e^{-i\xi}]_{2,1}+[\bar{\theta}\bar{b}+\theta ae^{ix}]_{2,3}+[\bar{\theta}\bar{b}+\theta ae^{-ix}]_{4,1}+[\theta b+\bar{\theta}\bar{a}e^{i\xi}]_{4,3}+\mathcal{O}(e^{-1/C_0h}).$$

Here it is understood that $\bar{a}$, $\bar{b}$, $\bar{\theta}$ denote the holomorphic extensions of the complex conjugates of a, b, θ on the real domain.

We have already observed in section 2, that Q_j may be replaced by any real linear combination of Q and Q_j with coefficients that do not depend on j. Since $\mathrm{Im}\theta$ is elliptic, this means that we may assume (4.94), where θ is any (convenient) complex number. Let us denote by Q^0 and Q_j^0 the explicit matrices appearing to the right in (4.93),(4.94). After modification of a,b, by exponentially small quantities, which do not affect the validity of the last proposition, we may assume that,

(4.95) $\arg(b)=\arg(a)+\pi/2$,

so that $b\bar{a}/i>0$, and $b^2-a^2=(|b|^2+|a|^2)e^{i2\arg(b)}$. Then we get,

(4.96) $\det Q^0(x,\xi)=2i(|b|^2+|a|^2)\sin(2\arg(b))+2b\bar{a}(\cos(\xi)+\cos(x))=$
$= 2b\bar{a}((b\bar{a}/i)^{-1}(|b|^2+|a|^2)\sin(2\arg(b))+\cos(\xi)+\cos(x))$,

(4.97) $\det Q_j^\theta(x,\xi)=$
$2|\theta|^2b\bar{a}((b\bar{a}/i)^{-1}(|b|^2+|a|^2)\sin(2\arg(\theta b))+\cos(\xi)+\cos(x))$.

Recalling that $b=\beta/i=(\lambda\beta)/(\lambda i)$, and similarly for a, we obtain from our earlier results,

(4.98) $b=(2\pi)^{-\frac{1}{2}}\Gamma(\frac{1}{2}-i\mu'/h)e^{i\mu'\log(1/h)/h+\pi\mu'/2h+f(\mu',h)/h}$,

(4.99) $a=(2\pi)^{-\frac{1}{2}}\Gamma(\frac{1}{2}-i\mu'/h)e^{i\mu'\log(1/h)/h-\pi\mu'/2h+f(\mu',h)/h-i\pi/2}$,

where f is a classical analytic symbol of order ≤ 0, such that Re(f) is of order ≤ -1. From this it follows that,

(4.100) $|a|^2+|b|^2=e^{2\mathrm{Re}(f)/h}$, $b\bar{a}/i=e^{2\mathrm{Re}(f)/h}/2\mathrm{ch}(\pi\mu'/h)$,

when μ is real. Using this in (4.96), we get:

(4.101) $\det(Q^0)=2b\bar{a}(2\mathrm{ch}(\pi\mu'/h)\sin(2\arg(b))+\cos(\xi)+\cos(x))$

The formulas (4.98),(4.99) are a little less precise than earlier corresponding ones, but as we shall see later, they carry enough information, in order to continue the renormalization procedure.

Let us finally formulate the main result of this section,

Proposition 4.4. Let (P,P_1,P_2) be a strong type 1 operator with $0<h\leq 2\pi$ and with $0\leq\varepsilon(P)\leq\varepsilon_0$, where $\varepsilon_0>0$ is sufficiently small. Define $\mu_0(P)=\mathcal{O}(\varepsilon(P))$ as in Proposition 3.4. There exists $h_1>0$, such that for $0<h\leq h_1$, we have an analytic diffeomorphism $\mu\longrightarrow\mu'$ from a neighborhood of μ_0 to a neighborhood of 0 such that $\mu'(\mu_0,h)=\mathcal{O}(h)$. μ' and its inverse are given by classical analytic symbols of order 0. Moreover, for $h_1>0$ and $\varepsilon_1>0$ sufficiently small, the μ'-spectra of P and Q coincide in the interval $[-\varepsilon_1,\varepsilon_1]$, if Q denotes the h'-quantization of the matrix symbol Q, given by (4.93). Here $2\pi/h\equiv h'/2\pi$ mod($\mathbb{Z}$), $0<h'\leq 2\pi$, and a, b are given by (4.98), (4.99) (satisfying also (4.100)). For every $\theta\in\mathbb{C}$, there exist symbols Q_j satisfying (4.94), such that (Q,Q_1,Q_2) is a type 2 operator in the sense of section 2. (Here Q_j also denote the corresponding h'-quantizations.)

5. Preliminaries for the renormalized operator of section 4.

In section 4, we proved that the renormalization of E_{-+} is given by:

(5.1) $$Q(x,\xi)=[b+\bar{a}e^{-i\xi}]_{2,1}+[\bar{b}+ae^{ix}]_{2,3}+[\bar{b}+ae^{-ix}]_{4,1}+[b+\bar{a}\,e^{i\xi}]_{4,3} +\mathcal{O}(e^{-1/C_0h}),$$

when $|\mathrm{Re}\mu'|<\varepsilon_0$, $|\mathrm{Im}\mu'|<\varepsilon_1 h$, for some sufficiently small ε_0 and $h>0$, when ε_1 is fixed in $]0,\frac{1}{2}[$. Here,

(5.2) $$b=(2\pi)^{-\frac{1}{2}}\Gamma(\tfrac{1}{2}-i\mu'/h)e^{i(\mu'/h)\log(1/h)+\pi\mu'/2h+if(\mu',h)/h},$$

(5.3) $$a=(2\pi)^{-\frac{1}{2}}\Gamma(\tfrac{1}{2}-i\mu'/h)e^{i(\mu'/h)\log(1/h)-\pi\mu'/2h+if(\mu',h)/h-i\pi/2},$$

where f is a classical analytic symbol of order ≤ 0, such that $\mathrm{Im}f$ is of order ≤ -1. After multiplication of Q by $e^{\mathrm{Im}f/h}$ times the identity matrix, we may assume from now on that f is real valued. Then for real μ':

(5.4) $$|a|^2+|b|^2=1,\ \arg(b)-\arg(a)=\pi/2,$$

(5.5) $$|b|=e^{\pi\mu'/2h}(e^{\pi\mu'/h}+e^{-\pi\mu'/h})^{-1/2},$$

(5.6) $$|a|=e^{-\pi\mu'/2h}(e^{\pi\mu'/h}+e^{-\pi\mu'/h})^{-1/2},$$

(5.7) $$|a|\,|b|=1/2\mathrm{ch}(\pi\mu'/h).$$

Using the complex version of Stirling's formula, we also had,

(5.8) $$b=e^{i\mu'/h-i(\mu'/h)\log(\mu')+i(h/\mu')F(\mu'/h)+if(\mu',h)/h},$$

for $\mu'/h\geq\mathrm{const.}>0$, where $F(z)$ is a bounded holomorphic function in a sector $|\mathrm{Im}z|<\mathrm{Re}(z)/C$, $\mathrm{Re}(z)>C$, and,

(5.9) $$\mathrm{Im}F(\mu'/h)=\mathcal{O}(1)(\mu'/h)e^{-2\pi\mu'/h},$$

when $\mu'/h>C$. This relation can also be differentiated with respect to μ'.

Consider the function $B=i^{-1}\log(b)$. Then for $|\mathrm{Re}\mu'/h|\leq\mathrm{const.}$, $|\mathrm{Im}\mu'/h|\leq\varepsilon_1$, we get from (5.2):

(5.10) $$\partial_{\mu'}B=h^{-1}\log(1/h)+\mathcal{O}(h^{-1})=(1+o(1))h^{-1}\log(1/h),\ h\to 0.$$

For $\mathrm{Re}\mu'/h\geq\mathrm{const.}>0$, $|\mathrm{Im}\mu'/h|\leq\varepsilon_1$, we can apply (5.8):

$$\partial_{\mu'}B=-h^{-1}\log\mu'+F'(\mu'/h)/\mu'-h(\mu')^{-2}F(\mu'/h)+\partial_{\mu'}f(\mu',h)/h.$$

Restricting this to the real axis, we get for $\mu'/h\geq\mathrm{const.}>0$:

(5.11) $$\partial_{\mu'}B=(h^{-1}\log(1/\mu')+\mathcal{O}(h^{-1}))+i\mathcal{O}(\mu'^{-1}+h^{-1}+\mu'h^{-2})e^{-2\pi\mu'/h}.$$

For complex μ' with $\mathrm{Re}\mu'/h\geq\mathrm{const.}>0$, we get,

$$\partial_{\mu'}B=-h^{-1}\log\mu'+\mathcal{O}(|\mu'|^{-1}+h|\mu'|^{-2}+h^{-1}),$$

which we can write as,

(5.12) $$\partial_{\mu'}B=-h^{-1}\log|\mu'|+\mathcal{O}(h^{-1}).$$

Combining (5.10) and (5.12), we get,

(5.13) $$\partial_{\mu'}B=h^{-1}\log((h+|\mu'|)^{-1})+\mathcal{O}(h^{-1}),$$

and this relation is also valid for $\mathrm{Re}\mu'/h\leq-\mathrm{const.}<0$, and hence everywhere in the rectangle $|\mathrm{Re}\mu'|<\varepsilon_0$, $|\mathrm{Im}\mu'|<\varepsilon_1 h$. In view of (5.11), we also know that $\partial_{\mu'}\mathrm{Im}B$ is $\mathcal{O}(h^{-1}e^{-\mu'/C_0h})$ for $\mu'/h\geq\mathrm{const.}>0$.

Put,

(5.14) $$Q_0(x,\xi)=[b+\bar{a}e^{-i\xi}]_{2,1}+[\bar{b}+ae^{ix}]_{2,3}+[\bar{b}+ae^{-ix}]_{4,1}+[b+\bar{a}\,e^{i\xi}]_{4,3}.$$

Then for real μ', we get from (4.101) and (5.7),(5.4), (5.15) $\det Q_0(x,\xi)=(i/\mathrm{ch}(\pi\mu'/h))[2\mathrm{ch}(\pi\mu'/h)\sin(2\arg(b))+\cos\xi+\cos x]$.

The real characteristic set of Q_0 is non-empty precisely when,

(5.16) $\mathrm{ch}(\pi\mu'/h)\sin(2\arg(b))\in[-1,1]$.

Cases of special interest appear when,

(5.17) $\mathrm{ch}(\pi\mu'/h)\sin(2\arg b)=-1$, then $(\det Q_0)^{-1}(0)=2\pi\mathbb{Z}^2$,

(5.18) $\mathrm{ch}(\pi\mu'/h)\sin(2\arg b)=1$, then $(\det Q_0)^{-1}(0)=2\pi\mathbb{Z}^2+(\pi,\pi)$,

(5.19) $\mathrm{ch}(\pi\mu'/h)\sin(2\arg b)=0$, then $(\det Q_0)^{-1}(0)$ is the union of the lines $\xi\pm x=(2k+1)\pi$, $k\in\mathbb{Z}$.

In order to understand the structure of the set of real μ' such that (5.16) holds, we split the discussion into two cases: 1^0, $|\mu'/h|\le C_0$, 2^0, $|\mu'/h|\ge C_0$. Here C_0 will be choosen sufficiently large in the discussion of case 2^0. In the discussion of the case 1^0 it may be arbitrarily large, provided that $h>0$ is sufficiently small.

1^0 $|\mu'/h|\le C_0>0$. Put $\zeta(\mu',h)=\mathrm{ch}(\pi\mu'/h)e^{2i\arg(b)}$. Since $\partial_{\mu'}\arg(b)=(1+o(1))h^{-1}\log(1/h)$, and by Cauchy's inequalities, $(\partial^j_{\mu'}\arg(b)=\mathcal{O}(h^{-j}\log(1/h))$, it follows that,

$$\partial_{\mu'}\zeta(\mu',h)/\zeta(\mu',h)=2i(1+o(1))h^{-1}\log(1/h),$$

$$\partial^2_{\mu'}\zeta(\mu',h)/\zeta(\mu',h)=-4(1+o(1))(h^{-1}\log(1/h))^2.$$

The curvature of $\mu'\to\zeta(\mu',h)\in\mathbb{C}\simeq\mathbb{R}^2$, is given by,

$$|\partial_{\mu'}\zeta|^{-2}\mathrm{Re}(\partial^2_{\mu'}\zeta\,i\overline{\partial_{\mu'}\zeta})=8(1+o(1))(h^{-1}\log(1/h))^3,$$

so for $h>0$ small enough the curvature is strictly positive. Moreover $|\zeta(\mu',h)|=\mathrm{ch}(\pi\mu'/h)\ge1$ with equality precisely for $\mu'=0$. Now (5.16) holds precisely when $\mathrm{Im}\zeta(\mu',h)\in[-1,1]$, so it follows that the set of μ' satisfying (5.16) is a union of closed intervals of length $\sim h/\log(1/h)$. All these intervals are disjoint except possibly two. This exceptional situation can appear only when $\sin(\arg(b))=\pm1$ for $\mu'=0$, i.e. when $\arg(b(0,h))=\pi/4+n\pi/2$, $n\in\mathbb{Z}$. Then the two intervals have $\mu'=0$ as a common boundary point. In a region, where $|\mu'/h|\ge$ const.>0, the distance between two consecutive intervals is of the same order of magnitude as the length of these intervals. In the region where $|\mu'/h|$ is small, if μ' is between two intervals, then the distance between the two intervals is of the order of magnitude, $(2h/\log(1/h))|\mathrm{sh}(\pi\mu'/h)|$. Also, notice that $\mu'=0$ always belongs to one of the intervals.

2^0 $|\mu'/h|\ge C_0>0$. Then $\mathrm{ch}(\pi\mu'/h)\ge$ const.>1, and we rewrite (5.15),(5.16), as,

(5.15)′ $\det Q_0=i[2\sin(2\arg(b))+(\mathrm{ch}(\pi\mu'/h))^{-1}(\cos\xi+\cos x)]$,

(5.16)′ $\sin(2\arg(b))\in[-1/\mathrm{ch}(\pi\mu'/h),1/\mathrm{ch}(\pi\mu'/h)]$.

In the region $|\mu'/h|\ge$ const.>0, $|\sin(2\arg b)|\le$ const.<1, we have, $|\partial_{\mu'}\sin(2\arg(b))|\sim h^{-1}\log(1/|\mu'|)>>\partial_{\mu'}(1/\mathrm{ch}(\pi\mu'/h))$, so the set of μ' satisfying (5.16)′ is a union of closed disjoint intervals, such that if μ' is in the separation of two such intervals, then the separation is of the order of magnitude $h/\log(1/|\mu'|)$, and if μ' is in such an interval, then the length of

that interval is $(2+o(1))he^{-\pi|\mu'|/h}(\log(1/|\mu'|))^{-1}$ when $h\to 0$, $|\mu'|\to 0$, $h/|\mu'|\to 0$.
Still in case 2^0, we notice that either $|b|/|a|\geq$const.>1, or $|a|/|b|\geq$const.>1, and $Q_0(x,\xi)$ can never vanish completely for real x,ξ. In other words, the worse that can happen is that $Q_0(x,\xi)$ (or $Q(x,\xi)$) has a 1-dimensional kernel for some real x,ξ.

In the case 1^0, let us look for the special situations when Q_0 may vanish completely for some real μ',x,ξ. Assuming we are at such a point, since both a and b are $\neq 0$, we get from $b+\bar{a}e^{-i\xi}=b+\bar{a}e^{i\xi}=0$, that $e^{-i\xi}=e^{i\xi}$, so $\xi=k\pi$, $k\in\mathbb{Z}$. Similarly, we must have $x=l\pi$, $l\in\mathbb{Z}$. We then get $b+(-1)^k\bar{a}=0$, $\bar{b}+(-1)^l a=0$, and comparing the last equation with the complex conjugate of the first one, we see that $(-1)^k=(-1)^l$, so k and l must have the same parity. Since Q_0 is 2π-periodic both in x and in ξ, it suffices to study the cases of complete degeneration at (0,0) and at (π,π). The complete degeneration at (0,0) appears iff $b+\bar{a}=0$, and it is easy to see that this happens precisely when $\mu'=0$, and $\arg(b(0,h))=3\pi/4+n\pi$, $n\in\mathbb{Z}$. Complete degeneration at (π,π) appears iff $b-\bar{a}=0$, which happens precisely when $\mu'=0$ and $\arg(b(0,h))=\pi/4+n\pi$. The complete degeneration is a rather exceptional case, but if $|\mu'/h|$ is small, then $|a|\approx|b|$, and $Q_0(0,0)$ and $Q_0(\pi,\pi)$ become small respectively, when $\arg(b)\approx 3\pi/4+n\pi$, and $\arg(b)\approx\pi/4+n\pi$. These cases appear near the end-points of the intervals given by (5.16)'.

We end this section by giving some qualitative statements, which are more easy to carry on in a general iteration scheme. Let $P_0(a,b;x,\xi)$ denote the matrix given by (5.14).

<u>Definition 5.1.</u> The triple of h-pseudodifferential operators (P,P_1,P_2) is of strong type 2, if it is of type 2 and if for the symbols, we have $|P(x,\xi)-P_0(a,b;x,\xi)|\leq\varepsilon$, $|P_j(x,\xi)-P_0(ia,ib;x,\xi)|\leq\varepsilon$ for $|\mathrm{Im}(x,\xi)|<1/\varepsilon$, $|\mu|<4$. Here it is further assumed that P,P_j,a,b depend holomorphically on μ, that $a=a(\mu)$ and $b=b(\mu)$ satisfy,

(5.20) $\quad |a|^2+|b|^2=1,\ |\arg(b)-\arg(a)|=\pi/2$, for μ real,

and

(5.21) $\quad b(\mu)=b(0)(1+\mathcal{O}(\varepsilon))e^{i\mu(1+\mathcal{O}(\varepsilon))}$,

where each $\mathcal{O}(\varepsilon)$ indicates a holomorphic term of modulus $\leq\varepsilon$, which is real when μ is real. We define $\varepsilon(P)$ to be the infimum of all ε satisfying the above inequalities, and we define $C(P)$ as $\max(1/|a(0)|,1/|b(0)|)$.

In this definition the number 4 could easily be replaced by any fixed strictly positive number.

From the discussion above and Proposition 4.4, it is easy to obtain the following result,

<u>Proposition 5.2.</u> Let (P,P_1,P_2) be a strong type 1 operator with $0<h\leq h_0$ and $\varepsilon(P)\leq\varepsilon_0$, with $h_0,\varepsilon_0>0$ sufficiently small, so that we can define the new

μ'-variable as in Proposition 4.4. Fix $0 \le C_0 < C_1$. For $C_0 h \le |\mu_0'| \le C_1 h$, we put $\mu'' = (\mu' - \mu_0')h^{-1}\log(h^{-1})$. Then for $0 < h \le h_1 > 0$ sufficiently small, depending on C_1 and for $|\mu''| < 4$, we can express $\mu - Sp(P)$ in the μ''-variables as $\mu'' - Sp(Q)$, where (Q, Q_j) is a strong type 2 h'-pseudodifferential operator (with μ replaced by μ'' in the definition above) with $\varepsilon(Q) \to 0$ as $h \to 0$ and with $C_-(C_0) \le C(Q) \le C_+(C_1)$, where $C_\pm$ only depend on C_0 and C_1 respectively, and where $C_-(C_0) \to +\infty$ when $C_0 \to +\infty$. We also recall that $2\pi/h \equiv h'/2\pi \bmod(\mathbb{Z})$, $0 < h' \le 2\pi$.

6. Reduction to type 1 operators.

Let Q be as in section 5. We shall here study the case when $|\mu'/h|$ is bounded from below by some sufficiently large constant. In the case when $\mu'>0$, we have $|b|>>|a|$ and Q_0 is close to a constant matrix. In the case $\mu'<0$, $|a|>>|b|$, and the variable part of Q_0 dominates. We shall first exhibit some symmetries that show that the second case is actually equivalent to the first one.

As before, we denote by $P_0(a,b;x,\xi)$ the expression (5.14) and we shall use the same letters to denote the corresponding h'-Weyl quantizations. We recall that the matrix P_0 maps $\mathbb{C}^2_{odd}$ into $\mathbb{C}^2_{even}$, and define L: $\mathbb{C}^2_{even} \to \mathbb{C}^2_{even}$, and M: $\mathbb{C}^2_{odd} \to \mathbb{C}^2_{odd}$, by

$L=[e^{i(x-\xi)/2}]_{2,4}+[e^{i(-x+\xi)/2}]_{4,2}$, $M=[e^{i(x+\xi)/2}]_{1,3}+[e^{i(-x-\xi)/2}]_{3,1}$.

Recalling the definition of T: $\mathbb{C}^2_{odd} \to \mathbb{C}^2_{even}$, $\mathbb{C}^2_{even} \to \mathbb{C}^2_{odd}$ in section 2, we check that,

(6.1) $\qquad \mathfrak{F}TM=L\mathfrak{F}T,\ \mathfrak{F}TL=M\mathfrak{F}T.$

A long but straightforward computation shows that for the h'-Weyl quantizations,

(6.2) $\qquad L\circ P_0(a,b)=P_0(e^{-ih'/4}\bar{b},e^{-ih'/4}\bar{a})\circ M.$

This gives in particular a unitary equivalence between the cases $|b|>>|a|$ and $|a|<<|b|$. In order to complete the symmetry discussion, we let P,P_1,P_2 be of type 2 as defined in section 2, and we define P',P_j', by:

(6.3) $\qquad LP=P'M,\quad LP_j=P_j'M.$

It is then straight forward to check that P', P_j' commute with the T_γ, that $P_1'^*P'$ and $P'P_2'^*$ are self-adjoint, and with the help of (6.1) and (2.7),(2.9), we also obtain (2.7),(2.9) for P',P_j'. Finally, we check that,

(6.4) $\qquad VL=LV,\ MVT^2=VT^2M,$

and using (2.8),(2.10), we obtain the same relations for P',P_j'. Hence P',P_j' is of type 2 and if $P=P_0(a,b)+\mathcal{O}(\varepsilon)$, $P_j=P_0(\theta a,\theta b)+\mathcal{O}(\varepsilon)$, then $P'=P_0(e^{-ih'/4}\bar{b},e^{-ih'/4}\bar{a})+\mathcal{O}(\varepsilon)$, $P_j'=P_0(e^{-ih'/4}\bar{\theta}\bar{b},e^{-ih'/4}\bar{\theta}\bar{a})+\mathcal{O}(\varepsilon)$. The conclusion of this discussion is that it suffices from now on to consider the case when $\mu'/h>0$.

Let Q be as in section 5. We shall now study Q for μ' in the region $Ch\le\mu'\le 1/C$, where C is some sufficiently large constant. Let μ'_0 be in this interval with $\sin(2\arg(b(\mu'_0)))=0$, so that

(6.5) $\qquad \arg(b(\mu'_0))=n\pi/2,$

for some $n\in\mathbb{Z}$. We rescale by introducing a new variable μ'',

(6.6) $\qquad \mu'-\mu'_0=(h/\log(\mu_0'^{-1}))\mu'',$

and restrict the attention to a region where $|\mu''|$ is bounded by a constant.

For 1/C and h small enough, we then have,

(6.7) $b(\mu',h)=(1+\mathcal{O}(\varepsilon\mu''))b(\mu'_0,h)e^{i(1+\mathcal{O}(\varepsilon))\mu''}$,

(6.8) $a(\mu',h)=(1+\mathcal{O}(\varepsilon\mu''))a(\mu'_0,h)e^{i(1+\mathcal{O}(\varepsilon))\mu''}$,

where $\varepsilon>0$ can be chosen arbitrarily provided that C and 1/h are sufficiently large. Put $\delta=|a(\mu'_0,h)|$. Then $|b(\mu'_0,h)|=(1-\delta^2)^{\frac{1}{2}}$, and choosing C (and 1/h) large enough, we may assume that,

(6.9) $Q(x,\xi)=Q_0(x,\xi)+\mathcal{O}(\delta^2)$, $|\mathrm{Im}(x,\xi)|\leq 1/\varepsilon$.

Recall (4.101),(5.7):

(6.10) $\det(Q_0)=2b\bar{a}(|ab|^{-1}\sin(2\arg(b))+\cos(\xi)+\cos(x))$.

This quantity is non vanishing for real (x,ξ) except when $\mu''=\mathcal{O}(\delta)$, and the same is true for $\det(Q)$, in view of (6.9).

Let $b_0=e^{in\pi/2}$, $P_{0,0}=P_0(0,b_0)$. Then $P_{0,0}$ is independent of (x,ξ) and $\mathrm{Ker}(P_{0,0})=(e_0)$, where $e_0=(2^{-\frac{1}{2}},(-1)^{n+1}2^{-\frac{1}{2}})$, and we also notice that $P_{0,0}$ is equal to i^n times a self adjoint matrix. We now define $R_+:L^2(\mathbb{R};\mathbb{C}^2)\to L^2(\mathbb{R})$, $R_-:L^2(\mathbb{R})\to L^2(\mathbb{R};\mathbb{C}^2)$, by $R_+u(x)=(u(x)|e_0)_{\mathbb{C}^2}$, $R_-u^-(x)=u^-(x)e_0$. Let $\mathcal{P}_0=[P_{0,0}]_{1,1}+[R_-]_{1,2}+[R_+]_{2,1}$: $L^2(\mathbb{R};\mathbb{C}^2)\times L^2(\mathbb{R})\to L^2(\mathbb{R};\mathbb{C}^2)\times L^2(\mathbb{R})$, and define $\mathcal{P}$ similarly with $P_{0,0}$ replaced by Q. In view of (6.7)-(6.9),

(6.11) $\mathcal{P}-\mathcal{P}_0=\mathcal{O}(\delta+|\mu''|)$,

in the sense of L^2 bounded operators, and also in the sense of Weyl symbols defined on $|\mathrm{Im}(x,\xi)|\leq 1/\varepsilon$. (Here we work with the h'-quantization.) Now $\mathcal{P}_0$ has a bounded inverse, $\mathcal{E}_0=[E_0]_{1,1}+[E_{0,+}]_{1,2}+[E_{0,-}]_{2,1}$, so it follows from (6.11) that the same is true for $\mathcal{P}$. We write $\mathcal{P}^{-1}=\mathcal{E}=[E]_{1,1}+[E_+]_{1,2}+[E_-]_{2,1}+[E_{-+}]_{2,2}$. As earlier, 0 belongs to the spectrum of P iff it belongs to the spectrum of E_{-+}. We have the Neumann series,

(6.12) $\mathcal{E}=\mathcal{E}_0(I-(\mathcal{P}-\mathcal{P}_0)\mathcal{E}_0+((\mathcal{P}-\mathcal{P}_0)\mathcal{E}_0)^2-\ldots)$.

In view of (6.11) we get:

(6.13) $\mathcal{E}-\mathcal{E}_0=-\mathcal{E}_0(\mathcal{P}-\mathcal{P}_0)\mathcal{E}_0+\mathcal{O}(\delta^2+|\mu''|^2)$,

for the symbols in the strip $|\mathrm{Im}(x,\xi)|\leq 1/\varepsilon$. In particular,

(6.14)
$$E_{-+}=-E_{0,-}(Q-P_{0,0})E_{0,+}+\mathcal{O}(\delta^2+|\mu''|^2)=$$
$$-E_{0,-}(P_0-P_{0,0})E_{0,+}+\mathcal{O}(\delta^2+|\mu''|^2)$$

Here $E_{0,+}: u\to u(x)e_0$, $E_{0,-}:v\to(v(x)|e_0)$, so the first term of the last member of (6.14) is equal to

(6.15)
$$-((P_0(x,\xi)-P_{0,0})e_0|e_0)=-(P_0(x,\xi)e_0|e_0)=$$
$$-[b+(-1)^{n+1}\bar{b}+\bar{a}\cos(\xi)+(-1)^{n+1}a\cos(x)].$$

Using that $b(\mu'_0)=(1-\delta^2)^{\frac{1}{2}}i^n$, $a(\mu'_0)=\delta i^{n-1}$, we get from (6.7),(6.8):

(6.16) $b(\mu')=(1+\mathcal{O}(\varepsilon\mu''))(1-\delta^2)^{\frac{1}{2}}i^n e^{i(1+\mathcal{O}(\varepsilon))\mu''}$,

(6.17) $a(\mu')=(1+\mathcal{O}(\varepsilon\mu''))\delta i^{n-1}e^{i(1+\mathcal{O}(\varepsilon))\mu''}$.

Here all the $\mathcal{O}$-terms are real for μ'' real, and the ones in the exponents are the same. Combining this with (6.15) and (6.14), we get,

(6.18) $$E_{-+}(x,\xi)=i^{n-1}[2\sin((1+\mathcal{O}(\varepsilon))\mu'')+(-1)^n\delta(\cos(\xi)+\cos(x))]+\mathcal{O}(\delta^2+|\mu''|^2),$$

and possibly after increasing ε, this relation remains valid in the complex strip, $|\mathrm{Im}(x,\xi)|\leq 1/\varepsilon$.

For $|\mu''|>>\delta$, the first term in (6.18) dominates, so E_{-+} is bijective in this region. When $|\mu''|\leq(\mathrm{Const.})\delta$, we have,

(6.19) $$E_{-+}=i^{n-1}\delta[(2/\delta)\sin((1+\mathcal{O}(\varepsilon))\mu'')+(-1)^n(\cos(\xi)+\cos(x))+\mathcal{O}(\delta)],$$

for $|\mathrm{Im}(x,\xi)|\leq 1/\varepsilon$. Introducing $\pm(2/\delta)\sin((1+\mathcal{O}(\varepsilon))\mu'')$ as a new spectral parameter, we see that $E_{-+}/(i^{n-1}\delta)$ satisfies the first part of the condition (3.2) for type 1 operators with a new parameter ε, that can be chosen arbitrarily small. As in section 2 we see that $(R_-^*P_1E_+)^*E_{-+}$ and $E_{-+}(E_-P_2R_+^*)^*$ are self-adjoint. Choosing $\theta=i$ for Q_j it is easy to check by the same argument as above, that the self-adjoining operators $i^{n-1}R_-^*Q_1E_+$ and $i^{n-1}E_-Q_2R_+^*$ after multiplication by a common real factor satisfy the second part of (3.2). Moreover, it is easy to see that all the invariance properties for a type 1 system are satisfied, so E_{-+} can again be studied by applying the results of section 3 and 4. More precisely, we have,

Proposition 6.1. There exist $h_0,\varepsilon_0,C_0>0$ such that the following holds: Let (P,P_1,P_2) be a strong type 1 operator with $\varepsilon(P)\leq\varepsilon_0$, and $0<h\leq h_0$. We can then introduce the new μ'-variable as in Proposition 4.4, and for $C>C_0$, we have $\mu'\text{-}\mathrm{Sp}(P)\cap\{\mu';Ch\leq\mu'\leq 1/C\}\subset\cup_{1\leq j\leq N_+}J_j$, where J_j are closed disjoint intervals, labelled in increasing order such that the width of J_j is of the order of magnitude $he^{-\pi\mu'/h}(\log(1/\mu'))^{-1}$ (,where $\mu'\in J_j$), and the separation between two consecutive intervals is of the order of magnitude $h(\log(1/\mu'))^{-1}$ (,where μ' is in the separation). Moreover, if κ_j: $\mu'\rightarrow\mu''$ is a suitable increasing affine map, then $\kappa_j(J_j\cap\mu'\text{-}\mathrm{Sp}(P))=\mu''\text{-}\mathrm{Sp}(Q)$, where $Q=Q_j$ is a strong type 1 h'-pseudodifferential operator with $\varepsilon(Q)\rightarrow 0$ as $h,1/C\rightarrow 0$, uniformly with respect to j and with respect to the choice of P as above, with $\varepsilon(P)\leq\varepsilon_0$. The analogous result holds in the region $-1/C\leq\mu'\leq -Ch$. Here $2\pi/h\equiv h'/2\pi\ \mathrm{mod}(\mathbb{Z})$, $0<h'\leq 2\pi$.

It also follows from the discussion in this section, that if Q is a strong type 2 h'-pseudodifferential operator, $0<h'\leq 2\pi$, (and writing μ' instead of μ), then if $C(Q)\geq C_0$, where C_0 is sufficiently large, and if $\varepsilon(Q)\leq\alpha(C(Q))$, for some strictly positive decreasing function, α on $[C_0,\infty[$, then $\mu'\text{-}\mathrm{Sp}(Q)=\cup J_j$, where J_j are closed disjoint intervals such that for each j, there is an affine map $\kappa_j:\mu'\rightarrow\mu''$, such that $\kappa_j(J_j\cap\mu'\text{-}\mathrm{Sp}(Q))=\mu''\text{-}\mathrm{Sp}(\tilde{Q})$, where $\tilde{Q}=\tilde{Q}_j$ is a strong type 1 operator with $\varepsilon(\tilde{Q})\leq\beta(C(Q))$. Here $\beta(C)\rightarrow 0$ as $C\rightarrow+\infty$.

Combining this with the propositions 3.4, 4.1 and 5.2, we get, Theorem 6.2. There exist $\varepsilon_0>0$ and functions, $F:]0,1]\to[1,\infty[$, $h_0:]0,1]\to]0,1]$, $\alpha:]0,1]^2\to]0,1]$ with $\alpha(\varepsilon,h)\to 0$ when $h\to 0$ for every fixed ε, such that if $\varepsilon\in]0,1]$, and P is of strong type 1 with $\varepsilon(P)\le\varepsilon_0$, $0<h\le h_0(\varepsilon)$, then $\mu-\mathrm{Sp}(P)\subset\cup_{-N_-\le j\le N_+} J_j$, where J_j are closed disjoint intervals, labelled in increasing order, such that for each j, there is an affine increasing map $\kappa_j:\mu\to\mu''$ such that:

a) For $j\neq 0$, $\kappa_j(J_j\cap\mu-\mathrm{Sp}(P))=\mu''-\mathrm{Sp}(Q)$, where $Q=Q_j$ is a strong type 1 h'-pseudodifferential operator with $\varepsilon(Q)\le\beta(C)$.

b) For $j=0$, $\kappa_0(J_0\cap\mu-\mathrm{Sp}(P))=\mu''-\mathrm{Sp}(Q)$, where for every $\mu_0''\in\kappa_0(J_0)$, Q is a strong type 2 h'-pseudodifferential operator with $\varepsilon(Q)\le\alpha(\varepsilon,h)$, $C(Q)\le F(\varepsilon)$, as a function of $\mu'''=\mu''-\mu_0''\in]-4,4[$.

This is is the main result, concerning type 1 operators. We also notice that in terms of the variables μ' introduced in Proposition 4.4, the length of the interval J_0 is of the order of magnitude h (when C is fixed) and that the lengths and separations of the other J_j's in the domain of the μ'-variables are given by Proposition 5.2. Finally outside the domain of the μ'-variables, the lengths are $e^{-C_j/h}$ with C_j of the order of magnitude 1 and the separations are of the order of magnitude h.

7. Type 2 operators close to the totally degenerate case.

Here and in the next two sections we shall consider a general h-pseudodifferential triple, (P,P_1,P_2) of strong type 2, and the corresponding (original) spectral parameter will be denoted by μ. Let $P_0(a,b;x,\xi)$ be as in Definition 5.1. Using that $|a|^2+|b|^2=1$, $|\arg(b)-\arg(a)|=\pi/2$ we get for real values of μ,

(7.1) $\det(P_0(a,b;x,\xi))=2i\alpha(\alpha^{-1}\sin(2\arg(b))+\cos(\xi)+\cos(x))$,

where $\alpha=b\bar{a}/i$ is real and of absolute value $\leq 1/2$.

In this section, we are interested in the case when P may come close to the zero matrix for some real (x,ξ). As we saw in section 5, $P_0(a,b;x,\xi)$ may vanish completely, only if $|a|=|b|$. In the case, $\arg(b)-\arg(a)=\pi/2$, we saw in section 5 that $P_0(a,b;x,\xi)$ vanishes completely at $(x,\xi)=(0,0)$, when $\arg(b)=3\pi/4+n\pi$, and at (π,π), when $\arg(b)=\pi/4+n\pi$. These are the only points of complete degeneration in that case. When $|a|=|b|=2^{-\frac{1}{2}}$, $\arg(b)-\arg(a)=-\pi/2$, the same discussion shows that $P_0(a,b;x,\xi)$ vanishes completely at $(x,\xi)=(0,0)$, when $\arg(b)=\pi/4+n\pi$, and at (π,π), when $\arg(b)=3\pi/4+n\pi$, and that these are the only points where P_0 vanishes.

The eight cases can be treated the same way and in order to fix the ideas, let us assume that

(7.2) $\arg(b(\mu))-\arg(a(\mu))=\pi/2$,

and we wish to study P in a region where $|b|\approx 2^{-\frac{1}{2}}$, $\arg(b)\approx 3\pi/4$. More precisely, we shall assume that,

(7.3) $\arg(b(0))=3\pi/4$, $||b(0)|-2^{-\frac{1}{2}}|\leq\varepsilon$,

and the discussion below will be uniformly valid for μ in some small fixed neighborhood of 0, provided that $\varepsilon(P),\varepsilon,h>0$ are sufficiently small. Without any loss of generality, we may assume that,

(7.4) $\varepsilon(P)\leq\varepsilon$.

Combining (7.3),(7.1) and Definition 5.1, we get,

(7.5) $\det P_0(x,\xi)=i(1+\mathcal{O}(\varepsilon))[\cos\xi+\cos x-2+4\mu^2+\mathcal{O}(\varepsilon)+\mathcal{O}(\mu^4)]$,

for $|(x,\xi)|,|\mu|\leq 1/C_0$, where C_0 is large but independent of ε. When μ is real, we also know that the expression inside the bracket is real. When μ^2/ε is larger than some constant, we see that $(x,\xi)\longrightarrow\det P_0(x,\xi)$ vanishes on a Jordan curve around (0,0) which is close to a circle of radius $8^{\frac{1}{2}}|\mu|$.

It is easy to check that $\det P_j\neq 0$, for $|\mu|\leq 1/C_0$, $|(x,\xi)|\leq 1/C_0$. Consider then the self-adjoint operator, $H_1=P_1^*P$. On the symbol level,

(7.6) $H_1(x,\xi)=P_1^*(x,\xi)P(x,\xi)+\mathcal{O}(h)$.

$H_1(x,\xi)$ is a selfadjoint matrix for μ and (x,ξ) real. Combining (7.5),(7.6) and the fact that $\det P-\det P_0=\mathcal{O}(\varepsilon)$, (which follows from (7.4),) we get,

(7.7) $\det H_1(x,\xi)=f(x,\xi,\mu)[\cos\xi+\cos x-2+4\mu^2+\mathcal{O}(\varepsilon)+\mathcal{O}(\mu^4)+\mathcal{O}(h)]$,

where f is a non-vanishing real-valued analytic function. Assuming from now on that $0<h\leq 1/C_0$, we see that when μ is real and $\mu^2/(\varepsilon+h)$ is large, then

$\det H_1$ vanishes on a Jordan curve around (0,0), close to a circle of radius $8^{\frac{1}{2}}|\mu|$. According to our results on normal forms for systems in appendix c, there exist classical analytic symbols $Z(x,\xi,\mu,h)$, $C_1(\mu,h)$, $C_2(\mu,h)$ of order 0, defined for $|\mu|\leq 1/C_0$, $|(x,\xi)|\leq 1/C_0$ (and varying in a bounded set of analytic symbols, if P,P_j depend on additional parameters, but remaining within the bounds that we have specified for P, and P_j), such that in the sense of classical analytic pseudodifferential operators,

(7.8) $\qquad U^*H_1U=[C_1]_{1,1}+[C_2]_{2,2}+[Z]_{1,2}+[Z^*]_{2,1}.$

Here U is an elliptic classical analytic Fourier integral operator of order 0 as in Proposition c.3, depending analytically μ in the sense that it can be realized with a phase $\varphi(x,y,\theta,\mu)$ which is analytic in all variables, and with an amplitude, $a(x,y,\theta,\mu,h)$, which is a classical analytic symbol. The principal part $\zeta(x,\xi,\mu)$, (of Z) satisfies $|\zeta|\sim|(x,\xi)|$ and $i^{-1}\{\zeta,\bar{\zeta}\}>0$, and C_j are real-valued. Again, if $\mu^2/(\varepsilon+h)$, is large, the determinant of the symbol of (7.8) vanishes roughly on a circle of radius $|\mu|$ around the origin, and we conclude that in the same region,

(7.9) $\qquad C_1(\mu,h)C_2(\mu,h)\approx\mu^2.$

Using that $P(0,0)=\mathcal{O}(\varepsilon)$, when $\mu=0$, we also get,

(7.10) $\qquad C_j(0,h)=\mathcal{O}(\varepsilon+h).$

Let $c_j(\mu)$ denote the principal part of C_j. Considering the Taylor expansions $c_j(\mu)=c_j(0)+c_j'(0)\mu+\mathcal{O}(\mu^2)$, we get from (7.9), that,

(7.11) $\qquad c_1'(0)c_2'(0)>\text{const.}>0.$

We may assume that $c_j'(0)>\text{const.}>0$, and we are then in the situation when C_j are both strictly increasing functions of μ changing sign somewhere in the interval $[-C(\varepsilon+h),C(\varepsilon+h)]$.

In order to study the operator (7.8), we first consider the operator Z^*Z. According to our results on normal forms for scalar pseudodifferential operators in appendix b, we can find a real valued analytic symbol $F=F(\mu,t,h)$ of order 0, and a unitary Fourier integral operator W, such that,

(7.12) $W^{-1}F(\mu,Z^*Z,h)W=P_0=\frac{1}{2}((hD)^2+x^2-h).$

Using this, we can define the k:th eigenvalue of Z^*Z (microlocally) by,

(7.13) $\qquad F(\mu,\lambda_k,h)=kh,$

provided that $kh\leq 1/C_0$ (,where $C_0>0$ is some new fixed constant). Since Z has a one-dimensional kernel (microlocally), it is clear that $\lambda_0=0$, and hence,

(7.14) $\qquad F(\mu,0,h)=0.$

Recalling that $\partial_tF(\mu,0,h)>0$, we let $G=G(\mu,s,h)$ be the inverse of the map $t\rightarrow F$, so that G is also a classical analytic symbol of order 0. Then (7.13), can also be written,

(7.15) $\qquad \lambda_k=G(\mu,kh,h).$

A normalized eigenfunction u_k of Z^*Z is obtained (microlocally) as a multiple of

Wu_k^0, where u_k^0 is the k-th normalized eigenfunction of the harmonic oscillator, P_0:

(7.16) $\quad Z^*Zu_k=\lambda_k u_k.$

Then,

(7.17) $\quad ZZ^*(Zu_k)=\lambda_k(Zu_k),$

and for $k\geq 1$, we have $\|Zu_k\|\geq(h/\text{Const.})^{\frac{1}{2}}$, so we conclude that $\lambda_1,\lambda_2,\ldots$ are also among the eigenvalues of ZZ^*. Conversely, let $ZZ^*v_k=\mu_k v_k$, microlocally, near (0,0). Then $(Z^*Z)Z^*v_k=\mu_k Z^*v_k$, and since $\|Z^*v\|\geq(h/\text{Const.})^{\frac{1}{2}}\|v\|$, we conclude that μ_k is an eigenvalue of Z^*Z. The conclusion of this discussion is that the low eigenvalues of ZZ^* (defined modulo $\mathcal{O}(e^{-1/Ch})$ by reduction of ZZ^* to P_0) are the values $\lambda_1,\lambda_2,..,\lambda_k,..$, $kh\leq 1/C_0$. From here to (7.38), our arguments are slightly heuristic. They could easily be made rigorous (to the price of a few more pages ...), by introducing microlocal parametrix operators, starting by introducing $\pi_k u=(u|u_k)u_k$ and E_k (self adjoint), such that $I=\pi_k+E_k(Z^*Z-\lambda_k)$. Let $J(ZZ^*,\lambda_k)$, $k\geq 1$, and $J(Z^*Z,\lambda_k)$, $k\geq 0$, denote the corresponding 1-dimensional eigenspaces. If v_k denotes a normalized element of $J(ZZ^*,\lambda_k)$, then,

(7.18) $\quad Zu_k=\alpha_k v_k,\ Z^*v_k=\beta_k u_k,\ \alpha_k\beta_k=\lambda_k,$

for $k\geq 1$. Since $\alpha_k=(Zu_k|v_k)=(u_k|Z^*v_k)=\bar{\beta}_k$, we may assume after changing v_k, that,

(7.19) $\quad \alpha_k=\beta_k=\lambda_k^{1/2}.$

Suppose now that for some value of μ, the kernel of the operator (7.8) is microlocally nonempty. In other words, there exist a normalized vector ${}^t(f,g)$, such that microlocally,

(7.20) $\quad C_1 f+Zg=0,\quad Z^*f+C_2 g=0.$

Here C_1 and C_2 are scalars, so if we apply Z^* to the first equation, and use the second one, we get, $Z^*Zg=C_1C_2 g$, and similarly, $ZZ^*f=C_1C_2 f$. This gives the necessary condition,

(7.21) $\quad C_1C_2=\lambda_k$

for some $k\geq 0$, and when this condition is verified with $k\geq 1$, we must have $f=x_k v_k$, $g=y_k u_k$ for some $x_k,y_k\in\mathbb{C}$. The problem (7.20) is then equivalent to the system,

(7.22) $\quad C_1x_k+\alpha_k y_k=0,\ \beta_k x_k+C_2 y_k=0,$

which has a 1-dimensional space of solutions, since $\alpha_k\beta_k=C_1C_2$ in this case. When $k=0$, we have $C_1C_2=0$, and we notice in this case that we must have $C_2=0$. In fact, otherwise $C_2\neq 0$, $C_1=0$, and (7.20) becomes,

(7.23) $\quad Zg=0,\ Z^*f+C_2 g=0.$

Applying Z to the second equation and using the first, we get $ZZ^*f=0$, and since ZZ^* is bijective, we get $f=0$. Then since $C_2\neq 0$, we get $g=0$ from the second part of (7.23).

On the other hand, if $C_2=0$, (7.20) becomes

(7.24) $C_1 f+Zg=0,\ Z^*f=0,$

and the one-dimensional solution space is generated by $f=0$, $g=u_0$.

It follows from the earlier discussion, and in particular from (7.11), that the function $\mu \longrightarrow C_1C_2$ has a strictly positive second derivative, and that the minimum is ≤ 0. Using also the fact that $\partial_\mu^j\lambda_k=\mathcal{O}(\lambda_k)$ uniformly for every fixed j, we see that the values of μ such that $C_1C_2=\lambda_k$ for some $k\geq 1$, or $C_2=0$, are isolated and the gap between two consecutive values is of an order of magnitude varying between h and $h^{\frac{1}{2}}$, with the exception of one gap (neighboring the value where $C_2=0$) whose order of magnitude is at least of the order of $h^{\frac{1}{2}}$. At these values the microlocal kernel of H_1 is 1-dimensional. From now on we shall work near one of these values, μ_0, but our arguments will be uniform, with respect to all such possible choices. For μ close to μ_0, H_1 has a simple isolated (microlocally defined) eigenvalue $E=E(\mu)$. We want to study $\partial_\mu E(\mu_0)$. The arguments in the section on reduction to normal forms of systems still work, if we add one more parameter $E\in\mathbb{R}$, and give,

(7.25) $U^*(H_1-E)U= [C_1]_{1,1}+[C_2]_{2,2}+[Z]_{1,2}+[Z^*]_{2,1},$

where now Z,C_j are analytic symbols in x,ξ,μ,E and U depends analytically on μ,E in the same way as explained after (7.8). Differentiating (7.25) with respect to E, and putting $(x,\xi)=(0,0)$, we get,

(7.26) $\partial_E C_j<0,$

and more precisely, that $\partial_E C_j$ are negative elliptic symbols of order 0.

When $k=0$, $E(\mu)=E_0(\mu)$ is determined by $C_2(\mu,E_0(\mu),h)=0$, and in view of (7.26) and the fact that $\partial_\mu C_2>0$, we see that,

(7.27) $\partial_\mu E_0(\mu,h)>0,$

and more precisely, that $\partial_\mu E_0$ is a positive elliptic analytic symbol of order 0.

When $k\geq 1$, $E=E_k(\mu,h)$ is determined by,

(7.28) $C_1(\mu,E,h)C_2(\mu,E,h)=\lambda_k(\mu,E,h).$

(Actually, there are two such solutions.) It is easy to see that $\partial_\mu\lambda_k,\partial_E\lambda_k=\mathcal{O}(\lambda_k)$, and if we differentiate (7.28), we get,

$$C_1\partial_\mu C_2+C_2\partial_\mu C_1+(C_1\partial_E C_2+C_2\partial_E C_1)\partial_\mu E_k=\mathcal{O}(\lambda_k).$$

Here C_1 and C_2 have the same sign, and we may assume for instance that they are both positive. Then $C_1\partial_\mu C_2+C_2\partial_\mu C_1$ and $-(C_1\partial_E C_2+C_2\partial_E C_1)$ are both positive and of the same order of magnitude as $C_1+C_2\geq\lambda_k^{\frac{1}{2}}/\text{Const.}$. It then follows that,

(7.29) $\partial_\mu E_k=-(C_1\partial_\mu C_2+C_2\partial_\mu C_1+\mathcal{O}(\lambda_k))/(C_1\partial_E C_2+C_2\partial_E C_1)\sim 1.$

In order to control also the domain of definition of E_k as a holomorphic function of μ, we add some arguments. Let $f(\mu,E)=C_1(\mu,E)C_2(\mu,E)-\lambda_k(\mu,E)$, which is a bounded holomorphic function, defined in a 2-disc of fixed radius centered at $(\mu_0,0)$, where $\mu_0=\mu_0(0)$ is one of the two real solutions of $C_1(\mu,0)C_2(\mu,0)-\lambda_k(\mu)=0$. The computation above is then valid for $\mu=\mu_0$, $E=0$, and shows that $a=_{def}|\partial_E f(\mu_0,0)|\sim|\partial_\mu f(\mu_0,0)|$ is at least of the same order of magnitude as $\lambda_k^{\frac{1}{2}}$. Moreover $|f''|\leq$const. . Then for $|\mu-\mu_0|\leq\delta_1 h^{\frac{1}{2}}$, $|E|\leq\delta_2 h^{\frac{1}{2}}$, where δ_1/δ_2, and δ_2 are sufficiently small, we have $|\partial_E f(\mu,E)-\partial_E f(\mu_0,0)|<<a$, $|f(\mu,0)|\leq$(Const.)$\delta_1 a h^{\frac{1}{2}}$, and it follows that there is a unique solution $E=E_k(\mu)$ with $|E_k|\leq\delta_2 h^{\frac{1}{2}}$ of the equation $f(\mu,E)=0$. Thus in addition to (7.29) we know that E_k is a well defined holomorphic function of μ in $|\mu-\mu_0|\leq h^{\frac{1}{2}}$/Const. such that $|E_k|\leq$(Const.)$h^{\frac{1}{2}}$.

If we let φ_0 be a normalized element (defined microlocally near (0,0)) of the kernel of H_1, then,

(7.30) $\quad \partial_\mu E(\mu_0)=(\partial_\mu H_1(\mu_0)\varphi_0|\varphi_0)$.

Next, we recall some invariance properties. From (2.7),(2.9), we get,

(7.31) $H_1\mathcal{F}T=\mathcal{F}TH_2,\ H_2\mathcal{F}T=\mathcal{F}TH_1$.

Using (2.8),(2.10), we also get,

(7.32) $\quad H_1VT^2=VT^2H_1,\ H_2V=VH_2$.

If $\varphi_0^*=\mathcal{F}T'\varphi_0$, then,

(7.33) $\quad H_2\varphi_0^*=0$, microlocally near (0,0),

and φ_0^* is also normalized. Here, we recall that $T'=T$, when the + sign is valid in the general identity $\varphi_0=\pm(\mathcal{F}T)^2\varphi_0$, and that $T'=iT$ in the − sign case. Since $P\varphi_0=0$, $P^*\varphi_0^*=0$, we have $P^*P_1\varphi_0=P_1^*P\varphi_0=0$, so

(7.34) $\quad P_1\varphi_0=\alpha\varphi_0^*$,

for some non-vanishing $\alpha\in\mathbb{C}$. Similarly,

(7.35) $\quad P_2^*\varphi_0^*=\beta\varphi_0$,

for some non-vanishing $\beta\in\mathbb{C}$. Combining this with (7.30), we get for $\mu=\mu_0$ (with $\partial=\partial_\mu$): $\partial E=((\partial P_1^*)P\varphi_0|\varphi_0)+(P_1^*(\partial P)\varphi_0|\varphi_0)=0+((\partial P)\varphi_0|P_1\varphi_0)=\bar{\alpha}((\partial P)\varphi_0|\varphi_0^*)$. Then using also (7.29), we conclude that,

(7.36) $\quad |\partial_\mu(P\varphi_0|\varphi_0^*)|\sim 1$, for $\mu=\mu_0$.

We now have to pass from microlocal results near (0,0) to global results. Still with $\mu=\mu_0$, let us define U_β, $\beta\in\mathbb{C}^2$ as the real component close to $2\pi\beta$ of $\det H_1(x,\xi)=0$. In order to define a suitable distance, let us recall that the "leading parts" of P,P_j are $P_0(a,b;x,\xi)$, and $P_{0,j}=P_0(ia,ib;x,\xi)$ respectively, and that with $b\bar{a}=i\alpha$, $\alpha>0$, we have,

$$\det P_0=2i(\sin(2\arg(b))+\alpha\cos\xi+\alpha\cos x),$$

$$\det P_{0,j} = 2i(-\sin(2\arg(b)) + \alpha\cos\xi + \alpha\cos x).$$

Since we are far from the "branching case" when $\sin(2\arg(b))=0$, we see that $\det P_j \neq 0$, near the complex characteristics of P. As in section 3, we then define $D(x,y)=|\Phi(x)-\Phi(y)|$, where $\Phi'(x)=0$ on the projections of the potential wells, and $\Phi'(x)=\mathrm{Im}\zeta(x)$, between these projections, where $\zeta(x)$ is the complex root with positive imaginary part of the equation $\det(H_1(x,\zeta))=0$, depending continuously on x, and such that $(x,\zeta(x))$ tends to a point in $U_{j,0}$, when x decreases towards the projection of $U_{j,0}$. Here $U_{j,0}$ is the closest projected well to the left of x. In this definition, we have privileged H_1 over H_2, but as in section 3, we use the fact that the roots of $\det(H_1)$ are close to those of $\det(H_2)$, and that H_1 and H_2 are intertwined by $\mathcal{F}T$, so defining D from H_2 would give no essential difference. Notice that $D(x,0)$ is an even function of x, since the characteristics of H_1 are invariant under reflection in $(0,0)$.

Because of the presence of possible real characteristics of H_1 far away from those of P, we shall avoid the use of spectral projections of H_1. The treatment will therefore be slightly different from that in section 3. Let φ_0, φ_0^* denote realizations of φ_0, φ_0^*, obtained by superpositions of Gaussians. (To be more precise, let $S: L^2(\mathbb{R}) \rightarrow H(\mathbb{C};e^{-2\Phi_0/h})$ be a globally defined unitary FBI-transform with a quadratic phase (see appendix a and [S1]). Then $S\varphi_0$ and $S\varphi_0^*$ are well defined (modulo exponentially small contributions) as elements of $H_{loc}(e^{-2\Phi_0/h})$ in a neighborhood of $\pi_x \circ \kappa_S(U_{0,0})$, and exponentially small outside that set. If $\chi \in C_0^\infty(\mathbb{C})$ has its support close to $\pi_x \circ \kappa_S(U_{0,0})$ and is equal to 1 near that set, then we can take as realizations, $S^*(\chi S\varphi_0)$ and $S^*(\chi S\varphi_0^*)$.) Then we may arrange so that,

$$\varphi_0, \varphi_0^*, \mathcal{F}\varphi_0, \mathcal{F}\varphi_0^* = \tilde{\mathcal{O}}(e^{-f/h}), \tag{7.37}$$

where f is a Lipschitz continuous non-negative function, vanishing precisely on $\pi_x(U_{0,0})$, such that $f(x) \geq v_0 D(0,x)/D(0,2\pi)$ and $|f'(x)| \leq |\partial_x D(x,0)|$. (To get this near $\pi_x(U_{0,0})$ we repeat the arguments of the proof of Proposition 5.1 in [HS1].) Here $0 < v_0 \leq D(0,2\pi)$. Recall that in section 3 we had a similar function f with the crucial property that $f(x)=D(0,x)$, in a neighborhood of $[-\pi,\pi]$.

Since $\varphi_0 = \pm(\mathcal{F}T)^2\varphi_0$ microlocally, we may also arrange that the realization φ_0 has the same property in the exact sense. This implies that $\varphi_0^* = \pm(\mathcal{F}T)^2\varphi_0^*$. We can also arrange so that $VT^2\varphi_0 = \varphi_0$, which implies that $V\varphi_0^* = \varphi_0^*$. Put $\varphi_\alpha = T_\alpha\varphi_0$, $\varphi_\alpha^* = T_\alpha\varphi_0^*$, $R_+u(\alpha) = (u|\varphi_\alpha)$, $R_-u^- = \Sigma\, u_\alpha^- \varphi_\alpha^*$. Then as in section 3, we see that the corresponding operator $\mathcal{P}$: $L^2 \times l^2 \rightarrow L^2 \times l^2$ is bijective for $|\mu - \mu_0| \leq h/C_0$, with an inverse $\mathcal{E} = \mathcal{O}(h^{-1})$. Let as usual E_{-+} denote the lower diagonal element of the square matrix of $\mathcal{E}$, and let $E_{-+}(\alpha,\beta)$ denote the corresponding matrix. From reflection and translation invariance, we then see that $E_{-+}(\alpha,\alpha)=z(\mu)$ is independent of α, and real valued when μ is real. Using (7.36), we obtain as in section 3, that,

(7.38) $|\partial_\mu z(\mu)|\sim 1$, for $\mu=\mu_0$,

and it also follows from the construction, that $z(\mu_0)$ is exponentially small. After an exponentially small real correction of μ_0, we may then assume that $z(\mu_0)=0$. Cauchy's inequalities imply that $z''(\mu)=\mathcal{O}(h^{-3})$, so for $|\mu-\mu_0|<<h^3$, we have $|\partial_\mu z(\mu)-\partial_\mu z(\mu_0)|<<1$, and it follows that μ_0 is the only zero of $z(\mu)$ in this region, which is the only interesting one since it follows from the results above that 0 may belong to the spectrum of P_0 only if μ is exponentially close to (one of the values) μ_0. Repeating the argument of section 3, we see that $\mathcal{E}=\mathcal{O}(h^{-1})$ as an operator in $L^2(\mathbb{R};e^{2F(x)/h})\times l^2(e^{2F^*/h})$, where

(7.39) $|F'(x)|\le(|\Phi'(x)|-\delta)_+$,

(7.40) $|F(x)-F(2\pi k)|\le(1-\delta)f(x-2\pi k)$, $k\in\mathbb{Z}$,

where $\delta>0$ may be chosen arbitrarily small, and where f was introduced in (7.37). Moreover $F^*(\alpha)=F(\pi_x(U_\alpha))$. Using the Grushin problem based on φ_0,φ_0^*, we shall now improve the functions φ_0,φ_0^*. Let $\mu=\mu_0$. As a new function φ_0, we take $\varphi_0^1=aE_+(\delta_0)$, where a is a normalization constant, exponentially close to 1. Then $\varphi_0=\varphi_0^1$ microlocally near $U_{0,0}$, and using the boundedness of $\mathcal{E}$ in weighted spaces plus the fact that,

(7.41) $P\varphi_0^1=-a\Sigma_{\beta\neq 0}E_{-+}(\beta,0)\varphi_\beta$,

we get:

(7.42) $\varphi_0^1,\mathcal{F}\varphi_0^1=\tilde{\mathcal{O}}(e^{-f^1/h})$,

where f^1 is even, and given by,

(7.43) $f^1(x)=\min(D(0,x),v_0+f(x),\min_{k\in\mathbb{Z}\setminus\{0\}}|k|v_0+f(x-2\pi k))$.

Here, we also use the fact that $E_{-+}(\alpha,\beta)=\tilde{\mathcal{O}}(e^{-v_0|\alpha-\beta|_\infty/h})$, which is a consequence of the boundedness in weighted spaces.

From φ_0^1 we generate φ_α^1, φ_α^{1*}, as before, and we then consider the new Grushin problem with R_+,R_- defined, using these new functions. Let $\mathcal{E}$, E_{-+} correspond to this new problem. After an exponentially small correction of μ_0, it is then clear that $\mathcal{E}$, E_{-+} will have the same properties as before, but with f replaced by f^1. Iterating this procedure, we obtain a sequence of functions φ_0^1, $\varphi_0^2,\dots$, with,

(7.44) φ_0^k, $\mathcal{F}\varphi_0^k=\tilde{\mathcal{O}}(e^{-f^k/h})$,

where,

(7.45) $f^{k+1}(x)=\min(D(0,x),v_0+f^k(x),\min_{l\in\mathbb{Z}\setminus\{0\}}|l|v_0+f^k(x-2\pi l))$.

Here $f^0=f=v_0D(0,x)/D(0,2\pi)$, and if we put $f^\infty(x)=\min_{l\in\mathbb{Z}}(|l|v_0+D(2\pi l,x))$, then it is easy to check that $f^0 \le f^k \le f^\infty$. We can rewrite (7.45) as

(7.46) $f^{k+1}(x) = \min(f^\infty(x), \Lambda(f^k))$,

where $\Lambda(f) = \min(v_0 + f(x), \min_{l \neq 0} |l| v_0 + f(x - 2\pi l))$. Notice that, $\Lambda(\min(\alpha, \beta)) = \min(\Lambda(\alpha), \Lambda(\beta))$. If $g^0 = f$, $g^{k+1} = \Lambda(g^k)$, then it is easy to see that, $g^k(x) = kv_0 + \min_{|l| \le k} f(x - 2\pi l)$. Now we claim that,

(7.47) $f^k = \min(f^\infty, g^k)$.

In fact, this is true for $k=0$, and if it is true for some k, then $f^{k+1} = \min(f^\infty, \Lambda(\min(f^\infty, g^k)) = \min(f^\infty, \Lambda(f^\infty), \Lambda(g^k)) = \min(f^\infty, g^{k+1})$, since $\Lambda(f^\infty) \ge f^\infty$. From the behaviour of g^k, we see that if $0 < v_1 < v_0$, and k is sufficiently large, then $f^k \ge \min_l (v_1 |l| + D(2\pi l, x))$. Then, after replacing v_0, by some smaller number, and φ_0 by φ_0^k, for some sufficienly large k, we may assume that (7.37) holds with,

(7.48) $f(x) = \min_{l \in \mathbb{Z}} |l| v_0 + D(2\pi l, x)$.

Putting $f_{0,1} = \varphi_0$, $f_{0,2} = \varphi_0^*$, we are now completely at the same point as in section 3 at (3.24), and we can now repeat the arguments of that section without any essential changes until (3.37). $g_{\alpha,j}$ have the same growth estimates as before, and WKB-representations, valid in the same intervals,

(7.49) $g_{0,1} = c_0(h) e^{i\varphi(x)/h} a(x,h)$, $g_{0,2} = c_1(h) e^{i\psi(x)/h} b(x,h)$,

where $\det P(x, \varphi'(x)) = \det P^*(x, \psi'(x)) = 0$, $c_j, c_j^{-1} = \mathcal{O}(e^{\delta/h})$, for every $\delta > 0$. Notice that since on the symbol level, $P^*(x,\xi) = P(\bar{x}, \bar{\xi})^*$, we have $\bar{\psi}'(x) = \varphi'(x)$ for real x, and hence we may assume that,

(7.50) $\operatorname{Im}(\varphi + \psi) = S_0 = D(0, 2\pi)$, $\operatorname{Re}\varphi = \operatorname{Re}\psi$.

As in section 3, a and b are classical analytic symbols of order 0, but now with values in $\mathbb{C}^2$. The leading parts a_0, b_0 are non-vanishing, and satisfy,

(7.51) $P(x, \varphi'(x))(a_0) = 0$, $P^*(x, \psi'(x))(b_0) = P(x, \varphi'(x))^*(b_0) = 0$.

We still have (3.39), where d is an analytic symbol of order 0. If $d_0(x)$ is the leading part, then $d_0(\pi) = q(\pi, \varphi'(\pi)) a_0(\pi) = C_0(\partial_\xi P(\pi, \varphi'(\pi)))(a_0(\pi))$, where C_0 is a non-vanishing constant. Instead of (3.41), we have,

(7.52) $(\psi Q_\pi \psi g_{0,1} | g_{0,2}) = c_3(h) k(h) e^{-S_0/h}$,

where k is an analytic symbol of order 0, with leading part,

(7.53) $(\partial_\xi P(\pi, \varphi'(\pi)) a_0(\pi) | b_0(\pi))_{\mathbb{C}^2}$.

Here $b_0(\pi) \in (\operatorname{Im}(P(\pi, \varphi'(\pi))))^\perp$. We claim that the expression (7.53) is non vanishing. In order to prove this, we have to verify that $\partial_\xi P(\pi, \varphi'(\pi))(a_0(\pi)) \notin \operatorname{Im}(P(\pi, \varphi'(\pi)))$. Working at the point $(\pi, \varphi'(\pi))$, we start from ${}^{co}P \circ P = \det(P) I$, which we differentiate, and get, $({}^{co}\partial_\xi P) \circ P + {}^{co}P \circ (\partial_\xi P) = (\partial_\xi \det(P)) I$. Here we know that $\partial_\xi \det(P) \neq 0$. Since $P a_0 = 0$, we get, ${}^{co}P \circ (\partial_\xi P) a_0 = (\partial_\xi \det(P)) a_0$. If $(\partial_\xi P)(a_0) \in \operatorname{Im}(P)$, there is a vector c_0, such that $(\partial_\xi P)(a_0) = P(c_0)$. Then, ${}^{co}P \circ P(c_0) = (\partial_\xi \det(P)) a_0$, or equivalently, $\det(P) c_0 = (\partial_\xi \det(P)) a_0$. Here the left hand side vanishes, since det(P)=0, while the right hand side does not, so we have a contradiction. This proves that k is an elliptic symbol of order 0. Then we obtain (3.43),(3.44), exactly as in section 3, and we can finish the discussion as in that section.

Summing up, we have,

Proposition 7.1. There exist $\varepsilon_0, h_0, \mu_0 > 0$, such that if P is a strong type 2 operator with $\varepsilon(P) \le \varepsilon_0$, $0 < h \le h_0$, $||b(0)| - 2^{-\frac{1}{2}}| \le \varepsilon_0$, $\arg(b(0)) = \pi/4 + n\pi/2$, for some $n \in \mathbb{Z}$, we have: $\mu - \mathrm{Sp}(P) \cap [-\mu_0, \mu_0] \subset \cup_{-N_- \le j \le N_+} J_j$, where J_j are closed disjoint intervals of width $e^{-1/C_j h}$, where $C_j \sim 1$, and the separation between two consecutive intervals is at least h/C for some fixed C>0. More precisely, J_j is exponentially close to a point $\tilde{J}_j$, where the points $\tilde{J}_j$ are given by $C_1(\tilde{J}_j) C_2(\tilde{J}_j) = G(\tilde{J}_j, |j|h, h)$ when $j \neq 0$, and by $C_2(\tilde{J}_0) = 0$ when j=0. Here C_1, C_2 are analytic functions defined in a fixed neighborhood of 0, (varying in a bounded set of such functions,) satisfying $C_j' \ge \mathrm{Const.} > 0$, and $G(\mu, \lambda, h)$ is a real valued analytic symbol of order 0, satisfying $G(\mu, 0, h) = 0$, $\partial_\lambda G > 0$. For each j, there is an affine increasing map $\kappa_j : \mu \to \mu'$, such that $\kappa_j(\mu - \mathrm{Sp}(P) \cap J_j) = \mu' - \mathrm{Sp}(Q)$, where $Q = Q_j$ is an h'-pseudodifferential operator of strong type 1 with $\varepsilon(Q) \to 0$ as $h \to 0$, uniformly with respect to the other parameters.

8. Simply degenerate type 2 operators in the potential well case.

Let (P,P_1,P_2) be a strong type two operator with $C(P)\le C_0$ where $C_0>0$ is fixed. Let $[\alpha,\beta]\subset[-3,3]$ be an interval such that,

(8.1) $\quad ||b(\mu)|-2^{-\frac{1}{2}}|+|\arg(b(\mu))-(\pi/4+n\pi/2)|\ge 1/C_0,$

(8.2) $\quad |\arg(b(\mu))-n\pi/2|\ge 1/C_0,$

for every $\mu\in[\alpha,\beta]$ and every $n\in\mathbb{Z}$. Assuming $\varepsilon(P)\le\varepsilon_0$, $0<h\le h_0$, where $\varepsilon_0>0$, $h_0>0$, we shall then study $\mu-Sp(P)\cap[\alpha,\beta]$. The assumption (8.1) implies that we are not in the totally degenerate case, studied in the preceding section, and the assumption (8.2) implies that we are not in the "branching case", that will be studied in the next section. In order to fix the ideas, we may assume that $\sin(2\arg(b(\mu)))<0$ on $[\alpha,\beta]$. Again P_j, $j=1,2$ are elliptic near the complex characteristics of P. Moreover, the selfadjoint operator H_1 has the property that if $\det(H_1(x,\xi))=0$ for some real (x,ξ) close to the characteristics of P, then $\mathrm{Ker}(H_1(x,\xi))$ is one dimensional. The same holds for H_2. The part of the real characteristic set of H_1 (for μ real) which is close to the characteristics of P, is then either empty, or of the form $\cup_{\alpha\in\mathbb{Z}^2}U_\alpha$, where U_α is either equal to $2\pi\alpha$, or equal to a simple Jordan curve around $2\pi\alpha$. We here assume that all points in $[\alpha,\beta]$ are close to an interval where $\det(P_0(a,b;x,\xi)$ may vanish for some real (x,ξ). Indeed, if μ is far from such a value then P is elliptic and μ is also far from $\mu-Sp(P)$. In the case when $\det(H_1)$ has no real zeros, we see that $\det(H_1(0,0))$ is small and we define U_α to be $\{2\pi\alpha\}$.

Microlocally, near $U_{0,0}$, we can apply the results on block-decomposition of systems, developed in [HS2] and in section a of the appendix, to see that there is unitary 2×2 system of classical analytic pseudodifferential operators, $U=U(\mu,x,hD,h)$, such that,

(8.3) $\quad U^*H_1U=[q]_{1,1}+[h_1]_{2,2},$

where q, h_1 are scalar classical analytic selfadjoint pseudodifferential operators of order 0. Moreover q is elliptic (near $U_{0,0}$).

We now restrict μ to a neighborhood of a real value μ_0, with the property that $U_{0,0}\ne\emptyset$. In the case when $U_{0,0}=(0,0)$ for $\mu=\mu_0$, then for that value of μ, we have $h_1(0,0)=\mathcal{O}(h)$, $h_1'(0,0)=\mathcal{O}(h)$, $h_1''(0,0)\sim -I$, and $\partial_\mu h_1\ne0$. We may assume for simplicity that $\partial_\mu h_1>0$. If $U_{0,0}$ is diffeomorphic to a circle around $(0,0)$ (for $\mu=\mu_0$), then h_1 vanishes on a circle of distance $\mathcal{O}(h)$ from $U_{0,0}$, that we shall from now on identify with $U_{0,0}$. On this circle $d_{x,\xi}h_1\ne0$, and h_1 is positive inside the circle. Moreover $\partial_\mu h_1\ne0$ on $U_{0,0}$ and in order to fix the ideas, we may assume that $\partial_\mu h_1>0$. (All this follows from the fact that h_1 is close to a non-vanishing factor times $\det(P_0)$.)

Microlocally, near $U_{0,0}$, the eigenfunctions of h_1, associated to small eigen-values can be obtained by WKB-constructions, and they are given by,

(8.4) $F(\mu,E,h)=kh,\ k\in\mathbb{N}$,

where F is an analytic symbol of order 0, with leading part $f(\mu,E)$, given by,

(8.5) $f(\mu,E)=\mathrm{Vol}(\{(x,\xi);h_1\geq E\})$.

(A more complete statement would require the introduction of microlocal spectral projections and partial parametrices, as we also mentioned in section 7 after (7.17).)

At this point, we recall, that if $T(E)=T(E,\mu)>0$ is the primitive period of the H_{h_1}-flow in $h_1=E$, then T is analytic in (E,μ) and,

(8.6) $\partial_E \mathrm{Vol}(h_1\geq E)=-T(E)$.

Hence, in the case when $U_{0,0}$ is close to (0,0), we see how (8.4) follows from the results in the end of Appendix b. In the case when $U_{0,0}$ is a circle, the formula (8.4) was obtained in the analytic case in [HS1]. In the C^∞ category, it was earlier proved by Helffer-Robert [HR1,2]. When $U_{0,0}=U_{0,0}(\mu_0)$ is a circle it is clear that $f(\mu,E)$ is analytic near $(\mu_0,0)$. When $U_{0,0}(\mu_0)$ is a point, we see that $f(\mu,E)$ is analytic near (μ_0,E_0) when $-E_0$ is small and positive. Integrating (8.6), and using the analyticity of $T(E,\mu)$, we conclude that $f(\mu,E)$ is analytic near $(\mu_0,0)$ in this case too. In both cases we have,

(8.7) $\partial_\mu f>0,\ \partial_E f<0$, for $\mu=\mu_0,\ E=0$.

Let us now assume that μ_0 is one of the interesting values, namely that,

(8.8) $F(\mu_0,0,h)=kh$, for some $k\in\mathbb{N}$.

For $|\mu-\mu_0|/h\leq 1/C$, with $C>0$ large enough, let $E=E(\mu,h)$ be the solution of,

(8.9) $F(\mu,E,h)=kh$,

where k is the same number as in (8.8). In view of (8.7) this definition makes sense, and $E(\mu,h)=\mathcal{O}(h)$ is holomorphic for $|\mu-\mu_0|<h/C$. Moreover,

(8.10) $\partial_\mu E>0$.

Since U is unitary, we get the same (microlocal) eigenvalues for H_1. Let $\varphi_{0,0}$ be a normalized function defined microlocally near $U_{0,0}$, such that $H_1\varphi_{0,0}=0$ for $\mu=\mu_0$. We then have (7.30), and from this point on the discussion of section 7 applies without any essential changes. We then obtain,

Proposition 8.1. Let $C_0>0$. Then there exist $\varepsilon_0>0$, $h_0>0$ such that if P is a strong type 2 h-pseudodifferential operator with $C(P)\leq C_0$, $\varepsilon(P)\leq\varepsilon_0$, $0<h\leq h_0$ and $[\alpha,\beta]\subset[-3,3]$ is an interval on which (8.1),(8.2) hold, (with the same constant, C_0,) then $\mu-\mathrm{Sp}(P)\cap[\alpha,\beta]\subset\cup_{1\leq j\leq N} J_j$, where J_j are closed disjoint intervals, labelled in increasing order. The separation between J_j and J_{j+1} is of the order of magnitude, h, and the width of J_j is e^{-1/C_jh}, where $C_j\sim 1$. Moreover, for each j, there is an increasing affine map $\kappa_j:\ \mu\rightarrow\mu'$, such that

$\varkappa_j(\mu - Sp(P) \cap J_j) = \mu' - Sp(Q)$, where Q is a strong type 1 h'-pseudodifferential operator (depending on j), such that $\varepsilon(Q) \longrightarrow 0$, when $h \longrightarrow 0$, uniformly with respect to the other parameters.

9. The branching problem for type 2 operators.

Let (P,P_1,P_2) be a strong type 2 operator such that,

(9.1) $\sin(2\arg(b(0)))=0$; $\arg(b(0))=n\pi/2,\ n\in\mathbb{Z}$.

If $C(P)\le C_1$, for some arbitrary but fixed constant C_1, then our study will be uniformly valid for $|\mu|\le\mu_0$, $\varepsilon(P)\le\varepsilon_0$, $0<h\le h_0$, where μ_0,ε_0,h_0 are strictly positive and depend on C_1.

We start by some general remarks on the reduction of certain 2×2 systems to scalar ones. Let $H=H(x,hD)$ be a selfadjoint 2×2 system of analytic pseudodifferential operators defined near $(x_0,\xi_0)\in T^*\mathbb{R}$. We assume that $H(x_0,\xi_0)$ is of rank 1, and to start with, we shall work microlocally near (x_0,ξ_0). Let us repeat some arguments developed for the block decomposition of systems of pseudodifferential operators. See [HS2]. Let γ be a small circle around 0 with positive orientation and put,

(9.2) $\Pi=(2\pi i)^{-1}\int_\gamma(z-H)^{-1}dz$.

Then Π is an analytic pseudodifferential operator whose Weyl symbol satisfies,

(9.3) $\Pi(x_0,\xi_0)+\mathcal{O}(h^2)=$ the orthogonal projection onto the kernel of $H(x_0,\xi_0)$.

Moreover,

(9.4) $\Pi^*=\Pi,\ \Pi^2=\Pi,\ H\Pi=\Pi H$.

Let R be a 2×1 system of pseudodifferential operators of order 0, such that ΠR is elliptic. Then $R^*\Pi R$ is a positive elliptic scalar pseudodifferential operator of order 0, and we put,

(9.5) $S=\Pi R(R^*\Pi R)^{-\frac{1}{2}}$.

Then,

(9.6) $S^*S=I,\ \ \Pi S=S$.

Similarly, we can construct a 2×1 system, S', such that

(9.7) $S'^*S'=I,\ \ (I-\Pi)S'=S'$.

Since $S'^*S=0$, $S^*S'=0$, we see that the 2×2 system $\mathcal{S}=(S\ S')$ is isometric, and hence elliptic and unitary. Hence $\mathcal{S}\,\mathcal{S}^*=I$,

(9.8) $S\,S^*+S'\,S'^*=I$.

Applying Π to this relation, we get, $\Pi SS^*=\Pi$. Since $(I-\Pi)SS^*=0$, we conclude that,

(9.9) $SS^*=\Pi$.

Similarly,

(9.10) $S'S'^*=I-\Pi$.

Put $\tilde{H}=S^*HS$. Then,

(9.11) $S\tilde{H}=HS,\ \ \tilde{H}S^*=S^*H$.

Here the two relations are equivalent and the first one follows from a straight forward computation: $S\tilde{H}=SS^*HS=\Pi HS=H\Pi S=HS$.

If u is a scalar function satisfying $\tilde{H}u=0$ (near (x_0,ξ_0)), then with $v=Su$, we get $Hv=S\tilde{H}u=0$. Conversely, if v is a $\mathbb{C}^2$ valued function such that

$Hv=0$, then $\Pi v=v$, $SS^*v=v$, and $u=S^*v$ satisfies $\tilde{H}u=0$. Hence the problems $Hv=0$ and $\tilde{H}u=0$ are equivalent (microlocally).

We now return to our type 2 operator P,P_1,P_2. We know that P is elliptic outside a small neighborhood of $\cup s(\alpha,j)$, and after replacing P_j by $(\mathrm{Re}\theta)P+(\mathrm{Im}\theta)P_j$ for a suitable θ, we can assume that P_j are elliptic in that region. Define H_1, H_2 as usual. We shall make a reduction of H_j to scalar operators in a neighborhood of $\cup s(\alpha,j)$. In this region $H_1(x,\xi)$ is of rank 1 whenever $\det(H_1(x,\xi))$ vanishes. The invariance properties of P,P_j imply that,

(9.12) $\qquad H_1\mathcal{F}T=\mathcal{F}TH_2,\ H_2\mathcal{F}T=\mathcal{F}TH_1,$

(9.13) $\qquad H_1VT^2=VT^2H_1,\ H_2V=VH_2,$

(9.14) $\qquad T_\gamma H_j=H_jT_\gamma,\ j=1,2,\ \gamma\in\mathbb{Z}^2.$

Notice that (9.12) implies,

(9.15) $\qquad [H_j,(\mathcal{F}T)^2]=0.$

We now concentrate on H_1 for a while. We shall construct S_1 in a neighborhood of $\cup s(\alpha,j)$ adapted to H_1 as above, such that,

(9.16) $\qquad (\mathcal{F}T)^2S_1=\iota S_1\mathcal{F}^2,$

(9.17) $\qquad VT^2S_1=\iota S_1V,$

(9.18) $\qquad T_\gamma S_1=S_1T_\gamma,\ \gamma\in\mathbb{Z}^2,$

where $\iota=\pm1$, will be determined below.

Suppose, that we have found a 2×1 system R of order 0, such that Π_1R is elliptic near $\cup s(\alpha,j)$, where Π_1 is the projection associated to H_1 as in (9.2), and such that,

(9.19) $\qquad (\mathcal{F}T)^2R=\iota R\mathcal{F}^2,$

(9.20) $\qquad VT^2R=\iota RV,$

(9.21) $\qquad T_\gamma R=RT_\gamma.$

Taking the adjoints of these relations, and using the fact that (9.13)-(9.15) carry over to Π_1, we see that $R^*\Pi_1R$ and consequently $(R^*\Pi_1R)^{-\frac{1}{2}}$ commute with $V,\mathcal{F}^2,T_\gamma$, and hence,

(9.22) $\qquad S_1=\Pi_1R(R^*\Pi_1R)^{-\frac{1}{2}},$

has all the desired properties.

The problem is then to construct R satisfying (9.19)-(9.21). Identifying the Weyl symbol $R(x,\xi)$ with $R(x,\xi)(1)\in\mathbb{C}^2$, we see that the problem is to construct an analytic vector $R(x,\xi)$ such that $\Pi_1(x,\xi)(R(x,\xi))\neq0$, and such that,

(9.23) $\qquad R=\iota T^2R\circ c_0,$

(9.24) $\qquad R=\iota T^2\Gamma(R\circ\delta),$

(9.25) $\qquad R\circ\tau_{2\pi\gamma}=R,$

where $c_0(x,\xi)=(-x,-\xi)$, $\delta(x,\xi)=(\xi,x)$, $\tau_{2\pi\gamma}(x,\xi)=(x,\xi)-2\pi\gamma$, and Γ denotes complex conjugation in $\mathbb{C}^2$. Recall that δ describes the reflection in where $c_0(x,\xi)=(-x,-\xi)$, $\delta(x,\xi)=(\xi,x)$, $\tau_{2\pi\gamma}(x,\xi)=(x,\xi)-2\pi\gamma$, and Γ denotes complex conjugation in $\mathbb{C}^2$. Recall that δ describes the reflection in the line $x=\xi$, and that $\tilde{\delta}=\delta\circ c_0=c_0\circ\delta$ is given by $\tilde{\delta}(x,\xi)=(-\xi,-x)$ and describes reflection in the line $x=-\xi$. (9.23),(9.24) imply,

(9.26) $\qquad R=\Gamma(R\circ\tilde{\delta})$,

and more generally, any two of (9.23),(9.24),(9.26) imply the third.

The first problem will be to construct a continuous vector, $R(x,\xi)$ satisfying (9.23)-(9.26). Let $\lambda(x,\xi)$ be the small eigenvalue of $H_1(x,\xi)$. The ellipticity requirement will then be fullfilled if we can find $R(x,\xi)$ non-vanishing in the kernel of $H_1(x,\xi)-\lambda(x,\xi)I$, satisfying (9.23)-(9.26).

On the level of Weyl symbols (9.13)-(9.15) give,

(9.27) $\qquad H_1=T^2\Gamma(H_1\circ\delta)\Gamma T^2$,

(9.28) $\qquad H_1=T^2(H_1\circ c_0)T^2$,

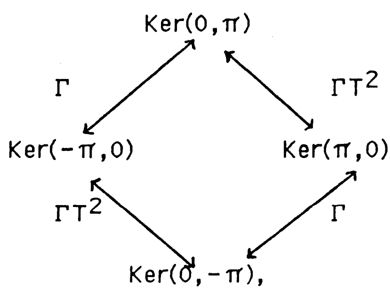
(9.29) $\qquad H_1\circ\tau_{2\pi\gamma}=H_1$.

We conclude that ΓT^2 maps $\mathrm{Ker}(H_1-\lambda_1)(x,\xi)$ onto $\mathrm{Ker}(H_1-\lambda)(\delta(x,\xi))$, and that T^2 maps $\mathrm{Ker}(H_1-\lambda)(x,\xi)$ onto $\mathrm{Ker}(H_1-\lambda)(c_0(x,\xi))$. From (9.27),(9.28), we also get,

(9.30) $\qquad H_1=\Gamma(H_1\circ\tilde{\delta})\Gamma$,

from which we conclude that Γ maps $\mathrm{Ker}(H_1-\lambda)(x,\xi)$ onto $\mathrm{Ker}(H_1-\lambda)(\tilde{\delta}(x,\xi))$. Let us shorten the notation by writing $\mathrm{Ker}(x,\xi)$ instead of $\mathrm{Ker}(H_1-\lambda)(x,\xi)$. By periodicity, $\mathrm{Ker}(0,\pi)=\mathrm{Ker}(0,-\pi)$, so T^2 maps $\mathrm{Ker}(0,\pi)$ onto itself. The eigenvalues of T^2 are 1 and -1, so the restriction of T^2 to $\mathrm{Ker}(0,\pi)$ is ± 1 and we define ι to be that number. We can compute ι by putting $\mu'=0$ and noticing that $\mathrm{Ker}(0,\pi)$ is then close to $\mathrm{Ker}(P_0(0,\pi))$. Hence ι is also given by the restriction of T^2 to this space. When $\mu=0$, we have, $b=|b|i^n$, $a=|a|i^{n\mp 1}$. An easy computation shows that $\mathrm{Ker}(P_0(0,\pi))$ is generated by ${}^t(1\ -1)$ when n is even, and by ${}^t(1\ 1)$, when n is odd. Hence $\iota=1$ when n is odd and equal to -1, when n is even.

From the commutative diagram,



where all maps commute with T^2, we conclude that $T^2=\iota$ also on the kernel of $H_1-\lambda$ at the points $(\pi,0),(0,-\pi),(-\pi,0)$.

We next notice that $\iota\Gamma T^2$ maps $\mathrm{Ker}(\pi/2,\pi/2)$ onto itself. This map is antilinear and conserves the $\mathbb{C}^2$ norm, so there is a non-vanishing element u_0 of $\mathrm{Ker}(\pi/2,\pi/2)$ such that $\iota\Gamma T^2 u_0=u_0$. Put $R(\pi/2,\pi/2)=u_0$. We extend the definition of $R(x,\xi)$ by continuity to the closed segment $s(0,1)$, in such way that (9.24) holds. Then there is a unique way of extending the definition to $s(0,3)$ so that (9.23) holds, and it is also easy to check that (9.24) extends to $s(0,3)$. We then also have (9.26) on $s(0,1)\cup s(0,3)$. Now let $v_0\in\mathrm{Ker}(-\pi/2,\pi/2)$ be a non-vanishing element such that $\Gamma v_0=v_0$. We put $R(-\pi/2,\pi/2)=v_0$ and extend the definition by continuity to $s(0,2)$ in such a way that (9.26) holds on $s(0,2)$ and so that $R(x,\xi)$ takes the already prescribed values at the endpoints of this segment. After that we extend to $s(0,4)$ by using (9.23). Then (9.23) holds on $\cup s(0,j)$, and we get (9.26) on $s(0,4)$ and hence on $\cup s(0,j)$. We have then obtained (9.23),(9.24),(9.26) on $\cup s(0,j)$, and by construction, we have $R(0,\pi)=R(0,-\pi)$, $R(\pi,0)=R(-\pi,0)$. We can then extend the definition of R to $\cup s(\alpha,j)$ in such a way that (9.25) holds. If $(x,\xi)=(y,\eta)+2\pi\gamma$, $(y,\eta)\in\cup s(0,j)$, then $c_0(x,\xi)\equiv c_0(y,\eta)$ and $\delta(x,\xi)\equiv\delta(y,\eta)$ modulo $2\pi\mathbb{Z}^2$, so we have (9.23)-(9.26) for the extension.

Let $\tilde{R}$ be an analytic function defined in a neighborhood of $\cup s(\alpha,j)$ such that $\tilde{R}-R$ is small on $\cup s(\alpha,j)$ and such that (9.25) holds. Then $\tilde{R}$ is close to $\hat{R}=\frac{1}{2}(\tilde{R}+\iota T^2(\tilde{R}\circ c_0))$, which satisfies (9.23) and (9.25). If $R'=\frac{1}{2}(\hat{R}+\iota T^2\Gamma(\hat{R}\circ\delta))$, then (9.23),(9.25) remain valid, and we also obtain (9.24). R' can be chosen arbitrily close to R (on $\cup s(\alpha,j)$) so $\Pi_1 R'$ is elliptic near $\cup s(\alpha,j)$. Writing R instead of R' we have then the required properties, and as we have seen, this gives an operator S_1 satisfying (9.16)-(9.18) and

(9.31) $\quad S_1S_1^*=\Pi_1,\ S_1^*S_1=I.$

Put $\tilde{H}_1=S_1^*H_1S_1$, so that,

(9.32) $\quad S_1\tilde{H}_1=H_1S_1,\ \tilde{H}_1S_1^*=S_1^*H_1.$

Combining this with (9.16)-(9.18) and (9.13)-(9.15), we get,

(9.33) $\quad [\mathcal{F}^2,\tilde{H}_1]=0,$

(9.34) $\quad [V,\tilde{H}_1]=0,$

(9.35) $\quad [T_\gamma,\tilde{H}_1]=0.$

In view of (9.12), we next define the isometry S_2 by,

(9.36) $\quad \mathcal{F}TS_1=S_2\mathcal{F}.$

Then $\mathcal{F}TS_2=(\mathcal{F}T)^2S_1\mathcal{F}^{-1}=\iota S_1\mathcal{F}$:

(9.37) $\quad \mathcal{F}TS_2=\iota S_1\mathcal{F}.$

If Π_2 is the projection associated to H_2 as in (9.2), we still have (9.12) after replacing H_1,H_2 by Π_1,Π_2. Then $S_2S_2^*=(\mathcal{F}TS_1\mathcal{F}^{-1})(\mathcal{F}TS_1\mathcal{F}^{-1})^*=$ $\mathcal{F}TS_1\mathcal{F}^{-1}\mathcal{F}S_1^*(\mathcal{F}T)^{-1}=\mathcal{F}TS_1S_1^*(\mathcal{F}T)^{-1}=\mathcal{F}T\Pi_1(\mathcal{F}T)^{-1}=\Pi_2$,

(9.38) $\quad S_2S_2^*=\Pi_2,\ S_2^*S_2=I.$

Using (9.16)-(9.18), we get,

(9.39) $(\mathcal{T}T)^2 S_2 = \iota S_2 \mathcal{T}^2$,

(9.40) $VS_2 = S_2 V$,

(9.41) $T_\gamma S_2 = S_2 T_\gamma$.

Put $\tilde{H}_2 = S_2^* H_2 S_2$, so that,

(9.42) $S_2 \tilde{H}_2 = H_2 S_2$, $S_2^* H_2 = \tilde{H}_2 S_2^*$.

In conclusion, $\tilde{H}_1$, $\tilde{H}_2$ have the same invariance properties as the operators H_1, H_2 of section 4.

We next recall that,

(9.43) $\det(P_0) = 2i|a||b|((\sin(2\arg(b)))/(|a||b|) \pm (\cos(\xi) + \cos(x)))$,

and that $\det(P) = \det(P_0) + \mathcal{O}(\varepsilon)$. It is clear that $\tilde{H}_j$ are modulo $\mathcal{O}(h)$ equal to elliptic factors times $\det(P)$, so if we consider $\mp(\sin(2\arg(b)))/(|a||b|)$ as the new spectral parameter, μ, then,

(9.44) $\tilde{H}_j = a_j(x,\xi)(\cos(\xi) + \cos(x) + \mathcal{O}(\varepsilon) + \mathcal{O}(h) - \mu)$,

where a_j are elliptic and real valued of order 0. It is no restriction to assume that $a_j > 0$. Repeating the argument of section 4, we see that the symbol $\tilde{H}_1$ is invariant under reflection in the point $(0,\pi)$, and that $(0,\pi)$, $(\pi,0)$, $(0,-\pi)$, $(-\pi,0)$ are non-degenerate saddle points for $\tilde{H}_1$ with the same critical value $d(\mu)$. We have,

(9.45) $\partial_\mu d(\mu) < 0$,

and $d(\mu)$ has a simple zero for some $\mu = \mathcal{O}(\varepsilon) + \mathcal{O}(h)$. After another change of the spectral parameter, we may assume that $d(\mu) = -\mu$. For $\mu = 0$, we also know that the real characteristics of $\tilde{H}_1$ close to the (complex) characteristics of P, are contained in an arbitrarily small neighborhood of $\cup s(\alpha,j)$, when ε and h are sufficiently small.

Let $\chi_{\alpha,j}$ be as in section 4, satisfying (4.26),(4.27),(4.34). Let $u_{0,1}$ be WKB solution of $H_1 u = 0$, defined microlocally near $\mathrm{Int}(s(0,1))$, and extended to neighborhoods of $(0,\pi)$ and $(\pi,0)$, as in section 4. The choice of extension is obvious, if we recall that,

(9.46) $u_{0,1} = S_1 \tilde{u}_{0,1}$,

where,

(9.47) $\tilde{H}_1 \tilde{u}_{0,1} = 0$.

We shall assume that $\tilde{u}_{0,1}$ is a normalized solution of (9.47). An easy computation then shows that,

$$(i[H_1, \chi_{0,1}]u_{0,1} | u_{0,1}) = (i[\tilde{H}_1, S_1^* \chi_{0,1} S_1]\tilde{u}_{0,1} | \tilde{u}_{0,1}) = 1,$$

since $\tilde{\chi}_{0,1} = S_1^* \chi_{0,1} S_1$ is equal to 1 near $(0,\pi)$ and equal to 0 near $(\pi,0)$. Hence $u_{0,1}$ is a normalized solution of $H_1 u = 0$. Notice that this notion has the same invariance properties as in the scalar case.

Imitating section 4, we define $f_{\alpha,j}$, $u_{\alpha,j}$ by, $f_{0,1}=i[H_1,\chi_{0,1}]u_{0,1}$,

(9.48) $f_{0,j}=(\mathcal{F}T)^{1-j}f_{0,1}$, $u_{0,j}=(\mathcal{F}T)^{1-j}u_{0,1}$,

(9.49) $f_{\alpha,j}=T_\alpha f_{0,j}$, $u_{\alpha,j}=T_\alpha u_{0,j}$.

Again, $u_{\alpha,j}$ is a microlocal solution of $H_1u=0$ if j is odd and of $H_2u=0$, if j is even. Repeating the arguments in section 4, (using the fact that we can always reduce ourselves to the scalar case near the characteristics of P,) we see that the corresponding Grushin problem for P is well posed. In order to be completely in the general framework of section 2, we need (2.39) rather than (4.32). We obtain this by using that $\chi_{0,1}\circ\delta=1-\chi_{0,1}$ near s(0,1) as in section 4: Since VT^2 commutes with H_1, we obtain:

$$VT^2f_{0,1}=-iVT^2[H_1,\chi_{0,1}]u_{0,1}=i[H_1,\chi_{0,1}]VT^2u_{0,1}.$$

Here $VT^2u_{0,1}$ is also a normalized microlocal solution of $H_1u=0$. The space of these solutions being one dimensional, we may assume after multiplying $u_{0,1}$ by a complex scalar, that $VT^2u_{0,1}=u_{0,1}$. Then $VT^2f_{0,1}=f_{0,1}$ microlocally, and as in section 4, we can turn this into an exact relation.

Introducing again the WKB problem, (4.36), we obtain Proposition 4.1, as well as (4.43),(4.44). Here λ has the same properties as in section 4, so again the problem is to find $\lambda\alpha$, $\lambda\beta$ satisfying (4.43). We recall here that $\gamma=\alpha$, $\delta=\beta$. Again $v_{0,1}$ is a normalized microlocal solution of $P^*v=0$, and $v_{\alpha,j}$ are generated from $v_{0,1}$ the same way as $u_{\alpha,j}$ were generated from $u_{0,1}$. Put $v_{0,1}=S_2\,\tilde v_{0,1}$, where $\tilde v_{0,1}$ is a normalized solution of $\tilde H_2v=0$. Then it is easy to check that,

(9.50) $u_{\alpha,1}=S_1\tilde u_{\alpha,1}$, $u_{\alpha,2}=\iota S_2\,\tilde u_{\alpha,2}$, $u_{\alpha,3}=\iota S_1\,\tilde u_{\alpha,3}$, $u_{\alpha,4}=S_2\,\tilde u_{\alpha,4}$,

(9.51) $v_{\alpha,1}=S_2\tilde v_{0,1}$, $v_{\alpha,2}=S_1\tilde v_{\alpha,2}$, $v_{\alpha,3}=\iota S_2\tilde v_{\alpha,3}$, $v_{\alpha,4}=\iota S_1\tilde v_{\alpha,4}$.

Substituting this into the problem (4.43), we get the equivalent problem,

(9.52) $\tilde u_{0,1}=\lambda\beta\,\tilde v_{0,2}$ near Int(s(0,2)),

$\tilde u_{0,1}=\iota\lambda\alpha\,\tilde v_{(0,1),4}$ near Int(s((0,1),4),

$\tilde u_{(0,1),3}=\iota\lambda\alpha\,\tilde v_{0,2}$ near Int(s(0,2)),

$\tilde u_{(0,1),3}=\lambda\beta\,\tilde v_{(0,1),4}$ near Int(s((0,1),4)).

It is then clear that the computation of $\lambda\beta$, $\lambda\alpha$ in section 4 applies without any change to $\lambda\beta$, $\iota\lambda\alpha$, and we see that Proposition 4.2 remains valid, with only one modification: We can no longer state that the + sign is valid in (4.71). The study of the new selfadjoining operators goes through almost without any change, we just have to replace V by VT^2 at certain places, and we see that Proposition 4.3 remains valid. So does the discussion after that proposition, with the obvious changes in (4.95),(4.99) due to the fact that now $\arg(b)-\arg(a)=\pm\pi/2$.

Summing up, we have,

<u>Proposition 9.1.</u> There exist functions, $\tilde\varepsilon_0:[1,\infty[\to]0,1]$, $\tilde\mu_0:[1,\infty[\to]0,1]$, $F:]0,1]\to[1,\infty[$, $\tilde h_0:]0,1]\times[1,\infty[\to]0,1]$, $\tilde\alpha:]0,1]\times[1,\infty[\times]0,1]\to]0,1]$, with

$\tilde{\alpha}(\varepsilon,C,h)\to 0$, when $h\to 0$ for every fixed (ε,C), such that if $0<\varepsilon\le 1$, $C\ge 1$, and P is a strong type 2 h-pseudodifferential operator with $C(P)\le C$, $\varepsilon(P)\le \tilde{\varepsilon}_0(C)$, $0<h\le \tilde{h}_0(\varepsilon,C)$, $\arg(b(0))=n\pi/2$ for some $n\in\mathbb{Z}$, then $\mu-Sp(P)\cap[-\tilde{\mu}_0(C),\tilde{\mu}_0(C)]\subset \cup_{-N_-\le j\le N_+} J_j$, where J_j are closed disjoint intervals, labelled in increasing order, such that for each j, there is an affine increasing map $\kappa_j:\mu\to\mu''$ such that one of the following holds:

a) $\kappa_j(J_j\cap\mu-Sp(P))=\mu''-Sp(Q)$, where $Q=Q_j$ is a strong type 1 h'-pseudodifferential operator with $\varepsilon(Q)\le\varepsilon$.

b) $\kappa_j(J_j\cap\mu-Sp(P))=\mu''-Sp(Q)$, where for every $\mu_0''\in\kappa_j(J_j)$, Q is of strong type 2 with $\varepsilon(Q)\le\tilde{\alpha}(\varepsilon,C,h)$, $C(Q)\le F(\varepsilon)$, as a function of $\mu'''=\mu''-\mu_0''\in]-4,4[$.

Combining this result with the propositions 7.1 and 8.1, we get the following result, which is the analogue of Theorem 6.2 for type 2 operators,

Theorem 9.2. There exist functions, $\tilde{\varepsilon}_0:[1,\infty[\to]0,1]$, $F:]0,1]\to[1,\infty[$, $\tilde{h}_0:]0,1]\times[1,\infty[\to]0,1]$, $\tilde{\alpha}:]0,1]\times[1,\infty[\times]0,1]\to]0,1]$, with $\tilde{\alpha}(\varepsilon,C,h)\to 0$, when $h\to 0$ for every fixed (ε,C), such that if $0<\varepsilon\le 1$, $C\ge 1$, and P is a strong type 2 h-pseudodifferential operator with $C(P)\le C$, $\varepsilon(P)\le\tilde{\varepsilon}_0(C)$, $0<h\le\tilde{h}_0(\varepsilon,C)$, then $\mu-Sp(P)\cap[-3,3]\subset\cup_{-N_-\le j\le N_+} J_j$, where J_j are closed disjoint intervals, labelled in increasing order, such that for each j, there is an affine increasing map $\kappa_j:\mu\to\mu''$ such that one of the following holds:

a) $\kappa_j(J_j\cap\mu-Sp(P))=\mu''-Sp(Q)$, where $Q=Q_j$ is a strong type 1 h'-pseudodifferential operator with $\varepsilon(Q)\le\varepsilon$.

b) $\kappa_j(J_j\cap\mu-Sp(P))=\mu''-Sp(Q)$, where for every $\mu_0''\in\kappa_j(J_j)$, Q is of strong type 2 with $\varepsilon(Q)\le\tilde{\alpha}(\varepsilon,C,h)$, $C(Q)\le F(\varepsilon)$, as a function of $\mu'''=\mu''-\mu_0''\in]-4,4[$.

The discussion in section 7-9 also gave information about the sizes of, and separations between the intervals: Let P be as in the theorem with ε and C fixed. Then the intervals J_j may be subdivided into groups, $\cup J_j\subset\cup I_k$, where I_k are closed disjoint intervals of width ~ 1 and at most $\pi+\mathcal{O}(\varepsilon(P))+\mathcal{O}(h)$. The separation between two neighboring intervals is at least $h^{\frac{1}{2}}/(\text{const.})$ and at most $\pi/(1+\text{const.})$, where "const." indicate two strictly positive constants, which only depend on C. In each I_k there is at most one $J_j=J_{j(k)}$, such that b) of the theorem applies, and $J_{j(k)}$ is of width $\sim h$ and situated at a distance $\mathcal{O}(\varepsilon(P))+\mathcal{O}(h)$ from the middle point of I_k. This interval exists if the middle of I_k is in $[-3,3]$ and at a distance $\gg\varepsilon(P)+h$ from ± 3. As for the other intervals in I_k, their sizes and separations are as described after Theorem 6.2, with only one modification: If $I_k=[a',a]$, and $I_{k+1}=[b,b']$ are two consecutive intervals such that $b-a$ is very small (but larger than $h^{\frac{1}{2}}/\text{const.}$), and a,b belong to some intervals J_j and J_{j+1} (,which we always can assume after shrinking I_k and I_{k+1}), then if $\mu>b$ is close to b and in the separation of two consecutive J-intervals, the size of that separation is of the order of magnitude, $h/((b-a)+(\mu-b))$. The analogous statement holds for the separation between two consecutive intervals slightly to the left of a; just replace $\mu-b$ by $a-\mu$.

The theorems 6.2 and 9.2 may be combined into an infinite iteration scheme. Without loss of generality, we may assume that the function F is the same in both theorems. Let $\varepsilon_0>0$ be as in Theorem 6.2 and let $0<h_1\le\min(h_0(\varepsilon_0),\tilde{h}_0(\varepsilon_0,F(\varepsilon_0)))$ be sufficiently small, so that if $0<h\le h_1$, then $\max(\alpha(\varepsilon_0,h),\tilde{\alpha}(\varepsilon_0,F(\varepsilon_0),h))\le\tilde{\varepsilon}_0(F(\varepsilon_0))$. Then we obtain:

<u>Theorem 9.3.</u> Let $0<h\le h_1$ and define $h'\in]0,2\pi]$ by $2\pi/h\equiv h'/2\pi \bmod(\mathbb{Z})$. Then,

(A) Let P be a strong type 1 h-operator with $\varepsilon(P)\le\varepsilon_0$. Then after restriction to suitable subintervals and after suitable affine maps, the study of μ-Sp(P) can be reduced to the study of μ''-Sp(Q)$\cap[-3,3]$, where Q is either a strong type 1 h'-operator with $\varepsilon(Q)\le\varepsilon_0$, or a strong type 2 h'-operator with $\varepsilon(Q)\le\tilde{\varepsilon}_0(F(\varepsilon_0))$, $C(Q)\le F(\varepsilon_0)$.

(B) Let P be a strong type 2 h-operator with $\varepsilon(P)\le\tilde{\varepsilon}_0(F(\varepsilon_0))$, $C(P)\le F(\varepsilon_0)$. Then for μ-Sp(P)$\cap[-3,3]$, we have the same conclusion as in (A).

This theorem, together with the more precise quantitative information, that we added after the theorems 6.2 and 9.2, give Theorem 0.1 in the introduction, in the case when all the q_j in (0.3) are positive. The following discussion shows how to extend this result to the case when q_j have arbitrary signs.

Let us first extend the definition of the auxiliary operators to the case case when $h<0$. The definition of $T_\alpha=T_{\alpha,h}$ is as before, and we check that,

(9.53) $$T_{\alpha,-h}=T_{\gamma(\alpha),h},$$

where we recall that $\gamma(\alpha)=(\alpha_1,-\alpha_2)$. The extention of $\mathcal{F}_h$ to negative h is given by:

(9.54) $$\mathcal{F}_h u(\xi)=(2\pi|h|)^{-\frac{1}{2}}\int e^{-ix\xi/h}\,u(x)\,dx.$$

Then,

(9.55) $$\mathcal{F}_{-h}=\mathcal{F}_h^{-1},$$

so if we put $R=h^2D_x^2+x^2-|h|$, then also for negative h, we have,

(9.56) $$\mathcal{F}_h=e^{-i\pi R/2h}=U_{-\pi/2,h}.$$

Here, we write $U_{t,h}=e^{itR/h}$. As before, we put:

(9.57) $$V_h=\Gamma\mathcal{F}_h=\mathcal{F}_h^{-1}\Gamma=U_{\pi/4,h}\Gamma U_{-\pi/4,h}\ ,$$

and we notice that,

(9.58) $V_{-h}=V_h\mathcal{F}_h^2=\mathcal{F}_h^2V_h$,

where $\mathcal{F}_h^2$ is independent of h. If $P(x,\xi)$ is a symbol, we still define the h-Weyl quantization of P as the standard Weyl quantization of $P(x,h\xi)$. By a change of variables:

(9.59) $P(x,hD)u(x)=(2\pi|h|)^{-1}\iint e^{i(x-y)\theta/h}P(\frac{1}{2}(x+y),\theta)u(y)dyd\theta$.

The commutation relations of section 1 then extend to the case, $h<0$. An h-pseudodifferential operator can also be viewed as an (−h)-pseudodifferential operator, and the relation between the corresponding symbols is simply: $\tilde{P}=P\circ\gamma$. The rules for computing the symbols of of $\Gamma P\Gamma$, VPV, $\mathcal{F}^{-1}P\mathcal{F}$ are the same for negative h (provided that the conjugating operators are quantized with the same h).

The definition of (strong) type 1 and 2 operators now extends naturally to the case when $h<0$, and for type 1 operators, we have:

Proposition 9.4. If (P,P_1,P_2) is of (strong) type 1 in the sense of h-pseudodifferential opertators, then the same is true in the sense of (−h)-pseudodifferential opertators.

Proof. Let us first check that the invariance properties are satisfied also in the (−h)-quantization: Using (9.53), it is obvious that P and P_j commute with $T_{\alpha,-h}$. Let $\mathcal{F}$, V denote the h-quantizations and $\mathcal{F}_-$, V_- the (−h)-quantizations. Then combining (2.2),(2.5) with (9.55), we see that (2.2),(2.5) hold with $\mathcal{F}$ replaced by $\mathcal{F}_-$. Since P,P_j commute with V (by (2.3),(2.6)) and with $\mathcal{F}^2$, they also commute with $V\mathcal{F}^2=V_-$. Hence, if (P,P_1,P_2) is of type 1 as h-operators, the same is true in the sense of (−h)-operators. Since $\cos(\xi)+\cos(x)$ is even in ξ, it is also clear that the notion of strong type 1 operators is independent of the sign of h. ∎

For type 2 operators, the situation is a little more complicated. Recall the definition of the operator T in section 2, and let S denote one of the two operators S_1: $\mathbb{C}^2_{odd}\to\mathbb{C}^2_{even}$, S_2: $\mathbb{C}^2_{even}\to\mathbb{C}^2_{odd}$, both given by the identity matrix, for the natural bases. Then S is real and,

(9.60) $TST=S$, $S^2=I$, $S^*=S^{-1}=S$.

Proposition 9.5. If (P,P_1,P_2) is a type 2 h-operator, then $(\tilde{P},\tilde{P}_1,\tilde{P}_2)=(SP^*S,SP_2^*S,SP_1^*S)$ is a type 2 (−h)-operator.

Proof. $\tilde{P}_1^*\tilde{P}=SP_2S^2P^*S=SP_2P^*S=SPP_2^*S$, which is self adjoint. Similarly, we see that $\tilde{P}\tilde{P}_2^*$ is self adjoint. It is obvious that $\tilde{P}$, $\tilde{P}_1$, $\tilde{P}_2$ commute with T_α. Let's check the Fourier invariance: $\tilde{P}_1\mathcal{F}_-T=SP_2^*S\mathcal{F}^{-1}T=SP_2^*\mathcal{F}^{-1}T^{-1}S=SP_2^*(\mathcal{F}T)^{-1}S=S(\mathcal{F}T)^{-1}P_1S=\mathcal{F}^{-1}TSP_1S=\mathcal{F}_-T\tilde{P}_2^*$. The remaining two Fourier invariance relations are proved the same way. The three reflection invariance relations are also proved the same way, so we only treat one case: $\tilde{P}_1V_-T^2=SP_2^*SV\mathcal{F}^2T^2=\mathcal{F}^2T^2SP_2^*SV=\mathcal{F}^2T^2VT^2SP_2^*S=V_-\tilde{P}_1$.

∎

Proposition 9.6. If (P,P_1,P_2) is of strong type 2 with respect to μ,h, then $(\tilde{P},-\tilde{P}_1,-\tilde{P}_2)$ (defined in the preceding proposition) is of strong type 2 with respect to $-\mu,-h$, and we have $\varepsilon(\tilde{P})=\varepsilon(P)$, $C(\tilde{P})=C(P)$.

Proof. If $P(x,\xi)$ denotes the h-symbol of P, then the (-h)-symbol of SP^*S is $SP(x,-\xi)^*S$. Here we recall that S is given by the identity matrix for the standard bases, so from: $P(x,\xi)=P_0(a,b;x,\xi)+\mathcal{O}(\varepsilon)$, we get $SP(x,-\xi)^*S=P_0(a,b;x,-\xi)^*+\mathcal{O}(\varepsilon)=P_0(\bar{a},\bar{b};x,\xi)+\mathcal{O}(\varepsilon)$. The same computations work for $-\tilde{P}_1$, $-\tilde{P}_2$, and the proposition follows. ∎

Propositions 9.4-9.6 permit to extend Theorems 6.2 and 9.2 to the case when h and h' are in $[-2\pi,0[$. (In the statements, we simply have to replace all upper bounds on h by upper bounds on $|h|$.) Also, if we start with $h>0$, and $2\pi/h\equiv h'/2\pi \mod(\mathbb{Z})$ with $|h'|<\pi$, $h'\neq 0$, then the arguments of sections 2-6 work without any changes, so Theorems 6.2 and 9.2 are valid also if we suppress the assumption that h' should be positive. This also holds for Theorem 9.3, and we then get Theorem 0.1.

a. Microlocal analysis.

In this section, we treat various generalities that will be needed in the main text. We shall first recall the general approach to (analytic) microlocal analysis, based on FBI (or generalized Bargmann) transforms in [S1]. Then we develop a simple functional calculus for analytic pseudodifferential operators, and finally, we make some remarks on the Weyl quantization of symbols.

a.1. FBI transforms and microlocal analysis. We shall essentially recall some of the theory of [S1], with some modifications, in order to treat global questions. As a general rule, the large parameter λ in [S1] will here be equal to $1/h$, with $0<h\leq 2\pi$.

Let $\Omega\subset\mathbb{C}^n$ and let φ be a continuous real valued function, defined on Ω. A function $u(x,h)$ on $\Omega\times]0,2\pi]$ (or possibly with 2π replaced by some smaller constant, >0) belongs by definition to the space $H_\varphi^{loc}(\Omega)$, if,

(a.1.1) u is holomorphic in x for every h,

(a.1.2) For every compact $K\subset\Omega$ and every $\varepsilon>0$, there is a constant $C>0$, such that $|u(x,h)|\leq Ce^{(\varphi(x)+\varepsilon)/h}$ on $K\times]0,2\pi]$.

We also define $H_\varphi(\Omega)$ to be the space of functions u on $\Omega\times]0,2\pi]$ satisfying (a.1.1), such that $u(.,h)$ belongs to $L^2(\Omega,e^{-2\varphi/h}L(dx))$ for every h. Here $L(dx)$ denotes the Lebesgue measure on Ω. Notice that the properties of belonging to H_φ^{loc} is an asymptotic property, while the property of belonging to $H_\varphi(\Omega)$ is of interest for every h, (and sometimes we shall use the property $u\in H_\varphi(\Omega)$ for individual h's or for a much smaller set of h's (by abuse of terminology)). We call $a(x,h)=\Sigma_0^\infty a_k(x)h^k$ a (formal) classical analytic symbol (c.a.s.) of order 0, if the a_k are holomorphic in the same open set, Ω and if for every compact $K\subset\Omega$, there is a constant $C>0$, such that $|a_k(x)|\leq C^{k+1}k^k$ for every $x\in K$, $k\in\mathbb{N}$. (Sometimes we may also allow k to vary in the half integers.) A classical analytic symbol of order $m\in\mathbb{R}$ is an expression of the form $h^{-m}a(x,h)$, where a is a c.a.s. of order 0. If a is a c.a.s. of order m, then for every open $\tilde{\Omega}\subset\subset\Omega$, we can define a realization, $\tilde{a}\in H_\varphi^{loc}(\tilde{\Omega})$, well defined modulo $\mathcal{O}(e^{-1/Ch})$ for some $C>0$. (This is done by summing the first (const.)/h terms in the formal sum giving a, and choosing the constant suitably. See [S1], page 3, for more details.) In the main text we will also have lots of parameters. A family of c.a.s. of order 0, $a_\alpha(x,h)$, $\alpha\in A$ on Ω, is by definition bounded if for every compact K in Ω, we can choose a constant C as above, which is independent of α. A c.a.s. a, of order 0 is <u>elliptic</u>, if the leading term, a_0 in its formal series expansion is non-vanishing.

We next define local FBI transforms (as in [S1], chapter 7). Let $\varphi(x,y)$ be a holomorphic function, defined in a neighborhood of $(x_0,y_0)\in\mathbb{C}^n\times\mathbb{R}^n$, such that,

(a.1.3) $\varphi_y'(x_0,y_0)=-\eta_0\in\mathbb{R}^n$,

(a.1.4) $\mathrm{Im}\varphi_y''{}_y(x_0,y_0)>0$, (in the sense of symmetric matrices),

(a.1.5) $\det(\varphi_x''{}_y(x_0,y_0))\neq 0$.

Let $t(x,y,h)$ be a realization of an elliptic c.a.s. . Let $Y\subset\mathbb{R}^n$, $X\subset\mathbb{C}^n$ be small neighborhoods of y_0 and x_0 respectively, and let $\chi\in C_0^\infty(Y)$ be equal to 1 in a neighborhood of y_0. Assuming these quantities chosen suitably, we can then define for $u\in\mathcal{D}'(Y)$, $x\in X$,

(a.1.6) $\quad Tu(x,h)=\int e^{i\varphi(x,y)/h}\, t(x,y,h)\chi(y)u(y)dy,$

and if u is independent of h, we get $Tu\in H_\Phi^{loc}(X)$, where,

(a.1.7) $\quad \Phi(x)=\sup_{y\in Y}(-\mathrm{Im}(\varphi(x,y)).$

The same conclusion is valid, if $u=u_h$ depends on h, but some Sobolev norm of u_h is of at most temperate growth when $h\to 0$ and bounded on every compact h-interval.

Viewing T as Fourier integral operator, we can associate to it the canonical transformation $\kappa_T:(y,-\varphi_y'(x,y))\to(x,\varphi_x'(x,y))$ from a complex neighborhood of (y_0,η_0) onto a complex neighborhood of (x_0,ξ_0). Here $\xi_0=-2i\partial\Phi(x_0)/\partial x$, and more generally, if we introduce $\Lambda_\Phi=\{(x,-2i\partial\Phi(x)/\partial x);\ x\in X\}$, then (locally near (x_0,ξ_0),) $\Lambda_\Phi=\kappa_T(\mathbb{R}^{2n})$.

Using FBI-transforms, we get a convenient setting for the microlocal theory of Fourier integral operators. We restrict the discussion to germs, but using classical theorems on the resolution of the $\bar\partial$-operator, it is easy to extend the discussion (by partitions of unity) to the case of pseudo-convex domains equipped with plurisubharmonic weight functions.

Let $\varphi(x,y,\theta)$ be a non-degenerate phase function defined near $(x_0,y_0,\theta_0)\in\mathbb{C}^n\times\mathbb{C}^n\times\mathbb{C}^N$, in the sense that,

1° $\quad \varphi$ is holomorphic,

2° $\quad d\varphi'_{\theta_1},..,d\varphi'_{\theta_N}$ are independent on the set C_θ defined by $d_\theta\varphi=0$.

We assume that $(x_0,y_0,\theta_0)\in C_\varphi$, and put $\eta_0=-\varphi_y'(x_0,y_0,\theta_0)$, $\xi_0=\varphi_x'(x_0,y_0,\theta_0)$. Then $\Lambda_\varphi=\{(x,\varphi_x'(x,y,\theta);y,-\varphi_y'(x,y,\theta));\ (x,y,\theta)\in C_\varphi\}\subset \mathbb{C}^n{}_{(x,\xi)}\times\mathbb{C}^n{}_{(y,\eta)}$ is a canonical relation, that is: a Lagrangian manifold for $\sigma_{(x,\xi)}-\sigma_{(y,\eta)}$, where $\sigma_{(x,\xi)}=\Sigma d\xi_j\wedge dx_j$ and similarly for $\sigma_{(y,\eta)}$. Let us assume that Λ_φ is the graph of a canonical transformation, and let $\Phi(y)$ be a plurisubharmonic C^2 function defined near y_0 satisfying, $\eta_0=-2i\partial\Phi(y_0)/\partial y$. Then $\kappa(\Lambda_\Phi)$ is of real dimension 2n, and if the projection of this manifold to $\mathbb{C}_x{}^n$ is a local diffeomorphism, then it has the form Λ_Ψ, for some C^2 function Ψ. If we assume that the hessian $\Psi''(x_0)$ is plurisubharmonic, then Ψ is plurisubharmonic everywhere, and after changing Ψ by a constant, we have $\Psi(x)=\mathrm{v.c.}_{(y,\theta)}(-\mathrm{Im}\varphi(x,y,\theta)+\Phi(y))$, where $\mathrm{v.c.}_{(y,\theta)}..$ means "critical value with respect to the variables (y,θ) of the function ..", and where it turns out that the critical value is taken at a non-degenerate critical point of signature 0, that is at a saddle point. Conversely, if we know that $(y,\theta)\to-\mathrm{Im}\varphi(x_0,y,\theta)+\Phi(y)$ has a saddle point at (y_0,θ_0), then the critical value Ψ, defined as above will be plurisubharmonic, and $\Lambda_\Psi=\kappa(\Lambda_\Phi)$.

If $a(x,y,\theta,h)$ is the realization of an c.a.s. in a neigborhood of (x_0,y_0,θ_0), then for suitable neighborhoods Y,X of y_0 and x_0 respectively, we can define the Fourier integral operator,

(a.1.8) $\quad Au(x,h)=\iint e^{i\varphi(x,y,\theta)/h}\, a(x,y,\theta,h)u(y,h)\, dyd\theta,$

as an operator $A:H_{\Phi}^{loc}(Y)\longrightarrow H_{\Psi}^{loc}(X)$, by choosing a suitable integration contour, and introducing suitable cut-off functions. A different choise of contour and cut-off will change Au only by a term which is exponentially small in a neighborhood of x_0. There is a natural composition result for these Fourier integral operators, and it is also possible to show that the class of operators we get by varying a essentially only depends on κ and on the normalizing value $\varphi(x_0,y_0,\theta_0)$. As a particular case, we get the classical analytic pseudodifferential operators, when $x_0=y_0$, $N=n$ and $\varphi(x,y,\theta)=(x-y)\theta$. The standard quantization of a symbol a is then,

(a.1.9) $\quad Au(x,h)=(2\pi h)^{-n}\iint e^{i(x-y)\theta/h}\, a(x,\theta,h)\, u(y)\, dyd\theta,$

while the Weyl quantization of a symbol b is

(a.1.10) $\quad Bu(x,h)=(2\pi h)^{-n}\iint e^{i(x-y)\theta/h}\, b(\tfrac{1}{2}(x+y),\theta,h)\, u(y)\, dyd\theta.$

The standard symbol, a and the Weyl symbol b are uniquely determined from the operators A,B, and we get the same operator, A=B if and only if

(a.1.11) $\quad b=e^{-ihD_xD_\theta/2}a,\ a=e^{ihD_xD_\theta/2}b,$

where we also know (by analytic stationary phase) that $e^{\pm iD_xD_\theta/h}$ are order preserving maps on the space of formal analytic symbols in an open set. We say that the pseudodifferential operator A is of order m if the corresponding symbol (either standard or Weyl) is of order m. In the natural composition result for analytic ps.d.o. we then also know that the order of the composition of two operators is equal to the sum of the orders of each factor. The (standard or Weyl) symbol of the composition, $C=A\circ B$ is also given by the standard composition formulas,

(a.1.12) $\quad c=\Sigma_0^{\infty}(k!)^{-1}(\partial_\xi D_y)^k(a(x,\xi)b(y,\eta))\,|_{(y,\eta)=(x,\xi)},$

(a.1.13) $\quad c=\Sigma_0^{\infty}(k!)^{-1}(\tfrac{1}{2}i\sigma(D_x,D_\xi;D_y,D_\eta))^k(a(x,\xi)b(y,\eta))\,|_{(y,\eta)=(x,\xi)},$

for respectively the standard symbols and the Weyl symbols.

If A is a Fourier integral operator, with an elliptic symbol, associated to the canonical transformation κ, with $\kappa(y_0,\eta_0)=(x_0,\xi_0)$, then we can find a Fourier integral operator B associated to κ^{-1} such that $B\circ A=I$, $A\circ B=I$, in the sense that the pseudodifferential operators $B\circ A$ and $A\circ B$, defined respectively near (y_0,η_0) and (x_0,ξ_0) have symbol 1. (This implies that the corresponding realizations on suitable H_{Φ}^{loc}-spaces, are simply the identity operator modulo exponentially small errors). Applying this to the FBI-transform, T in (a.1.6), we can find an "inverse" $S=T^{-1}$, of the form,

(a.1.14) $\quad Sv(y,h)=\int e^{-i\varphi(x,y)/h}s(x,y,h)v(x)\, dx,$

and if we denote by m(t) and m(s) the orders of t and s respectively, then $m(t)+m(s)=n$. If $\tilde{T}$ is a second FBI transform with $\kappa_{\tilde{T}}(y_0,\eta_0)=(\tilde{x}_0,\tilde{\xi}_0)$, and if Φ and $\tilde{\Phi}$ denote the weightfunctions associated to T and $\tilde{T}$ respectively, by (a.1.7) and it's analogue for $\tilde{T}$, then if $u=u_h \in \mathcal{D}'(Y)$ is a family of distributions of temperate growth, as specified after (a.1.7), and if U denotes the Fourier integral operator composition $\tilde{T}\circ S$, we have near $\tilde{x}_0$, $\tilde{T}u_h(x,h) \equiv UTu_h(x,h)$ modulo an exponentially small error in H_Φ^{loc}. Naturally, since the behaviour of $Tu_h(x,h)$ near x_0 only tells us anything about the phase space behaviour of u_h near (y_0,η_0), there is no hope to be able to recover u_h from only that information. This information is however sufficient, to predict the behaviour of $\tilde{T}u_h$ near $\tilde{x}_0$, and it justifies the terminology:"Let $u=u_h$ be a distribution defined microlocally near (y_0,η_0)". By this we mean simply, that each time we choose an FBI transformation T as above, then up to an exponentially small uncertainty, we have an element Tu of class H_Φ^{loc} defined in a neighborhood of x_0, the various elements being related by $\tilde{T}u \equiv (\tilde{T}\circ T^{-1})(Tu)$.

If $a(y,\eta,h)$ is a c.a.s. defined near (y_0,η_0), then we can associate a formal pseudodifferential operator A, using either the standard or the Weyl quantization. By "Egorov's theorem", which is valid in our setting, $B=TAT^{-1}$ is then again a pseudodifferential operator with a c.a.s. (either for the standard or for the Weyl quantization) defined near (x_0,ξ_0), so we can define the action of $B: H_\Phi^{loc}(\Omega_1) \to H_\Phi^{loc}(\Omega_2)$, where Ω_1, and Ω_2 are suitably chosen neighborhoods of x_0. In other words, we have then defined the action of A on functions that are defined microlocally near (y_0,η_0). Similarly, if κ is a real canonical transformation from a neighborhood of (y_0,η_0) onto a neighborhood of (z_0,ζ_0), given by the non-degenerate phase function, $\varphi(z,y,\theta)$, defined near (z_0,y_0,θ_0), and if a is a c.a.s. defined near the same point, then we can define the action of the corresponding Fourier integral operator A, mapping functions that are defined microlocally near (y_0,η_0) to functions defined near (z_0,ζ_0). Again, we have the natural composition results, including Egorov's theorem.

Somewhere in the proofs of these results in [S1], it is made use of a certain resolution of the identity, which permits to represent a distribution locally and not only microlocally as a superposition of Gaussians. Instead it would have been possible to use a global FBI-transform, (which is essentially a Bargmann transform). Such a transform is given by,

(a.1.15) $\quad Tu(x,h)=\int e^{i\varphi(x,y)/h}\, t(h)\, u(y)dy, \quad x\in\mathbb{C}^n,\ u\in L^2(\mathbb{R}^n),$

where φ is a second order polynomial, satisfying (a.1.3)-(a.1.5) (which now automatically become global conditions), and where $t(h)$ is of the form Ch^{-m}, $C\neq 0$. The function Φ, defined as before, now becomes a strictly plurisubharmonic second order polynomial on $\mathbb{C}^n$. Choosing $m=-3n/4$ and C suitably, we can arrange so that T becomes isometric: $L^2(\mathbb{R}^n) \to H_\Phi(\mathbb{C}^n)$ (by verifying that T^*T is the identity operator). Using also that the orthogonal projection $\Pi: L^2(\mathbb{C}^n; e^{-2\Phi/h}L(dx)) \to H_\Phi(\mathbb{C}^n)$ has a simple explicit integral kernel, (that can be obtained by choosing a suitable contour when writing the identity as a pseudodifferential operator, see [S1]), it is also easy to check that $TT^*=\Pi=I$ on $H_\Phi(\mathbb{C}^n)$, so T is unitary. We also notice both in the global and in the local case, that if T is an FBI transform, then $T\mathcal{F}$ is also an FBI transform. Here $\mathcal{F}$ denotes the unitary Fourier transform, defined in section 1. In the global case, it is possible to choose φ in such a way that Φ becomes rotation invariant, and $T\mathcal{F}=\mathcal{R}T$, where $\mathcal{R}v(x)=v(ix)$.

a.2. Functional calculus for analytic pseudodifferential operators.

Let P be an $m_0\times m_0$ matrix of formal analytic pseudodifferential operators of order 0, with a c.a.s. defined in a neighborhood of $(x_0\xi_0)\in\mathbb{R}^{2n}$. Let $p=p_0(x,\xi)$ be the principal symbol, that is the leading term in the asymptotic expansion in powers of h, of the symbol of P. (Notice that p does not depend on whether we take the standard or the Weyl quantization.) If $z\in\mathbb{C}\setminus Sp(p(x_0,\xi_0))$, then $(z-P)^{-1}$ is a formal analytic pseudodifferential operator of order 0 whose symbol is defined in a neighborhood of (x_0,ξ_0). If $F(z,h)$ is a c.a.s. of order m, defined in a neighborhood of $Sp(p(x_0,\xi_0))$, then we define $F(P,h)$ as the formal analytic pseudodifferential operator of order m, given by,

(a.2.1) $\quad F(P,h)=(2\pi i)^{-1}\int_{\partial\Omega} F(z,h)(z-P)^{-1}dz,$

where $\partial\Omega$ is the oriented C^∞ boundary of a small neigborhood, Ω, of $Sp(p(x_0,\xi_0))$. When F is a polynomial in z, it is easy to check that the definition coincides with what one expects in that case. In the general case, let $f(z)h^{-m}$ be the leading contribution to F (i.e; the principal symbol of F). Then the principal symbol of $F(P,h)$ is $f(p(x,\xi))h^{-m}$. If G is a second symbol with the same properties as F, we check, by choosing a smaller Ω in one of the representations of $F(P)$ or $G(P)$, that,

(a.2.2) $\quad F(P,h)\circ G(P,h)=(FG)(P,h).$

Finally, if F is of order 0, and if G is a c.a.s. defined in a neighborhood of $Sp(f(p(x_0,\xi_0))$, then we have,

(a.2.3) $\quad G(F(P,h))=(G\circ F)(P,h).$

In fact, let α, β be contours around $Sp(f(p(x_0,\xi_0)))$ and $Sp(p(x_0,\xi_0))$ respectively, such that if $z\in\beta$, then $f(z)$ is in the interior of α. Then,

$$G(F(P))=(2\pi i)^{-1}\int_\alpha G(w)(w-F(P))^{-1}\,dw=$$
$$(2\pi i)^{-2}\int_\alpha\int_\beta G(w)(w-F(z))^{-1}dw\,(z-P)^{-1}dz=$$
$$(2\pi i)^{-1}\int_\beta G\circ F(z)(z-P)^{-1}dz=(G\circ F)(P).$$

We finally notice that if the various symbols depend on parameters, but vary in bounded sets, then the same is true for the resulting symbols.

The Weyl composition formula (a.1.13) implies that $c=ab+(h/i)\{a,b\}+\mathcal{O}(h^2)$, when a and b are of order 0. Using this, it is easy to verify that the symbol of $(z-P)^{-1}$ is $(z-P(x,\xi))^{-1}+\mathcal{O}(h^2)$, in the case when P is a scalar symbol, so if F is a symbol of order 0, then the Weyl symbol of

$F(P,h)$ is $F(P(x,\xi),h)+\mathcal{O}(h^2)$.

a.3. Weyl quantization and conjugation by Fourier integral operators.

The results of this subsection are valid either in the category of ordinary classical C^∞ symbols on the real domain, when the corresponding phase functions are real, or in the category of analytic symbols, either on the real domain with real phase functions, or on the complex domain, with complex phases. The assumptions in each case will be local and the obvious ones. To fix the ideas we choose the real case (analytic or C^∞). To start with, we examine the simple case of conjugation by an elliptic factor. Let $P(x,\xi)$ be defined near $(x_0,\xi_0)\in\mathbb{R}^{2n}$. (We could also let P be a classical symbol of order 0). Let $\varphi(x)$ be a real valued smooth function , defined near x_0, such that $\varphi'(x_0)=\zeta_0$. Let P also denote the Weyl quantization of P. We are then interested in the Weyl symbol of the pseudodifferential operator $Q=e^{-i\varphi(x)/h}\circ P\circ e^{i\varphi(x)/h}$, which is defined near $(x_0,\xi_0-\zeta_0)$. We proceed formally, by first writing the integral kernel of Q as,

(a.3.1) $$K_Q(x,y)=\int e^{i((x-y)\theta-(\varphi(x)-\varphi(y)))/h}\,P((x+y)/2,\theta)\,\tilde{d}\theta,$$

$$\tilde{d}\theta=d\theta/(2\pi h)^n.$$

Using the standard trick, we write,

$\varphi(x)-\varphi(y)=(x-y)(\varphi_x'((x+y)/2)+\mathcal{O}((x-y)^2)$. After a change of variables, we then get,

(a.3.2) $$K_Q(x,y)=\int e^{i(x-y)\theta/h}\,P(\tfrac{1}{2}(x+y),\varphi'(\tfrac{1}{2}(x+y))+\theta)+\mathcal{O}((x-y)^2)\,\tilde{d}\theta.$$

Here the $\mathcal{O}$-term contributes to the Weyl symbol of Q by $\mathcal{O}(h^2)$ (i.e. by a classical symbol of order -2), so if we denote by Q also the Weyl symbol, then we get,

(a.3.3) $$Q(x,\xi)=P(x,\xi+\varphi'(x))+\mathcal{O}(h^2).$$

If $a=a(x,h)\sim a_0(x)+a_1(x)h+\dots$ is a classical symbol defined near x_0, then by formal stationary phase,

(a.3.4) $$Q(a)\sim Q(x,0)a(x)+(h/i)(Q'_\xi(x,0)a'(x)+\tfrac{1}{2}\Sigma Q''_{x_j\xi_j}(x,0)a(x))+\mathcal{O}(h^2).$$

Assume now that P is real valued, and that $P(x_0,\xi_0)=0$, $P'_\xi(x_0,\xi_0)\neq 0$. Let φ be real valued and satisfy the eiconal equation,

(a.3.5) $$P(x,\varphi'(x))=0.$$

We then want to construct a such that $e^{-i\varphi/h}P(e^{i\varphi/h}a)=0$ (in the sense of formal classical symbols). Then combining (a.3.3) and (a.3.4), we get the leading transport equation,

(a.3.6) $$q'_\xi(x,0)a_0'(x)+\tfrac{1}{2}\Sigma q''_{x_j\xi_j}(x,0)a_0(x)=0,$$

where $q(x,\xi)=P(x,\varphi'(x)+\xi)$ is real valued. It follows that a_0 has constant argument along each bicharacteristic curve, associated to P,φ.

We now attack the more general question of conjugation with F.I.O.'s. Recall that if A,B are classical pseudodifferential operators of order 0, defined microlocally near (0,0), and if $C=[A,B]$, then for the corresponding Weyl symbols (denoted by the same letters), we have,

(a.3.7) $$C=(h/i)\{A,B\}+\mathcal{O}(h^3).$$

We shall exploit this fact in order to give a refinement of Egorov's theorem.

Let κ be a canonical transformation from a neighborhood of $(0,0)\in\mathbb{R}^{2n}$ onto another neighborhood of the same point, and assume in order to fix the ideas that $\kappa(0,0)=(0,0)$. Then, we can find a smooth 1-parameter family of canonical transformations, κ_t, $t\in[0,1]$, with the same properties, such that $\kappa_0=\mathrm{id}$ and $\kappa_1=\kappa$. We have,

(a.3.8) $\quad \partial_t\kappa_t(x,\xi)=H_{Q_t}(\kappa_t(x,\xi))$,

where Q_t is a smooth family of smooth real valued functions, defined in a neighborhood of $(0,0)$. (Here H_Q denotes the Hamilton field of Q.) Let Q_t also denote the corresponding Weyl quantization, so that $Q^*_t=Q_t$, where the formal adjoint is taken with respect to the standard inner product on $L^2(\mathbb{R}^n)$. We can then define a family of unitary Fourier integral operators, U_t associated to κ_t, defined microlocally near $(0,0)$, and given by,

(a.3.9) $\quad hD_tU_t+Q_tU_t=0,\ \ U_0=I,\ \ \text{where } D_t=i^{-1}\partial/\partial t$

Let P be a pseudodifferential operator of order 0, defined near $(0,0)$, and define P_t by,

(a.3.10) $\quad P_tU_t=U_tP$.

For the corresponding Weyl symbols, we claim that,

(a.3.11) $\quad P_t\circ\kappa_t=P+\mathcal{O}(h^2)$.

The interesting fact here (as well as in (a.3.3)) is that we have $\mathcal{O}(h^2)$ and not just $\mathcal{O}(h)$.

To prove (a.3.11), we start by differentiating (a.3.10) with respect to t: $(\partial_tP_t)U_t+P_t(\partial_tU_t)=(\partial_tU_t)P_t$. Then (a.3.9) gives, $\partial_tP_t=(i/h)[P_t,Q_t]$, and using (a.3.7), we get for the corresponding symbols, $\partial_tP_t=\{P_t,Q_t\}+\mathcal{O}(h^2)$, which we can also write as,

(a.3.12) $\quad \partial_tP_t+H_{Q_t}P_t=\mathcal{O}(h^2)$.

Since $\partial_t(P_t(\kappa_t(x,\xi)))=(\partial_tP)(\kappa_t(x,\xi))+(H_{Q_t}P_t)(\kappa_t(x,\xi))$, we conclude that,

(a.3.13) $\quad \partial_t(P_t(\kappa_t(x,\xi)))=\mathcal{O}(h^2)$,

and after integration of this estimate, we get (a.3.11).

We now consider the converse question of associating a suitable canonical transformation to a given unitary Fourier integral operator. The basic idea is that if

(a.3.14) $\quad Uu(x,h)=\iint e^{i\varphi(x,y,\theta)/h}a(x,y,\theta,h)u(y)\,dy\,d\theta$,

is such an operator, where a is a classical symbol of order m, then we can always replace φ by $\varphi+h\psi(x,y,\theta,h)$, where ψ is a classical symbol of order 0, since the new amplitude, $e^{-i\psi}a$ is again a classical symbol of order 0. One then gets an associated canonical transformation, κ which depends on h and satisfies $\kappa=\kappa_0+\mathcal{O}(h)$, where κ_0 is the standard one. In the main text, the phase functions, will already depend on h in some complicated way (but varying in a bounded set) so κ will be as natural as κ_0.

Let U be a unitary F.I.O. whose associated standard canonical transformation, $\tilde{\kappa}$ maps $(0,0)$ to $(0,0)$. Taking an intermediate family of canonical transformations, $\tilde{\kappa}_t$ as above, we can then construct a smooth family of unitary F.I.O.'s, U_t such that $U_0=I$, $U_1=U$. We then use (a.3.9) to define the pseudodifferential operators Q_t of order 0. Then $Q_t=(h/i)(\partial_t U_t)U^*_t$, so $Q_t-Q_t^*=(h/i)((\partial_t U_t)U_t^* + U_t(\partial_t U^*_t))=(h/i)\partial_t(U_t U_t^*)=0$, so Q_t is self adjoint. We can then define κ_t by (a.3.8), with initial condition, $\kappa_0=\mathrm{id}$. Then we still have (a.3.11), if P_t is given by (a.3.10), and for $t=1$, this shows that if we associate to U the canonical transformation, $\kappa=\kappa_1$, then if $\tilde{P}$ and P are two pseudodifferential operators of order 0, related by,

(a.3.15) $\tilde{P}U=UP$,

then, for the Weyl symbols,

(a.3.16) $\tilde{P}\circ\kappa=P+\mathcal{O}(h^2)$.

By varying $\tilde{P}$, we see that κ is uniquely determined modulo $\mathcal{O}(h^2)$. (This means that in the correction of the phase φ in (a.3.14) it is enough to take ψ independent of h.

It would be interesting to give a more explicit description of the relation between a unitary F.I.O. and the corresponding canonical transformation, κ given (modulo $\mathcal{O}(h^2)$) by the relations (a.3.15),(a.3.16). A reasonable guess would be that if we write a given unitary FIO in such a way that the leading amplitude has constant argument, (by adding a term $h\psi(x,\theta)$ to the phase,) then the corresponding phase gives the more precise canonical transformation. We have not checked this however.

Let us here only investigate to what extent U is uniquely determined by κ. In the following we shall say that κ is associated to U, if (a.3.15),(a.3.16) hold. Let us first notice that if κ_j is associated to U_j for $j=1,2$, then $U_1\circ U_2$ is associated to $\kappa_1\circ\kappa_2$. Now let U and V be associated to the same canonical transformation κ. Then $A=U\circ V^{-1}$ is a unitary pseudodifferential operator, which is associated to the identity:

(a.3.17) $[P,A]=\mathcal{O}(h^2)$,

for every pseudodifferential operator of order 0. This means on the symbol level, that $\nabla A=\mathcal{O}(h)$, so $A=\omega+\mathcal{O}(h)$, for some $\omega\in\mathbb{C}$ of modulus 1. Hence two unitary FIO are associated to the same canonical transformation (in the precise sense) iff there is a number $\omega\in\mathbb{C}$ with $|\omega|=1$, such that $U=AV$, where $A=\omega+\mathcal{O}(h)$.

b. Normal forms for some scalar pseudodifferential operators.

In this section, we shall give normal forms for some scalar, classical analytic pseudodifferential operators, valid near a critical point of the symbol. Contrary to the usual case in the theory of local solvability and hypoellipticity and so on, we do not (,at least not to start with,) admit multiplication by elliptic factors, but only conjugation by Fourier integral operators and passage to a function of the operator. Thus the general question here is: "Given a certain selfadjoint operator P, find a real function f and a unitary operator U, such that $U^*f(P)U$ takes a simple form." There will be two cases (closely parallel): the case when the symbol has a saddle point, and the case when the symbol has a non-degenerate minimum. The second case has already been treated in [HS2], but here we give a second approach especially adapted to the analytic case.

We start with the saddle point (or "branching") case, and we consider first on the level of principal symbols, the model symbol $p_0=x\xi$ on $\mathbb{C}^2$. The associated Hamilton field is $H_{p_0}=x\partial/\partial x-\xi\partial/\partial\xi$, and $\exp(tH_{p_0})(x,\xi)=(e^t x,e^{-t}\xi)$, $t\in\mathbb{C}$. The flow is periodic with period $T_0=2\pi i$. If $p_0(x,\xi)=E\neq 0$, then $(x,\xi)=\exp(tH_{p_0})(y,y)$, where $y^2=E$ and $t=\log(x/y)+2\pi ik$, $k\in\mathbb{Z}$. The complex curve Γ_0: $\xi=x$ has the property that $\exp(\frac{1}{2}T_0H_{p_0})(\Gamma_0)=\Gamma_0$. and as we just saw, every point with $p_0(x,\xi)\neq 0$, is of the form $\exp(tH_{p_0})(y,y)$ for suitable t and y. We also notice that Γ_0 is the complexification of its intersection with $\mathbb{R}^2$. Let now $p(x,\xi)$ be a real-valued analytic function defined in a neighborhood of $(0,0)\in\mathbb{R}^2$ and having a saddle point at (0,0), with $p(0,0)=0$. If we let Γ_+ and Γ_- be the stable outgoing and incoming manifolds for the H_p flow (see [AM]), we know (see for instance [S2]), that Γ_+ and Γ_- are analytic curves, intersecting transversally at (0,0), and after composing p with a suitable real and analytic canonical transformation, we may assume that Γ_+ is the x-axis and that Γ_- is the ξ-axis. Then, it is easy to see that we have reduced the problem to the case when

(b.1) $\qquad p(x,\xi)=a(x,\xi)x\xi,$

where $a>0$ is a real-valued analytic function. Replacing p by f(p), where $f(t)=Ct$, $C>0$, we may assume that $a(0,0)=1$. If we replace x by (const.)$a(x,\xi)x$, we get an analytic diffeomorphism, that transforms p onto (const.)p_0. In particular, we know that the surfaces $p=E_1$ and $p_0=E_2$ have the same topological structure when intersected with suitable neighborhoods of (0,0), provided that E_1 and E_2 are non-zero and small. In particular $p=E$ is a connected curve for $E\neq 0$, so if (x_0,ξ_0) belongs to this curve and $\exp TH_p(x_0,\xi_0)=(x_0,\xi_0)$ for some $T\in\mathbb{C}$, then $\exp TH_p(x,\xi)=(x,\xi)$ for all (x,ξ) on this curve, and $T=T(E)$ is a period for the H_p flow, restricted to the energy curve $p=E$. To see that such non-trivial periods exist, let $E=\mu^2$ be small but non-vanishing and make the change of variables $(x,\xi)=(\mu y,\mu\eta)$. Put

$p_\mu=\mu^{-2}p(\mu y,\mu\eta)$. Then (x,ξ) belongs to $p^{-1}(E)$ iff (y,η) belongs to $p^{-}{}_\mu{}^{1}(1)$. Moreover, H_p is the direct image under the map $(y,\eta)\to(x,\xi)$ of H_{p_μ}, so an equivalent problem is then to study periods of the Hamilton flow of p_μ in the surface of energy 1. Since $p_\mu=p_0+\mathcal{O}(\mu)$ (uniformly in every fixed compact set) we conclude that the H_p-flow in $p^{-1}(E)$ has the period $T(E)=T_0+\mathcal{O}(E^{1/2})$, for small non-vanishing values of $E\in\mathbb{C}$. Since $T(E)$ is holomorphic in E outside the origin, we have a removable singularity at $E=0$. Hence E is holomorphic in a full complex neighborhood of 0, and

(b.2) $\qquad T(E)=T_0+\mathcal{O}(E)$.

Since p is real-valued, it is easy to see that $\bar{T}(E)=-T_0+\mathcal{O}(E)$ is a period when E is real. On the other hand $-T(E)=\bar{T}(E)+\mathcal{O}(E)$ is also a period, and since two different periods cannot be too close to each other, they have to agree; $\bar{T}(E)=-T(E)$, for E real, so $T(E)$ is purely imaginary for real energies. By the unique continuation property we get more generally,

(b.3) $\qquad \bar{T}(\bar{E})=-T(E)$,

when E is complex. We now look for a real canonical transformation, that transforms p into p_0. An obvious necessary condition is that $T(E)=T_0$, for all E, and we can easily reduce ourselves to this situation, by replacing p by $f\circ p$, where f is the real-valued analytic function given by,

(b.4) $\qquad df(E)/dE=T(E)/T_0,\ f(0)=0$.

Hence, we shall assume from now on, that

(b.5) $\qquad T(E)=T_0$.

We next study the H_p trajectories for real times. For such a trajectory, we have $x'=(a(x,0)+\mathcal{O}(\xi))x$, $\xi'=-(a(0,\xi)+\mathcal{O}(x))\xi$, and we first see that the flow is expansive in x and contractive in ξ, in the sense that for $t\ge s$: $|x(t)|\ge e^{(t-s)/C}|x(s)|$, $|\xi(t)|\le e^{-(t-s)/C}|\xi(s)|$. (Here, we restrict the attention to a trajectory which stays in a region: $|x|\le\varepsilon_0$, $|\xi|\le\varepsilon_0$, for some sufficiently small ε_0.) Consider now such a trajectory $(x(t),\xi(t))$, $0\le t\le T$, with $|x(0)/\xi(0)|\in[1/2,2]$. Then $|\xi(t)|\le 2|x(t)|$, and we get, $x'=(1+\mathcal{O}(x(t)))x(t)$. Using that $|x(t)|\le\varepsilon_0 e^{-(T-t)/C}$, we can integrate the earlier relation, and obtain,

$$x(t)=\exp(t+\mathcal{O}(1)\varepsilon_0\int_0^t e^{-(T-s)/C}ds)\,x(0),$$

and hence,

(b.6) $\qquad x(t)=e^{(t+\mathcal{O}(\varepsilon_0))}x(0)$.

Since the flow is $2\pi i$-periodic, this relation extends to the case when $t\in\mathbb{C}$, $0\le\mathrm{Re}\,t\le T$. We also have an analogous estimate for $\xi(t)$, when $\mathrm{Re}\,t\le 0$.

We next observe that, locally near (0,0),

(b.7) $\qquad \exp(\frac{1}{2}T_0H_p)(\mathbb{R}^2)=\mathbb{R}^2$.

In fact, the set of conjugate points of the left hand side is equal to $\exp(-\frac{1}{2}T_0H_p)(\mathbb{R}^2)=\exp(\frac{1}{2}T_0H_p)(\mathbb{R}^2)$, since T_0 is a period. Hence the left hand side of (b.7) is invariant under complex conjugation. Since it is a twodimensional surface, close to $\mathbb{R}^2$ in the C^1 topology, the only possibility is that we have equality in (b.7).

Let $\Gamma_0=\{(x,\xi);x=\xi\}$ and recall that $\exp(\frac{1}{2}T_0H_{p_0})(\Gamma_0)=\Gamma_0$. Let f be a real valued analytic function vanishing on Γ_0, with $df(0,0)\neq 0$, for instance $f=\xi-x$. Then $g=f\circ\exp(\frac{1}{2}T_0H_p)$ satisfies $dg(0,0)=-df(0,0)$, and if we put $h=f-g$, then $dh(0,0)=2f(0,0)\neq 0$, and $h\circ\exp(\frac{1}{2}T_0H_p)=-h$. Hence $\Gamma=h^{-1}(0)$ is a complex curve, equal to the comlexification of a real analytic curve, tangent to Γ_0 at (0,0). We have,

(b.8) $\quad p|_\Gamma=q^2,\ p_0|_{\Gamma_0}=(q_0)^2,$

where q,q_0 are holomorphic on Γ and Γ_0 respectively, satisfying $dq(0,0)=dq_0(0,0)$, $q_0(x,x)=x$.

Let $U_0=\{(x,\xi)\in\mathbb{C}^2;\ |x|,|\xi|\le\varepsilon_0\}$ for some sufficiently small $\varepsilon_0>0$. Then every $(x,\xi)\in U_0\setminus p_0^{-1}(0)$ can be written $(x,\xi)=\exp(tH_{p_0})(\rho)$, $\rho\in\Gamma_0\cap U_0$.Here ρ is well-defined up to a choice of sign; we can replace ρ by $-\rho$ if we change t to $t+\pi i$. Once the choice of ρ is fixed, then t is well defined up to a multiple of $2\pi i$. We now define the map κ: $U_0\setminus p_0^{-1}(0)\longrightarrow U\setminus p^{-1}(0)$, where U is a suitable small neighborhood of (0,0), by:

1^0 If $\rho\in\Gamma_0$, then $\kappa(\rho)\in\Gamma$ is given by $q(\kappa(\rho))=q_0(\rho)$.

2^0 If $(x,\xi)=\exp(tH_{p_0})(\rho)$, as above, we put $\kappa(x,\xi)=\exp(tH_p)(\kappa(\rho))$.

Since $|\mathrm{Re}\,t|$ may be very large here, we have to verify (A); that $\exp(tH_p(\rho))$ is well-defined and belongs to a small neighborhood of (0,0), and (B); that the definition of $\kappa(x,\xi)$ does not depend on the choice of ρ,t in the representation of (x,ξ). Assuming (A), the verification of (B) is immediate, using that $\exp(\frac{1}{2}T_0H_p)(\Gamma)=\Gamma$, and that T_0 is a period for the H_p flow. (A) follows from (b.6) and its analogue for $\xi(t)$, which imply that $\kappa(x,\xi)$ is well-defined, and even that $|\kappa(x,\xi)|\le e^{\mathcal{O}(\varepsilon_0)}|(x,\xi)|$ (if we use l^∞-norms). We have constructed a single-valued bounded holomorphic map κ: $U_0\setminus p_0^{-1}(0)\longrightarrow U\setminus p^{-1}(0)$, which then clearly has a holomorphic extension to: $U_0\longrightarrow U$. Moreover, by construction,

(b.9) $\quad p\circ\kappa=p_0,$

and κ is symplectic, mapping the real domain into the real domain, and we have $d\kappa(0,0)=\mathrm{id}$.

Let now $P(x,\xi,h)\sim\Sigma_{j\ge 0}\,p_jh^j$ denote a formal classical analytic symbol, defined in a neighborhood of (0,0), with leading part $p_0=p$, and assume that the corresponding formal pseudo-differential operator $P(x,hD,h)$ is formally self-adjoint.(We here take the classical quantization, but nothing changes in the arguments, if instead we take the h-Weyl quantization.)

From what we have done so far, it follows that there is a formal unitary Fourier integral operator, associated to the canonical transformation κ, and a real-valued analytic function $f(t)$, with $f(0)=0$, $f'(0)>0$, such that $U^{-1}f(P)U$

has the principal symbol $x\xi$. From now on, we assume that already $p=p_0=x\xi$. Since the H_p-flow is T_0-periodic, we find formally that,

(b.10) $$e^{-iT_0P/h}=R,$$

where R is a formal analytic elliptic pseudodifferential operator of order 0. Since $T_0=2\pi i$ is non-real, we have to be a little careful, in giving a sense to (b.10). We do this in the following way. Let Φ_0 be a strictly subharmonic quadratic form on $\mathbb{C}$, close to 0, and define (as in [S1]); $\Lambda_{\Phi_0}=\{(x,(2/i)\partial\Phi_0/\partial x);\ x\in\mathbb{C}\}$. Then for t imaginary, we can define $\Lambda_{\Phi_t}=\exp(tH_p)(\Lambda_{\Phi_0})$, and Φ_t is again a strictly subharmonic quadratic form close to 0. Using the theory developped in the first sections in [S1], we can then define unambiguously a classical Fourier integral operator $U_t=e^{-itP/h}$, that can be realized as an operator from $H_{\Phi_0}{}^{loc}(\Omega_0)$, to $H_{\Phi_t}{}^{loc}(\Omega_t)$, where Ω_0 and Ω_t are suitable complex neighborhoods of 0. As a matter of fact, this is just the standard construction, except for the fact, that due to an embarrassing imaginary part of the phase-function, there is no reasonable way to define this operator acting microlocally in the usual L^2 space. Nevertheless we have $\Phi_{T_0}=\Phi_0$, and we see that R is a pseudodifferential operator. If $\alpha=(\alpha_x,\alpha_\xi)\in\mathbb{R}^2$ belongs to a small neighborhood of the origin, we put $u_\alpha(x,h)=e^{i((x-\alpha_x)\alpha_\xi+i(x-\alpha_x)^2/2)/h}$. Then for α,β sufficiently close to (0,0), the scalar products $(U_tu_\alpha|u_\beta)_{L^2}$ are well-defined up to exponentially small terms, and up to such a term, we have $(U_tu_\alpha|u_\beta)_{L^2}=(u_\alpha|U_tu_\beta)_{L^2}$, as we can see, using the selfadjointness of P and the fact that we now restrict the attention to imaginary values of t. Since the functions u_α fill up a microlocal neighborhood of (0,0) (,in a sense that can be made precise by using an FBI-transform), it follows that R is formally self-adjoint, so the principal symbol, r is either >0 or <0 on the real domain. As already mentionned, U_t is formally obtained by the standard WKB-procedure, which in this case gives us,

(b.11) $$U_tu(x)=(2\pi h)^{-1}\iint e^{i(\varphi(t,x,\eta)-y\eta)/h}a(t,x,\eta,h)u(y)\,dyd\eta,$$

where $\varphi(t,x,\eta)=e^{-t}x\eta$, and where a is a classical analytic symbol of order 0, determined by the standard transport equations. Notice that $\varphi(T_0,x,\eta)=x\eta$ as we could expect. Analyzing the first transport equation (as we shall do below), we see that the leading part, a_0 of a, satisfies, $a_0(T_0,0)<0$. This supports the belief that $r<0$, but we have to be careful, since the realization of the operators U_t involves the choice of certain contours, and this may be responsible for an additional factor -1. However, we can deform P continuously into $P_0=\frac{1}{2}(xhD+hDx)=xhD+h/2i$, and in the case of P_0 the corresponding group is given by $U_tu(x)=e^{-t/2}u(e^{-t}x)$, so we see in that case that $R=-\mathrm{id}$. Hence, in general,

(b.12) $$r<0.$$

It is also easy to check that $[P,R]=0$. Microlocally, where P is of principal type (and hence reducible to hD), it is easy to see that $R=F(P,h)$, where F is an analytic symbol of order 0, elliptic and real valued on the real domain. Since R and $F(P,h)$ are both well defined analytic symbols on a neighborhood of $(0,0)$, which agree on some open set, they have to agree on a full neighborhood of $(0,0)$, and hence,

(b.13) $\quad R=F(P,h).$

Since F is negative on the real domain, there is a real valued analytic symbol $G(t,h)$, of order 0, such that,

(b.14) $\quad e^{iT_0 G(t,h)}=-F(t,h)$,

and hence,

(b.15) $\quad e^{-iT_0(P+hG(P,h))/h}=-\mathrm{id}.$

Replacing P by $P+hG(P,h)$, we have now reduced the problem of normal forms to the case when,

(b.16) $\quad e^{-iT_0 P/h}=-\mathrm{id}.$

Solving the first transport equation for a_0 (cf (b.11)),

(b.17) $\quad \partial_t a_0+x\partial_x a_0+ip_{-1}(x,\partial_x\varphi)a_0=0,\ a_0(0)=1,$

we obtain, since we now know that $a_0(T_0,x)=-1$:

(b.18) $\exp(i\int_0^{T_0} p_{-1}(x(t),\xi(t))dt)= -1,$

or equivalently,

(b.19) $\quad \int_0^{T_0} p_{-1}(x(t),\xi(t))dt=(2k+1)\pi,\ k\in\mathbb{Z}.$

In the case of P_0, we have $p_{-1}=1/2i$, and $k=0$ in (b.19). This is then also true in the general case, since by self-adjointness the subprincipal symbol $p_{-1}-(1/2i)\partial^2 p_0/\partial x\partial\xi= p_{-1}-1/2i$, is real, and T_0 is imaginary. Hence in general,

(b.20) $\quad \int_0^{T_0} p_{-1}(x(t),\xi(t))dt=\pi,$

and in particular $p_{-1}(0,0)=1/2i$.

We next look for an elliptic pseudodifferential operator R_0 of order 0, such that,

(b.21) $\quad P_0R_0\equiv R_0P$ modulo an operator of order -2.

For the principal symbol r_0, we then obtain the condition,

(b.22) $\quad i^{-1}H_{p_0}(r_0)-(p_{-1}-1/2i)r_0=0,$

so we can take

(b.23) $\quad r_0=e^{iq},$

where q solves,

(b.24) $\quad H_{p_0}(q)=(p_{-1}-1/2i),$

Here we make a general remark on the equation $H_{p_0}(a)=b$. If b is given, we put $a= -\int_0^{T_0}(1-t/T_0)\, b\circ\exp tH_{p_0}\,dt$, then

$H_{p_0}(a)= -\int_0^{T_0}(1-t/T_0)(d/dt)(b\circ\exp tH_{p_0})dt=b-T_0^{-1}\int_0^{T_0} b\circ\exp tH_{p_0}\,dt$. We have then a solution, if the last integral vanishes. The vanishing of this integral is also obviously a necessary condition for solvability. In the case of (b.24) the condition is fulfilled, in view of (b.20), so a solution q exists. Moreover, since $p_{-1}-1/2i$ is real valued, we may choose q real. And this means that we may choose R_0 unitary in (b.21). We are now reduced to the case when $P-P_0$ is of order -2, and (b.16) holds. We then look for a pseudodifferential operator R of order -1, such that,

(b.25) $\quad P_0(I+R)=(I+R)P.$

We rewrite this equation as

(b.26) $\quad \mathfrak{L}R=P-P_0,$

where $\mathfrak{L}R=P_0R-RP$. By a formal WKB construction, we can define $e^{-it\mathfrak{L}/h}$, acting on formal analytic pseudodifferential operators, and since,

(b.27) $\quad e^{-it\mathfrak{L}/h}R = e^{-itP_0/h}R\, e^{itP/h},$

we see that $e^{-it\mathfrak{L}/h}R$ is again a classical analytic pseudodifferential operator, and that

(b.28) $\quad e^{-iT_0\mathfrak{L}/h}=\mathrm{id}.$

If B is an analytic pseudodifferential operator of order k, then,

(b.29) $\quad A=(i/h)\int_0^{T_0}(1-t/T_0)\, e^{-it\mathfrak{L}/h}B\, dt$

is an analytic pseudodifferential operator of order k+1, and we have,

(b.30) $\quad \mathfrak{L}A= B-(1/T_0)\int e^{-it\mathfrak{L}/h}B\, dt.$

As before, a necessary and sufficient condition, in order to solve $\mathfrak{L}A=B$, is that the integral in (b.30) vanishes. Let us verify this condition in the case of (b.26):

$$\int_0^{T_0} e^{-it\mathfrak{L}/h}(P-P_0)dt= \int_0^{T_0} e^{-itP_0/h}(P-P_0)e^{itP/h}\, dt= \int_0^{T_0}(d/dt)(e^{-itP_0/h}\, e^{itP/h})\, dt= 0,$$

where the last equality follows from (b.16), which is also verified by P_0. Hence, it is clear that we can find R of order -1, solving (b.26). Changing notations, we have now found an elliptic analytic pseudo-differential operator R of order 0, such that,

(b.31) $\quad RP=P_0R.$

It remains to see that we can take R to be unitary. If we let R^* denote the complex adjoint of R, then, since P and P_0 are formally selfadjoint, we get from (b.31),

(b.32) $\quad PR^*=R^*P_0,$

and hence,

(b.33) $\quad (R^*R)P=P(R^*R).$

The operators $(R^*R)^{\pm 1/2}$, defined by the functional calculus of appendix a, then commute with P, and we put $U=R(R^*R)^{-1/2}$. Then,

(b.34) $\quad U^*U=(R^*R)^{-1/2}R^*R(R^*R)^{-1/2}= I,$

so U is unitary, and an easy computation gives,

(b.35) $\quad P_0U=UP.$

Let us sum up what we have proved so far:

<u>Theorem b.1.</u> Let $P(x,hD,h)$ be a formal classical analytic pseudodifferential operator, of order 0, formally selfadjoint, with symbol defined in a neighborhood of (0,0). Let p be the principal symbol, and assume that p has a non-degenerate saddle point at (0,0) with critical value 0. Then there is a real-valued analytic symbol; $F(t,h)\sim\Sigma_0^\infty f_j(t)h^j$, defined for t in a neighborhood of 0, and a formal unitary analytic Fourier integral operator, whose associated canonical transformation (in the classical sense) is defined in a neighborhood of (0,0), and maps this point onto itself, such that

(b.36) $\quad U^*F(P,h)U=P_0=\frac{1}{2}(xhD+hDx)$.

To end this section, we study the case when P has additional symmetries. We start with the most important case, when P commutes with $C_0=\mathcal{F}^2$, which is a Fourier integral operator, whose associated canonical transformation is given by $c_0(x,\xi)=(-x,-\xi)$, and which satisfies $C^2{}_0=I$. We notice that P_0 also commutes with C_0, and the natural problem is then to choose U in (b.36) so that U commutes with C_0. Examining the proof of the theorem, we first observe that it is possible to choose U, such that the associated canonical transformation, κ_U commutes with c_0. We now fix such a canonical transformation κ_0 and we let $\mathcal{U}(\kappa_0)$ be the set of unitary Fourier integral operators satisfying (b.36) (with a fixed F) with κ_0 as their associated canonical transformation. If $U,V\in\mathcal{U}(\kappa_0)$, then $U^{-1}V$ is a pseudodifferential operator which commutes with P_0, so by an argument already given above,

(b.37) $\quad V=Ue^{iG(P_0,h)}$,

where G is a real valued analytic symbol of order 0. (Here we work with associated canonical transformations in the classical sense. We could also have chosen the more precise correspondence of appendix a. The we would have G=const.+(operator of order ≤ -1).) Conversely, if $U\in\mathcal{U}(\kappa_0)$, and V is given by (b.37) with G as above, then $V\in\mathcal{U}(\kappa_0)$.

We now fix $U\in\mathcal{U}(\kappa_0)$. Then $C_0UC_0\in\mathcal{U}(\kappa_0)$, and hence $C_0UC_0=Ue^{iG(P_0,h)}$, with G as above. Iterating this relation and using that $C_0{}^2=I$, we get $U=C_0UC_0e^{iG(P_0,h)}=Ue^{i2G(P_0,h)}$, and hence there are only two possibilities, either G=0, or $G=\pi$ modulo a multiple of 2π. In the first case, we get $C_0UC_0=U$, or rather $C_0U=UC_0$, which is the desired commutation relation. In the second case, we obtain $C_0U=-UC_0$. Let us apply this equation to the function $u_0(x)=e^{-x^2/2h}$, noticing that $C_0u(x)=u(-x)$. We then have $C_0u_0=u_0$, and $Uu_0=a(x,h)e^{i\varphi(x)/h}$, where a is an elliptic symbol of order 0, and φ is analytic with $\mathrm{Im}(\varphi)\ge 0$. We then get, $a(-x,h)e^{i\varphi(-x)/h}=-a(x,h)e^{i\varphi(x)/h}$, which for x=0 contradicts the ellipticity of a. Hence only the first case can appear, and we have proved, Proposition b.2. In addition to the assumptions of the preceding theorem, we also assume that P commutes with $C_0=\mathcal{F}^2$. Then it is possible to find U as in Theorem b.1, which commutes with C_0.

At least in the case of the first step in our general iteration procedure, the operator P will satisfy more elaborate symmetry relations, that we also want U to respect. We shall now assume in addition to the assumptions of Theorem .1., that,

(b.38) $\mathcal{F}P=-P\mathcal{F}$, and $\Gamma P=-P\Gamma$.

where $\Gamma=\Gamma_0$ denotes the operator of complex conjugation. This assumption implies that the principal symbol p changes sign after composition by $\kappa=\kappa_{-\pi/2}$ or by reflection in one of the axes. It follows that p is invariant under reflection in the line $x=\xi$, that is: $p\circ\delta=p$, where $\delta(x,\xi)=(\xi,x)$. Using that $p\circ\kappa=-p$, it is easy to see that the period T(E), introduced in the beginning of this section, is even, and hence the function f, such that the period of f(p) is constant, is odd. This implies that the symmetries for p deduced from (b.38), are also valid for f(p).In the earlier construction of $\kappa=\kappa_U$, we can then take the curve Γ to be the curve $\Gamma_0:x=\xi$. It then becomes clear that κ_U commutes with reflection in $x=\xi$ and with $c_0=\kappa^2$. If we also impose the condition that $d\kappa_U(0,0)=\mathrm{id}$, then we see that κ_U is the unique canonical transformation determined by this property and $f(p)\circ\kappa_U=p_0$, $\kappa_U\circ\delta=\delta\circ\kappa_U$. Now write: $f(p)\circ\kappa^{-1}\circ\kappa_U\circ\kappa= -f(p)\circ\kappa_U\circ\kappa= -p_0\circ\kappa=p_0$, $\kappa^{-1}\circ\kappa_U\circ\kappa\circ\delta= \kappa^{-1}\circ\kappa_U\circ c_0\circ\delta\circ\kappa= \kappa^{-1}\circ c_0\circ\delta\circ\kappa_U\circ\kappa= \delta\circ\kappa^{-1}\circ\kappa_U\circ\kappa$, which shows that $\kappa^{-1}\circ\kappa_U\circ\kappa$ has the same properties as κ_U, and hence is equal to κ_U. In other words,

(b.39) $\kappa_U\circ\kappa=\kappa\circ\kappa_U$.

We next reexamine the construction of the real valued analytic symbol F(t,h) with leading part f, such that $\exp(-iT_0F(P,h)/h)=I$. In fact, the requirement that F be real-valued, implies that F is uniquely determined with this leading part. If we supress the - sign in the last relation, and conjugate with $\mathcal{F}$, we get in view of (b.38): $\exp(-iT_0(-F(-P_0,h))/h)=I$, and knowing that f is odd, the earlier remark on the uniqueness of F, shows that $F(-E,h)=-F(E,h)$. It follows that F(P,h) satisfies (b.38). In the following, we may then assume that $F(E,h)=E$, and work directly with P instead of F(P,h). We fix $\kappa_0=\kappa_U$ as above, and define $\mathcal{U}(\kappa_0)$ to be the set of unitary Fourier integral operators U associated to κ_0, and such that $P\circ U=U\circ P_0$. In view of the properties of κ_0, we see that if $U\in\mathcal{U}(\kappa_0)$, then $\mathcal{F}U\mathcal{F}^{-1}\in\mathcal{U}(\kappa_0)$, so by an earlier argument,

(b.40) $\mathcal{F}U\mathcal{F}^{-1}=U\, e^{iG(P_0,h)}$,

where G is real and of order 0. Testing this relation on the same function u_0 as before, we see that the leading part $G_0(E)$ of G satisfies $G(0)=0$. Iterating (b.40), and using that P_0 satisfies (b.38), we get

$\mathcal{F}^2 U\mathcal{F}^{-2}=\mathcal{F}Ue^{iG(P_0,h)}\mathcal{F}^{-1}=\mathcal{F}U\mathcal{F}^{-1}e^{iG(-P_0,h)}=Ue^{i(G(P_0,h)+G(-P_0,h))}$.
Conjugating this with $\mathcal{F}^2$ and using that $\mathcal{F}^4=1$, we get
$U=Ue^{2i(G(P_0,h)+G(-P_0,h))}$. Hence $G(P_0,h)+G(-P_0,h)=\pi k$, $k\in\mathbb{Z}$. Since $G_0(0)=0$, we have necessarily that $k=0$, so

(b.41) $\quad G(-t,h)=-G(t,h)$.

We now look for $V=Ue^{iJ(P_0,h)}\in\mathcal{U}(\kappa_0)$ which commutes with $\mathcal{F}$. A straightforward calculation shows that $\mathcal{F}V\mathcal{F}^{-1}=V$ if $G(t,h)=J(t,h)-J(-t,h)+2\pi k$ for some $k\in\mathbb{Z}$. In view of (b.41), we can take $J=\frac{1}{2}G$ and $k=0$. Thus, V commutes with $\mathcal{F}$ and in order to save notations, we shall assume that already U commutes with $\mathcal{F}$. Let $V=Ue^{iJ(P_0,h)}\in\mathcal{U}(\kappa_0)$ be a second element which commutes with $\mathcal{F}$. Then a simple computation gives the sufficient condition on J: $0=J(t,h)-J(-t,h)+2\pi k$ for some $k\in\mathbb{Z}$. Taking $t=0$, we see that k must be 0, so the condition on J is simply that J be even. Now look for $V=Ue^{iJ(P_0,h)}$, with J even, such that

(b.42) $\quad \Gamma V\Gamma=V$.

First, we notice that $\Gamma U\Gamma\in\mathcal{U}(\kappa_0)$, so $\Gamma U\Gamma=Ue^{iK(P_0,h)}$ with K real. Iterating this and using that $\Gamma^2=I$, we get $K(t,h)-K(-t,h)=2\pi n$. Again $t=0$ implies that $n=0$, so K is even. A simple computation shows that (b.42) holds if $K(t,h)=J(t,h)+J(-t,h)+2\pi n$ for some $n\in\mathbb{Z}$. Since K is even, we can take $J=\frac{1}{2}K$. We have proved,

Proposition b.3. In addition to the assumptions of Theorem b.1, we assume (b.38). Then in the theorem it is possible to take F to be odd, and U to commute with $\mathcal{F}$ and Γ.

Remark b.4. If we assume that $P=P_\mu$ depends on the real parameter $\mu\in$neighborhood of 0, in such a way that $P_\mu(x,\xi,h)$ is an analytic symbol in μ,x,ξ . Then, possibly after shrinking the neighborhood in μ, we can construct U, F, depending analytically on μ in the sense that all phase-functions and canonical transformations are analytic in μ, and all symbols remain analytic symbols after addition of μ as an independent variable. This is valid for all the results above. Moreover, if P depends on additional parameters α, in such a way that P or P_μ varies in a bounded set of analytic symbols, then all symbols, phases, and transformations, will also vary in bounded sets.

Remark b.5. Assume that $P=P_\mu$ depends analytically on μ as in the preceding remark, that P_μ is selfadjoint for real μ, but that the assumptions of the theorem are satisfied only for $\mu=0$. Let p_μ denote the principal symbol of P_μ. Then p_μ has a saddle point (x_μ,ξ_μ) depending analytically on μ, and after conjugation by a unitary Fourier integral operator, we may assume that $(x_\mu,\xi_\mu)=(0,0)$. Then $P_\mu-p_\mu(0,0)$ satisfies all the assumptions of the theorem, so there is a real-valued analytic symbol $F(\mu,t,h)$ and unitary operators U_μ depending analytically on μ, such that
$U_\mu^{-1}F(\mu,P_\mu-p_\mu(x_\mu,\xi_\mu),h)U_\mu=P_0$. If $u=U_\mu v$ is a microlocal solution of the

equation $P_\mu u=0$, then $P_0 v=F(\mu,-p_\mu(x_\mu,\xi_\mu),h)v$, and conversely.

We shall now look at the second case when (0,0) is a non degenerate minimum, so we keep all the assumptions of Theorem b.1, except that we replace the assumption that (0,0) be a (non degenerate) saddle point by the assumption that (0,0) is a non degenerate minimum for p, again with critical value 0. Consider first the symbol $p_0=\frac{1}{2}(\xi^2+x^2)$, $H_{p_0}=\xi\partial_x-x\partial_\xi$. Then the Hamilton flow is periodic with period $T_0=2\pi$. If $\Gamma_0=\{(x,x)\}$, then every real point (x,ξ) can be written $(x,\xi)=\exp(tH_{p_0})(y,y)$, for some real (y,y) and some $t\in\mathbb{R}$. Here y is unique up to the sign and once y has been fixed, t is unique up to a multiple of 2π. If we take (x,ξ) complex, we have the analogous result, provided that $p_0(x,\xi)\neq 0$. Also notice that $\exp(\frac{1}{2}T_0H_{p_0})(\Gamma_0)=\Gamma_0$.

Returning to the symbol p, we see as in the beginning of this section, that there is a real and analytic function T(E), such that T(E) is a period for the H_p flow restricted to the energy surface $p^{-1}(E)$, and we may assume that $T(E)=T_0+\mathcal{O}(E)$. Replacing p by $f(p)$, where $f'(t)=T(t)/T_0$, we may assume that,

(b.43) $\quad T(E)=T_0$.

As before, we construct an analytic curve Γ tangent to Γ_0 at (0,0), such that $\exp(\frac{1}{2}T_0H_p)(\Gamma)=\Gamma$, and we write,

(b.44) $\quad p|_\Gamma=q^2,\ p_0|_{\Gamma_0}=q^2{}_0,$

where q and q_0 are analytic on Γ and Γ_0 respectively, and $dq(0,0)=dq_0(0,0)$, $q_0(x,x)=2^{-\frac{1}{2}}x$. Using this, we can proceed as before, and construct a real valued analytic canonical transformation κ, such that,

(b.45) $\quad p\circ\kappa=p_0,\ d\kappa(0,0)=\mathrm{id}.$

(In the estimates of the flows, the roles of Re(t) and Im(t) are now permuted, and the stable manifolds for the H_{p_0} flow are now given by $\xi\pm ix=0$.) This means that we may assume that $p=p_0$.

Consider now the full operators. As a model operator, we take $P_0=\frac{1}{2}((hD)^2+x^2-h)$, and we notice that,

(b.46) $\quad e^{-iT_0P_0/h}=I,$

where we no more have to use the H_Φ spaces to justify our arguments. As before, we can construct a real valued analytic symbol G(t,h) of order 0, such that,

(b.47) $\quad e^{-iT_0(P+hG(P,h))/h}=I.$

Contrary to the earlier case, G is uniquely determined only up to an integer. Replacing P by P+hG(P,h), we have then reduced the problem to the case when,

(b.48) $\quad e^{-iT_0P/h}=I$

Comparing the first transport equations, given by (b.46) and (b.47), we get,

(b.49) $\quad \int_0^{T_0} p_{-1}(x(t),\xi(t))\,dt-\int_0^{T_0}-1/2\,dt=2\pi k,$

for some $k\in\mathbb{Z}$. After modifying G by an integer, we can assume that, (b.50) $k=0.$

We can then repeat the earlier arguments with almost no changes, and obtain,

<u>Theorem b.6.</u> Let $P(x,hD,h)$ be a formal classical analytic pseudodifferential operator, of order 0 ,formally selfadjoint, with symbol defined in a neighborhood of (0,0). Let p be the principal symbol, and assume that p has a non-degenerate minimum at (0,0) with critical value 0. Then there is a real-valued analytic symbol; $F(t,h)\sim\Sigma_0{}^{\infty}f_j(t)h^j$, defined for t in a neighborhood of 0, and a formal analytic Fourier integral operator, which is unitary, and whose associated canonical transformation is defined in a neighborhood of (0,0), and maps this point onto itself, such that

(b.36) $U^*F(P,h)U=P_0=\frac{1}{2}((hD)^2+x^2-h).$

c. Normal forms for 2×2-systems.

For typographical reasons, we denote by $[A]_{j,k}$, $j,k=1,2$, the 2×2-matrix $(a_{\alpha,\beta})_{1\le\alpha,\beta\le2}$, with $a_{\alpha,\beta}=A$ if $(\alpha,\beta)=(j,k)$, and $a_{\alpha,\beta}=0$ otherwise. We shall first discuss reductions to normal forms on the level of principal symbols, and we start with the case of an analytic hermitian 2×2-matrix $p(x,\xi)$, defined for (x,ξ) in a neighborhood of $(0,0)$ in $\mathbb{R}^2$, which satisfies,

(c.1) $\qquad -\det(p(x,\xi))\sim x^2+\xi^2,\ p(0,0)=0.$

Here $\sim$ denotes "of the same order of magnitude as". On the principal symbol level, we allow transformations $p\longrightarrow a^*pa$, where a is invertible and analytic, as well as real, analytic canonical changes of variables which preserve the origin. Applying the reductions of [HS2], we then have a first reduction to the case when $p=[\zeta]_{1,2}+[\bar\zeta]_{2,1}+\mathcal{O}((x,\xi)^2)$, $\zeta=\xi+ix$. After replacing ζ by $\zeta+\mathcal{O}((x,\xi)^2)$, we can then write,

(c.2) $\qquad p(x,\xi)=[c_1(x,\xi)]_{1,1}+[c_2(x,\xi)]_{2,2}+[\zeta]_{1,2}+[\bar\zeta]_{2,1},$

with $c_j=\mathcal{O}((x,\xi)^2)$. We then look for $a_1,a_2=\mathcal{O}((x,\xi))$, such that,

(c.3) $\qquad (I+[a_1]_{1,2}+[a_2]_{2,1})p(I+[a_1]_{1,2}+[a_2]_{2,1})^*=[\zeta']_{1,2}+[\bar\zeta']_{2,1},$

with $\zeta'=\zeta+\mathcal{O}((x,\xi)^3)$. To obtain (c.3), it is enough to solve the system,

(c.4) $\qquad c_1+a_1\bar\zeta+\bar a_1\zeta+|a_1|^2c_2=0,$

$\qquad\qquad c_2+a_2\zeta+\bar a_2\bar\zeta+|a_2|^2c_1=0,$

and we will then have $\zeta'=\zeta+a_1c_2+a_1\bar a_2\bar\zeta+c_1\bar a_2$. The equations in (c.4) are independent, so we may concentrate on the first one. We write,

(c.5) $\qquad c_1=f\zeta^2+\overline{f\zeta^2}+2g\zeta\bar\zeta,\ \ a_1=h\zeta+k\bar\zeta.$

It is then enough to find analytic function h,k such that,

(c.6) $\qquad f+k+c_2hk=0,$

$\qquad\qquad 2g+h+\bar h+c_2(h\bar h+k\bar k)=0.$

Here g and c_2 are real and since c_2 is very small, if we shrink the neighborhood of $(0,0)$ under consideration, the implicit function theorem gives a unique analytic solution of (c.6), if we impose the additional assumption that h should be real.

Thus, a symbol p satisfying (c.1) can be reduced to,

(c.7) $\qquad p_0(x,\xi)=[\zeta]_{1,2}+[\bar\zeta]_{2,1},\ \zeta=\xi+ix+\mathcal{O}((x,\xi)^2).$

We now consider a new selfadjoint analytic symbol $p=p_0+\mathcal{O}(\varepsilon)$, where $\mathcal{O}(\varepsilon)$ refers to a perturbation whose L^∞ norm over a fixed complex neighborhood of $(0,0)$ is $\mathcal{O}(\varepsilon)$. Changing ζ by $\mathcal{O}(\varepsilon)$, we can then write,

(c.8) $\qquad p(x,\xi)=[c_1(x,\xi)]_{1,1}+[c_2(x,\xi)]_{2,2}+[\zeta]_{1,2}+[\bar\zeta]_{2,1},\ c_j=\mathcal{O}(\varepsilon),$

and we look for $a_j=\mathcal{O}(\varepsilon)$ such that,

(c.9) $\qquad (I+[a_1]_{1,2}+[a_2]_{2,1})p(I+[a_1]_{1,2}+[a_2]_{2,1})^*=$

$$[c_1']_{1,1}+[c_2']_{2,2}+[\zeta']_{1,2}+[\bar\zeta']_{2,1},$$

with $\zeta-\zeta'=\mathcal{O}(\varepsilon^2)$, $c_j-c_j'=\mathcal{O}(\varepsilon)$, and c_j' <u>is constant</u>. More explicitly, we have to find $a_j=\mathcal{O}(\varepsilon)$, such that,

$$c_1+a_1\bar{\zeta}+\bar{a}_1\zeta+|a_1|^2c_2=c_1'=\text{const.} \qquad (c.10)$$
$$c_2+a_2\zeta+\bar{a}_2\bar{\zeta}+|a_2|^2c_1=c_2'=\text{const},$$

and again we may concentrate on one of the equations, for instance the first one. Write $\zeta=x+iy$, and notice that x,y can be used as local coordinates. If we write $a_1=f(x,y)+ig(x,y)$, then $a_1\bar{\zeta}+\bar{a}_1\zeta=2(fx+gy)$, and the first equation in (c.10) takes the form,

$$c_1(x,y)+2(f(x,y)x+g(x,y)y)+c_2(f^2+g^2)=d_1=\text{const.} \qquad (c.11)$$

Let us assume for simplicity, that c_j are holomorphic and bounded in $D=\{(x,y)\in\mathbb{C}^2; |x|<1, |y|<1\}$, and that $\|c_j\|_{L^\infty(D)}\le\varepsilon$. If F is a bounded holomorphic function on D, we obtain by successive divisions and the maximum-principle that $F(x,y)-F(0,0)=A(x,y)y+B(x,y)x$, where $\|A\|_{L^\infty}\le 2\|F\|_{L^\infty}$, $\|B\|\le 2\|F\|_{L^\infty}$. As a first approximate solution of (c.11), we can take f_1,g_1 holomorphic on D, with $2\|f_1\|, 2\|g_1\|\le\|c_1(x,y)-c_1(0,0)\|\le 2\varepsilon$, such that,

$$c_1(x,y)+2(f_1(x,y)x+g_1(x,y)y)=c_1(0,0). \qquad (c.12)$$

Then,

$$c_1+2(f_1x+g_1y)+c_2(f^2{}_1+g^2{}_1)=c_1(0,0)+c_2(f^2{}_1+g^2{}_1), \text{ with} \qquad (c.13)$$

$\|c_2(f^2{}_1+g^2{}_1)\|\le 2\varepsilon^3$. Assume by induction, that we have found f_k,g_k, with

$$c_1+2(f_kx+g_ky)+c_2(f_k{}^2+g_k{}^2)=d_k+r_k, \qquad (c.14)$$

where d_k is constant (and real), $2\|f_k\|, 2\|g_k\|\le M_k$, $\|r_k\|\le m_k$. Then we choose f_{k+1}, g_{k+1} real with,

$$2((f_{k+1}-f_k)x+(g_{k+1}-g_k)y)=-(r_k(x,y)-r_k(0,0)), \qquad (c.15)$$

$$2\|f_{k+1}-f_k\|, 2\|g_{k+1}-g_k\|\le 2\|r_k\|\le 2m_k, \qquad (c.16)$$

so $\|f_{k+1}\|, \|g_{k+1}\|\le M_k+m_k=_{\text{def.}}M_{k+1}$. Then,

$$c_1+2(f_{k+1}x+g_{k+1}y)+c_2((f_{k+1})^2+(g_{k+1})^2)=d_{k+1}+r_{k+1}, \qquad (c.17)$$

where $d_{k+1}=d_k+r_k(0,0)$, $r_{k+1}=c_2((f^2{}_{k+1}-f^2{}_k+g^2{}_{k+1}-g^2{}_k)$. Here, $\|f^2{}_{k+1}-f^2{}_k\|\le(\|f_{k+1}\|+\|f_k\|)\|f_{k+1}-f_k\|\le 2(M_k+m_k)m_k$, and similarly for $g^2{}_{k+1}-g^2{}_k$ and hence, $\|r_{k+1}\|\le 4\varepsilon(M_k+m_k)m_k=_{\text{def.}}m_{k+1}$. With the initial choice $f_0=g_0=0$, we have $M_0=0$, $m_0=2\varepsilon$, and we are led to study the recurrence relations,

$$m_{k+1}=4\varepsilon(M_k+m_k)m_k, \qquad (c.18)$$
$$M_{k+1}=M_k+m_k\ .$$

Assume by induction that $m_j\le 2^{1-j}\varepsilon$, $0\le j\le k$, which is certainly true for $k=0$. Then $M_k=m_0+..+m_k\le 4\varepsilon$, and $m_{k+1}\le 4\varepsilon 6\varepsilon 2^{1-k}\varepsilon=(48\varepsilon^2)2^{1-(k+1)}\varepsilon\le 2^{1-(k+1)}\varepsilon$, if $48\varepsilon^2\le 1$, so the induction procedure goes through, if $\varepsilon>0$ is small enough, and f_k,g_k converge to f, g; analytic solutions of the problem (c.11) with $f,g=\mathcal{O}(1)(\|c_1\|+\|c_2\|)$.

This implies a solution of the problem (c.10).

Now let P be a formally selfadjoint classical 2×2 pseudodifferential operator of order 0. We assume that the principal symbol p satisfies $|p-p_0|\leq\varepsilon$ in a fixed complex neighborhood of $(0,0)$, where p_0 is hermitian, and satisfies (c.1). If $\varepsilon>0$ is sufficiently small, the preceding discussion implies that there is an elliptic scalar Fourier integral operator U of order 0 and an elliptic 2×2-system of zero order pseudodifferential operators, A, such that $(AU)^*P(AU)$ has the principal symbol

(c.19) $\quad [c_1]_{1,1}+[c_2]_{2,2}+[\zeta]_{1,2}+[\bar{\zeta}]_{2,1}$, where $c_j=\mathcal{O}(\varepsilon)$ are constants, and $\zeta=\xi+ix+\mathcal{O}((x,\xi)^2)$.

By further conjugations, we shall now see that we can arrange so that the full symbol also has diagonal elements which are independent of (x,ξ). We may assume that already P has the principal symbol (c.19). Then, $(I+[A_1]_{1,2}+[A_2]_{2,1})P(I+[A_1]_{1,2}+[A_2]_{2,1})^*$ will have this property, if we can find classical analytic pseudodifferential operators, A_j of order -1, such that

(c.20)
$$C_1+A_1Z^*+ZA_1^*+A_1C_2A^*{}_1=\text{const.}\ ,$$
$$C_2+A_2Z+Z^*A^*{}_2+A_2C_1A^*{}_2=\text{const.}\ \ .$$

Here, we have written $P=[C_1]_{1,1}+[C_2]_{2,2}+[Z]_{1,2}+[Z^*]_{2,1}$, so the principal symbol c_j of C_j is constant, and Z has the principal symbol ζ. Again the two equations can be treated separately, and it is quite easy to see that there are classical pseudodifferential operators, A_j which satisfy these equations. The difficulty, is to verify that A_j are analytic pseudodifferential operators, i.e. we have to show the usual growth conditions on the asymptotic expansions of A_j.

Let Ω_t, $t_0\leq t\leq t_1$ be an increasing family of relatively compact open sets in $\mathbb{C}^2$, such that $\mathrm{dist}(\Omega_t,\mathbb{C}\setminus\Omega_s)\geq s-t$, for $s\geq t$. If $a(x,\xi,h)=\Sigma_0^\infty a_k(x,\xi)h^k$ is an analytic symbol defined on Ω_{t_1}, we put $A(x,\xi,D_x,h)=a(x,\xi+hD_x,h)=\Sigma_0^\infty h^kA_k(x,\xi,D_x)$ (as in [S1]), so that A_k is a differential operator of order $\leq k$. Let $f_k(A)\geq 0$, be the smallest constant such that $\|A_k\|_{t,s}\leq f_k(A)k^k(s-t)^k$, $t_0\leq t<s\leq t_1$, where $\|A_k\|_{t,s}$ denotes the operator norm of A_k: $\mathrm{Hol}(\Omega_s)\cap L^\infty(\Omega_s)\rightarrow\mathrm{Hol}(\Omega_t)\cap L^\infty(\Omega_t)$, and $\mathrm{Hol}(\Omega)$ denotes the space of holomorphic functions on Ω. If a is an analytic function defined in a neighborhood of $\bar{\Omega}_{t_1}$, then for $\rho>0$ sufficiently small, $\|a\|_\rho=_{\mathrm{def.}}\Sigma f_k(A)\rho^k$ is finite, and conversely, if the a_k are all holomorphic in Ω_{t_1} and $\|a\|_\rho<\infty$ for some $\rho>0$, then a is an analytic symbol in Ω_{t_1}. We recall from [S1], that if a and b are analytic symbols, then $\|a\circ b\|_\rho\leq\|a\|_\rho\|b\|_\rho$. Here $a\circ b$ is the symbol of the composition of the corresponding pseudodifferential operators.(In this section we do not use the Weyl quantization, but rather the "classical" one.)

Before attacking (c.20), we analyze two simpler division problems. Let $\Omega_t=\{x;|x|<t\}\times\{\xi;|\xi|<t\}$. We start by looking at division to the left by x. If

$a(0,\xi,h)=0$, we can write $a(x,\xi,h)=xb(x,\xi,h)$, where for the corresponding operators A, B, we have $A_k=xB_k$. By the maximum principle, if $u\in \mathrm{Hol}(\Omega_t)\cap L^\infty$, and $\|u\|_t$ denotes the corresponding L^∞ norm, then $\|A_k u\|_t=t\|B_k u\|_t$, which implies that $\|B_k\|_{t,s}=t^{-1}\|A_k\|_{t,s}\le t_0^{-1}\|A_k\|_{t,s}$, and hence,

(c.21) $\|b\|_\rho\le t_0^{-1}\|a\|_\rho$, when $a=xb$.

Next we look at division to the right by hD. Assume that $a(x,0,h)=0$. Then we can write; $a(x,hD,h)=b(x,hD,h)hD$, or simply, $a(x,\xi,h)= b(x,\xi,h)\xi$, which gives, $A= B\circ(\xi+hD_x)= \Sigma_0^\infty h^k B_k(x,\xi,D_x)(\xi+hD_x)= \Sigma_0^\infty h^k(\xi B_k+B_{k-1}D_x)$, with the convention that $B_{-1}=0$. In other words, $A_k= \xi B_k+B_{k-1}D_x$, or rather, $\xi B_k=A_k-B_{k-1}D_x$, and as in the case of division by x, we obtain, $\|B_k\|_{t,s}\le t_0^{-1}(\|A_k\|_{t,s}+\|B_{k-1}D_x\|_{t,s})$. Here, by Cauchy's inequality, $\|B_{k-1}D_x\|_{t,s}\le f_{k-1}(B)(k-1)^{k-1}(r-t)^{-(k-1)}(s-r)^{-1}$, for $t<r<s$. Choosing r such that $r-t=(k-1)(s-t)/k$, we get $\|B_{k-1}D_x\|_{t,s}\le f_{k-1}(B)k^k/(s-t)^k$, so $t_0 f_k(B)\le f_k(A)+f_{k-1}(B)$. Multiplying this by ρ^k and summing over ρ, we get $t_0\|b\|_\rho\le\|a\|_\rho+\rho\|b\|_\rho$, and hence,

(c.22) $\|b\|_\rho\le(t_0-\rho)^{-1}\|a\|_\rho$, if $\rho<t_0$ and $a=b\xi$.

<u>Lemma c.1.</u> *If a is an analytic symbol on Ω_{t_1} of order 0, with $\|a\|_\rho<\infty$, for some sufficiently small $\rho>0$, then we have a decomposition,*

(c.23) $a=a(0,0,h)+xc+b\xi$,

where b and c are analytic symbols of order 0, with $\|b\|_\rho\le 2(t_0-\rho)^{-1}\|a\|_\rho$, $\|c\|_\rho\le 2t_0^{-1}\|a\|_\rho$. We also have $\|a(0,0,h)\|_\rho\le\|a\|_\rho$, and the choice of b, c is unique if we impose the additional assumption, that $c=c(x,h)$ is independent of ξ.

<u>Proof.</u> The estimate on $\|a(0,0,h)\|_\rho$ follows easily from the definitions. We also have $\|a(x,0,h)\|_\rho\le\|a\|_\rho$. With the additional condition that c is independent of ξ, it is clear that the unique choice of c, b is given by,

(c.24) $x\,c(x,h)=a(x,0,h)-a(0,0,h)$, $a(x,\xi,h)-a(x,0,h)=b(x,\xi,h)\xi$.

Again, it follows from the definitions that $\|c\|_\rho\le 2t_0^{-1}\|a\|_\rho$. (Notice that the operator C associated to c is simply multiplication by $(a(x,0,h)-a(0,0,h))/h$, so $C_k=a_k(x,0)=A_k(1)(x,0)$. Hence $\|C_k\|_{t,s}=\|a_k(x,0)\|_t\le\|a_k\|_t=\|A_k(1)\|_t\le \|A_k\|_{t,s}$.) The estimate of $\|b\|_\rho$ follows from this and from (c.22). ∎

As a preparation for the handling of complex adjoints, we also need to study the behaviour of the "ρ-norms", when we take commutators with x and with hD_x. Let us first estimate $b=[x,a]$, where the composition is that of the corresponding operators, and where a is an analytic symbol of order 0 with $\|a\|_\rho<\infty$ for some $\rho>0$. Let $A=\Sigma h^k A_k$, $B=\Sigma h^k B_k$, be the corresponding operators introduced before. Using the explicit form, $A_k=\Sigma_{j+\alpha=k}(\alpha!)^{-1}\partial_\xi^\alpha a_j(x,\xi)\, D_x^\alpha$, and similarly for B, one finds that

$B_k=i[\partial_\xi,A_{k-1}]=i(\partial_\xi\circ A_{k-1}-A_{k-1}\circ\partial_\xi)$. Using that $\|\partial_\xi\|_{t,s}\leq 1/(s-t)$, we get $\|B_k\|\leq 2u^{-1}f_{k-1}(A)(k-1)^{k-1}v^{-(k-1)}$, for all $u,v>0$ with $u+v=s-t$. choosing $u=(s-t)/k$, $v=(k-1)(s-t)/k$, we obtain $f_k(B)\leq 2f_{k-1}(A)$. (Notice that $B_0=0$, since b is of order -1.) This implies that,

(c.25) $\quad \|[x,a]\|_\rho\leq 2\rho\|a\|_\rho$.

The estimate of $b=[hD,a]$, works the same way, since for the corresponding operators A, B, we have, $B_k=[D_x,A_{k-1}]$, and we obtain,

(c.26) $\quad \|[hD,a]\|_\rho\leq 2\rho\|a\|_\rho$.

Now return to the decomposition (c.23). We write $a=a(0,0,h)+xc+hDb-[hD,b]$, where $[hD,b]$ is of order -1 and $\|[hD,b]\|_\rho\leq 2\rho\|b\|_\rho\leq 4(\rho/(t_0-\rho))\|a\|_\rho$. Taking ρ so small that $4\rho/(t_0-\rho)\leq 1/2$, we redecompose $[hD,b]$ as in the lemma, et c. . Eventually we then obtain the general decomposition,

(c.27) $\quad a=d(h)+xc'(x,h)+hD\circ b'(x,hD,h)$,

where d, c′, b′ are analytic symbols of order 0, such that,

(c.28) $\quad \|d\|_\rho,\|c'\|_\rho,\|b'\|_\rho\leq C_0\|a\|_\rho$, when $\rho\leq 1/C_0$.

Here $C_0>0$ only depends on t_0. Again, it is easy to see that d,c′,b′ are unique (with the requirement that c′ should be independent of ξ).

We can now combine decompositions and adjoints. If a is an analytic symbol, we let a^* denote the symbol corresponding to the complex adjoint, and we put, $\|a\|_\rho'=\|a\|_\rho+\|a^*\|_\rho$. Notice the general inequality: $\|a\circ b\|_\rho'\leq\|a\|_\rho'\|b\|_\rho'$. Taking the adjoints of the decomposition (c.23), we get $a^*=\bar{a}(0,0,h)+xc^*(x,h)+hD\circ b^*(x,hD,h)$, which is of the form (c.28), and hence we get for the decomposition (c.23):

(c.29) $\quad \|a(0,0,h)\|_\rho'+\|b\|_\rho'+\|c\|_\rho'\leq C_0\|a\|_\rho'$.

We now return to the problem (c.20). There are analytic pseudodifferential operators F,G of order 0 and F_{-1},G_{-1} of order -1, such that,

(c.30) $\quad hD=FZ+Z^*F^*+F_{-1},\ x=GZ+Z^*G^*+G_{-1}$.

Moreover F_{-1} and G_{-1} are selfadjoint. Let a be an analytic symbol of order 0, and let b,c be as in the decomposition (c.23). Substituting (c.30) into (c.23), we get,

(c.31) $a-\text{const.}=(bF+Gc)Z+Z^*(bF+Gc)^*+R(a)$,

where,

(c.32) $\quad R(a)=G[Z,c]+[b,Z^*]F^*+bF_{-1}+G_{-1}c+Z^*[b,F^*]+$
$\qquad\qquad Z^*F^*(b-b^*)+Z^*[G^*,c]+Z^*(c-c^*)G^*$.

Now assume that a is selfadjoint. Then $b-b^*$ and $c-c^*$ are of order -1, and hence R(a) is of order -1 and we obtain,

<u>Lemma c.2.</u> Let a be an analytic symbol of order $m\leq 0$, defined in Ω_{t_1} with $\|a\|_\rho'<\infty$ for $\rho<\rho_0$, where $\rho_0>0$. If a is selfadjoint, we can find analytic symbols A of order $\leq m$ and R(a) of order $\leq m-1$, such that

(c.33) $\quad a=AZ+Z^*A^*+R(a)+\text{const.},$

and $\|A\|_\rho'+\|R(a)\|_\rho\le C_0\|a\|_\rho$, for $\rho<\min(\rho_0,1/C_0)$. Here C_0 is independent of a, m, ρ.

If C, D are selfadjoint of order 0, and D has a constant principal part, we next want to find an analytic symbol A of order −1, such that

(c.34) $\quad AZ+Z^*A^*+ACA^*=D+\text{const.},$

which is nothing but the second equation of (c.20) (with $A=A_2$, $D=-C_2$, $C=C_1$). Trying $A=A_{-1}+A_{-2}+..$ with A_{-j} of order −j, we get by successive use of Lemma c.2 the following recursive system:

$$A_{-1}Z+Z^*A^*_{-1}=D_{-1}+R(D_{-1})+\text{const.},\ (\text{where } D=D_{-1}+\text{const.}),$$

$$A_{-2}Z+Z^*A_{-2}=-R(D_{-1})-A_{-1}CA^*_{-1}-R(R(D)+A_{-1}CA^*_{-1})+\text{const.},$$

....

$$A_{-N}Z+Z^*A^*_{-N}=D_{-N}+R(D_{-N})+\text{const.},$$

where,

$$D_{-N}=-R(D_{-N+1})-(A_{-1}CA^*_{-N+1}+A_{-2}CA^*_{-N+2}+..+A_{-N+1}CA^*_{-1}).$$

Put $m(n,\rho)=\|D_{-n}\|_\rho'$. Then for $\rho\le 1/C_0$: $\|A_{-n}\|_\rho'\le C_1m(n,\rho)$,

(*) $\quad m(N,\rho)\le\ C_0m(N-1,\rho)+$

$C_0\|C\|_\rho'(m(1,\rho)m(N-1,\rho)+m(2,\rho)m(N-2,\rho)+..+m(N-1,\rho)m(1,\rho))$, and we make the induction hypothesis,

(N−1) $\quad m(k,\rho)\le Ek^{-2}F^k,\ k\le N-1.$

This hypothesis is fulfilled for $N-1=1$, if $EF\ge\|D_{-1}\|_\rho'$.

Substitution into (*) gives,

$$m(N,\rho)\le C_0(\|C\|_\rho'E^2F^N(\Sigma_{1\le k\le N-1}k^{-2}(N-k)^{-2})+E(N-1)^{-2}F^{N-1})\le$$
$$C_0C_1\|C\|'_\rho E^2F^N/N^2+C_0EF^{N-1}(N-1)^{-2},$$

where C_1 is a universal constant. The hypothesis (N) will then be satisfied, if $C_0C_1E\|C\|_\rho+4C_0/F\le 1$. This can be achieved, if we first fix E sufficiently small so that $C_0C_1E\|C\|'_\rho\le\frac{1}{2}$, and then choose F sufficiently large. (We then also get the induction hypothesis (1).)

Hence there is a choice of E,F, such that $m(k,\rho)\le Ek^{-2}F^k$ for all $k\ge 1$. It follows that $A=A_{-1}+A_{-2}+..$ is a well defined analytic symbol satisfying (c.34). We have thus solved the second equation of (c.20). The first equation can be handled the same way, and this completes the proof of:

Proposition c.3. Let p_0 be hermitian and satisfy (c.1). Let P be a formally selfadjoint 2×2 system of classical analytic pseudodifferential operators, whose complete symbol is defined in some fixed complex neighborhood of (0,0) in $\mathbb{R}^2$. Assume further that the principal symbol p satisfies, $\|p-p_0\|\le\varepsilon$, in that neighborhood. Then if $\varepsilon>0$ is sufficiently small, there exist an elliptic 2×2 system, A of classical analytic pseudodifferential operators of order 0, (with symbol defined in a fixed ε-independent neighborhood of (0,0)), and an elliptic scalar analytic Fourier integral operator of order 0, (whose associated canonical graph is closed in an ε-independent neighborhood of $((0,0),(0,0)),)$ such that,

(c.35) $\quad (AU)^*P(AU)=[C_1]_{1,1}+[C_2]_{2,2}+[Z]_{1,2}+[Z^*]_{2,1},$

where C_1 and C_2 are analytic symbols of order 0, independent of (x,ξ), and Z is of order 0 with principal symbol $\zeta=\xi+ix+\mathcal{O}((x,\xi)^2)$.

d. Parametrix for the Grushin problem of section 4.

We shall first develop a simple theory of WF and WF′ in the microlocal setting of section a.1. Most of this material is already in [S1] and in [GrS]. If $\Omega\subset\mathbb{C}^n$ is open, and $\Phi:\Omega\longrightarrow\mathbb{R}$ is continuous, we slightly modify our earlier definition, by letting $H_\Phi^{loc}(\Omega)$ denote the space of functions, $u(x,h)$ on $\Omega\times]0,2\pi]$, such that,

1° u is of class C^∞ with respect to x, and for every compact set, $K\subset\Omega$, there exist $\varepsilon=\varepsilon_K>0$, and $C=C_K>0$ such that $|\bar\partial_x u|\le C\,e^{(\Phi(x)-\varepsilon)/h}$ on $K\times]0,2\pi]$.

2° For every compact set $K\subset\Omega$ and all $\varepsilon>0$, and $\alpha,\beta\in\mathbb{N}^n$, there exists a constant $C=C_{K,\varepsilon,\alpha,\beta}$, such that, $|\partial^\alpha\bar\partial^\beta u|\le C\,e^{(\Phi(x)+\varepsilon)/h}$ on $K\times]0,2\pi]$.

We also say that two elements u_1, $u_2\in H_\Phi^{loc}(\Omega)$ are equivalent, and write $u_1\equiv u_2$, if for every compact set $K\subset\Omega$, there exist $C,\varepsilon>0$, such that,

(d.1) $$|u_1-u_2|\le C\,e^{(\Phi(x)-\varepsilon)/h}\text{ on }K\times]0,2\pi].$$

If Φ is strictly plurisubharmonic, and if $\Omega'\subset\subset\Omega$ is strictly pseudoconvex, then for every $u\in H_\Phi^{loc}(\Omega)$, there exists $u'\in H_\Phi^{loc}(\Omega')$, holomorphic in x, such that $u\equiv u'$ in $H_\Phi^{loc}(\Omega')$. A family $\{u_\alpha\}$ of elements in $H_\Phi^{loc}(\Omega)$ is said to be bounded, if for $u=u_\alpha$, we can choose all constants in 1° and 2° independent of α. Similarly, we define the notion of equivalent families.

If $\{u_\alpha\}$ is such a bounded family, we define $\Omega(\{u_\alpha\})$, to be the largest open subset $\Omega'\subset\Omega$ such that $\{u_\alpha\}\equiv 0$ in $H_\Phi^{loc}(\Omega')$, and we put $WF(\{u_\alpha\})=\Omega\setminus\Omega(\{u_\alpha\})$. Restricting to a single element, we also get a definition of WF(u), when $u\in H_\Phi^{loc}(\Omega)$.

We next extend these notions to the case of kernels, and for simplicity, we only discuss the case of single elements, the extension to the case of families being immediate. Let $\Phi_j\in C(\Omega_j,\mathbb{R})$, $j=1,2$, where Ω_j is an open subset of $\mathbb{C}^{n_j}$. Let ρ denote the map $y\longrightarrow\bar y$, and put $(\Phi_1+\Phi_2\circ\rho)(x,y)=\Phi_1(x)+\Phi_2\circ\rho(y)$. If $K(x,y,h)\in H^{loc}_{\Phi_1+\Phi_2\circ\rho}(\Omega_1\times\rho(\Omega_2))$, then we put $WF'(K)=\{(x,\bar y);\ (x,y)\in WF(K)\}$. WF′(K) is the smallest closed set in $\Omega_1\times\Omega_2$, such that $e^{-(\Phi_1(x)+\Phi_2(y))/h}K(x,\bar y,h)$ is locally uniformly of exponential decrease in it's complement. We can associate to A a formal integral operator,

(d.2) $$Au(x,h)=\int K(x,\bar y,h)u(y,h)e^{-2\Phi_2(y)/h}L(dy).$$

We shall write $K=K_A$ and $WF'(A)=WF'(K)$. If $u\in H^{loc}_{\Phi_2}(\Omega_2)$ and if the projection $WF'(A)\cap(\Omega_1\times WF(u))\ni(x,y)\longrightarrow x\in\Omega_1$ is proper, then Au is well defined up to equivalence in $H^{loc}_{\Phi_1}(\Omega_1)$, and $WF(Au)\subset WF'(A)(WF(u))$, where WF′(A) is interpreted as a relation. The equivalence class of Au only depends on the equivalence classes of A and of u. Similarly, if B is given by a kernel $K=K_B\in H^{loc}_{\Phi_2+\Phi_3\circ\rho}(\Omega_2\times\rho(\Omega_3))$, and if the projection,

$(WF'(A)\times WF'(B))\cap\Omega_1\times \mathrm{diag}(\Omega_2\times\Omega_2)\times\Omega_3\rightarrow\Omega_1\times\Omega_3$ is proper, then we can define the composed kernel,

(d.3) $\qquad K_{A\circ B}(x,\bar{z},h)=\int K_A(x,\bar{y},h)K_B(y,\bar{z},h)e^{-2\Phi_2(y)/h}L(dy),$

and we get $WF'(A\circ B)\subset WF'(A)\circ WF'(B)$.

From now on, we assume that all domains are bounded, and that the weight functions are analytic and strictly plurisubharmonic in a neighborhood of the closure of the corresponding domains. For such a couple, (Ω,Φ), it is well known that there exists a unique holomorphic function $\Psi(x,y)$, defined in a neighborhood of $\{(x,\bar{x});\ x\in\bar{\Omega}\}$, such that $\Psi(x,\bar{x})=\Phi(x)$ and that this function has the property that,

(d.4) $\qquad \frac{1}{2}(\Phi(x)+\Phi(y))-\mathrm{Re}\Psi(x,\bar{y})\sim|x-y|^2.$

A classical analytic pseudodifferential operator A of order m with symbol defined in a neighborhood of $\Lambda_{\Phi}|_{\bar{\Omega}}$, can be realized by a kernel of the form,

(d.5) $\qquad K_A(x,\bar{y},h)=a(x,\bar{y},h)e^{2\Psi(x,\bar{y})/h}\chi(x,y),$

where a is a classical analytic symbol of order $m-\frac{1}{2}n$, defined in a neighborhood of $\{(x,\bar{x});\ x\in\bar{\Omega}\}$. Here $\chi\in C^\infty(\mathbb{C}^{2n})$ is equal to one near the set, $\{(x,\bar{x});\ x\in\bar{\Omega}\}$. This is obtained (see [S1]) by representing A as a formal Fourier integral operator with a suitable phase function, and then choosing a suitable integration contour. In particular, when $A=I$, (d.5) become an approximate Bergmann kernel. Using such an approximate Bergmann kernel, we can extend the result above to the case when A is a Fourier integral operator of the type considered in section a.1, with an associated canonical transformation κ mapping Λ_{Φ_2} to Λ_{Φ_1} and such that $\Phi_1(x_0)=-\mathrm{Im}\varphi(x_0,y_0,\theta_0)+\Phi_2(y_0)$ if $\kappa(y_0,\eta_0)=(x_0,\xi_0)$, and if φ is the phase used in the description of A. If Ω_1 and Ω_2 are suitable neighborhoods of x_0 and y_0 respectively, then we can realize A by a kernel of the form (d.5), where now Ψ and a are defined in a neighborhood of Γ_κ, which by definition is the (x,y)-projection of the graph of the restriction of κ to Λ_{Φ_1}, where χ is equal to 1 near Γ_κ and where $\frac{1}{2}(\Phi_1(x)+\Phi_2(y))-\mathrm{Re}\Psi(x,\bar{y})\sim \mathrm{dist}((x,y),\Gamma_\kappa)^2$. In particular, we see that $WF'(A)\subset\Gamma_\kappa$.

Using this observation, and the fact that different representatives of a microlocally defined function (as in section a.1) are related by elliptic Fourier integral operators, we see that if $u=u_h$ is a function, which is defined microlocally in an open set $V\subset\mathbb{R}^{2n}$, then we can define in a natural way, its wavefront set $WF(u)$ as a closed subset of V. In the case when u is a distribution independent of h, defined in some open set $X\subset\mathbb{R}^n$, and we take V to be $X\times\mathbb{R}^n$, then $WF(u)=(\mathrm{Supp}(u)\times\{0\})\cup WF_a(u)$, where WF_a is the classical analytic wavefront set. (See [S1].) Similarly, let $V_j\subset\mathbb{R}^{2n_j}$ be open and let T_j be FBI-transforms, permitting to represent microlocally defined functions in Ω_j as T_ju_j in $H_{\Phi_j}^{loc}(\Omega_j)$. If $K(x,\bar{y},h)$ is a kernel as above, then we can think of K as a kernel of an operator, A, which takes certain functions, with compact wavefront set in V_2 into functions, defined microlocally in V_1. and we get a corresponding definition of WF'(A) in $V_1 \times V_2$. Using the natural change of K under changes of FBI-transforms, we see that this notion is independent of the choice of such transforms, and concerning the possibility of defining Au or $A \circ B$ we have the obvious analogous results to the one stated above for Kernels and in the frame work of the H_Φ^{loc} spaces.

As an example, we shall compute $WF'(E_{\mu'})$, where $E=E_{\mu'}$ is the parametrix of $P_0-\mu'$, given by (4.21). Put,

(d.6) $$U_{0,t}u(x)=e^{-t/2}u(e^{-t}x),$$

so that $U_{0,t}$ is unitary, and $U_t=e^{i\mu' t/h}U_{0,t}=_{def.} U_{\mu',t}$. We choose the global unitary FBI-transform,

(d.7) $$Tu(x,h)=C_0h^{-3/4}\int e^{-(x-y)^2/2h}\, u(y)\, dy,$$

where $C_0>0$ is a suitable normalization constant, and where the associated weight is,

(d.8) $$\Phi_0(x)=\tfrac{1}{2}(\mathrm{Im}x)^2,$$

and the associated canonical transformation is,

(d.9) $$\kappa_T:(y,\eta)\rightarrow(y-i\eta,\eta).$$

The adjoint of T is then,

(d.10) $$T^*v(y,h)=\bar{C}_0\, h^{-3/4}\int e^{-(\bar{z}-y)^2/2h}\, v(z)\, e^{-2\Phi_0(z)/h}\, L(dz),$$

so the kernel of $TU_{0,t}T^*$ is,

(d.11) $$K_t(x,\bar{z},h)=|C_0|^2\, e^{-t/2}\int e^{-F(x,\bar{z},y)/h}\, dy,$$

where,

(d.12) $$F(x,\bar{z},y)=\tfrac{1}{2}((x-y)^2+(\bar{z}-e^{-t}y)^2).$$

The gaussian integral can be computed, and we get,

(d.13) $$K_t(x,\bar{z},h)=C_1h^{-1}e^{-t/2}(1+e^{-2t})^{-\frac{1}{2}}\, e^{-G(t,x,\bar{z})/h},$$

where,

(d.14) $$G(t,x,\bar{z})=\tfrac{1}{2}(1+e^{-2t})^{-1}(\bar{z}-e^{-t}x)^2.$$

We get,

(d.15) $$\Phi_0(x)+\Phi_0(z)+\mathrm{Re}(G(t,x,\bar{z})= \tfrac{1}{2}(1+e^{-2t})^{-1}((e^{-t}\mathrm{Re}x-\mathrm{Re}z)^2+(\mathrm{Im}x-e^{-t}\mathrm{Im}z)^2),$$

from which it follows that,

(d.16) $$WF'(TU_{0,t}T^*)\subset\{(x,z);\ \mathrm{Re}x=e^t\mathrm{Re}z,\ \mathrm{Im}x=e^{-t}\mathrm{Im}z\}.$$

Using (d.9), we conclude that,

(d.17) $$WF'(U_{0,t})\subset\{(x,\xi;y,\eta)\in\mathbb{R}^2\times\mathbb{R}^2;\ x=e^ty,\ \xi=e^{-t}\eta\}.$$

Combining (d.13),(d.15), we also get,

(d.18) $$WF'(\mathcal{K}_{\mu'})\subset\text{Closure of }\{(x,z)\in\mathbb{C}^2;\text{ there exists } t\ge0 \text{ with } \mathrm{Re}x=e^t\mathrm{Re}z,\ \mathrm{Im}x=e^{-t}\mathrm{Im}z\},$$

where, $\mathcal{K}_{\mu'}$ is defined to be the kernel of,

(d.19) $$(i/h)\int_0^\infty TU_{\mu',t}T^*\, dt.$$

By contour deformation, we see that if

(d.20) $\mathrm{Re}x_0 = e^{t_0}\mathrm{Re}z_0$, $\mathrm{Im}x_0 = e^{-t_0}\mathrm{Im}z_0$, for some $t_0 > 0$,

and if at that point,

(d.21) $\partial_t(i\mu' t - G(t,x,\bar{z})) \neq 0$,

then $(x_0,z_0) \notin WF'(\mathcal{K}_{\mu'})$. In order to study this condition, we first notice that the map $t \to \mathrm{Re}(G(t,x_0,\bar{z}_0))$, has a critical point at t_0, so it is enough to study $\partial_t \mathrm{Im}(G)$ at $(t_0,x_0,\bar{z}_0)$. Making use of the relations (d.20), we get,

(d.22) $\partial_t \mathrm{Im}(G(t_0,x_0,\bar{z}_0)) = -(\mathrm{Im}x_0)(\mathrm{Re}x_0)$,

so $(x_0,z_0) \notin WF'(\mathcal{K}_{\mu'})$ if $\mu' \neq -(\mathrm{Im}x_0)(\mathrm{Re}x_0)$. Similarly, for $\mu' \neq 0$, we can eliminate any point (x_0,z_0) outside the diagonal with $(\mathrm{Im}x_0)(\mathrm{Re}x_0) = (\mathrm{Im}z_0)(\mathrm{Re}z_0) = 0$, by contour deformation near ∞. For $\mu' \neq 0$ (d.18) improves to,

(d.23) $WF'(\mathcal{K}_{\mu'}) \subset \mathrm{diag}(\mathbb{C}^2) \cup \{(x,z) \in \mathbb{C}^2;\ \mu' = -(\mathrm{Im}x)(\mathrm{Re}x)$, and there exists $t \geq 0$ with $\mathrm{Re}x = e^t \mathrm{Re}z$, $\mathrm{Im}x = e^{-t}\mathrm{Im}z\}$,

and for $\mu' = 0$, we get,

(d.24) $WF'(\mathcal{K}_{\mu'}) \subset \mathrm{diag}(\mathbb{C}^2) \cup \{(x,z) \in \mathbb{C}^2;\ (\mathrm{Im}x)(\mathrm{Re}x) = (\mathrm{Im}z)(\mathrm{Re}z) = 0,$ $|\mathrm{Im}x| \leq |\mathrm{Im}z|,\ |\mathrm{Re}x| \geq |\mathrm{Re}z|,\ (\mathrm{Im}x)(\mathrm{Im}z) \geq 0,\ (\mathrm{Re}x)(\mathrm{Re}z) \geq 0\}$.

Since κ_T maps $y\eta_{|\mathbb{R}^2}$ to $-(\mathrm{Im}x)(\mathrm{Re}x)$, we finally get for $\mu' \neq 0$,

(d.25) $WF'(E_{\mu'}) \subset \mathrm{diag}(\mathbb{R}^2 \times \mathbb{R}^2) \cup \{(x,\xi;y,\eta);\ x\xi = y\eta = \mu'$ and there exists $t > 0$, such that $(x,\xi) = \exp(tH_{p_0})(y,\eta)\}$.

Here $p_0(x,\xi) = x\xi$. For $\mu' = 0$, we get,

(d.26) $WF(E_0) \subset \mathrm{diag}(\mathbb{R}^2 \times \mathbb{R}^2) \cup \{(x,\xi;y,\eta); x\xi = y\eta = 0,\ |\xi| \leq |\eta|,$ $|x| \geq |y|,\ xy \geq 0,\ \xi\eta \geq 0\}$.

Denote the right hand sides of (d.25),(d.26) for $\mu' \neq 0$ and $\mu' = 0$ respectively by $\Gamma(\mu')$. If we use the notion of WF' for bounded families, we get,

(d.27) $WF'(\{E_{\mu'}\}_{\mu' \in I}) \subset \cup_{\mu' \in I} \Gamma(\mu')$,

for every compact interval $I \in \mathbb{R}$. These results can easily be transported by Fourier integral operators, to give (unique) microlocal parametrices for $P_1{}^*P$ and $PP_2{}^*$ in section 4, near the branching point $(0,\pi)$ (as well as the other branching points).

As a second preparation, we consider a formally self adjoint analytic classical pseudodifferential operator, P of order 0, defined microlocally in an open set $\Omega \subset \mathbb{R}^2$, and such that the real principal symbol, p has the property that $p^{-1}(0) = \gamma(]a,b[)$, is a bicharacteristic strip. Microlocally in $\Omega \times \Omega$, we can then define the forward and the backward parametrices, by,

(d.28) $E_f = (i/h)\int_0^\infty e^{-itP/h}\,dt$,

(d.29) $E_b = -(i/h)\int_{-\infty}^0 e^{-itP/h}\,dt$.

We also put,

(d.30) $G = E_f - E_b = (i/h)\int_{-\infty}^\infty e^{-itP/h}\,dt$.

Here $e^{-itP/h}$ is a microlocally defined unitary Fourier integral operator, associated to the canonical transformation $\kappa_t=\exp(tH_p)$. Working on the FBI-side, it is also easy to see (and well known) that G is a Fourier integral operator, of the form,

(d.31) $\quad Gu(x,h)=\int a(x,y,h)e^{i(\varphi(x)-\varphi(y))/h}\,u(y)dy,$

where the phase φ satisfies $p(x,\varphi'(x))=0$. The relations $PG=0$, $GP=0$ give two transport equations for a, and we conclude that $a(x,y,h)=b(x,h)c(y,h)$, where b,c are classical analytic symbols of order $\frac{1}{2}$, satisfying $P(ae^{i\varphi/h})=0$, ${}^tP(be^{-i\varphi/h})=0$. Undoing the FBI-transform, we see that G is a Fourier integral operator associated to the canonical relation,

$\Gamma=\{(\exp(tH_p)(\rho),\rho)\in\Omega\times\Omega;\ p(\rho)=0\}$, and,

(d.32) $\quad Gu=iC(u|u_0)u_0$, where

u_0 is a microlocal normalized solution to $Pu_0=0$, and $C=C(h)$ is an elliptic c.a.s. of order 0. It is easy to see that $G^*=-G$, so it follows that C is real valued.

Writing, $\Gamma=\Gamma_f\cup\Gamma_b$, where Γ_f corresponds to $t\geq 0$ and Γ_b corresponds to $t\leq 0$, we also have,

(d.33) $\quad WF'(E_f)\subset\mathrm{diag}(\mathbb{R}^2\times\mathbb{R}^2)\cup\Gamma_f,\ WF'(E_b)\subset\mathrm{diag}(\mathbb{R}^2\times\mathbb{R}^2)\cup\Gamma_b.$

If $\rho_0=\exp(t_0H_p)(\mu_0)$, with $p(\mu_0)=0$ and $t_0>0$, this implies that $E_f=G$ and $E_b=0$ microlocally, near (ρ_0,μ_0). In order to determine C, we let χ be a pseudodifferential operator of order 0, such the symbol is equal to 1 in a neighborhood of ρ_0 and equal to 0 near μ_0. (We may work with a gaussian quantization of this symbol, so that χu is well defined microlocally, and $WF(\chi u)\subset WF(u)$ for every microlocally defined function, u.) Put, $\tilde{E}_f=(1-\chi)E_f+E_f[P,\chi]E_f$. If $WF(u)\subset\subset\Omega$, we know that E_f can be applied to u, and $WF(\tilde{E}_fu)\subset WF(u)\cup\Gamma_f(WF(u))$. We also know that E_fP and PE_f reduce to the identity on such functions. Hence, $\tilde{E}_fu=E_fP\tilde{E}_fu$. Since $P\tilde{E}_f=(1-\chi)-[P,\chi]E_f+[P,\chi]E_f=(1-\chi)$, we get, $\tilde{E}_f=E_f(1-\chi)$. Hence microlocally near (ρ_0,μ_0), we have $\tilde{E}_f=E_f=G$. On the other hand we see from the definition of $\tilde{E}_f$, that $\tilde{E}_f=G[P,\chi]G$ near the same point, so there we have $G=G[P,\chi]G$. This means that if $WF(u)$ is close to μ_0, then near ρ_0, we have, $iC(u|u_0)u_0=(iC)^2(u|u_0)([P,\chi]u_0|u_0)u_0$. This means that $iC=iC^2(i[P,\chi]u_0|u_0)$, and since u_0 is normalized, $(i[P,\chi]u_0|u_0)=1$. Hence $C=C^2$, and since C is elliptic, we must have $C=1$. Hence,

(d.34) $\quad Gu=i(u|u_0)u_0,$

if u_0 is a normalized solution of $Pu_0=0$.

We shall now establish the wellposedness of the Grushin problem of section 4, by using a priori estimates. The first step will be to establish microlocal estimates near $\mathrm{Int}(s(0,1))$, then near $(0,\pi)$, then near $\mathrm{Int}(s(0,2))$. Patching together these estimates, we get an estimate in a neighborhood of the closed square with corners, $(-\pi/2\pm\pi,\pi/2\pm\pi)$, and combining translates of this we finally get a global a priori estimate.

Let $D(r)=D((\pi/2,\pi/2),r)$ be the open disc with center $(\pi/2,\pi/2)$ and radius $r<\pi/2^{\frac{1}{2}}$, sufficiently large so that $WF(f_{0,1})\subset D(r)$. In $D(r)$ we then know that H_j, $j=1,2$, are of principal type and as above we can define the corresponding forward and backward parametrices, $E_f(j)$, $E_b(j)$, $j=1,2$. Also put $G^{(j)}= E_f(j)-E_b(j)$. Once and for all, we fix a global FBI-transform, say (d.7) and let Φ_0 be the corresponding weight function. For simplicity, we write $H(\Omega)$ instead of $H_{\Phi_0}(\Omega)$ and the corresponding norm will be denoted simply by $\| \ \|_\Omega$. By means of $\pi_x\circ\kappa_T$, (where $\pi_x:\mathbb{C}^2(x,\xi)\to\mathbb{C}_x$ is the natural projection,) we shall identify sets in $\mathbb{R}^2$ with sets in $\mathbb{C}$. For simplicity, we shall identify operators, (A) with their conjugates under T, (TAT^{-1}). In the case of pseudodifferential operators, if $\Omega''\subset\subset\Omega'$ are open with smooth boundary, (and automatically pseudoconvex since we work in one complex dimension,) and Q is a formal analytic pseudodifferential operator with Weyl symbol defined in a neighborhood of $\bar\Omega'$ (i.e. near $\Lambda_{\Phi_0}|_{\bar\Omega'}$ after conjugation by T), then we can find a natural realization, $H(\Omega')\to H(\Omega'')$, (see [S1],[GrS],) and two such realizations will differ by a term which is $\mathcal{O}(e^{-1/Ch})$ in norm for some constant $C>0$. Such a natural realization will always (tacitly) be chosen and will be denoted by the same letter. At a later stage in our estimates, we will have to specify the relation between the globally defined operator, P and it's local realizations.

As a microlocal approximation near $\mathrm{Int}(s(0,1))$ of the full Grushin problem, we consider the problem,

(d.35) $\quad Pu=v,\ (u|f_{0,1})=v_+,$

where $u\in H(D(r))$, $v\in H(D(r'))$, $v\in\mathbb{C}$ and $r'<r$ with $r-r'$ small. Composing the first equation by $P_1{}^*$, we get with a new slightly smaller r':

(d.36) $\quad H_1(u-v_+u_{0,1})=P_1{}^*v+\mathcal{O}(|v_+|e^{-1/Ch})$ in $H(D(r'))$,

$\quad (u-v_+u_{0,1}|f_{0,1})=0.$

Now $H_1\chi_{0,1}(u-v_+u_{0,1})=\chi_{0,1}P_1{}^*v+[H_1,\chi_{0,1}](u-v_+u_{0,1})+\mathcal{O}(|v_+|e^{-1/Ch})$, (where we work with the Gaussian realization of $\chi_{0,1}$, which has a natural local realization,) and in in view of the properties of $WF(\chi_{0,1}(u-v_+u_{0,1}))$, we can apply $E_f=E_f{}^{(1)}$ to the last equation, and get,

(d.37) $\quad \chi_{0,1}(u-v_+u_{0,1})=$

$\quad E_f\chi_{0,1}P_1{}^*v+E_f[H_1,\chi_{0,1}](u-v_+u_{0,1})+\mathcal{O}(e^{-1/Ch})(\|u\|_{D(r)}+|v_+|)$,

in $H(D(r'))$, with a new slightly smaller r'. Here we notice that,

(d.38) $\quad E_f[H_1,\chi_{0,1}](u-v_+u_{0,1})\equiv G_f[H_1,\chi_{0,1}](u-v_+u_{0,1})$,

modulo the same error as in (d.37), in any subdomain of $D(r')$ whose closure is compact and disjoint from $\Gamma_1([H_{10},\chi_{0,1}])$, where the last set by definition is the union of $WF([H_1,\chi_{0,1}])$ and the largest bicharacteristic segment of H_1 with both end points in $WF([H_1,\chi_{0,1}])$. Here the last set is defined by:

$WF'([H_1,\chi_{0,1}])=\{(x,x): x\in WF([H_1,\chi_{0,1}])\}$. Now modulo the same type of errors, the right hand side of (d.38) is,
$(i[H_1,\chi_{0,1}](u-v_+u_{0,1})|u_{0,1})u_{0,1}\equiv(u-v_+u_{0,1}|f_{0,1})\equiv 0$, so in any compact set disjoint from $\Gamma_1([H_1,\chi_{0,1}])$, we get,

(d.39) $\qquad \chi_{0,1}(u-v_+u_{0,1})\equiv E_f^{(1)}\chi_{0,1}P_1{}^*v.$

Similarly,

(d.39′) $\qquad \chi_{0,4}(u-v_+u_{0,1})\equiv E_b^{(1)}\chi_{0,4}P_1{}^*v,$

in the same set. Since $E_f^{(1)},E_b^{(1)}$ (when suitably realized) have norms, $\mathcal{O}(1/h)$, we get after combining (d.39),(d.39′) with simple elliptic estimates outside the characteristics of H_1, that

(d.40) $\qquad \|u-v_+u_{0,1}\|_{D(r'')\setminus D(r''')}\le(C/h)(\|Pu\|_{D(r')}+e^{-1/Ch}\|u\|_{D(r)}),$

for all solutions of (d.35), if $r'''<r''<r'<r$ and r''' is sufficiently large. Since $\|u_{0,1}\|=\mathcal{O}(h^{-\frac{1}{2}})$, we conclude that,

(d.41) $\|u\|_{D(r'')\setminus D(r''')}\ge C(h^{-1}\|Pu\|_{D(r')}+h^{-\frac{1}{2}}|v_+|+e^{-1/Ch}\|u\|_{D(r)}).$

Controling u in the annulus, we can apply a cut off operator χ with support in $D(r'')$, equal to I near the closure of $D(r''')$:

(d.42) $\qquad \|[H,\chi]u\|_{D(r')}\le C(h\|u\|_{D(r'')\setminus D(r''')}+e^{-1/Ch}\|u\|_{D(r)}),$

and if we write,

$$H_1\chi u=\chi H_1u+[H_1,\chi]u,$$

we deduce after applying E_f or E_b:

$$\|\chi u\|_{D(r'')}\le C(h^{-1}\|Pu\|_{D(r')}+h^{-\frac{1}{2}}|v_+|+e^{-1/Ch}\|u\|_{D(r)}),$$

for all solutions of (d.35), when r'' is large enough and $0<r''<r'<r<\pi/2\frac{1}{2}$, $D(r)=D((\pi/2,\pi/2),r)$.

We next write down an easy estimate near the branching point, which follows from the fact that H_1 and hence P has a left parametrix of norm $\mathcal{O}(h^{-3/2})$ whose WF′ can be obtained from the WF′ of the parametrix of $P_0-\mu'$ above, by applying the canonical transformation of U_1. (Here the estimate on the norm follows from (4.25).) We now let $D(r)=D((0,\pi),r)$, $0<r<2\frac{1}{2}\pi$. If r' is slightly smaller than r, we let P denote a realization, $H(D(r)\to H(D(r'))$. Let W be the intersection of $D(r)\setminus D(r')$ and a small neughborhood of $s(0,1)\cup s((0,1),3)$. Then if $r''<r'$, we easily get,

(d.44) $\qquad \|u\|_{D(r'')}\le C(h^{-3/2}\|Pu\|_{D(r')}+h^{-\frac{1}{2}}\|u\|_W+e^{-1/Ch}\|u\|_{D(r)}),$

for all $u\in H(D(r))$.

We next move to Int(s(0,2)), so we now put $D(r)=D((-\pi/2,\pi/2),r)$, with $0<r<\pi/2$, and r sufficiently large so that $WF(f_{0,1})$ is contained in D(r). If r' is slightly smaller than r, we consider the problem,

(d.45) $\qquad Pu+u^-f_{0,2}=v$ in $D(r')$, $u\in H(D(r))$, $u^-\in\mathbb{C}$.

Let χ be a cut off operator with support in $D(r'')$ and equal to I near $\bar{D}(r''')$, where $r'''<r''<r'<r$ and r''' is only slightly smaller tnan r. From (d.45) we get,

(d.46) $\qquad (Pu|\chi u_{0,2})+u^-(f_{0,2}|\chi u_{0,2})=(v|\chi u_{0,2}),$

where $(f_{0,2}|\chi u_{0,2})=1+\mathcal{O}(e^{-1/Ch})$, and
$(Pu|\chi u_{0,2})=(u|[P^*,\chi]u_{0,2})+\mathcal{O}(e^{-1/Ch})\|u\|_{D(r)}$. Hence,

$$|(Pu|\chi u_{0,2})|\leq C(h^{\frac{1}{2}}\|u\|_{D(r'')\setminus D(r''')}+e^{-1/Ch}\|u\|_{D(r)}).$$

Using also that, $|(v|\chi u_{0,2})|\leq Ch^{-\frac{1}{2}}\|v\|_{D(r')}$, we get from (d.46),

(d.47) $$|u^-|\leq C(h^{-\frac{1}{2}}\|v\|_{D(r')}+h^{\frac{1}{2}}\|u\|_{D(r'')\setminus D(r''')}+e^{-1/Ch}\|u\|_{D(r)}).$$

Using this estimate and the fact that $\|f_{0,2}\|=\mathcal{O}(h^{\frac{1}{2}})$, we can return to (d.45), truncate and apply $E_f P^*$ or $E_b P^*$, which gives,

(d.48) $$\|u\|_{D(r'')}+h^{-\frac{1}{2}}|u^-|\leq C(h^{-1}\|v\|_{D(r)}+\|u\|_{D(r'')\setminus D(r''')}+e^{-1/Ch}\|u\|_{D(r)}).$$

for the solution (u,u^-) of the problem (d.45).

Let Ω be the open square with corners at $(-\pi/2\pm(\pi+\delta),\pi/2\pm(\pi+\delta))$, where $\delta>0$ is so large that $WF(f_{\alpha,j})\subset\Omega$, when $(\alpha,j)=(0,1),(0,3),((0,1),1),((0,1),3)$, but so small that $WF(f_{\alpha,j})\cap\bar{\Omega}=\emptyset$, for all other values of (α,j) with j odd, and so that $(0,\pi)$ and $(-\pi,0)$ are the only branching points in $\bar{\Omega}$. Combining the three a priori estimates (d.43),(d.44),(d.48) with simpler estimates in the elliptic region, we see that if $\Omega''\subset\subset\Omega'\subset\subset\Omega$ are slightly smaller squares, and if P is realized as an operator $H(\Omega)\longrightarrow H(\Omega')$, then if $u\in H(\Omega)$ and $v^+(0,1)$, $v^+(0,3)$, $v^+((0,1),1)$, $v^+((0,1),3)\in\mathbb{C}$, $u^-(0,2)\in\mathbb{C}$, and

(d.49) $$Pu+u^-(0,2)f_{0,2}=v \text{ in } \Omega',\ (u|f_{\alpha,j})=v^+(\alpha,j) \text{ for } (\alpha,j)=(0,1),(0,3),((0,1),1),((0,1),3),$$

then we have the a priori estimate,

(d.50) $$\|u\|_{\Omega''}+h^{-\frac{1}{2}}|u^-(0,2)|\leq C(h^{-3/2}\|v\|_{\Omega'}+h^{-1}\Sigma|v^+(\alpha,j)|+\mathcal{O}(e^{-1/Ch})\|u\|_{\Omega}).$$

If we now consider the full Grushin problem,

(d.51) $Pu+R_-u^-=v$, $R_+u=v_+$,

and make an FBI-transform and restrict to Ω', we have to take into account two facts:

1° The full Weyl quantization of P can be realized first by taking Λ_{Φ_0} as an integration contour, then using the fact that the symbol of P is holomorphic in a neighborhood of Λ_{Φ_0}, we see that if $P_{\Omega',\Omega}$ is the realization used in (d.49), then we get,

(d.52) $$\|P_{\Omega',\Omega}(u_{|\Omega})-Pu\|_{\Omega'}\leq Ce^{-1/Ch}\|u\|_{\Phi_0+\mathrm{dist}(.,\Omega)/C},$$

where in general, $\|u\|_f$ (when f is a function) will denote the L^2-norm over $\mathbb{C}$, with respect to the measure $e^{-2f/h}L(dx)$. (It will be clear from the context whether the subscript ... in $\|.\|_{...}$ denotes a domain or a function.

2° The terms $u^-(\alpha,j)f_{\alpha,j}$ with j even $(\alpha,j)\neq(0,2)$ will have some exponentially small influence in Ω'.

We then get,

(d.53) $\quad P_{\Omega',\Omega}u+u^-(0,2)f_{0,2}=v+w$ in Ω',

$(u|f_{\alpha,j})=v^+(\alpha,j)+w^+(\alpha,j)$, $(\alpha,j)=(0,1),(0,3),((0,1),1),((0,1),3)$,

where,

$$\|w\|_{\Omega'}\le Ce^{-1/Ch}(\|u\|_{\Phi_0+\mathrm{dist}(.,\Omega)/C}+\|u^-\|_{l^2,|\alpha|/C}),$$

$$|w^+|\le Ce^{-1/Ch}\|u\|_{\Phi_0+\mathrm{dist}(.,\Omega)/C}.$$

Here we write, $(\|u^-\|_{l^2,f})^2=\Sigma_{\alpha,j}\,e^{-2f/h}\,|u^-(\alpha,j)|^2$. Applying (d.50) with (v,v^+) replaced by $(v+w,v^++w^+)$ gives,

(d.54) $\quad \|u\|_{\Omega''}+h^{-\frac{1}{2}}|u^-(0,2)|\le$

$$C(h^{-1\frac{1}{2}}\|v\|_{\Omega'}+h^{-1}\Sigma|v^+(\alpha,j)|+e^{-1/Ch}(\|u\|_{\Phi_0+\mathrm{dist}(.,\Omega)}+\|u^-\|_{l^2,|\alpha|/C}))$$

Now write, $\Omega=\Omega(0,2)$ and similarly for Ω', Ω''. Let $m(\alpha,j)$ denote the middle point of $s(\alpha,j)$. For j even we put, $\Omega(\alpha,j)=\Omega(0,2)+\{m(\alpha,j)-m(0,2)\}$, and similarly for Ω', Ω''. The estimate (d.54) remains valid, if we replace Ω, Ω', Ω'' by $\Omega(\beta,k)$, $\Omega'(\beta,k)$, $\Omega''(\beta,k)$, and thake the norm, $\|\ \|_{|\alpha-\beta|/C}$ of u^- to the right. Squaring all these inequalities and summing with respect to (β,k), we get, (since the $\Omega''(\beta,k)$ will cover $\mathbb{C}$,)

(d.55) $\quad \|u\|^2+h^{-1}\|u^-\|^2\le$

$$C(h^{-3}\|v\|^2+h^{-2}\|v^+\|^2+e^{-1/Ch}(\|u\|^2+\|u^-\|^2),$$

where the norms are now the standard L^2 and l^2-norms over $\mathbb{R}$ and $\mathbb{Z}^2$ respectively. When $h>0$ is sufficiently small, we can absorb the last two terms to the right, and get,

(d.56) $\quad \|u\|^2+h^{-1}\|u^-\|^2\le C(h^{-3}\|v\|^2+h^{-2}\|v^+\|^2)$.

This shows that $\mathcal{P}$ is injective: $L^2\times l^2\rightarrow L^2\times l^2$ with closed range. Since all our estimates work equally well for $\mathcal{P}^*$ which has the same structure, and since $\mathrm{Ker}\mathcal{P}^*=(\mathrm{Im}\mathcal{P})^\perp=\{0\}$, we conclude that $\mathcal{P}$ is bijective with bounded inverse,

$$\mathcal{E}=\begin{pmatrix} E & E_+ \\ E_- & E_{-+}\end{pmatrix},$$

satisfying,

(d.57) $\quad \|E\|=\mathcal{O}(h^{-3/2})$, $\|E_+\|=\mathcal{O}(h^{-1})$, $\|E_-\|=\mathcal{O}(h^{-1})$, $\|E_{-+}\|=\mathcal{O}(h^{-\frac{1}{2}})$.

Before Proposition 4.1, we constructed an approximate solution of $Pu+R_-u^-=0$, $R^+u=\delta_{0,1}$, satisfying these equations with exponentially small errors. In view of (d.57), the approximate solution differs from the exact one with exponentially small errors, in particular the computation of $E_{-+}(\alpha,j;0,1)$ for $(\alpha,j)=(0,2),((0,1),4)$ is correct up to an exponentially small error.

It remains to establish the exponential decrease estimates for $E(\alpha,j;\beta,k)$ in Proposition 4.1, when $s(\alpha,j)\cap s(\beta,k)=\emptyset$. Let (u,u^-) be a solution of,

(d.58) $\quad Pu+R_-u^-=0$, $R_+u=\delta_{0,1}$.

If $\Omega(\beta,k)$ does not contain $m(0,1)$, we get from (d.54):

(d.59) $\|u\|^2_{\Omega''(\beta,k)}+h^{-1}|u^-(\beta,k)|^2 \leq$
$$Ce^{-1/Ch}(\|u\|^2_{\Phi_0+\mathrm{dist}(.,\Omega(\beta,k))/C}+\|u^-\|^2_{l^2,|.-\beta|/C}),$$
so after increasing C, we get,

(d.60) $\|u\|^2_{\Omega''(\beta,k)}+h^{-1}|u^-(\beta,k)|^2 \leq$
$$Ce^{-1/Ch}\Sigma_{\alpha,j}e^{-|(\beta,k)-(\alpha,j)|/Ch}(\|u\|^2_{\Omega''(\alpha,j)}+|u^-(\alpha,j)|^2).$$
Write, $\tilde{\alpha}=(\alpha,j)$, $\tilde{\beta}=(\beta,k)$, $f(\tilde{\alpha})=\|u\|^2_{\Omega''(\tilde{\alpha})}+|u^-(\tilde{\alpha})|^2$, $g(\tilde{\alpha})=f(\tilde{\alpha})$, if $(0,1)\in\Omega(\tilde{\alpha})$ and $=0$ otherwise. Then (d.60) gives,

(d.61) $f(\tilde{\beta})\leq g(\tilde{\beta})+\mathcal{K}f(\tilde{\beta})$,

where $\mathcal{K}$ is the positivity preserving linear operator,

(d.62) $\mathcal{K}u(\tilde{\beta})=Ce^{-1/Ch}\Sigma_{\tilde{\alpha}}\, e^{-|\tilde{\beta}-\tilde{\alpha}|/Ch}u(\tilde{\alpha}).$

Since $\|\mathcal{K}\|_{\mathcal{L}(l^1,l^1)}\leq C_1e^{-1/Ch}$, we get for $h>0$ sufficiently small,

(d.63) $f\leq g+\mathcal{K}g+\mathcal{K}^2g+\ldots,$

and in particular,

(d.64) $\|(f-g)_+\|_{l^1}\leq C_2e^{-1/Ch}\|g\|_{l^1}$. If $\psi(\tilde{\alpha})$ is a function such that $|\psi(\tilde{\alpha})-\psi(\tilde{\beta})|\leq|\tilde{\alpha}-\tilde{\beta}|/2C$, then we also get,

(d.65) $\|(f-g)_+\|_{l^1,\psi}\leq C_3e^{-1/Ch}\|g\|_{l^1,\psi}.$

Recalling that $E_{-+}(\tilde{\beta};0,1)=u^-(\tilde{\beta})$, we get the required exponential decrease estimates from (d.64),(d.65). This completes the proof of Proposition 4.1.

e. Application to the magnetic Schrödinger operator.

The main results of this article apply to h-Weyl quantizations of symbols, $P(x,\xi)$ which are 2π-periodic in x and in ξ, close to $\cos(\xi)+\cos(x)$ (in the sense of strong type 1 operators) and which satisfy the following invariance properties,

(e.1) P is real valued (so that the corresponding operator, P is self adjoint).

(e.2) $P(x,\xi)=P(\xi,-x)$ (so that the operator P commutes with the Fourier transform, $\mathcal{F}$).

(e.3) $P(x,-\xi)=P(x,\xi)$ (so that the operator P commutes with Γ, where $\Gamma u(x)=\bar{u}(x)$).

In [HS1],§9.4 we saw that the study of the spectrum of the Schrödinger operator with periodic electric potential, V and periodic magnetic field, B could be reduced to the study of an operator P satisfying (e.1) and (e.2) under the following assumptions,

(e.4) (0) $V(x+a_je_j)=V(x)$, $B(x+a_je_j)=B(x)$, $e_1=(1,0)$, $e_2=(0,1)$, $a_j\in\mathbb{R}$.

(a) $\alpha_*V=V$,

(b) $\alpha_*B=B$,

where $\alpha(x_1,x_2)=(x_2,-x_1)$ and $\alpha_*u=u\circ\alpha^{-1}$. A more intrinsic formulation of (e.4.b) is:

(e.4.b)' $\alpha_*\sigma_B=\sigma_B$,

where σ_B is the 2-form, $B(x_1,x_2)dx_1\wedge dx_2$, and $\alpha_*=(\alpha^*)^{-1}$, where * denotes the standard pull-back operation. If f is a function such that $\alpha_*\omega_A-\omega_A=df$, where $\omega_A=A_1dx_1+A_2dx_2$, then we saw in [HS1], that the magnetic Schrödinger operator $P_A(h)=\Sigma_{j=1,2}(hD_{x_j}-A_j)^2+V(x)$ commutes with the operetor,

(e.5) $\mathbb{F}=e^{if/h}\alpha_*$.

We also saw that P_A commutes with the two "translation operators", T_1 and T_2, given by,

(e.6) $T_j=e^{i\varphi_j(x)/h}\tau_{j*}$, $j=1,2$,

where,

(e.7) $\tau_{j*}g(x)=g(x-a_je_j)$, $\tau_j(x)=x+a_je_j$

(e.8) $d\varphi_j=(\tau_j)_*\omega_A-\omega_A$,

and we also had,

(e.9) $T_1T_2=e^{i\Phi/h}T_2T_1$.

Here Φ is the magnetic flux through a base cell. After a modification of φ_j and f by adding suitable constants, we obtained the following properties,

(e.10) (a) $[T_j,P_A]=0$,

(b) $[\mathbb{F},P_A]=0$,

(c) $(P_A)^*=P_A$ (expressing the self-adjointness of P_A),

(d) $T_1T_2=e^{ih'}T_2T_1$, $h'\equiv\Phi/h \bmod(2\pi\mathbb{Z})$,

(e) $\mathbb{F}^4=I$,

(f) $T_2^{-1}\mathbb{F}=\mathbb{F}T_1$, $T_1\mathbb{F}=\mathbb{F}T_2$.

As in [HS1], we concentrate on a suitable interval containing an isolated piece of the spectrum of P_A, and we let Π denote the corresponding spectral projection. We then constructed an orthonormal basis in $F=\Pi(L^2)$ of the form $\{\Pi(\psi_\alpha)\}$, where ψ_0 is a suitable (approximate eigen-) function satisfying

(e.11) $\mathbb{F}\psi_0=\omega\psi_0$, with $|\omega|=1$,

and,

(e.12) $\psi_\alpha=T^\alpha\psi_0$, where $T^\alpha=T_1^{\alpha_1}T_2^{\alpha_2}$.

The matrix of P_A restricted to F is then of the form, $m_{\alpha,\beta}=(P_A\psi_\beta|\psi_\alpha)=\mu\delta_{\alpha,\beta}+w_{\alpha,\beta}$. The analysis for this has already been treated in [HS1], and we shall here mainly discuss the additional symmetries that will permit us to obtain (e.3). It follows from (e.10)-(e.12), that $m_{\alpha,\beta}$ and $w_{\alpha,\beta}$ satisfy,

(e.13)(a) $m_{\alpha,\beta}=\bar{m}_{\beta,\alpha}$ (by (e.10.c),

(b) $w_{\alpha,\beta}=e^{ih'\gamma_2(\beta_1-\alpha_1)}\,w_{\alpha+\gamma,\beta+\gamma}$ (cf (4.26),(4.27) in [HS1]).

(c) $w_{\alpha,\beta}=e^{ih'(\alpha_1\alpha_2-\beta_1\beta_2)}w_{\kappa(\alpha),\kappa(\beta)}$, where $\kappa(\alpha)=(\alpha_2,-\alpha_1)$ (consequence of (e.10,b,f)).

We conclude that,

(e.14) $w_{\alpha,\beta}=e^{ih'\beta_2(\alpha_1-\beta_1)}f(\alpha-\beta)$,

with,

(e.15) (a) $f(-j,-k)=\bar{f}(j,k)e^{ijkh'}$,

(b) $f(\alpha)=e^{ih'\alpha_1\alpha_2}f(\kappa(\alpha))$.

Iterating (b) we get $f(-\alpha)=f(\alpha)$ and consequently, $f(\alpha)=\bar{f}(\alpha)e^{ih'\alpha_1\alpha_2}$. Finally, we saw that the operator $\Pi P_A\Pi$ is isospectral to P, the h'-Weyl quantization of the symbol $P(x,\xi)$ defined by,

(e.16) $P(x,\xi)=\Sigma\Sigma f(j,k)e^{-ijkh'/2}e^{-i(kx+j\xi)}$.

In addition to the 2π-periodicity of $P(x,\xi)$, we get from (e.15), that P satisfies (e.1),(e.2). The purpose of this appendix is to add a natural symmetry assumption on B and V which will imply that P satisfies (e.3). The natural idea is to find a suitable antilinear quantization of the map,

(e.17) $\gamma: (x_1,x_2)\to(x_1,-x_2)$.

$e=Id$, α, γ generate a finite subgroup, G of $\mathbb{O}_2(\mathbb{R})$ (the orthogonal 2×2-matrices) of 8 elements: $\alpha^k\gamma^l$, $0\le k\le3$, $0\le l\le1$. In addition to (e.4) we shall now assume,

(e.18) (a) $\gamma_*V=V$, (b) $\gamma_*B=B$.

The intrinsic formulation of (b) is,

(e.18) (b') $\gamma_*\sigma_B=-\sigma_B$.

The assumptions on α, γ in (e.4),(e.18) can be reformulated, by using the following two representations of G in $\mathbb{O}_2(\mathbb{R})$,

(e.19) $g \to M_0(g)=g, \quad g \to M_1(g)=(-1)^{k(g)}M_0(g)$,

where k(g) is defined modulo $2\mathbb{Z}$ by $g=\alpha^l\gamma^{k(g)}$. The reformulation is then that,

(e.20) $M_0(g)_*V=V, \ M_1(g)_*\sigma_B=\sigma_B$, for all $g\in G$.

The aim is then to find a representation, $\pi_1(g)$ of G on $L^2(\mathbb{R}^2)$, which is "pseudo linear" in the sense that $\pi_1(g)$ is linear when k(g) is even and antilinear when k(g) is odd, and such that all the $\pi_1(g)$ commute with P_A and satisfy suitable commutation relations with T_1 and T_2. After a gauge transform we can assume from now on that:

(e.21) $M_1(g)_*\omega_A=\omega_A$.

We put $\pi_1(\alpha)=\mathbb{F}$, $\pi_1(\gamma)=\Gamma$, where,

(e.22) $\mathbb{F}u(x_1,x_2)=u(-x_2,x_1)$,

(e.23) $\Gamma u(x_1,x_2)=\bar{u}(x_1,-x_2)$.

Also define T_j by (e.6),(e.7), where φ_j is the unique solution of (e.8) with

(e.24) $\varphi_j(0)=0$.

This is a natural normalization in view of the fact that 0 is a fixed point for $\pi_1(G)$. Essentially as in [HS1],(9.4.24), we verify that,

(e.25) $\varphi_2=\alpha_*\varphi_1, \ \varphi_1=-\tau_{1*}\alpha_*\varphi_2, \ \gamma_*\varphi_1=-\varphi_1, \ \varphi_2=\tau_{2*}\gamma_*\varphi_2$,

and this with (e.20) shows that we have (e.10) and,

(e.25) (a) $[\Gamma,P_A]=0$, (b) $\Gamma T_1=T_1\Gamma$, $\Gamma T_2=T_2{}^{-1}\Gamma$, (c) $\Gamma\mathbb{F}=\mathbb{F}^{-1}\Gamma$.

We recall how ψ_0 was constructed in [HS1]. With a suitable function $\varphi_0=\varphi_{(0,0)}$ associated to a potential well, U_0 we put $u_0=\Pi\varphi_0$ (and by the choice of φ_0, that we do not recall here, we know that u_0 is very close to φ_0 and of exponential decrease outside U_0). Putting $u_\alpha=T^\alpha u_0$, where $T^\alpha=T_1{}^{\alpha_1}T_2{}^{\alpha_2}$ we then obtained $\{\psi_\alpha\}$ as the orthonormalization of the basis $\{u_\alpha\}$ in the image of Π. We then had $\psi_\alpha=T^\alpha\psi_0$. Now we may assume that $\Gamma\varphi_0=\omega_1\varphi_0$, $|\omega_1|=1$, since P_A commutes with Γ and since we may choose the reference operator (in [HS1]) with only the well U_0, having the same property. As in [HS1] we then have $\Gamma\psi_0=\omega_1\psi_0$ and in view of the antilinearity of Γ we may assume that,

(e.26) $\Gamma\psi_0=\psi_0$,

without destroying the properties (e.11), (e.12). Using (e.25,b), we get,

(e.27) $\Gamma\psi_\alpha=\psi_{\gamma(\alpha)}$.

Now observe that for $u=\Sigma z_\alpha\psi_\alpha\in \mathrm{Im}(\Pi)$, we have,

$P_A\Gamma u= \Sigma\, m_{\beta,\alpha}\bar{z}_{\gamma(\alpha)}\psi_\beta$, $\Gamma P_A u= \Sigma\,\bar{m}_{\beta,\alpha}\bar{z}_\alpha\psi_{\gamma(\beta)}=$ $\Sigma\,\bar{m}_{\gamma(\beta),\gamma(\alpha)}\bar{z}_{\gamma(\alpha)}\psi_\beta$, and from this and (e.25,a) we deduce,

(e.28) $\bar{m}_{\gamma(\alpha),\gamma(\beta)}=m_{\alpha,\beta}$.

Putting $\beta=0$ and combining with (e.14), we get,

(e.29) $\qquad f=\bar{f}\circ\gamma,$

This is exactly the condition that implies (e.3) in view of (e.16). We have then proved,

Proposition e.1. Under the assumptions (e.4),(e.18), (as well as the other technical assumptions of Theorem 9.4.1 in [HS1],) the study of the spectrum of P_A in a suitable interval containing the ground level of the modified 1-well operator can be reduced by an affine transformation to the study of the spectrum of a strong type 1 self adjoint h'-pseudodifferential operator, P with $\varepsilon(P)\to 0$, when $h\to 0$.

This means that the results of the present paper are applicable, and if $h>0$ is sufficiently small, then the spectrum of P_A near the ground level is a Cantor set of Lebesgue measure 0. For the sake of completeness, we formulate this as a theorem,

Theorem e.2. Let V, A_1, $A_2\in C^\infty(\mathbb{R}^2;\mathbb{R})$ satisfy:

(H.1) $\qquad V(x+ae_j)=V(x),\ V(-x_2,x_1)=V(x_1,x_2),\ V(x_1,-x_2)=V(x_1,x_2)$

where $a>0$ is fixed and e_1, e_2 is the canonical basis in $\mathbb{R}^2$,

(H.2) $\qquad$ The same relations for $B=\partial_{x_1}A_2-\partial_{x_2}A_1$.

(H.3) $\qquad$ V has only one minimum mod($a\mathbb{Z}^2$), namely 0, and this minimum is non-degenerate.

Without loss of generality, we may assume that $V(0)=0$.

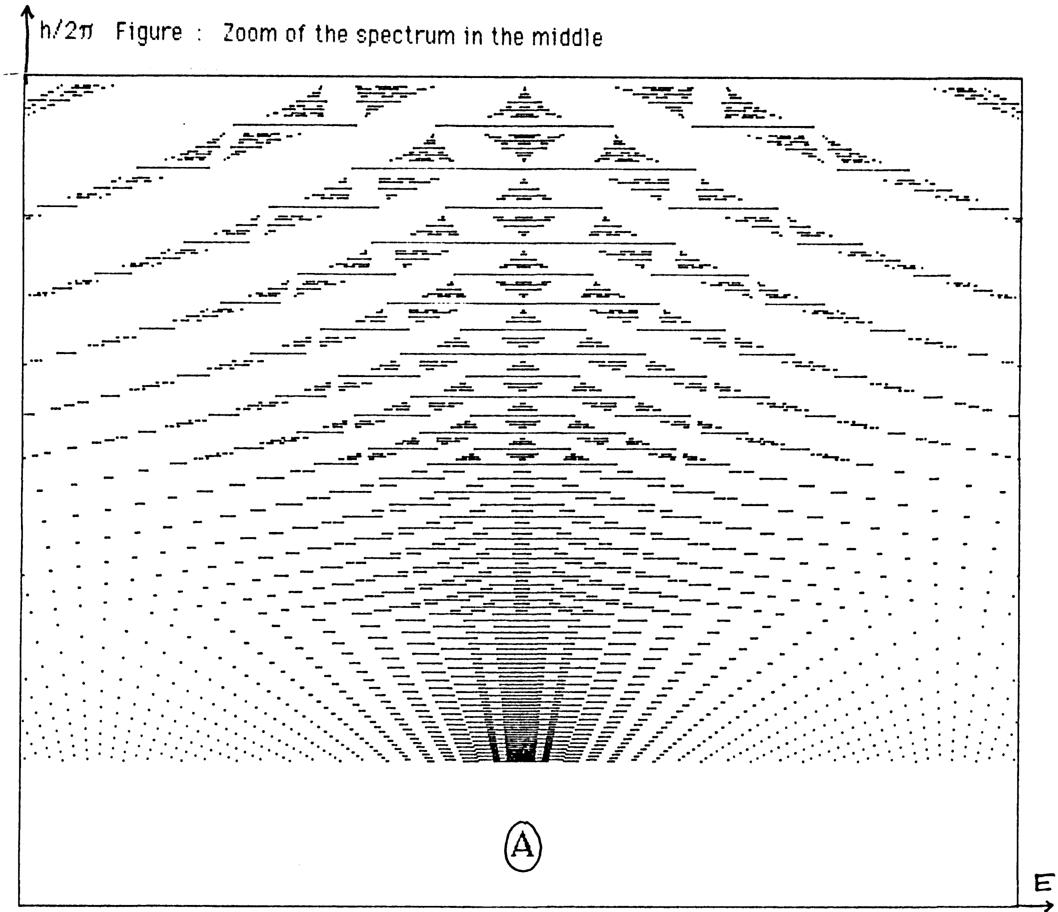
(H.4) $\qquad$ Let d_V be the Agmon distance associated to Vdx^2. Then the points $a\alpha$ in $a\mathbb{Z}^2\setminus\{0\}$, which are closest to 0 with respect to this distance, are precisely the ones with $|\alpha|=|\alpha_1|+|\alpha_2|=1$. Moreover, between 0 and each such point, there is only one minimal geodesic $\gamma=\gamma_\alpha$, which is non-degenerate, and near this geodesic, V and B are analytic.

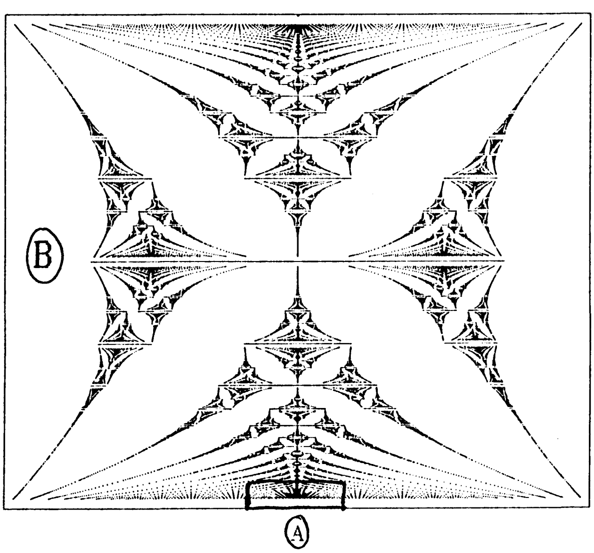
Let Φ denote the flux of the 2-form σ_B through the cell $[0,a]^2$, and let $P_{tA}(h)=(hD_{x_1}-tA_1)^2+(hD_{x_2}-tA_2)^2+V$ with the magnetic field tB. Let $\lambda_t(h)$ be the first eigenvalue of the harmonic oscillator (approximating P_{tA} at 0), $(hD_{x_1}-t\langle A'_1(0),x\rangle)^2+(hD_{x_2}-t\langle A'_2(0),x\rangle)^2+\frac{1}{2}\langle V''(0)x,x\rangle$.

If $\varepsilon_0>0$ is sufficiently small, and $|t|\leq\varepsilon_0$, $|h|\leq h_0$ (with $h_0>0$ sufficiently small), then the the spectrum of $P_{tA}(h)$ in the interval $\{E;\ |E-\lambda_t(h)|\leq h^{3/2}/\varepsilon_0\}$ can after an affine transformation be identified with the spectrum of a strong type 1 self adjoint h'-pseudodifferential operator, P, with $h'\equiv -t\Phi/h \bmod(2\pi\mathbb{Z})$. In particular, the results of this work apply, so if $h'/2\pi$ is irrational and has an expansion as in (0.3),(0.4), with $C_0>0$ sufficiently large, then the piece of the spectrum under consideration is a Cantor set of Lebesgue measure 0.

Remark e.3. The study of symmetries in a closely related setting appears in the work of Wilkinson [W2-4].

h/2π Figure : Zoom of the spectrum in the middle





Spectrum near $h = 0$. For each h with $h/2\pi \in \mathbb{Q}$ ($h/2\pi = p/q$; $q \leqslant 90$, $h/2\pi \in]0, 0.0625[$) , we present in (A) the spectrum of $P_0 = \cos hD_x + \cos x$ in $]-0.5, 0.5[$ on a horizontal line of ordinate $h/2\pi$. This is a part of the celebrated Hofstadter's butterfly (presented in (B)) . The length of the bands in the middle do not decrease exponentially rapidly with h due to a strong tunneling effect .

References.

[AM] R.Abraham,J.Marsden, Foundations of Mechanics, Benjamin/Cumming publ. Co. (1978),

[AuAn] S.Aubry,C.André, Proc. Israel Phys. Soc. Ed.C.G.Kuper 3 (Adam Hilger, Bristol)(1979),133-,

[AvSi] J.Avron,B.Simon, Stability of gaps for periodic potentials under a variation of a magnetic field, J.Phys.A, 18(1985),2199-2205,

[Az] Ya. Azbel, Energy spectrum of a conduction electron in a magnetic field, Soviet Physics JETP, 19, n°3, Sept 1964.

[Be] J.Bellissard, Schrödinger operators with almost periodic potentials, Springer Lecture Notes in Physics, 153,

[BeSi] J.Bellissard,B.Simon, Cantor spectrum for the almost Mathieu equation, J.Funct.An.,48,n°3,Oct. 1982,

[GeS] C.Gérard,J.Sjöstrand, Semi-classical resonances generated by a closed trajectory of hyperbolic type, Comm.Math.Phys.,108(1987),391-421.

[GrS] A.Grigis,J.Sjöstrand, Front d'onde analytique et sommes de carrés de champs de vecteurs, Duke Math.J., 52,n°1(1985),35-51,

[HR1] B.Helffer,D.Robert, Calcul fonctionnel par la transformée de Mellin et applications, J.Funct.An.,53,n°3,oct.1983.

[HR2] B.Helffer,D.Robert, Puits de potentiel généralisés et asymptotique semiclassique, Ann.I.H.P.(section Phys.théorique),41,n°3,(1984), 291-331,

[HS1] B.Helffer,J.Sjöstrand, Analyse semi-classique pour l'équation de Harper, Preprint(1987)

[HS2] B.Helffer,J.Sjöstrand, Analyse semi-classique pour l'équation de Harper II-Comportement semi-classique près d'un rationnel, Preprint.

[HS3] B.Helffer,J.Sjöstrand, Structure Cantorienne du spectre de l'opérateur de Harper, Sém. des Equations aux Dérivées partielles, Ecole polytechnique, March 1988,

[HS4] B.Helffer,J.Sjöstrand, Multiple wells in the semiclassical limit I, Comm.PDE,9(4)(1984),337-408,

[HS5] B.Helffer,J.Sjöstrand, Multiple wells in the semiclassical limit II, Ann.I.H.P.(section Phys.théorique),42,n°2,(1985), 127-212,

[HS6] B.Helffer,J.Sjöstrand, Effet tunnel pour l'équation de Schrödinger avec champ magnétique, Ann.Sc.Norm.Sup.,to appear,

[HS7] B.Helffer,J.Sjöstrand, Résonances en limite semi-classique, Bull. de la SMF,114,fasc.3,(mémoire n°24-25).

[Ho] D.Hofstadter, Energy levels and wave functions for Bloch electrons in rational and irrational magnetic fields, Phys.Rev.B 14(1976), 2239-2249,

[L] J.Leray, Lagrangian analysis and quantum mechanics, MIT Press (1981).

[Mo] P. van Mouche, The coexistence problem for the discrete Mathieu operator, Preprint (1988),

[No] S.P.Novikov, Two dimensional operators in periodic fields, J.Sov.Math.,28,n°1,January 1985,

[O] F.W.J.Olver, Asymptotics and special functions, Academic Press, New York, LOndon, 1974.

[Si] B.Simon, Almost periodic Schrödinger operators: a review, Advances in applied math. 3,(1982),463-490,

[S1] J.Sjöstrand, Singularités analytiques microlocales, Astérisque 95 (1982),

[S2] J.Sjöstrand, Resonances generated by non-degenerate critical points, Springer Lecture Notes in Mathematics,n°1256,402-429,

[So] J.B.Sokoloff, Unusual band structure, wave functions and electrical conductance in crystals with incommensurate periodic potentials, Physics reports (review section of Physics letters), 126,n°4(1985),189-244,

[W1] M.Wilkinson, Critical properties of electron eigenstates in incommensurate systems, Proc.R.Soc.London A391(1984),305-350,

[W2] M.Wilkinson, Von Neumann lattices of Wannier functions for Bloch electrons in a magnetic field, Proc.R.Soc.London A403(1986),135-166,

[W3] M.Wilkinson, An exact effective Hamiltonian for a perturbed Landau level, Journal of Phys.A, 20,n°7, 11-May 1987,1761-,

[W4] M.Wilkinson, An exact renormalisation group for Bloch electrons in a magnetic field, Journal of Physics A, to appear.