B. EH. KUNYAVSKIJ
A. N. SKOROBOGATOV
M. A. TSFASMAN

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DEL PEZZO SURFACES OF DEGREE FOUR

B.È. KUNYAVSKIĬ, A.N. SKOROBOGATOV, M.A. TSFASMAN

Résumé.– Les intersections de deux quadriques dans $\mathbb{P}^4_k$ (c'est–à–dire les surfaces de Del Pezzo de degré 4, soit lisses, soit "singulières") constituent la première classe de surfaces rationnelles dont l'arithmétique est non–triviale. L'arithmétique de telles surfaces X dépend de leurs propriétés algébriques (combinatoires) et géométriques, propriétés que l'on peut lire sur l'action du groupe de Galois $\mathrm{Gal}(\bar{k}/k)$ sur le groupe de Picard $\mathrm{Pic}\,\bar{X}$ (ici $\bar{k}$ est une clôture séparable de k et $\bar{X} = X \times_k \bar{k}$). Pour étudier ces propriétés, nous donnons des formules générales pour certains invariants cohomologiques importants. Ces formules nous permettent d'établir la liste des cas "intéressants", c'est–à–dire des cas où ces invariants sont non–triviaux. Nous étudions les équivalences birationnelles entre divers types de surfaces rationnelles de degré 4, tant en termes géométriques que combinatoires. Puis nous exhibons de nombreux exemples explicites (y compris tous les cas "intéressants") et nous donnons une méthode générale de construction de tels exemples. Nous étudions aussi les propriétés de rationalité du tore de Néron–Severi.

Summary.–Intersections of two quadrics in $\mathbb{P}^4_k$ (i.e. smooth and "singular" Del Pezzo surface of degree four) form the first non–quasi–trivial class of rational surfaces. The arithmetic of such surfaces depends on their algebraic (combinatorial) and geometric properties, reflected in the Galois action on $\mathrm{Pic}\,\bar{X}$. To study these properties we obtain general formulae for some important cohomological invariants. Using these formulae we list all the interesting cases (when the invariants are non–trivial). We investigate birational interrelations between various types of rational surfaces of degree four, both in geometric and combinatorial terms. Then we provide a lot of explicit examples (including all interesting cases), and give a general method to construct such examples. We also establish rationality properties of Néron–Severi tori.

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B.È. KUNYAVSKIĬ, ul. Radichtcheva, 18–A, kv. 48, 410028 SARATOV, USSR.

A.N. SKOROBOGATOV and M.A. TSFASMAN, Institute for Problems of Information Transmission, Academy of Sciences of the USSR, 19 Ermolovoy ul., MOSCOW 101447, USSR.

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CONTENTS

INTRODUCTION.

Various mathematical objects are presented in this paper, such as root systems, integral representations of Weyl groups, Galois cohomology, algebraic tori in semisimple groups, conic bundles, and some elements of the inverse Galois problem. Any of these is interesting enough to deserve a special study, but their role in this paper is due to their direct or indirect relationship to the main hero – the Del Pezzo surface of degree four – to its geometry, algebra, combinatorics and arithmetic.

A Del Pezzo surface of degree four is a smooth intersection of two quadrics in $\mathbb{P}^4_k$ (i.e. a system of two homogeneous quadratic equations in five variables). This surface is rational, i.e. the field of rational functions on it over the algebraic closure of the ground field is a purely transcendental extension. Therefore, its study naturally belongs in the context of the theory of rational varieties. Moreover (cf. for example the review [29]), we suppose Del Pezzo surfaces of degree four to be a proving ground for almost all the modern methods in this theory. We also hope that in the near future the theory of Del Pezzo surfaces of degree four will be able to answer all its main questions, a too daring hope for more complicated surfaces (such as, for example, cubic surfaces in $\mathbb{P}^3_k$).

Some results of this paper were previously announced in [25] and [24]. Let us describe its contents.

In section 0 we introduce some necessary definitions, give some motivations, and formulate main results. The first section is purely algebraic, here we calculate cohomology groups for Weyl groups and their subgroups with coefficients in weight lattices and their sublattices generated by roots. The results obtained in section 1 are applied in section 2 to the study of some algebraic tori and their stable invariants. These tori include maximal tori in classical semisimple groups of adjoint type and Néron–Severi tori of rational surfaces. The third section is mostly geometric : here we study quadric pencils in $\mathbb{P}^4_k$, conic bundles, birational transformations, and so on. The fourth section is more of a combinatorial style : in the first of two possible situations we present the complete list of all possible cases, and for each of them we calculate the main cohomological invariants. In the second situation, which is much more complicated, we do not give a complete list of cases, but in section 5 we actually present all the cases when the most interesting invariant does not vanish. Section 6 is devoted to explicit constructions of many examples of Del Pezzo surfaces of degree four; we give a general method to produce such examples, as well as examples of conic bundles. Section 7 deals with the problem of stable rationality of Néron–Severi tori of Del Pezzo surfaces of degree four.

Our notation is mostly standard. By $^{\circ}$ we denote the duality of free $\mathbb{Z}$–modules, i.e. $N^{\circ} = \mathrm{Hom}(N,\mathbb{Z})$, and by $*$ we denote the duality of finite abelian groups, i.e. $M^{*} = \mathrm{Hom}(M,\mathbb{Q}/\mathbb{Z})$. The ground field k is always perfect, char. $k \neq 2$. If k is finite, we suppose that its cardinality is not too small.

We wish to express our gratitude to J.–L. Colliot–Thélène, D.F. Coray, A.A. Klyachko, Yu.I. Manin and V.E. Voskresenskiĭ for their interest in our work and for many stimulating discussions.

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0
DEFINITIONS, MOTIVATIONS AND RESULTS.

Here are some definitions (for details see [29]).

A surface X over a field k is *rational* if the field of rational functions on $\bar{X} = X \times_k \bar{k}$ is purely transcendental ($\bar{k}$ being the algebraic closure of k). The integer $d = (K_X^2)$ is called the *degree* of X, its *rank* is defined as the rank of Pic X (which is a free $\mathbb{Z}$–module). The free $\mathbb{Z}$–module Pic $\bar{X}$ of *rank* $(10-d)$ is equipped with the natural action of the Galois group $\mathfrak{g} = \mathrm{Gal}(\bar{k}/k)$, preserving the scalar product (the intersection pairing) and the canonical class K_X. This action defines a representation $\rho : \mathfrak{g} \longrightarrow \mathrm{Aut}_0(\mathrm{Pic}\ \bar{X})$, Aut_0 being the group of automorphisms preserving the scalar product and K_X.

The *splitting group* of X is $G = \mathrm{Im}\ \rho$; the *splitting field* K is the field of invariants of the kernel :

$$K = \bar{k}^{\mathrm{Ker}\ \rho}\ ; \quad G = \mathrm{Gal}\ (K/k).$$

The Enriques–Manin–Iskovskih classification shows that there are two types of k–minimal rational surfaces. A *Del Pezzo surface* X is a smooth complete geometrically integral surface with ample anticanonical class $-K_X$; the degree of a Del Pezzo surface always satisfies $1 \leq d \leq 9$. A smooth rational curve on X with self–intersection -1 (an exceptional curve) is called a *line*; if $-K_X$ is very ample, such a curve actually becomes a line after the anticanonical embedding. A rational surface of the second type is a *conic bundle*, i.e. a surface X having a k–morphism $f : X \longrightarrow C$, whose base C and generic fibre are rational curves. For a Del Pezzo surface X of degree $d \leq 6$ there is an isomorphism $\mathrm{Aut}_0(\mathrm{Pic}\ \bar{X}) \simeq W(E_{9-d})$, where $W(E_{9-d})$ is the Weyl group of a root system E_ℓ, $\ell = 9-d$ (we set $E_5 = D_5$, $E_4 = A_4$, $E_3 = A_2 \times A_1$; root systems E_7, $E_6, \ldots, E_3$ are obtained from E_8 by deleting one by one the vertices from the long end of its Dynkin diagram). For a conic bundle $f : X \longrightarrow C$ the class of a fibre is obviously $\mathfrak{g}$–invariant, thus $\mathrm{Im}\ \rho$ belongs in fact to the subgroup $\mathrm{Aut}_1(\mathrm{Pic}\ \bar{X})$ consisting of all automorphisms preserving the scalar product, the canonical class, and the class of a fibre. The conic bundle is called *standard* if every degenerate geometric fibre is a pair of exceptional curves meeting transversally. A standard conic bundle X has $(8-d)$ degenerate geometric fibres (therefore, $-\infty < d \leq 8$) and $\mathrm{Aut}_1(\mathrm{Pic}\ \bar{X}) \simeq W(D_{8-d})$. For a Del Pezzo surface the action of $\mathfrak{g}$ on Pic $\bar{X}$ is uniquely determined by its action on the set of exceptional lines. For a standard conic bundle this action is determined by the action on the set of components of degenerate geometric fibres.

The main goal of this paper is to squeeze out everything possible of what the "algebra" of rational surfaces (in particular, of Del Pezzo surfaces of degree four) can give for the understanding of their arithmetic. By "algebra" we mean such properties of a surface over a non–closed field which depend neither on the ground field k nor on the splitting field K of X, but depend only on the group $G = \mathrm{Gal}\,(K/k)$ and on its action on divisors and other geometric objects. It is well known [28] that this action quite often determines the Brauer group of the surface : $Br\,X/Br\,k \simeq H^1(\mathfrak{g}, \mathrm{Pic}\,\bar{X})$ (it is so, for example, when k is local or global of characteristic zero). Besides, this action determines some very important cohomological birational invariants, namely

$$Ш^i_\omega(G, \mathrm{Pic}\,\bar{X}) = \mathrm{Ker}\,[H^i(G, \mathrm{Pic}\,\bar{X}) \longrightarrow \prod_{g\in G} H^i(<g>, \mathrm{Pic}\,\bar{X})], \quad i = 1,2.$$

In section 1 we find explicit formulae for these invariants. For a global field k the cardinalities of $Ш^i_\omega(G, \mathrm{Pic}\,\bar{X})$ give upper bounds for several purely arithmetical invariants such as the kernels of restriction maps $CH_0(X) \longrightarrow \prod_v CH_0(X_v)$ and $Br\,X/Br\,k \longrightarrow \prod_v Br\,X_v/Br\,k_v$, v ranging over all the places of k. A condition of k–minimality of X, a necessary condition of stable k–rationality of X, necessary conditions for X to be k–birationally equivalent to some (nonisomorphic) k–minimal surfaces, and the like can also be naturally described in terms of the action of $\mathfrak{g}$ on $\mathrm{Pic}\,\bar{X}$. In interesting cases the conjugacy class of the splitting group G of X in the group $\mathrm{Aut}_0(\mathrm{Pic}\,\bar{X}) \simeq W(R)$ is itself "almost" a birational invariant. To be precise, whenever $d \leq 4$, the conjugacy class of G (modulo the action of such automorphisms of $W(R)$ which are induced by automorphisms of the root system) is in fact a birational invariant [40]. All these facts induce a thorough study of subgroups of $W(R)$, especially while the same cohomological invariants play a significant role in the arithmetic of algebraic tori [43].

The study of the arithmetic of rational varieties with the help of the descent theory of Colliot–Thélène and Sansuc leads to "standard conjectures" (see [9], [29]). In particular, k being a number field, the set of k–points on a rational surface X is conjecturally described in the following way :

1) X has a k–point iff

   a) X has a k_v–point for any place v of k, and

   b) the Brauer–Manin obstruction to the Hasse principle is trivial.

Now we suppose $X(k) \neq \emptyset$.

2) $X(k)$ consists of a finite number of R–equivalence classes.

3) Each class is the image of the set of k–points of some k–rational variety Y of greater dimension under a k–morphism $f : Y \longrightarrow X$.

4) $X(k)$ is dense in $\prod_v X(k_v)$ iff the Brauer obstruction to weak approximation vanishes.

5) There is an injection from the set of R–equivalence classes into the group $A_0(X)$ of classes of zero–cycles of degree zero modulo rational equivalence.

6) There is an exact sequence of Colliot–Thélène and Sansuc :

$$0 \longrightarrow Ш^1(S_X) \longrightarrow A_0(X) \longrightarrow \bigoplus_v A_0(X_v) \longrightarrow H^1(k, \mathrm{Pic}\, \bar{X})^*,$$

where S_X is the torus dual to the g–module $\mathrm{Pic}\, \bar{X}$ (Néron–Severi torus), $Ш^1(S_X) = \mathrm{Ker}\, [H^1(G, S_X) \longrightarrow \bigoplus_v H^1(G_v, S_X)]$, $*$ denoting the duality of finite abelian groups.

In some cases these conjectures are trivially satisfied. Namely, rational surfaces of degree more than four form a quasi–trivial class (in the sense of [29]) : if $X(k)$ is not empty, X is k–rational; for a number field k the smooth Hasse principle holds. The same is of course valid for non–k–minimal surfaces of degree 4. The less the degree, the more complicated is the study of rational surfaces. Thus k–minimal surfaces of degree four are the simplest non–quasi–trivial varieties. That is why they are interesting and deserve thorough study.

Nowadays there are only few classes of non–quasi–trivial k–minimal rational surfaces of degree four for which the "standard conjectures" are known to hold (see, however, the note at the end of this section). Let us state it in more detail. There are two types of k–minimal surfaces of degree four [20] : namely, Del Pezzo surfaces of degree 4 of rank 1 and standard conic bundles of degree 4 of rank 2. The latter are anticanonically mapped to $\mathbb{P}^4_k$, either this map is an isomorphism having a smooth Del Pezzo surface of degree 4 of rank 2 for its image, or its image is a singular intersection of two quadrics X' having two conjugate singularities and such that the line joining the singularities does not lie on X' (following [14] such a surface is called an *Iskovskih surface*). Let Y be a smooth model of X'. Due to a remarkable paper [12] we know that "standard conjectures" hold on Y.

THEOREM A. ***Every k-minimal Iskovskih surface is k–birational to a k-minimal Del Pezzo surface of degree 4 of rank 2.***

Thus "standard conjectures" are also proved for some Del Pezzo surfaces. Whenever $X(k)$ is non–empty we are able to find out necessary and sufficient conditions for a k–minimal Del Pezzo surface to be birational to an Iskovskih surface. It is convenient to formulate these conditions in terms of the action of G on the graph of lines Γ and on the graph of pencils of conics Δ, which we are now going to describe.

A Del Pezzo surface $\bar{X}$ of degree 4 has 16 lines. The vertices of the intersection graph Γ of these 16 lines are drawn in Figure 1.

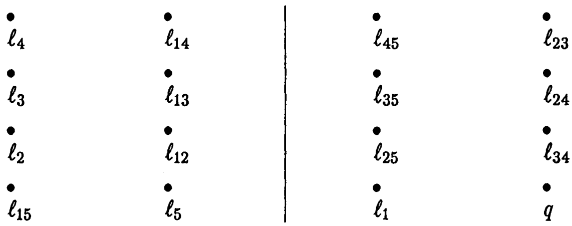

Figure 1 : Graph Γ

Each vertex is joined to the other vertex in the same row on the same side of the vertical line, and to just one vertex of each pair on the other side, the left–hand (right–hand) vertex of each pair being joined to the left–hand (right–hand) vertex in the same row and to the right–hand (left–hand) vertex in other rows. Thus ℓ_{15} is joined to ℓ_5 , ℓ_1 , ℓ_{23} , ℓ_{24} and ℓ_{34}. The notation of vertices comes from the fact that $\bar{X}$ is the blow–up of the plane $\mathbb{P}^2$ in five points $x_1,\dots,x_5$ of which no three are collinear. The 16 lines are : the five blown–up curves which we denote by ℓ_i , the 10 lines of $\mathbb{P}^2$ joining pairs of points (x_i,x_j) denoted by ℓ_{ij} , and the inverse image of the conic q through all five points.

The automorphism group $\mathrm{Aut}\,\Gamma \simeq W(E_5) = W(D_5)$ acts transitively on joined pairs of vertices. In Γ there are 10 subgraphs of type Σ (Figure 2).

Figure 2 : Graph Σ

The complement to a subgraph of type Σ is of the same type. Therefore the set of such subgraphs is itself a graph Δ (Figure 3).

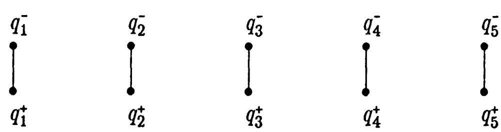

Figure 3 : Graph Δ

Two vertices of Δ are joined iff the corresponding graphs of type Σ have no vertex in common. A vertex of Δ is denoted q_i^+ if the corresponding subgraph Σ_i^+ contains the joined pair (q, ℓ_i) (it determines Σ_i^+ uniquely), q_i^- corresponds to the complementary subgraph Σ_i^-. It is easy to see that $\mathrm{Aut}\,\Delta \simeq W(B_5) = (\mathbb{Z}/2)^5 \rtimes S_5$, S_5 acting on joined pairs of vertices (i.e. on lower indices of $q_i^{\pm}$), and $(\mathbb{Z}/2)^5$ being generated by $c_1,\dots,c_5$, where c_i interchanges q_i^+ with q_i^- leaving the remaining 8 vertices invariant.

Let us remark that substituting e_i for q_i^+ and $-e_i$ for q_i^- we get the action of $W(B_5)$ on the standard realization of the root system B_5 in $\mathbb{R}^5$ ($e_1,\dots,e_5$ being the basis of $\mathbb{R}^5$).

There is an embedding $\mathrm{Aut}\,\Gamma \hookrightarrow \mathrm{Aut}\,\Delta$; an element of $\mathrm{Aut}\,\Delta$ belongs to the image of $\mathrm{Aut}\,\Gamma$ iff it is a product of an even number of c_i's and an element of S_5, i.e. the image coincides with $W(D_5)$ embedded into $W(B_5)$ in the usual way.

These combinatorial designs can be interpreted geometrically. A Del Pezzo surface X of degree four is anticanonically embedded into $\mathbb{P}^4_k$. There is a pencil of quadrics Q_λ parametrized by $\mathbb{P}^1_k$ such that the image of X lies in any Q_λ for $\lambda \in \bar{k}$ and coincides with the intersection of any two quadrics of the pencil. Let Q_0 and Q_∞ be smooth, $Q_\lambda = Q_0 + \lambda\, Q_\infty$. A quadric Q_λ is singular iff λ is a root of the characteristic polynomial $P(\lambda) = \det Q_\lambda$. Since X is smooth, $P(\lambda)$ has only simple roots in $\bar{k}$, corresponding to 5 quadratic cones $Q_i \subset \mathbb{P}^4_{\bar{k}}$, $i = 1,\dots,5$. Let v_i be the vertex and $Q_i^b \subset \mathbb{P}^3_{\bar{k}}$ be the base of Q_i (each Q_i is defined over $k(\lambda_i)$). Over the algebraic closure $\bar{k}$ of the ground field $Q_i^b \simeq \mathbb{P}^1_{\bar{k}} \times \mathbb{P}^1_{\bar{k}}$ has two pencils of lines, therefore Q_i has two pencils of projective planes; intersecting with $\bar{X}$ each pencil produces a pencil of conics on $\bar{X}$. Denote these pencils q_i^+ and q_i^-; the components of the singular fibres of the pencil q_i^+ (respectively q_i^-) form the subgraph Σ_i^+ (respectively Σ_i^-).

Let us define another graph Λ. The vertices of Λ are those of Γ, those of Δ, and one more vertex e. If two vertices both belong either to Γ, or to Δ, and are joined there, then they are joined in Λ. The vertex e is joined with all the vertices of Δ (and is not joined with any vertex of Γ), $q_i^+ \in \Delta$ (respectively q_i^-) is joined exactly with the vertices of $\Sigma_i^- \subset \Gamma$ (respectively Σ_i^+). Now we see that Λ is just the intersection graph of the 27 lines on a cubic surface. In fact, let us fix any line on a cubic surface (denote it by e). Then all the remaining lines are divided into two subgraphs, one formed by lines meeting e, the other by the rest. Contracting e we see that the second subgraph is just Γ (16 lines on a Del Pezzo surface of degree four). Now on a Del Pezzo surface of degree four blow up a point which does not lie on any line. This point lies on exactly one conic of each pencil $q_i^{\pm}$; their inverse images on the cubic are lines. The intersection graph of these 10 lines is just Δ. It can be checked that Aut $\Lambda \simeq W(E_6)$ (cf. [28], IV, 1.9).

Now let us show how different properties of a Del Pezzo surface of degree 4 can be described in terms of G–action on Γ, Δ, or Λ. According to [20] a k–minimal Del Pezzo surface of degree four is either of rank 1, or of rank 2. The latter occurs iff there exist two complementary subgraphs of type Σ such that each G–orbit belongs to one of them. It is clearly equivalent to the existence of two G–invariant vertices q_i^+ and q_i^- in Δ (for some i). Let such an i be equal to 1. In our notation it corresponds to the decomposition of Γ into two subgraphs $\overset{+}{\Sigma}_1$ and $\overset{-}{\Sigma}_1$, each of them consists of vertices lying on one side of the vertical line. G–invariant pencils of conics q_1^+ and q_1^- equip X with two different structures of a conic bundle. In particular, $G \subseteq \mathrm{Aut}_1(\mathrm{Pic}\,\bar{X}) \simeq W(D_4)$. Let Σ_i be a subgraph of Δ obtained by deleting two vertices q_i^+ and q_i^-. In terms of Λ the condition $rk\,X = 2$ reads as follows : there exists i, $1 \le i \le 5$, such that the G–orbit decomposition of Λ is a subdecomposition of the decomposition into $\overset{+}{\Sigma}_i$, $\overset{-}{\Sigma}_i$, Σ_i and three one–vertex graphs q_i^+, q_i^- and e. Note that each vertex of $\overset{+}{\Sigma}_i$ (respectively of $\overset{-}{\Sigma}_i$, of Σ_i) is joined to q_i^- (respectively q_i^+, e) and is not joined to q_i^+ and e (respectively to q_i^- and e, to q_i^+ and q_i^-). According to [30] the maximal subgroup of $W(E_6)$ leaving the set $\{q_i^+, q_i^-, e\}$ invariant is isomorphic to $W(F_4) \simeq W(D_4) \rtimes S_3$, S_3 being the automorphism group of the Dynkin diagram D_4.

If $X(k) \neq \emptyset$ and there exists a "good" k–point x on X (i.e. x does not lie on any line), then the G–graph Λ is realized as the intersection graph of the 27 lines on the blown–up surface. If, moreover, $rk\,X = 2$, contracting G–invariant line q_i^+ or q_i^- we get a surface X'; the splitting group G' of X' is the image of G under some automorphism of $W(D_4)$ induced from an automorphism of the root system D_4.

It is worth noticing that because of its purely combinatorial definition the G–graph Λ can be considered for any Del Pezzo surface X of degree 4 (including those having no good k–point).

THEOREM B. *Let X be a k-minimal Del Pezzo surface of degree 4. If X is k-birationally equivalent to an Iskovskih surface, then $rk\,X = 2$ and the following equivalent conditions hold (q_i^+ and q_i^- being the pencils defined over k) :*

i) *in Λ there is a G-orbit consisting of two joined vertices, other than q_i^+, q_i^-, and e;*

ii) *the group $h(G)$, h being an automorphism of $W(D_4)$ induced from an automorphism of the root system D_4, has an orbit of two joined vertices in Γ;*

iii) *the group $h(G)$, h being as above, has an orbit of two joined vertices, other than q_i^+, and q_i^-, in Δ.*

If $X(k)$ is non-empty, these conditions are also sufficient for X to be k-birationally equivalent to an Iskovskih surface.

COROLLARY. *Let X be a k-minimal Del Pezzo surface of degree four having no k-points. If X is k-birationally equivalent to an Iskovskih surface, then $rk\, X = 2$ and in Δ there is a G-orbit of two joined vertices, other than q_i^+ and q_i^-.*

In fact, the vertices of Λ are those of Γ, Δ, and $\{e\}$ (the three latter subgraphs being G–invariant). Therefore the G–orbit of two joined vertices is either in Γ, or in Δ. In the first case, joined vertices correspond to two intersecting lines on X, and the intersection point is defined over k, which contradicts the condition $X(k) = \emptyset$.

In the case $X(k) = \emptyset$ we do not know any necessary and sufficient condition. Let us give another version of Theorem B.

THEOREM B′. *Let $X = Q_0 \cap Q_\infty$ be a smooth k-minimal Del Pezzo surface of degree 4, such that $X(k) = \emptyset$. If X is k-birationally equivalent to an Iskovskih surface, then the polynomial $P(\lambda) = \det\,(Q_0 + \lambda Q_\infty)$ has at least two roots $\lambda_1, \lambda_2 \in k$, and the discriminant of the quadratic form corresponding to one of these roots, say $Q_{\lambda_1} = Q_0 + \lambda_1 Q_\infty$, restricted to a hyperplane such that this restriction is not degenerate, is a square in k. If, in addition, the second quadric Q_{λ_2} has a smooth k-point, this condition is also sufficient.*

A Del Pezzo surface X of degree 4 is k–minimal iff Γ has no G–orbit consisting of disjoint vertices. Such groups G are called *minimal.* For a minimal G all the possible orbit decompositions of Γ are classified [28]. There are 19 types (see Figure 4 on the next page).

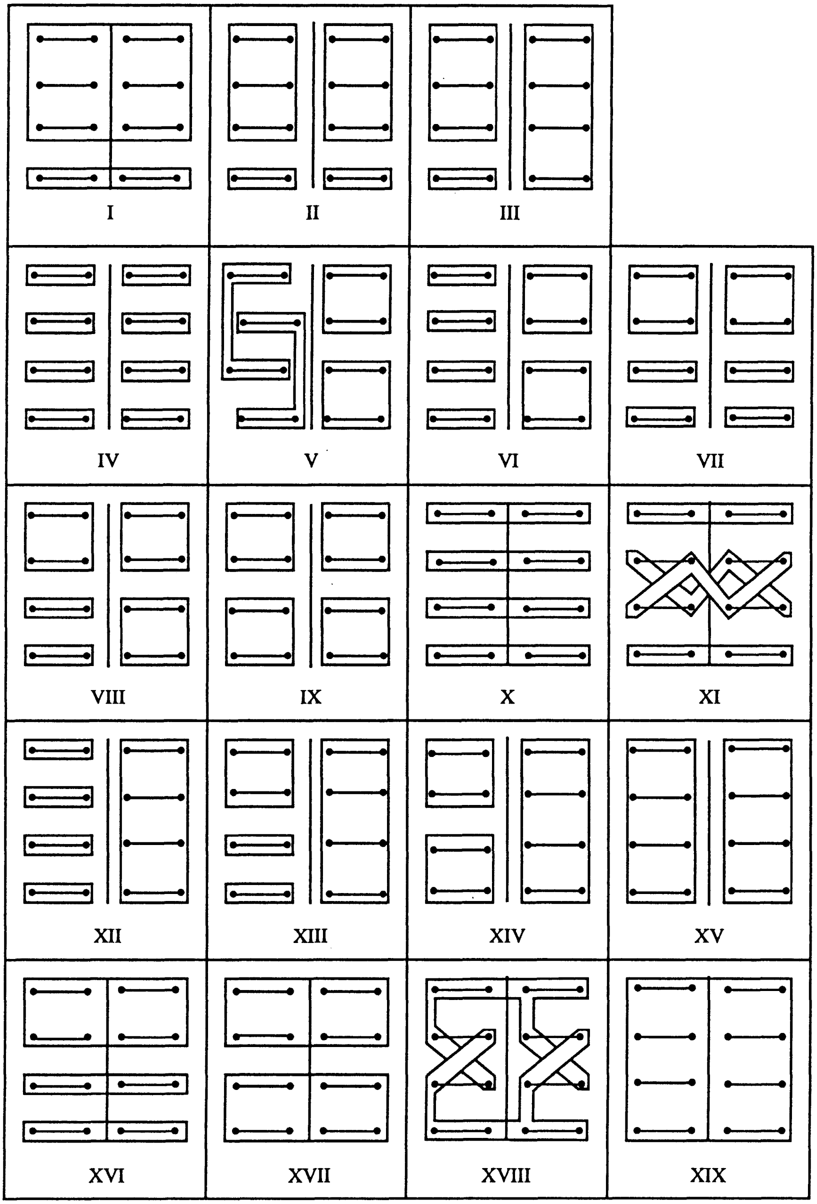
I
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Figure 4

Now we return to the "standard conjectures". The extension class of the obvious exact sequence $1 \to \bar{k}^* \to \bar{k}(X)^* \to \bar{k}(X)^*/\bar{k}^* \to 1$ is an obstruction to the non–emptiness of $X(k)$, since a k–point enables one to define a section $\bar{k}(X)^* \to \bar{k}^*$ ("value at the point"). This is the so–called elementary obstruction [10].

Let k be a number field. If $X(k_v) \neq \emptyset$ for all places v of k, then the elementary obstruction is equivalent to the Brauer–Manin obstruction attached to a subgroup of $Br\,X/Br\,k \simeq H^1(\mathfrak{g}, \text{Pic}\,\bar{X})$, namely to $Б(X) = \text{Ker}\,[Br\,X/Br\,k \to \prod_v Br\,X_v/Br\,k_v]$.

If this obstruction vanishes, there exists a universal torsor on X. Chebotarev's density theorem shows that $Б(X) \subseteq Ш^1_\omega(G, \text{Pic}\,\bar{X})$.

Now let $A_0(X)$ be the group of zero–cycles of degree zero modulo rational equivalence. From the results of Bloch [2], Colliot–Thélène and Sansuc [9], and Colliot–Thélène [4] it follows that $Ш\,A_0(X) = \text{Ker}\,[A_0(X) \to \prod_v A_0(X_v)]$ is a subfactor of the group dual to $Ш^2_\omega(G, \text{Pic}\,\bar{X})$ (see section 2).

Thus we see that the calculation of $Ш^1_\omega(G, \text{Pic}\,\bar{X})$ and $Ш^2_\omega(G, \text{Pic}\,\bar{X})$, carried out further on, gives significant information about $Б(X)$ and $Ш\,A_0(X)$, the latter invariants being of arithmetic nature. In several papers [25], [26] and [29] the group dual to $Ш^2_\omega(G, \text{Pic}\,\bar{X})$ was denoted by $A\,Ш(S_X)$, S_X being the algebraic k–torus dual to the $\mathfrak{g}$–module $\text{Pic}\,\bar{X}$ (the Néron–Severi torus of the surface X). This notation is due to the Voskresenskiĭ exact sequence [43] :

$$0 \to A(S_X) \to A\,Ш(S_X) \to Ш^1(S_X) \to 0,$$

where $Ш^1(S_X) = \text{Ker}\,[H^1(G, S_X) \to \bigoplus_v H^1(G_v, S_X)]$, and $A(S_X)$ is the weak approximation defect. The group $A\,Ш(S_X)$ can be also defined as dual to $H^1(k, \text{Pic}\,\overline{V(S_X)})$, $V(S_X)$ being a smooth compactification of the Néron–Severi torus S_X. This fact can be interpreted in terms of Galois modules. The groups $Ш^1_\omega(G, \text{Pic}\,\bar{X})$ and $Ш^2_\omega(G, \text{Pic}\,\bar{X})$ are invariants of the class of stable equivalence of the Néron–Severi torus.

THEOREM C. *Let* $G \subseteq W(D_4)$. *If* $Ш^1_\omega(G, \mathrm{Pic}\,\bar{X}) \neq 0$ *or* $Ш^2_\omega(G, \mathrm{Pic}\,\bar{X}) \neq 0$, *then* G *is conjugate to one of the following groups*:

N°	Group	Decomposition type	$Ш^1_\omega$	$Ш^2_\omega$
1	$G_{3,6} = \langle c_3c_4(12), c_1c_2(34)\rangle \simeq (\mathbb{Z}/2)^3$	VIII	$\mathbb{Z}/2$	$\mathbb{Z}/2$
	$G'_{3,6} = \langle c_1c_2c_3c_4(12), c_1c_2(34)\rangle \simeq (\mathbb{Z}/2)^3$	VIII	$\mathbb{Z}/2$	$\mathbb{Z}/2$
	$G_{1,5} = \langle c_1c_2, c_3c_4(12)\rangle \simeq (\mathbb{Z}/2)^3$	IX	$\mathbb{Z}/2$	$\mathbb{Z}/2$
2	$G_{7,5} = \langle (12)(34), c_1c_2(13)(24)\rangle \simeq \mathbb{Z}/2 \times \mathbb{Z}/4$	XIV	0	$\mathbb{Z}/2$
	$G'_{7,5} = \langle c_1c_2(12)(34), c_2c_3(13)(24)\rangle \simeq \mathbb{Z}/2 \times \mathbb{Z}/4$	XIV	0	$\mathbb{Z}/2$
	$G_{2,3} = \langle c_1c_2, c_1c_3(12)(34)\rangle \simeq \mathbb{Z}/2 \times \mathbb{Z}/4$	XV	0	$\mathbb{Z}/2$
3	$G_{3,5} = \langle c_1c_2, c_3c_4(12), (34)\rangle \simeq (\mathbb{Z}/2)^3$	IX	$\mathbb{Z}/2$	0
4	$G_{7,3} = \langle c_1c_2, c_1c_3(12)(34), (13)(24)\rangle$	XV	0	$\mathbb{Z}/2$
5	$G_{7,6} = \langle c_1c_3(12)(34), c_2c_3(13)(24)\rangle \simeq Q_8$	XV	0	$(\mathbb{Z}/2)^2$

Here Q_8 is the quaternion group of order 8. The first three groups are united under N° 1, and the second three groups under N° 2, since their conjugacy classes are obtained from one another by automorphisms of $W(D_4)$ induced from automorphisms of the root system D_4. The proof of Theorem C is based on listing all conjugacy classes of minimal subgroups in $W(D_4)$. For the group $W(D_5)$ we fell short providing the complete list. However it is possible to find all the groups G with $Ш^2_\omega(G, \mathrm{Pic}\,\bar{X}) \neq 0$.

THEOREM D. *There are* 8 *conjugacy classes of subgroups* $G \subseteq W(D_5)$, $G \not\subseteq W(D_4)$, *such that* $Ш^2_\omega(G, \mathrm{Pic}\,\bar{X}) \neq 0$. *For all these groups* $Ш^2_\omega(G, \mathrm{Pic}\,\bar{X}) = \mathbb{Z}/2$; *four of them have orbit decomposition of type* XVII, *the other four of type* XIX.

In a remarkable paper [1] stably k–rational but not k–rational Del Pezzo surfaces of degree four were constructed. It is in fact possible to give a precise description of such surfaces of rank 2.

THEOREM E. *Every stably k-rational but not k-rational Del Pezzo surface of degree four of rank 2 is k-birationally equivalent to a cubic (from [1]) given by*

$$y^2 - az^2 = P(x),$$

$P(x)$ being an irreducible polynomial of degree 3, whose discriminant equals a, and a being not a square in k.

For the surfaces of rank 1 the result is weaker :

THEOREM F. *If a Del Pezzo surface of degree four of rank 1 is stably k-rational but not k-rational, then its splitting group is conjugate to one of the three groups I_1, I_2, I_3. All these groups have orbit decomposition of type I.*

Actually, we prove that for a minimal $G \subsetneq W(D_5)$, $G \not\subset W(D_4)$, a G–module $\mathrm{Pic}\,\bar{X}$ is a direct summand of a permutation G–module iff G is conjugate to one of I_1, I_2, I_3. Conjecturally, it follows that $A_0(X)$ and $X(k)/R$ are trivial, and $X \times_k Y$ is k–rational for some k–variety Y.

It is natural to ask when a Néron–Severi torus is k–rational for a surface of a given type (in such a case all invariants are trivial). For the surfaces of rank 2 here is the answer :

THEOREM G. a) *Néron-Severi tori of types* II, IV, V, VI, VII *are always k–rational;*

b) *Néron-Severi tori of types* III, VIII, XII, XIII, XV *are never stably k-rational;*

c) *for types* IX *and* XIV *there exist both k-rational and not stably k-rational tori.*

All the calculations of this paper leave us a bit up in the air without concrete examples of Del Pezzo surfaces of degree four having prescribed splitting groups. To construct such examples we develop the method given in [26] (unfortunately, the exposition there lacks detail).

To construct a surface over k with a prescribed splitting group G we must first construct a Galois extension K/k such that $\mathrm{Gal}\,(K/k) \simeq G$. In fact, this is enough to construct a conic bundle :

THEOREM H. *Let G be a subgroup of $W(D_n)$, and let k be an infinite field,* Char. $k \not| \, 2 \, \# G$. *If there exists a Galois extension K/k such that $Gal\,(K/k) \simeq G$, then there exists a conic bundle X/k of degree $8-n$ with splitting field K (and splitting group G), and $X(k) \neq \emptyset$.*

To construct a k–minimal Del Pezzo surface of degree 4 with a prescribed (up to a conjugation) splitting group $G \subsetneq W(D_5)$ we act in the following way.

First we construct a k–minimal conic bundle Y (with 4 degenerate fibres if G is conjugate to a subgroup of $W(D_4)$ and with 5 degenerate fibres otherwise) such that the Galois action on the graph of components of degenerate fibres coincides with the action of $G \subsetneq W(D_5)$ on the graph Δ.

This can be done by the help of Theorem H, but sometimes it is better to do it in some other way (one should look that Y has a k–point). The equation for Y is usually either of the form

$$y^2 - az^2 = P(x)$$

or of the form

$$y^2 - xz^2 = P(x)$$

(the latter occurs if we use Theorem H).

Now we want to transform Y into a Del Pezzo surface of degree four. It is always possible.

If Y has 5 degenerate fibres, we use a result of Iskovskih (cf. [21], Theorem 5) saying that Y is anticanonically embedded into $\mathbb{P}^3_k$ as a cubic V with a k–line ℓ lying on it, such that the projection from ℓ gives us the conic bundle structure.

Blowing down ℓ we obtain a Del Pezzo surface X of degree 4. As we have explained above, G–action on Γ (the graph of lines on X) corresponds to G–action on Δ (the graph of components of degenerate fibres of Y), and everything is all right.

If Y has 4 degenerate fibres (i.e. $G \subsetneq W(D_4)$), as we have already explained there are two cases : either Y embeds anticanonically into $\mathbb{P}^4_k$ as a smooth Del Pezzo surface, or its anticanonical image is an Iskovskih surface. Unfortunately, even in the first case the surface Y is not the one we are looking for, since the action of G on the lines can differ from the prescribed action of $G \subsetneq W(D_5)$ on Γ. Blowing up a good k–point we get a cubic V with a k–line ℓ' on it. The projection from ℓ' gives us a pencil of conics, the components of its degenerate fibres form a graph Δ'. The group $G \subsetneq W(D_4)$ must act on Δ' leaving two vertices invariant, thus V has two additional k–lines. Projecting from one of them we get the pencil we started with, G acts on the components of its degenerate fibres as on Δ. Blowing down this line we "spoil" the original pencil and get a Del Pezzo surface X of degree four. The G–action on its graph of lines Γ corresponds to the prescribed G–action on Δ. In the case of an Iskovskih surface Theorem A is not enough, while the G–action may be wrong, but its proof in the simple case $Y(k) \neq \emptyset$ is enough. Blow up a good k–point and blow down the fibre passing through it. We get a smooth Del Pezzo surface of degree 4 with the "same" (from the point of view of G–action on the components of degenerate fibres) pencil of conics. Then we act just as in the smooth case.

In this way we obtain

THEOREM I. *Let* G *be a minimal subgroup of* $W(D_5)$, *and let* k *be an infinite field,* Char. $k \not| \, 2 \# G$. *If there exists a Galois extension* K/k *such that* $\mathrm{Gal}\,(K/k) \simeq G$, *then there exists a Del Pezzo surface* X/k *of degree* 4 *with the splitting field* K *(and splitting group* G*).*

The developed method enables us to construct examples of Del Pezzo surfaces of degree 4 over $\mathbb{Q}$, whose splitting groups are the maximal groups for each of the 19 types. The same can be done for all the "interesting" groups (i.e. groups mentioned in Theorems C, D and F). The same technic leads to an example of a conic bundle which is not split by any extension of degree 2^n.

Note. After this paper had been completed several significant results concerning standard conjectures were obtained by different authors. Let us give a brief account here. All the standard conjectures listed at the beginning of this section were proved for Del Pezzo surfaces of degree 4 of rank 2. Conjectures 1, 4, 6 were proved by Salberger in a series of papers [32], [33], [34] and [35]. Conjecture 2 is proved in [11], conjectures 3 and 5 in [13]. In a letter [34] Salberger also states that he can prove Conjecture 6 for arbitrary Del Pezzo surfaces of degree 4. See also a forthcoming paper of Colliot–Thélène, Salberger, and Skorobogatov on weak approximation for intersection of two quadrics, were Conjecture 4 is proved for arbitrary Del Pezzo surfaces of degree 4.

1
COHOMOLOGICAL PROPERTIES OF ROOT SYSTEMS.

Let us briefly recall the basic properties of root systems. Let $V = \mathbb{R}^n$, and let V' be its dual vector space. By $<,>$ we denote the canonical pairing $V \otimes V' \longrightarrow \mathbb{R}$. Let R be a reduced root system, i.e. a finite subset of V, generating V, and satisfying the following properties :

1) for any $\alpha \in R$ there exists $\alpha^\vee \in V'$ such that $< \alpha, \alpha^\vee > = 2$, and the reflection $s_\alpha : x \longmapsto x - < x, \alpha^\vee > \alpha$ preserves R,

2) $\alpha^\vee(R) \subset \mathbb{Z}$ for any $\alpha \in R$,

3) if $\alpha \in R$, then $2\alpha \notin R$.

Elements of R are called roots; the group of automorphisms of V preserving R is denoted by $A(R)$; its normal subgroup $W(R)$ generated by the reflections s_α is called the Weyl group. The set $\{\alpha^\vee | \alpha \in R\}$ forms the dual root system $R^\vee$. Let $S = \{\alpha_i \mid 1 \leq i \leq n\}$ be a basis of R, i.e. a set of linearly independent roots, such that any root is an integral linear combination of some elements of S with coefficients either all positive or all negative. The additive subgroup of V generated by R is denoted by $Q(R)$: elements of the $\mathbb{Z}$–module $P(R) = Q(R^\vee)^\circ$ are called weights, elements ω_i of the basis of $P(R)$ dual to the basis $\{\alpha_i^\vee \mid 1 \leq i \leq n\}$ of $Q(R^\vee)$ are called fundamental weights. Let $\pi(R) = P(R)/Q(R)$; the cardinality of $\pi(R)$ equals the determinant of the Cartan matrix $\det(< \alpha_i, \alpha_j^\vee >_{1 \leq i,j \leq n})$. In what follows we use the notation for roots and weights introduced in [3], ch. VI, § 4.

We define $E_5 = D_5$, $E_4 = A_4$, $E_3 = A_1 \times A_2$, these symbols being vacant; the root systems E_7 ,..., E_3 are obtained from E_8 by consecutive intersecting with hyperplanes given by $< \alpha, \omega_8 > = 0$, $< \alpha, \omega_7 > = 0$, and so on (cf. [3]). This procedure is equivalent to deleting of consecutive vertices in the Dynkin diagram starting from its long end.

PROPOSITION 1.1. *$\pi(R)$ is canonically isomorphic to $H^1(W(R), Q(R))$.*

We give a simple proof of this statement due to A.A. Klyachko.

Proof : Let φ be a crossed homomorphism $W(R) \longrightarrow Q(R)$, it is uniquely determined by its values $\varphi(s_{\alpha_i})$ for $\alpha_i \in S$. Since $s_{\alpha_i}^2 = 1$ we have $0 = \varphi(s_{\alpha_i}^2) = s_{\alpha_i} \varphi(s_{\alpha_i}) + \varphi(s_{\alpha_i})$, i.e. $\varphi(s_{\alpha_i}) = m_i \alpha_i$ for some integer m_i. Conversely, to each n–tuple $\{m_i\}$, $m_i \in \mathbb{Z}$, we associate the crossed homomorphism $\varphi(g) = (1 - g) \sum_{i=1}^{n} m_i \omega_i$, ω_i being fundamental weights. Let

$\omega_\varphi = \sum_{i=1}^{n} m_i \omega_i$. It is clear that this correspondence between crossed homomorphisms and weights is a bijection, and φ is a coboundary iff $\omega_\varphi \in Q(R)$.

Let us recall that $\pi(R)$ is trivial iff R is E_8, F_4 or G_2. The two latter systems will not be considered in this paper.

DEFINITION 1.2. *Let R be a root system such that $\pi(R)$ is cyclic (i.e. $R \neq D_{2k}$). Define the $W(R)$-module $M_\varphi(R)$ as the extension of the trivial module $\mathbb{Z}$ by $Q(R)$ given by a generator φ of* $\mathrm{Ext}^1_{W(R)}(\mathbb{Z}, Q(R)) = H^1(W(R), Q(R)) = \pi(R)$. *In particular, $M(E_8) = Q(E_8) \oplus \mathbb{Z}$.*

LEMMA 1.3. *If $\pi(R)$ is isomorphic to $\mathbb{Z}/2$, $\mathbb{Z}/3$ or $\mathbb{Z}/4$ (i.e. R is one of the root systems $A_1, A_2, A_3, B_n, C_n, D_{2n+1}, E_6, E_7$), then the isomorphism class of $M(R) = M_\varphi(R)$ does not depend on φ.*

Proof : Under the assumptions of the lemma the generator φ is unique up to a sign. Hence it suffices to prove that $\varphi_1 = -\varphi_2$ implies $M_{\varphi_1}(R) = M_{\varphi_2}(R)$. The exact sequence

$$0 \longrightarrow Q(R) \longrightarrow M_\varphi(R) \longrightarrow \mathbb{Z} \longrightarrow 0 \tag{1}$$

is split as a sequence of $\mathbb{Z}$–modules, and there exists an isomorphism of $\mathbb{Z}$–modules $M_\varphi(R) \simeq Q(R) \oplus \mathbb{Z}$. An element $g \in W(R)$ acts on (α,n), $\alpha \in Q(R)$, $n \in \mathbb{Z}$, as follows : $g(\alpha,n) = (g\alpha + n\varphi(g),n)$. An isomorphism $f : Q(R) \oplus \mathbb{Z} \longrightarrow Q(R) \oplus \mathbb{Z}$ such that $f(\alpha,n) = f(\alpha,-n)$ is an isomorphism of $W(R)$–modules :

$$f(g(\alpha,n)) = f(g\alpha + n\,\varphi_1(g),n) = (g\alpha + n\,\varphi_1(g), -n) = (g\alpha - n\,\varphi_2(g), -n) = g(f(\alpha,n)).$$

PROPOSITION 1.4. $H^1(W(R), M_\varphi(R)) = 0$.

Proof : Let $\delta : \mathbb{Z} \longrightarrow H^1(W(R), Q(R))$ be the boundary homomorphism in the long cohomological exact sequence provided by (1). Then $\delta(1) = \varphi$ (up to a sign), hence δ is surjective. It follows that $H^1(W(R), M_\varphi(R)) \hookrightarrow H^1(W(R),\mathbb{Z}) = 0$.

We would also like to have an analog of $M_\varphi(R)$ for $R = D_n$, in particular when n is even, $\pi(D_{2k})$ being isomorphic to $(\mathbb{Z}/2)^2$. Recall that $W(D_n)$ is a subgroup of $W(C_n)$, and the $W(D_n)$–module $Q(D_n)$ is obtained from the $W(C_n)$–module $Q(C_n)$ by the restriction of the group (cf. [3]).

LEMMA 1.5. $H^1(W(D_n), M(C_n)) = \mathbb{Z}/2$.

Proof : Consider (1) for $R = C_n$ and let the exact sequence of $W(D_n)$–modules

$$0 \longrightarrow Q(D_n) \longrightarrow M(C_n) \longrightarrow \mathbb{Z} \longrightarrow 0$$

be obtained from it by restriction to $W(D_n)$. The "restriction–inflation" exact sequence implies that $\mathrm{res} : H^1(W(C_n), Q(C_n)) \longrightarrow H^1(W(D_n), Q(D_n))$ is an injection (there are no $W(R)$–invariant elements in $Q(R)$). Therefore the boundary homomorphism maps $\mathbb{Z}$ onto a subgroup of $H^1(W(D_n), Q(D_n))$ isomorphic to $\mathbb{Z}/2$. Since $|\pi(D_n)| = 4$ we have $H^1(W(D_n), M(C_n)) = \mathbb{Z}/2$.

DEFINITION 1.6. *Define a $W(D_n)$-module $N(D_n)$ as the non-trivial extension of $\mathbb{Z}$ by the $W(D_n)$-module $M(C_n)$.*

PROPOSITION 1.7. $H^1(W(D_n), N(D_n)) = 0$.

Proof : Proceed as in the proof of 1.4.

Convention 1.8. Let $R = A_n$ or E_n. We omit the subscript φ in $M_\varphi(R)$, assuming that φ is always chosen so that its corresponding weight ω_φ coincides with ω_n (in the notation of Bourbaki [3]). In view of Lemma 1.3 we also omit the subscript φ in $M_\varphi(R)$, R being one of the root systems B_n, C_n, D_{2n+1}.

Remark 1.9. Let $R = A_n$, then $\omega_1 + \omega_n \equiv 0 \pmod{Q(R)}$. Hence $M(A_n)$ is isomorphic to the extension of $\mathbb{Z}$ by $Q(A_n)$ such that its cocycle corresponds to the class of ω_1 via the isomorphism of 1.1. This extension can be easily described. Indeed, let $\mathbb{Z}[S_{n+1}/S_n]$ be the $W(A_n)$–module induced from the trivial $W(A_{n-1})$–module $\mathbb{Z}$. The group $W(A_n) \simeq S_{n+1}$ permutes the elements of the canonical basis $e_1,\dots,e_{n+1}$: $g(e_i) = e_{g(i)}$. It is easy to see that the submodule of $\mathbb{Z}[S_{n+1}/S_n]$ consisting of the elements with the zero sum of coordinates is isomorphic to $Q(A_n)$. We claim that the class of the extension $\mathbb{Z}[S_{n+1}/S_n]$ corresponds to ω_1. Indeed, if $n \longmapsto n\, e_1$ is a section of the augmentation map $\mathbb{Z}[S_{n+1}/S_n] \longrightarrow \mathbb{Z}$, then $\varphi(g) = e_{g(1)} - e_1$, hence $\omega_\varphi = \omega_1$ (see the proof of 1.1). In particular, the $W(A_n)$–modules $M(A_n)$ and $\mathbb{Z}[S_{n+1}/S_n]$ are isomorphic.

THEOREM 1.10. *There exist exact bilinear $W(R)$-invariant pairings $M(R) \times M(R^\vee) \longrightarrow \mathbb{Z}$ and $N(D_n) \times N(D_n) \longrightarrow \mathbb{Z}$ extending the natural pairing $Q(R) \times Q(R^\vee) \longrightarrow \mathbb{Z}$ given by the Cartan matrix.*

Proof : First let R be A_n, B_n, C_n, D_{2n+1}, or E_n, then the $\mathbb{Z}$–module $M(R)$ is isomorphic to $Q(R) \oplus \mathbb{Z}$. The explicit action of $W(R)$ is given by $g(\alpha,n) = (g\alpha + n\, \varphi(g),n)$, where $\varphi(g) = \omega_\varphi - g\, \omega_g$. Let $\psi \in \mathrm{Ext}^1_{W(R)}(\mathbb{Z}, Q(R^\vee)) = H^1(W(R), Q(R^\vee))$ be the class of the extension

$$(2) \qquad 0 \longrightarrow Q(R^\vee) \longrightarrow M(R^\vee) \longrightarrow \mathbb{Z} \longrightarrow 0$$

and let $\omega_\psi \in P(R^\vee)$ be the corresponding weight. To make our pairing b $W(R)$–invariant one has to set $b\,((0,1), (\alpha,0)) = -<\omega_\psi,\alpha>$, $b\,((\alpha^\vee,0), (0,1)) = -<\alpha^\vee,\omega_\varphi>$. It remains to define $m = b\,((0,1),(0,1))$ so that the determinant of the pairing equals ± 1. Let us do it in each particular case. Let $R = A_n$, $\omega_\varphi = \omega_\psi = \omega_n$. Set $m = 1$, then

$$(3) \qquad \det \left(\begin{array}{cccccc|c} 2 & -1 & 0 & \dots & 0 & 0 & 0 \\ -1 & 2 & -1 & \dots & 0 & 0 & 0 \\ 0 & -1 & 2 & \dots & 0 & 0 & 0 \\ & & & \dots & & & \\ 0 & 0 & 0 & \dots & 2 & -1 & 0 \\ 0 & 0 & 0 & \dots & -1 & 2 & -1 \\ \hline 0 & 0 & 0 & \dots & 0 & -1 & 1 \end{array}\right) = 1$$

Let $R = C_n$, then by Lemma 1.3 we can choose $\omega_\varphi = \omega_1$ and $\omega_\psi = \omega_n$ (cf. [3], ch. VI, § 4, n° 5,6). Set $m = 1$, then

$$(4)\qquad \det\left(\begin{array}{cccccc|c} 2 & 0 & -1 & \dots & 0 & 0 & -1 \\ 0 & 2 & 0 & \dots & 0 & 0 & 0 \\ -1 & 0 & 2 & \dots & 0 & 0 & 0 \\ & & & \dots & & & \\ 0 & 0 & 0 & \dots & 2 & -2 & 0 \\ 0 & 0 & 0 & \dots & -1 & 2 & 0 \\ \hline 0 & 0 & 0 & \dots & 0 & -1 & 1 \end{array}\right) = 1$$

Let $R = E_n$, then $\omega_\varphi = \omega_\psi = \omega_n$. Again set $m = 1$.

$$(5)\qquad \det\left(\begin{array}{ccccccc|c} 2 & 0 & -1 & 0 & \dots & 0 & 0 & 0 \\ 0 & 2 & 0 & -1 & \dots & 0 & 0 & 0 \\ -1 & 0 & 2 & 0 & \dots & 0 & 0 & 0 \\ 0 & -1 & 0 & 2 & \dots & 0 & 0 & 0 \\ & & & & \dots & & & \\ 0 & 0 & 0 & 0 & \dots & 2 & -1 & 0 \\ 0 & 0 & 0 & 0 & \dots & -1 & 2 & -1 \\ \hline 0 & 0 & 0 & 0 & \dots & 0 & -1 & 1 \end{array}\right) = -1$$

Finally let $R = D_{2n+1}$ and $\omega_\varphi = \omega_\psi = \omega_{2n+1}$. Then

$$(6)\qquad \det\left(\begin{array}{ccccccc|c} 2 & -1 & 0 & \dots & 0 & 0 & 0 & 0 \\ -1 & 2 & -1 & \dots & 0 & 0 & 0 & 0 \\ 0 & -1 & 2 & \dots & 0 & 0 & 0 & 0 \\ & & & \dots & & & & \\ 0 & 0 & 0 & \dots & 2 & -1 & -1 & 0 \\ 0 & 0 & 0 & \dots & -1 & 2 & 0 & 0 \\ 0 & 0 & 0 & \dots & -1 & 0 & 2 & -1 \\ \hline 0 & 0 & 0 & \dots & 0 & 0 & -1 & m \end{array}\right) = 4m - (2n+1) = \pm 1$$

$4m$ being the multiple of 4 closest to $2n + 1$.

Now let $R = D_n$ (for arbitrary n). Define a quadratic form on $N(D_n)$ by the matrix

$$(7)\qquad \left(\begin{array}{cccccc|cc} 2 & -1 & \dots & 0 & 0 & 0 & -1 & 0 \\ -1 & 2 & \dots & 0 & 0 & 0 & 0 & 0 \\ & & & & & & & \\ & & \dots & & & & & \\ 0 & 0 & \dots & 2 & -1 & -1 & 0 & 0 \\ 0 & 0 & \dots & -1 & 2 & 0 & 0 & 0 \\ 0 & 0 & \dots & -1 & 0 & 2 & 0 & -1 \\ \hline -1 & 0 & \dots & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & \dots & 0 & 0 & -1 & 0 & 0 \end{array}\right)$$

using the fact that (for any n) $\pi(D_n)$ is generated by the classes of ω_1 and ω_n ([3], ch. VI, § 4, n° 8). The determinant of (7) equals -1.

Remark 1.11. The signatures of the quadratic forms (5) and (7) equal $(n,1)$ and $(n+1,1)$, respectively.

PROPOSITION 1.12. *The* $W(D_{2n+1})$*-modules* $N(D_{2n+1})$ *and* $M(D_{2n+1}) \oplus \mathbb{Z}$ *are isomorphic.*

Proof : Consider the diagram of $W(D_{2n+1})$–modules :

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & & & \\
 & & \uparrow & & \uparrow & & & & \\
 & & \mathbb{Z} & = & \mathbb{Z} & & & & \\
 & & \uparrow & & \uparrow & & & & \\
0 & \longrightarrow & M(C_{2n+1}) & \longrightarrow & N(D_{2n+1}) & \longrightarrow & \mathbb{Z} & \longrightarrow & 0 \\
 & & \uparrow a & & \uparrow & & & & \\
0 & \longrightarrow & Q(D_{2n+1}) & \longrightarrow & M(D_{2n+1}) & \longrightarrow & \mathbb{Z} & \longrightarrow & 0 \\
 & & \uparrow & & \uparrow & & & & \\
 & & 0 & & 0 & & & &
\end{array}$$

Here the left column is obtained from (1) by the restriction to the group $W(D_{2n+1})$, the middle row is the definition of $N(D_{2n+1})$, and the bottom row is just (1). Let $\varphi \in H^1(W(D_{2n+1}), Q(D_{2n+1})) \simeq \mathbb{Z}/4$ and $\psi \in H^1(W(D_{2n+1}), M(C_{2n+1})) \simeq \mathbb{Z}/2$ be the classes of the row extensions. Since $H^1(W(D_{2n+1}),\mathbb{Z}) = 0$ we have $\psi = a_*(\varphi)$. There exist splittings of $\mathbb{Z}$–modules : $M(D_{2n+1}) \simeq Q(D_{2n+1}) \oplus \mathbb{Z}$ and $N(D_{2n+1}) \simeq M(C_{2n+1}) \oplus \mathbb{Z}$, where $g(\alpha,n) = (g\alpha + n\varphi(g),n)$ and $g(\alpha',n) = (g\alpha' + n\psi(g),n)$, $g \in W(D_{2n+1})$, $\alpha \in Q(D_{2n+1})$, $\alpha' \in M(C_{2n+1})$, $n \in \mathbb{Z}$. It follows that $f : (\alpha,n) \longmapsto (a(\alpha),n)$ is a homomorphism of $W(D_{2n+1})$–module; f is injective, and $\mathrm{coker}(f) \simeq \mathbb{Z}$. The corresponding short exact sequence of $W(D_{2n+1})$–modules splits since

$$\mathrm{Ext}^1_{W(D_{2n+1})}(\mathbb{Z},M(D_{2n+1})) = H^1(W(D_{2n+1}), M(D_{2n+1})) = 0$$

by 1.4. The proposition is proved.

Let $Q(D_{2n})$ be the subgroup of $Q(D_{2n+1})$ generated by the roots $\alpha_2,\dots,\alpha_{2n+1}$, i.e. by all elements of the basis excepting the root α_1, which corresponds to the extreme vertex of the long end of the Dynkin diagram ([3], ch. VI, tables). In other terms, $Q(D_{2n}) = \mathrm{Ker}\ \epsilon_1$, $\epsilon_1 : Q(D_{2n+1}) \longrightarrow \mathbb{Z}$ being defined by $\alpha \longmapsto \epsilon_1(\alpha) = < \alpha, \omega_1 >$. The corresponding embedding $W(D_{2n}) \hookrightarrow W(D_{2n+1})$ is called *standard.* In what follows by $W(D_{2n}) \hookrightarrow W(D_{2n+1})$ we mean the standard embedding.

LEMMA 1.13. *The $W(D_{2n})$–modules $M(C_{2n})$ and $Q(D_{2n+1})$, obtained by the restriction of the groups to $W(D_{2n})$, are isomorphic.*

Proof : Since both modules are extensions of $\mathbb{Z}$ by $Q(D_{2n})$, it is enough to check that their cocycles coincide. The exact sequence of $W(D_{2n})$–modules

$$0 \longrightarrow Q(C_{2n}) \longrightarrow M(C_{2n}) \longrightarrow \mathbb{Z} \longrightarrow 0$$

corresponds to the unique nontrivial cocycle $\varphi \in H^1(W(C_{2n}),Q(C_{2n})) \simeq \mathbb{Z}/2$. Its restriction $\mathrm{res}\ \varphi \in H^1(W(D_{2n}), Q(D_{2n})) \simeq (\mathbb{Z}/2)^2$ corresponds to the weight ω_1 (it follows from the fact that $\pi(C_{2n})$ is generated by the fundamental weight ω_1 of C_{2n}, which coincides with the fundamental weight ω_1 of D_{2n}, cf. [3], ch. VI, tables). On the other hand let $n \longmapsto n\ \alpha_1$ be a section of the homomorphism $\epsilon_1 : Q(D_{2n+1}) \longrightarrow \mathbb{Z}$ (α_1 being a root of D_{2n+1}). The class of the extension

$$0 \longrightarrow Q(D_{2n}) \longrightarrow Q(D_{2n+1}) \xrightarrow{\epsilon_1} \mathbb{Z} \longrightarrow 0$$

is thus given by $\psi(g) = \alpha_1 - g\ \alpha_1 = -(\omega_1 - g\omega_1)$, where ω_1 is the first fundamental weight of D_{2n}.

PROPOSITION 1.14. *The $W(D_{2n})$–module $N(D_{2n})$ is isomorphic to the $W(D_{2n})$-module $M(D_{2n+1})$ obtained by the restriction to the group $W(D_{2n})$.*

Proof : Both modules are extensions of $\mathbb{Z}$ by $M(C_{2n}) \simeq Q(D_{2n+1})$ (Lemma 1.13). In view of 1.5 and 1.6 it is enough to check that the extension of $W(D_{2n})$–modules

$$(8) \qquad 0 \longrightarrow Q(D_{2n+1}) \longrightarrow M(D_{2n+1}) \longrightarrow \mathbb{Z} \longrightarrow 0$$

is non–trivial. By functoriality the class of the exact sequence (8) is the restriction of the extension class φ of (1). By definition φ generates $H^1(W(D_{2n+1}), Q(D_{2n+1}) = P(D_{2n+1})/Q(D_{2n+1})$ (see 1.1). Using the 1–1–correspondence between crossed homomorphisms and weights we can assume $\varphi(g) = (1-g)\omega_{2n+1}$. If the restriction of φ to $W(D_{2n})$ is a coboundary, then $\varphi(g) = (1-g)r$ for $g \in W(D_{2n})$, $r \in Q(D_{2n+1})$. Then $\omega_{2n+1} - r \in P(D_{2n+1})^{W(D_{2n})} = <\omega_1>$. Now notice that the class of $\omega_{2n+1} + m\,\omega_1$ in $P(D_{2n+1})/Q(D_{2n+1}) \simeq \mathbb{Z}/4$ is of the form $1 + 2m \pmod 4$ ([3], Ch. VI, tables), thus, it is not zero.

PROPOSITION 1.15. *Let* $R = R^{\vee}$. *The group of automorphisms of* $M(R)$ *(and of* $N(D_n)$*) preserving the unimodular quadratic* $W(R)$*-invariant form obtained in Theorem* 1.10, *and acting trivially on the orthogonal complement to* $Q(R)$, *is isomorphic to the Weyl group* $W(R)$.

Proof : The group of orthogonal automorphisms of $Q(R)$ is $A(R)$. If $R = A_1$, E_7 or E_8, then $A(R) = W(R)$. Let $R = A_n$, E_6 or D_{2n+1}, then $A(R)/W(R) \simeq \mathbb{Z}/2$. Suppose for a moment that (1) is exact as a sequence of $A(R)$–modules. Then the class of the extension (1) (considered as an exact sequence of $W(R)$–modules) belongs to $H^1(W(R), Q(R))^{A(R)}$. By Proposition 1.1 this group is isomorphic to $(P(R)/Q(R))^{A(R)/W(R)}$ (since $W(R)$ acts trivially on $P(R)/Q(R)$ ([3], Ch. VI, § 1, proposition 27)). According to the tables in [3], Ch. VI, $A(R)/W(R) \simeq \mathbb{Z}/2$ acts on the cyclic group $P(R)/Q(R)$ via multiplication by -1. Hence, if the order of $\pi(R) = P(R)/Q(R)$ is greater than 2, then $(P(R)/Q(R))^{A(R)/W(R)}$ is a proper subgroup of $(P(R)/Q(R))$, and does not contain its generator. This contradicts Definition 1.2. If the order of $\pi(R)$ equals 2, then $R = A_1$ or E_7, but in these cases $A(R) = W(R)$.

Now let $R = D_n$, then $A(D_n)/W(D_n) \simeq \mathbb{Z}/2$ (if $n \neq 4$), and $A(D_4)/W(D_4) \simeq S_3$ ([3], Ch. VI, § 4). Since $N(D_n)^{W(D_n)} \simeq \mathbb{Z} \oplus \mathbb{Z}$ we have an exact sequence of $W(D_n)$–modules :

$$(10) \qquad 0 \longrightarrow Q(D_n) \longrightarrow N(D_n) \longrightarrow \mathbb{Z} \oplus \mathbb{Z} \longrightarrow 0$$

(the map $N(D_n) \longrightarrow \mathbb{Z} \oplus \mathbb{Z}$ is dual to the injection $\mathbb{Z} \oplus \mathbb{Z} \hookrightarrow N(D_n)$, $N(D_n)$ being autodual by Theorem 1.10). From $H^1(W(D_n), N(D_n)) = 0$ it follows that the boundary map δ is surjective, hence the class of the extension (10)

$$\psi \in \operatorname{Ext}^1_{W(D_n)}(\mathbb{Z} \oplus \mathbb{Z}, Q(D_n)) = H^1(W(D_n), Q(D_n)) \oplus H^1(W(D_n), Q(D_n))$$

is of the form $\psi = (\varphi_1, \varphi_2)$, where φ_1 and φ_2 generate $H^1(W(D_n), Q(D_n))$. If (10) were an exact sequence of $A(D_n)$–modules, then ψ would be a restriction of some class from $H^1(A(D_n), Q(D_n)) \oplus H^1(A(D_n), Q(D_n))$, i.e.

$$\varphi_1, \varphi_2 \in H^1(W(D_n), Q(D_n))^{A(D_n)} = (P(D_n)/Q(D_n))^{A(D_n)/W(D_n)}.$$

If n is even then $P(D_n)/Q(D_n) \simeq (\mathbb{Z}/2)^2$. If $n = 4$, then $A(D_4)/W(D_4) \simeq S_3$ acts on $P(D_4)/Q(D_4)$ permuting nonzero elements, and $(P(D_4)/Q(D_4))^{A(D_4)/W(D_4)} = 0$. If $n \neq 4$, then $A(D_n)/W(D_n) \simeq \mathbb{Z}/2$ permutes ω_n and ω_1, i.e. $(P(D_n)/Q(D_n))^{A(D_n)/W(D_n)} \simeq \mathbb{Z}/2$. If n is odd, then $(P(D_n)/Q(D_n)) \simeq \mathbb{Z}/4$, and $(P(D_n)/Q(D_n))^{A(D_n)/W(D_n)} \simeq \mathbb{Z}/2$. It follows that φ_1 and φ_2 do not generate $P(D_n)/Q(D_n)$. It is a contradiction, and the proposition is proved.

In what follows the group G is always assumed to be finite.

DEFINITION 1.16. *Let N be a G-module of finite type, then for $i = 1,2$ by $Ш^i_\omega(G,N)$ we denote the kernel of the restriction to all cyclic subgroups:*

$$Ш^i_\omega(G,N) = \mathrm{Ker}[H^i(G,N) \longrightarrow \prod_{g \in G} H^i(<g>,N)].$$

Recall that a *permutation* G–module is a direct sum of G–modules $\mathbb{Z}[G/H]$, H being a subgroup of G; the G–modules N_1 and N_2 are called *similar* if $N_1 \oplus M_1$ and $N_2 \oplus M_2$ are isomorphic for some permutation modules M_1 and M_2 [43].

LEMMA 1.17. *If M is a permutation G-module, then $Ш^i_\omega(G,M) = 0$, $i = 1,2$.*

Proof : The assumption implies $H^1(G,M) = 0$ (Shapiro's lemma), thus it is enough to prove the second assertion in case $M = \mathbb{Z}[G/H]$. Note that the restriction of the G–module $\mathbb{Z}[G/H]$ to H (respectively to $<h>$) always contains the trivial module $\mathbb{Z}$ as a direct summand (it corresponds to the trivial coset H). Hence the kernel of the restriction map $H^2(G, \mathbb{Z}[G/H]) \longrightarrow \prod_{h \in H} H^2(<h>, \mathbb{Z}[G/H])$ is embedded into the kernel of the restriction map $H^2(H, \mathbb{Z}) \longrightarrow \prod_{h \in H} H^2(<h>, \mathbb{Z})$, the latter being trivial because of the canonical isomorphism $H^2(G, \mathbb{Z}) = \mathrm{Hom}(G, \mathbb{Q}/\mathbb{Z})$.

COROLLARY 1.18. *The groups* $H^1(G,N)$, $Ш^1_\omega(G,N)$ *and* $Ш^2_\omega(G,N)$, *as well as the groups* $H^1(G,N^\circ)$, $Ш^1_\omega(G,N^\circ)$ *and* $Ш^2_\omega(G,N^\circ)$ *are invariants of the similarity class of the G-module N.*

This digression being over we come back to the modules related to root systems.

PROPOSITION 1.19. *Let* $G \subseteq W(R)$, *then*

$$Ш^1_\omega(G,M(R)) = Ш^1_\omega(G,P(R)), \qquad Ш^2_\omega(G,M(R)) = Ш^2_\omega(G,Q(R)).$$

Likewise, if $G \subseteq W(D_n)$, *then*

$$Ш^1_\omega(G,N(D_n)) = Ш^1_\omega(G,P(D_n)), \qquad Ш^2_\omega(G,N(D_n)) = Ш^2_\omega(G,Q(D_n)).$$

Proof : Dualizing the exact sequence (2) we obtain

$$(11) \qquad 0 \longrightarrow \mathbb{Z} \longrightarrow M(R) \longrightarrow P(R) \longrightarrow 0.$$

By the last lemma $Ш^2_\omega(G,\mathbb{Z}) = 0$, and we get the following diagram :

$$\begin{array}{ccccccc}
 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \\
 & & Ш^1_\omega(G,M(R)) & \longrightarrow & Ш^1_\omega(G,P(R)) & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow \\
 & 0 \longrightarrow & H^1(G,M(R)) & \longrightarrow & H^1(G,P(R)) & \longrightarrow & H^2(G,\mathbb{Z}) \\
 & & \downarrow & & \downarrow & & \downarrow \\
0 \longrightarrow & & \prod_{g\in G} H^1(\langle g\rangle,M(R)) & \longrightarrow & \prod_{g\in G} H^1(\langle g\rangle,M(R)) & \longrightarrow & \prod_{g\in G} H^2(\langle g\rangle,\mathbb{Z}).
\end{array}$$

The first statement follows immediately. In the case of $M(R)$ the second statement is proved in the same way. In the case of $N(D_n)$ the proof is the same with the exact sequence (11) replaced by (10).

COROLLARY 1.20. *Let* $G \subseteq W(A_n) \simeq S_{n+1}$, *then* $Ш^2_\omega(G,Q(A_n)) = Ш^1_\omega(G,P(A_n)) = 0$.

Proof : According to Remark 1.9 $M(A_n)$ is a permutation module.

The remaining part of this section is devoted to the computation of $Ш^1_\omega(G,P(R))$ and $Ш^2_\omega(G,Q(R))$ for $R = B_n$, C_n or D_n.

The pair $(Q(B_n), W(B_n))$ can be identified with a free $\mathbb{Z}$–module of rank n and the automorphism group of a quadratic form, which can be written as $x_1^2 + ... + x_n^2$ in some basis $e_1,...,e_n$. We call such a basis *standard*.

The group $W(B_n)$ is generated by S_n (permuting the coordinates) and reflections $c_i : x \longmapsto x - 2(x,e_i)e_i$. Therefore $W(B_n)$ is a semidirect product $W(B_n) = (\mathbb{Z}/2)^n \rtimes S_n$, where S_n acts on c_i by permuting the subscripts. There exists a character $\sigma : W(B_n) \longrightarrow \{\pm 1\}$ having $W(D_n)$ for its kernel, $\sigma(c_{i_1} c_{i_2} ... c_{i_k} \cdot \alpha) = (-1)^k$, where $\alpha \in S_n$. Let us denote the natural epimorphism $W(B_n) \longrightarrow S_n$ by pr, and set $I = \{1,2,...,n\}$.

Let $G \subseteq W(B_n)$, then I is a union of orbits of $pr(G)$, we denote them by $I_{G,k}$, $|I_{G,k}| = n_k$, $k = 1,...,t$. The G–module $Q(B_n)$ is a direct sum of $Q(B_{n_k})$, $Q(B_{n_k})$ being generated by e_i for $i \in I_{G,k}$. The representation of G in $Q(B_{n_k})$ yields a homomorphism $\theta_{G,k} : G \longrightarrow W(B_{n_k})$. Finally we define $\chi_{G,k} = \sigma \circ \theta_{G,k} \in \mathrm{Hom}(G,\mathbb{Z}/2)$.

LEMMA 1.21. *Let* $<g>$ *be the cyclic subgroup of* $W(B_n)$ *generated by* g. *All the characters* $\chi_{<g>,k}$ *are trivial iff* g *is conjugate in* $W(B_n)$ *to an element of* S_n.

Proof : Let $I = \bigcup_{k=1}^{s} I_{<g>,k}$ be the partition of I into $pr(g)$–orbits, $pr(g) = \rho_1 ... \rho_s$ being the decomposition of $pr(g)$ into independent cycles. Then g is of the form $g = \prod_{k=1}^{s} C_k \rho_k$, where $C_k = c_{i_1} ... c_{i_m}$ for some elements $i_1,...,i_m \in I_{<g>,k}$. Therefore $\theta_{<g>,k} = C_k \rho_k$, and $\chi_{<g>,k} = (-1)^m$. Let us check that if m is even then $c_{i_1} ... c_{i_m} (12...n_k)$ is conjugate to $(12...n_k)$ in $W(B_{n_k})$. Consider the action of $c_{i_1} ... c_{i_m} (12...n_k)$ on the set $\{e_1,...,e_{n_k}, -e_1,...,-e_{n_k}\}$. There is one orbit if m is odd, and two orbits if m is even. In the latter case one can choose any orbit for a new basis of $Q(B_{n_k})$. If h transforms $\{e_1,...,e_{n_k}\}$ into the new basis, then $h^{-1}(12...n_k)h = c_{i_1} ... c_{i_m} (12...n_k)$.

Before we go over to the main results of this section, let us comment on some properties of the $W(B_n)$–module $Q(B_n)$. The above description of this module implies that it is induced from the one–dimensional G–module $<e_1>$, $G = < c_1,\dots,c_n > \rtimes S_{n-1}$. In particular, by Shapiro's lemma, $H^1(W(B_n),Q(B_n)) = H^1(G,<e_1>) = H^1(<c_1>,<e_1>) = \mathbb{Z}/2$. Note also, that the $W(B_n)$–module $Q(B_n)$ is autodual since the quadratic form $x_1^2+\dots x_n^2$ is unimodular. Also $\text{Ш}^1_\omega(G,Q(B_n)) = \text{Ш}^2_\omega(G,Q(B_n)) = 0$ (cf. Remark 2.7).

THEOREM 1.22. *Let $G \subseteq W(B_n) = W(C_n)$, and let G_0 be the minimal normal subgroup of G containing every $g \in G$ conjugate in $W(B_n)$ to an element of S_n. Then*

$$\text{Ш}^2_\omega(G,Q(C_n)) = \text{Hom}\,(G/G_0, \mathbb{Z}/2)/< \chi_{G,1},\dots, \chi_{G,t} >.$$

Note, that the case of $G \subseteq W(D_n)$ is included in the theorem, since the $W(D_n)$–modules $Q(C_n)$ and $Q(D_n)$ are isomorphic.

Proof : Consider the exact sequence

$$0 \longrightarrow \mathbb{Z} \longrightarrow M(C_n) \longrightarrow Q(B_n) \longrightarrow 0$$

dual to (1) for $R = B_n$ (here we use the autoduality of $Q(B_n)$). Using the canonical isomorphism $H^2(G,\mathbb{Z}) = \text{Hom}\,(G,\mathbb{Q}/\mathbb{Z})$ we get the commutative diagram whose columns, middle row and bottom row are exact :

$$\begin{array}{ccccccc}
 & & 0 & \longrightarrow & \text{Ш}^2_\omega(G,M(C_n)) & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow \\
H^1(G,Q(B_n)) & \xrightarrow{\delta} & \text{Hom}(G,\mathbb{Q}/\mathbb{Z}) & \longrightarrow & H^2(G,M(C_n)) & \longrightarrow & H^2(G,Q(B_n)) \\
\gamma\downarrow & & \alpha\downarrow & & \downarrow & & \downarrow \\
\prod_{g\in G} H^1(<g>,Q(B_n)) & \xrightarrow{\beta} & \prod_{g\in G} \text{Hom}(<g>,\mathbb{Q}/\mathbb{Z}) & \longrightarrow & \prod_{g\in G} H^2(<g>,M(C_n)) & \longrightarrow & \prod_{g\in G} H^2(<g>,Q(B_n))
\end{array}$$

We put zero in the right hand column since $\text{Ш}^2_\omega(G,Q(B_n)) = 0$. Looking at the diagram we see that

$$\text{Ш}^2_\omega(G,M(C_n)) \simeq \mathrm{Im}\ \alpha \cap \mathrm{Im}\ \beta/\mathrm{Im}\ (\beta\circ\gamma) = \mathrm{Im}\ \alpha\cap \mathrm{Im}\ \beta/\mathrm{Im}\ (\alpha\circ\delta) \subseteq \prod_{g\in G} \mathrm{Hom}\ (<g>, \mathbb{Q}/\mathbb{Z}).$$

It is not difficult to compute $H^1(G,Q(B_n))$ using the decomposition of the G–module $Q(B_n)$ into the direct sum of G–modules $Q(B_{n_k})$ for all $pr(G)$–orbits $I_{G,k}$. Every G–module $Q(B_{n_k})$ is induced from any one–dimensional module $< e_{i_k} >$, $i_k \in I_{G,k}$. The largest subgroup preserving $< e_{i_k} >$ acts on $< e_{i_k} >$ either trivially or not. In the former case we have

$$H^1(G,Q(B_{n_k})) = H^1(\theta_{G,k}(G),\ Q(B_{n_k})) = 0,$$

whereas in the latter

$$H^1(G,Q(B_{n_k})) = H^1(\theta_{G,k}(G),\ Q(B_{n_k})) \simeq \mathbb{Z}/2.$$

If the latter case takes place for $k = 1,...,s$, and the former one for $k = s+1,...,t$, then surely $H^1(G,Q(B_n)) \simeq (\mathbb{Z}/2)^s$.

Now let us compute the map δ. Let $a_{G,k}$ be the generator of $H^1(G,Q(B_{n_k}))$, $k = 1,...,s$.

LEMMA 1.23. $\delta(a_{G,k}) = \chi_{G,k}$.

Proof : The restriction map $H^1(W(B_{n_k}),\ Q(B_{n_k})) \longrightarrow H^1(\theta_{G,k}(G),\ Q(B_{n_k}))$ is an isomorphism, hence by functoriality it is enough to prove the assertion for $\theta_{G,k}(G) = W(B_{n_k})$. In the rest of the proof we fix k, and omit the subscript k when it is not misleading. Let

$$H^1(W(B_n),\ M(C_n)) \longrightarrow H^1(W(B_n),\ Q(B_n)) \xrightarrow{\delta_W} \mathrm{Hom}\ (W(B_n),\ \mathbb{Q}/\mathbb{Z})$$

be a piece of the long exact sequence. According to Proposition 1.5, $H^1(W(B_n),\ M(C_n)) = 0$, hence δ_W is injective. Let a_W be the generator of $H^1(W(B_n), Q(B_n)) \simeq \mathbb{Z}/2$. We claim that the character $\delta_W(a_W)$ of $W(B_n)$ is trivial on S_n. Indeed, the following square commutes :

$$\begin{array}{ccc} H^1(W(B_n), Q(B_n)) & \xrightarrow{\delta_W} & \mathrm{Hom}(W(B_n), \mathbb{Q}/\mathbb{Z}) \\ \mathrm{res}\downarrow & & \mathrm{res}\downarrow \\ H^1(S_n, Q(B_n)) & \longrightarrow & \mathrm{Hom}(S_n, \mathbb{Q}/\mathbb{Z}) \end{array}$$

Since the S_n–module $Q(B_n)$ is induced from the trivial S_{n-1}–module $\mathbb{Z}$ we have $H^1(S_n, Q(B_n)) = 0$, thus $\delta_W(a_W)$ is trivial on S_n. Recall that $W(B_n) = (\mathbb{Z}/2)^n \rtimes S_n$. The triviality of $\delta_W(a_W)$ on S_n implies that $\delta_W(a_W)(c_i)$ does not depend on i. Since δ_W is injective, $\delta_W(a_W)$ is a nonzero character. From $c_i^2 = 1$ we conclude that $\delta_W(a_W) \in \mathrm{Hom}(W(B_n), \mathbb{Z}/2)$ and $\delta_W(a_W)(c_i) = -1$, i.e. $\delta_W(a_W) = \sigma$. Therefore $\delta(a_{G,k}) = \chi_{G,k}$.

Completion of the proof of the theorem : Let $\chi \in \mathrm{Hom}(G,\mathbb{Z}/2)$ and $\alpha(\chi) \in \mathrm{Im}\,\beta = \prod_{g\in G} \mathrm{Im}\,\beta_g$ (cf. (12)). Lemma 1.23 shows that $\mathrm{Im}\,\beta_g = 0$ iff all $\chi_{<g>,k}$ are trivial. Hence $\alpha(\chi)(g) = 0$ for all g conjugate to elements of S_n. If g does not satisfy this property then $\mathrm{Im}\,\beta_g \simeq \mathbb{Z}/2$, and for any $\chi \in \mathrm{Hom}(G, \mathbb{Z}/2)$ we have $\alpha(\chi)_g \in \mathrm{Im}\,\beta_g$. Thus we have proved that $\chi \in \mathrm{Hom}\,(G, \mathbb{Q}/\mathbb{Z})$ is subject to the condition $\alpha(\chi) \in \mathrm{Im}\,\beta$ if $\chi \in \mathrm{Hom}\,(G, \mathbb{Z}/2)$ and $\chi(g) = 1$ for any g conjugate in $W(B_n)$ to an element of S_n. Let G_0 be the normal subgroup generated by such elements $g \in G$, then χ is a pullback of some $\chi' \in \mathrm{Hom}(G/G_0, \mathbb{Z}/2)$. Therefore

$$Ш^2_\omega(G, M(C_n)) = \mathrm{Hom}\,(G/G_0, \mathbb{Z}/2)/\mathrm{Im}\,\delta = \mathrm{Hom}\,(G/G_0,\mathbb{Z}/2)/<\chi_{G,1},\ldots,\chi_{G,t}>.$$

The Weyl group $W(B_n)$ injects naturally into $W(B_{n+1})$ as the stabilizer of e_{n+1}. There also exists a natural injection $\varphi : W(B_n) \hookrightarrow W(D_{n+1})$. In fact, if $g \in W(B_n)$ and $\sigma(g) = 1$ let us set $\varphi(g) = g$, and if $\sigma(g) = -1$ let $\varphi(g) = g\, .\, c_{n+1}$.

PROPOSITION 1.24. *For any* $G \subseteq W(B_n)$ *we have* $Ш^2_\omega(G, Q(C_n)) = Ш^2_\omega(\varphi(G), Q(D_{n+1}))$.

Proof: Clearly $\varphi(G)_0 = \varphi(G_0)$ since any $g \in G_0$ satisfies $\sigma(g) = 1$. Furthermore, the set of characters $\{\chi_{\varphi(G),k}\}$ coincides with $\{\chi_{G,k}\}$ plus one more character χ_0 corresponding to the $\varphi(G)$–action on $< e_{n+1}>$. By definition $\chi_0(\varphi(g)) = \sigma(g) = \prod_{k=1}^{t} \chi_{G,k}(g)$, i.e. $<\chi_{G,1},\dots,\chi_{G,t}> = <\chi_{\varphi(G),1},\dots,\chi_{\varphi(G),t},\chi_0>$. Now the proposition follows from Theorem 1.22.

EXAMPLE 1.25. Let the set $\{1,\dots,4n\}$ be the union of 4 nonintersecting sets $I = \{i_s\}$, $J = \{j_s\}$, $M = \{m_s\}$, and $L = \{\ell_s\}$, $s = 1,\dots,n$. Let $G_n \subset W(B_{4n})$ be generated by $\alpha_s = C_{i_s} C_{j_s} (m_s \ell_s)$ and $\beta_s = C_{m_s} C_{\ell_s} (i_s j_s)$, $G_n \simeq (\mathbb{Z}/2)^{2n}$. We claim that $Ш^2_\omega(G_n, Q(C_{4n})) \simeq (\mathbb{Z}/2)^n$. In fact, $(G_n)_0$ is generated by $\alpha_s\beta_s$ and $G_n/(G_n)_0 \simeq (\mathbb{Z}/2)^n$. All $\chi_{G_n,i}$ related to orbits $\{i_s,j_s\}$ and $\{m_s,\ell_s\}$, $s = 1,\dots,n$, are trivial on G, hence $Ш^2_\omega(G_n, Q(C_{4n})) \simeq (\mathbb{Z}/2)^n$.

Our next goal is to compute $Ш^1_\omega(G,P(B_n))$. This includes the computation of $Ш^1_\omega(G,P(B_n))$ since the $W(D_n)$–modules $P(D_n)$ and $P(B_n)$ are isomorphic. Note that $P(C_n) = Q(B_n)^\circ = Q(B_n)$, and hence $Ш^1_\omega(G,P(C_n)) = 0$.

Let us introduce some notation. Let $a_{G,k}$ denote the generator of $H^1(G, Q(B_{n_k}))$ if this group is nonzero, and let a_G be the sum of all $a_{G,k}$ in $H^1(G, Q(B_n)) = \oplus H^1(G, Q(B_{n_k}))$. By I_g we denote the union of $pr(g)$–orbits $I_{<g>,k}$ such that $\chi_{<g>,k}$ is a nontrivial character of $<g>$.

THEOREM 1.26. *Let* $G \subseteq W(B_n)$, *and let* $I = \{1,2,\dots,n\} = \bigcup_{k=1}^{t} I_{G,k}$ *be the partition of* I *into* $pr(G)$*-orbits. Let* F *be the subgroup of* $H^1(G,Q(B_n))$ *consisting of* $\sum_{k\in J} a_{G,k}$, $J \subseteq \{1,2,\dots,t\}$, *such that for any* $g \in G$ *either* $(\bigcup_{k\in J} I_{G,k}) \cap I_g = \emptyset$ *or* $I_g \subset \bigcup_{k\in J} I_{G,k}$. *Then*

$$Ш^1_\omega(G,P(B_n)) = F/< a_G>.$$

Proof : The exact sequence (1) generates the diagram

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \coprod^1_\omega(G,M(B_n)) \\
 & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
 & & H^0(G,M(B_n)) & \longrightarrow & \mathbb{Z} & \xrightarrow{\delta_1} & H^1(G,Q(B_n)) & \longrightarrow & H^1(G,M(B_n)) & \longrightarrow 0 \\
 & & \downarrow & & \gamma_1\downarrow & & \downarrow & & \downarrow \\
 & & \prod_{g\in G} H^0(\langle g\rangle,M(B_n)) & \longrightarrow & \prod_{g\in G}\mathbb{Z} & \xrightarrow{\beta_1} & \prod_{g\in G} H^1(\langle g\rangle,Q(B_n)) & \longrightarrow & \prod_{g\in G} H^1(\langle g\rangle,M(B_n)) & \longrightarrow
\end{array}$$

It follows that $\coprod^1_\omega(G,M(B_n))$ is isomorphic to the subgroup of $H^1(G,Q(B_n))$ consisting of $\sum_{k\in J} a_{G,k}$, $J \subseteq \{1,2,\dots,s\}$, such that $\gamma\left[\sum_{k\in J} a_{G,k}\right] \in \operatorname{Im}\beta_1$, modulo $\operatorname{Im}\delta_1$. In order to compute δ_1 and β_1 let us first investigate the case of $G = W(B_n)$. Then $\operatorname{Im}\delta_1 = H^1(W(B_n),Q(B_n)) = <a_G>$. Now let $I = \bigcup_{k=1}^{t} I_{G,k}$ be a partition of I into $pr(G)$–orbits. The restriction map

$$\gamma : H^1(W(B_n),Q(B_n)) \longrightarrow H^1(W(B_{n_k}),Q(B_n)) = H^1(W(B_{n_k}),Q(B_{n_k}))$$

is an isomorphism. Indeed, the following diagram commutes :

$$\begin{array}{ccc}
H^1(W(B_n),Q(B_n)) & \xrightarrow{\ \delta\ } & \operatorname{Hom}(W(B_n),\mathbb{Q}/\mathbb{Z}) \\
\gamma\downarrow & & \alpha\downarrow \\
H^1(W(B_{n_k}),Q(B_n)) & \xrightarrow{\ \beta\ } & \operatorname{Hom}(W(B_{n_k}),\mathbb{Q}/\mathbb{Z})
\end{array}$$

(cf. (12)). According to Lemma 1.23, $\operatorname{Im}\delta = <\sigma>$, and $\alpha\circ\delta$ is injective, thus γ is also injective, hence it is an isomorphism. It follows that $H^1(W(B_n),Q(B_n)) \longrightarrow \prod_{k=1}^{t} H^1(W(B_{n_k}),Q(B_{n_k}))$ is just the diagonal injection. Now let $G \subseteq W(B_n)$ be an arbitrary subgroup, $I_{G,k}$ being $pr(G)$–orbits. The restriction map $\prod_{k=1}^{t} H^1(W(B_{n_k}),Q(B_n)) \longrightarrow H^1(G,Q(B_n))$ is surjective, hence $\operatorname{Im}\delta_1 = <a_G>$. Likewise $(\operatorname{Im}\beta_1)_g = <a_{<g>}>$.

It follows that $\gamma(\sum_{k\in J} a_{G,k}) \in \operatorname{Im}\beta$ iff $(\gamma(\sum_{k\in J} a_{G,k}))_g = a_{<g>}$ or 0. Now it is enough to note that if $I_{G,k}$ is a union of $pr(g)$–orbits $I_{<g>,k_i}$, $i = 1,\dots,m$, then $(\gamma(a_{G,k}))_g = \sum_{i=1}^{m} a_{<g>,k_i}$.

ALGEBRAIC TORI AND RATIONAL SURFACES.

V.E. Voskresenskiĭ has suggested the following approach to the classification of rational varieties ([43], Ch. IV, see also [29], § 4). Consider the natural action of the group $\mathfrak{g} = \mathrm{Gal}\,(\bar{k}/k)$ on $\mathrm{Pic}\,\bar{X}$, which is a free $\mathbb{Z}$–module of finite rank (in the discrete topology this action is continuous). If two smooth projective varieties are k–birationally equivalent, the corresponding Picard modules are similar. Let us associate to X the similarity class of the $\mathfrak{g}$–module $\mathrm{Pic}\,\bar{X}$. The varieties X and Y are called *stably equivalent* if $X \times_k \mathbb{P}^n_k$ is k–birationally equivalent to $Y \times_k \mathbb{P}^m_k$ for some $m, n \in \mathbb{N}$. Let $\mathscr{S}$ be the commutative semigroup consisting of stable equivalence classes of smooth projective rational varieties with the semigroup law given by $X \times_k Y$. Then the map $X \longmapsto [\mathrm{Pic}\,\bar{X}]$ can be extended to a homomorphism $\mathscr{S} \longrightarrow \mathscr{N}$, $\mathscr{N}$ denoting the semigroup of similarity classes of $\mathbb{Z}$–free $\mathfrak{g}$–modules of finite rank.

Let T be an algebraic torus, i.e. a k–form of the algebraic group $(G_{m,k})^n$, let $\hat{T} = \mathrm{Hom}(T \times_k \bar{k}, G_{m,\bar{k}})$ be its character module, and let X_T be a smooth projective compactification of T. Associating $\hat{T}$ to T we obtain the (anti)–duality between the category of algebraic k–tori and that of continuous (in the discrete topology) $\mathbb{Z}$–free $\mathfrak{g}$–modules of finite rank. We say that a k–torus T is *split* by an extension L/k if $T \times_k L \simeq (G_{m,L})^n$. Let $G = \mathrm{Gal}\,(L/k)$, then the category of algebraic k–tori split by L is dual to that of torsion–free G–modules of finite type. In what follows we shall (by a natural abuse of language) speak about a torus dual to a G–module N without pointing out either the ground field, or the splitting field. We denote the dual torus by $\hat{N}$.

According to [43], Ch. III, § 6 there always exists the minimal field L splitting a given torus. Moreover, L/k is a Galois extension.

DEFINITION 2.1. ***The minimal extension L/k splitting a torus T is called the minimal splitting field of T or, if there is no ambiguity, the splitting field of T. The group $G = \mathrm{Gal}\,(L/k)$ is called the splitting group of T.***

It is clear that G is nothing but the image of the representation of $\mathfrak{g}$ acting on $\hat{T}$.

Let $\mathscr{M}$ be the semigroup of stable equivalence classes of k–tori split by L/k, let $G = \mathrm{Gal}\,(L/k)$, and let ρ be the homomorphism $\mathscr{M} \longrightarrow \mathscr{N}$ such that $\rho(\hat{T}) = [\mathrm{Pic}\,\bar{X}_T]$. The main theorem ([43], 4.60; [6]) states that ρ maps $\mathscr{M}$ isomorphically to the subsemigroup of $\mathscr{N}$ consisting of flasque G–modules (a module F is called *flasque* if $H^{-1}(G',F) = 0$ for any subgroup $G' \subseteq G$). Thus, a hard geometric problem is reduced to an algebraic one. In particular, any invariant of the similarity class $\rho(\hat{T}) = [\mathrm{Pic}\,\bar{X}_T]$ is also a k–birational invariant of the torus T. This is the case, for example, for the invariant $H^1(G,\rho(\hat{T}))$.

The class $\rho(N)$ can be defined without any reference to the dual torus. There exists a flasque resolution of G–modules, i.e. a short exact sequence

$$0 \longrightarrow N \longrightarrow S \longrightarrow F \longrightarrow 0, \tag{13}$$

where S is a permutation G–module, and F is a flasque one. Then $\rho(N) = [F]$.

PROPOSITION 2.2. ***Let N be a torsion-free G-module of finite type, then $H^1(G,\rho(N)) = \text{Ш}^2_\omega(G,N)$.***

Proof : (See also [8]). Let $0 \to N \to S \to F \to 0$ be any flasque resolution. We have a commutative diagram :

$$\begin{array}{ccccccc}
 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \\
 & & \text{Ш}^1_\omega(G,F) & \longrightarrow & \text{Ш}^2_\omega(G,N) & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow \\
0 & \longrightarrow & H^1(G,F) & \longrightarrow & H^2(G,N) & \longrightarrow & H^2(G,S) \\
 & & \downarrow & & \downarrow & & \downarrow \\
0 & \longrightarrow & \prod_{g\in G} H^1(\langle g\rangle,F) & \longrightarrow & \prod_{g\in G} H^2(\langle g\rangle,N) & \longrightarrow & \prod_{g\in G} H^2(\langle g\rangle,S).
\end{array}$$

It follows that $\text{Ш}^2_\omega(G,N) = \text{Ш}^1_\omega(G,F)$. By periodicity, we get $H^1(\langle g\rangle,F) = H^{-1}(\langle g\rangle,F)$; F being flasque, the latter group vanishes. Consequently, $\text{Ш}^2_\omega(G,N) = H^1(G,F) = H^1(G,\rho(N))$.

The G–module F is defined up to addition of a permutation G–module, the same is also true for the dual G–module F°. Let us write down a flasque resolution for $F^\circ : 0 \to F^\circ \to S_1 \to F_1 \to 0$. Then F_1 is also defined up to addition of a permutation G–module, i.e. the similarity class $[\rho(\rho(N)^\circ)]$ is well–defined. The following result was communicated by A.A. Klyachko.

PROPOSITION 2.3. $H^1(G,\rho(\rho(N)^\circ)) = \text{Ш}^1_\omega(G,N^\circ)$.

Proof : By Proposition 2.2 we have $\text{Ш}^2_\omega(G,F^\circ) = H^1(G,F_1)$. Dualizing (13) we get the exact triple of G–modules : $0 \to F^\circ \to S^\circ \to N^\circ \to 0$; acting as in the proof of Proposition 2.2, we see that $\text{Ш}^2_\omega(G,F^\circ) = \text{Ш}^1_\omega(G,N^\circ)$, as required.

COROLLARY 2.4. *The groups* $Ш^2_\omega(G,N)$ *and* $Ш^1_\omega(G,N^\circ)$ *are* *k-birational invariants of the torus dual to a* *G-module* *N.*

Remark 2.5. One can construct many other k–birational invariants of the dual torus applying Corollary 1.18. However, these two ones have some advantage being expressed in terms of the G–module N itself. Besides, as we explain below, $Ш^2_\omega(G,N)$ contains some information on arithmetic properties of the dual torus (in the case of a global ground field).

Remark 2.6. Let A be a linear algebraic group defined over k, and let T be a maximal k–torus in A. The Galois group $\mathfrak{g} = \mathrm{Gal}\,(\bar{k}/k)$ acts on $\hat{T}$ fixing the set of roots of A corresponding to T. If A is semi–simple and R is the corresponding root system, then $\hat{T}$ is a subgroup of finite index in the weight lattice $P(R)$, moreover $\hat{T}$ contains the lattice $Q(R)$ generated by the roots of R. Under the additional assumption that A is an inner k–form, G can be embedded into the Weyl group $W(R)$, and we are in the situation of § 1. Theorems 1.22 and 1.26 compute the birational invariants of a maximal k–torus T in a classical group of adjoint type (an inner form), since in this case $\hat{T} = Q(R)$.

Remark 2.7. Let us explain how to prove that $Ш^1_\omega(G,\ Q(B_n)) = Ш^2_\omega(G,\ Q(B_n)) = 0$. To this end it is enough to establish the k–rationality of the torus T dual to the $W(B_n)$–module $Q(B_n)$, and to apply Corollary 2.4. However, it is clear that T can be embedded into a k–form of $(\mathbb{P}^1_k)^n$ as an open subvariety. V.E. Voskresenskiĭ has shown ([44]) that this variety is k–rational iff it has a k–point.

Now let us go over to the main object of the paper – to rational surfaces. The study of rational surfaces has three different aspects. The first one is their geometry over an algebraic closure of the ground field. This is one of the well–understood themes of classical algebraic geometry. The second problem to deal with is to study the action of the Galois group on divisors or on some other geometric objects associated to a surface. This part is of algebraic (or combinatorial) nature. It is somewhat parallel to Voskresenskiĭ's approach to tori. The results of this theory are essential for the third stage of the study, this one of arithmetic nature. On this stage one studies rational points on a surface (computing Manin's obtruction to the Hasse principle, computing the group $CH^2(X)$, etc...). One can find a more detailed account of this subject in [28], [29], [7], [9], [10]. Here we are going to show how the results of § 1 can be interpreted from the geometric point of view, and point out some consequences, useful for the study of zero–cycles on a surface; for the details see the next section.

The set of all rational surfaces is, in general, too numerous for a more or less explicit description. However, many properties (and among them the most interesting ones) do not depend on a particular choice of a surface in its k–birational class. Therefore we can restrict ourselves to studying k–minimal models, i.e. smooth complete surfaces for which any birational k–morphism to a smooth surface is an isomorphism. All the minimal models over an algebraically closed field are well–known : these are $\mathbb{P}^2$ and ruled surfaces F_n with $n \geq 0$, $n \neq 1$. If the ground field is not closed, then the classification is much more complicated. Let us recall some necessary definitions.

DEFINITION 2.8. *A smooth rational surface X such that $-K_X$ is ample is called a Del Pezzo surface. The number $n = (K_X^2)$ is called the degree of X; for $n \geq 3$ the divisor $-K_X$ is very ample and embeds X into $\mathbb{P}^n_k$. The surface $\bar{X} = X \times_k \bar{k}$ is $\mathbb{P}^2_{\bar{k}}$ with $(9-n)$ points in general position blown-up (it means that no three points lie on a line, no five points lie on a conic, etc..., cf.* [29], 2.1.1).

Examples : A Del Pezzo surface of degree 9, i.e. a k–form of $\mathbb{P}^2_k$, is called a Severi–Brauer surface. Among Del Pezzo surfaces there are some complete intersections : that is the case when the degree is equal to 3 (a smooth cubic in $\mathbb{P}^3_k$), or to 4 (a smooth intersection of two quadrics in $\mathbb{P}^4_k$).

If X is a Del Pezzo surface of degree n, the structure of the Picard group Pic $\bar{X}$ is clear : as an abelian group it is just a direct sum of $(10-n)$ copies of $\mathbb{Z}$; a generator ℓ_0 is the proper transform of the class of a line in $\mathbb{P}^2_{\bar{k}}$, and other generators ℓ_i are the classes of the blown–up points.

The quadratic form given by the intersection pairing is diagonal in this basis : $x_0^2 - \sum_{i=1}^{9-n} x_i^2$.

The Galois group $\mathfrak{g}$ acts on $\bar{X}$, hence it also acts on Pic $\bar{X}$. The action preserves K_X (which is defined over k) and the intersection pairing. It is clear that if D can be contracted by a k–morphism, then its class belongs to the $\mathfrak{g}$–invariant part of Pic $\bar{X}$. Therefore the condition $(\text{Pic } \bar{X})^{\mathfrak{g}} \simeq \mathbb{Z}$ guarantees k–minimality of X. This is one family of k–minimal rational surfaces.

DEFINITION 2.9. *Let Y be a surface endowed with a morphism f onto a rational curve C, and suppose that Y is a k-form of F_n with m points blown-up, no fibre of f containing more than one of these points. Then Y is called a (standard) conic bundle with m degenerate fibres. The integer $(K_Y^2) = 8 - m$ is called the degree of the conic bundle Y.*

Let $\Theta_{Y/C}$ be the relative tangent bundle, then $R^0 f_* \Theta_{Y/C}$ is a locally free sheaf of rank 3 embedding Y into $\mathbb{P}(R^0 f_* \Theta_{Y/C})$. Each irreducible fibre is a conic, and a degenerate fibre is a pair of lines meeting transversally. Recall that a conic bundle $f : Y \longrightarrow C$ is called *relatively k–minimal* if its fibres contain no contractible lines. In other words, there exists no k–morphism $h : Y \longrightarrow Y'$ such that the surface Y' is smooth and is not isomorphic to Y, where $f' : Y' \longrightarrow C$ is a conic bundle over the same base C, and $f = f' \circ h$.

The Picard group of F_n is isomorphic to $\mathbb{Z} \oplus \mathbb{Z}$, let us denote its generators by ℓ_0 and s, the first one being the class of a fibre of the projection onto $\mathbb{P}^1$, and the second one being the class of the exceptional section. We have $(\ell_0^2) = 0$, $(\ell_0, s) = 1$, $(s^2) = -n$. Therefore $\operatorname{Pic} \bar{X}$ is the direct sum of $\operatorname{Pic} F_n$ and m copies of $\mathbb{Z}$, the generators of the latter groups being the classes of the blown–up points. Denote them by $\ell_1, \ldots, \ell_m$; we have $(\ell_i, \ell_j) = -\delta_{ij}$.

LEMMA 2.10. *For* $m \geq 1$ *there exists a basis* $\ell_{-1}, \ell_0, \ell_1, \ldots, \ell_m$ *in the lattice* $\operatorname{Pic} \bar{Y}$ *such that* ℓ_0 *is the class of a fibre of* f, *and* $\ell_1, \ldots, \ell_m$ *are the classes of the components of degenerate fibres, one from each. We have* $(\ell_{-1}^2) = (\ell_{-1}, \ell_i) = 0$ *for* $i \geq 1$, *and* $(\ell_{-1}, \ell_0) = 1$.

Proof : Set $\ell'_i = \ell_0 - \ell_i$, then all the intersection numbers are as before except for $(s, \ell'_i) = 1 - (s, \ell_i)$. Therefore we can set $(s, \ell_i) = 0$ for $i \geq 1$. Set $s' = s + \ell_0 + \ell_1$, then all the intersection numbers are as before except for $(s'^2) = (s^2) + 1$. Iterating this substitution we obtain the required result.

PROPOSITION 2.11 ([21]). *Any k-minimal rational surface is isomorphic to one of the following list*: $\mathbb{P}^2_k$, *a quadric* $Q \subset \mathbb{P}^3_k$ *with* $(\operatorname{Pic} \bar{Q})^{\mathfrak{g}} \simeq \mathbb{Z}$, *Del Pezzo surfaces* X *of degree* n $(n \neq 7)$ *with* $(\operatorname{Pic} \bar{X})^{\mathfrak{g}} = \mathbb{Z} \cdot K_X$, *conic bundles* Y *of degree* n $(n \neq 3, 5, 6, 7,$ *and* $Y \neq F_1$, for $n = 8)$ with $(\operatorname{Pic} \bar{Y})^{\mathfrak{g}} \simeq \mathbb{Z} \oplus \mathbb{Z}$.

Let us remark that if $(\operatorname{Pic} \bar{Y})^{\mathfrak{g}} \simeq \mathbb{Z} \oplus \mathbb{Z}$, then for $n \neq 8$ this group is generated by ℓ_0 and K_Y.

The condition $(\operatorname{Pic} \bar{Y})^{\mathfrak{g}} \simeq \mathbb{Z} \cdot \ell_0 \oplus \mathbb{Z} \cdot K_Y$ guarantees relative k–minimality of Y, although, in general, such a surface must not be k–minimal. However, by Proposition 2.11 a relatively k–minimal but non–k–minimal conic bundle can appear only if its degree is equal to 3, 5, 6, 7 or 8.

THEOREM 2.12. a) *Let X be a Del Pezzo surface of degree $(9-n)$ with $3 \leq n \leq 8$, then there exists an isomorphism* $\sigma : \operatorname{Pic} \bar{X} \longrightarrow M(E_n)$ *mapping the intersection form to the quadratic form* (5) *multiplied by* -1, *and such that the orthogonal complement to* $\sigma(K_X)$ *in* $M(E_n)$ *is* $Q(E_n)$.

b) *Let Y be a conic bundle with n degenerate fibres, then there exists an isomorphism* $\sigma : \operatorname{Pic} Y \longrightarrow N(D_n)$ *mapping the intersection form to the quadratic form* (7) *multiplied by* -1, *and such that the orthogonal complement in* $N(D_n)$ *to the sublattice generated by* $\sigma(K_Y)$ *and* $\sigma(\ell_0)$ *is* $Q(D_n)$.

Proof: According to Remark 1.11 the quadratic forms on $M(E_n)$ and $N(D_n)$ are indefinite and, by Theorem 1.10, unimodular. The signature of each form is equal (up to a sign) to the signature prescribed by the Hodge index theorem. Therefore they are isomorphic (as quadratic forms) to the intersection forms on $\operatorname{Pic} \bar{X}$ and $\operatorname{Pic} \bar{Y}$, respectively.

Using the Gram–Schmidt orthogonalization we can construct the isomorphism σ satisfying the required condition. Let $\alpha_1,\dots,\alpha_n$ be a basis of the root system E_n, and let $\{\alpha_1,\dots,\alpha_n,\beta\}$ be a basis of $M(E_n)$ such that the matrix of the quadratic form coincides with (5). Set $\sigma(\ell_n) = \beta$, $\sigma(\ell_{n-1}) = \alpha_n + \beta$, ... ,$\sigma(\ell_2) = \alpha_3 + \dots + \alpha_n + \beta$, $\sigma(\ell_1) = \alpha_1 + \alpha_3 + \dots + \alpha_n + \beta$, $\sigma(\ell_0) = \alpha_1 + \alpha_2 + 2\alpha_3 + 3(\alpha_4 + \dots + \alpha_n) + 3\beta$.

Induction shows that σ is an isomorphism of quadratic forms. Besides, $K_X = -3\ell_0 + \sum_{i=1}^{n} \ell_i$,

$$\sigma(K_X) = (n-9)\beta - 2\alpha_1 - 3\alpha_2 - 4\alpha_3 - \sum_{i=4}^{n} (10-i)\alpha_i = (n-9)(\beta + \omega_n)$$

(see the formulae for fundamental weights in [3], Ch. VI). Since $<\beta, \alpha_i> + <\omega_n, \alpha_i> = 0$ for all i, it follows that $\sigma(K_X)$ is orthogonal to $Q(E_n)$.

The construction of an isomorphism $\sigma : \operatorname{Pic} \bar{Y} \longrightarrow N(D_n)$ is similar. Let $\alpha_1,\dots,\alpha_n$ be a basis of the root system D_n, and let $\{\alpha_1,\dots,\alpha_n, \beta, \gamma\}$ be a basis of $M(D_n)$ such that $<\beta, \alpha_i> = -<\omega_1, \alpha_i>$, $<\gamma, \alpha_i> = -<\omega_n, \alpha_i>$, i.e. in this basis the matrix of the quadratic form coincides with (7). Set $\sigma(\ell_1) = \beta$, $\sigma(\ell_2) = \beta = \alpha_1$, $\sigma(\ell_3) = \beta + \alpha_1 + \alpha_2, \dots$, $\sigma(\ell_n) = \beta + \alpha_1 + \dots + \alpha_{n-1}$, $\sigma(\ell_0) = 2\beta + 2(\alpha_1 + \dots + \alpha_{n-2}) + \alpha_{n-1} + \alpha_n$, $\sigma(\ell_{-1}) = \gamma$. A straightforward computation shows that σ is an isomorphism of quadratic forms. By the adjunction formula we have $(K_Y, \ell_0) = -2$ and $(K_Y, \ell_i) = -1$ for $i \geq 1$, on the other hand,

$(K_Y^2) = 8 - n$, therefore $K_Y = -2(\ell_{-1}+\ell_0) + \sum_{i=1}^{n} \ell_i$. It follows that $\sigma(-2\ell_{-1} + \sum_{i=1}^{n} \ell_i) = -2\gamma + n\beta + (n-1)\alpha_1 + \ldots + \alpha_{n-1}$. Note that $\sigma(\ell_0) = 2(\beta+\omega_1)$, $\sigma(-2\ell_{-1} + \sum_{i=1}^{n} \ell_i) = n(\beta+\omega_1) - 2(\gamma+\omega_n)$. The theorem is proved.

COROLLARY 2.13. *The group of automorphisms of the lattice* $\operatorname{Pic}\bar{X}$ *(respectively of* $\operatorname{Pic}\bar{Y}$*) preserving the intersection form and* K_X *(respectively* K_Y *and* ℓ_0*) is isomorphic to the Weyl group* $W(E_{9-n})$ *with* n *equal to the degree of the Del Pezzo surface* X *(respectively to* $W(D_m)$ *with* m *equal to the number of degenerate fibres of the conic bundle* Y*).*

Proof : Follows from Proposition 1.15.

DEFINITION 2.14. *The image of the Galois group* $\mathfrak{g}$ *under the action on* $\operatorname{Pic}\bar{X}$ *is called the splitting group of the rational surface* X*. The fixed field of the kernel of this action is called the splitting field of* X*.*

This definition is analogous to Definition 2.1.

According to Corollary 2.13 the splitting group G is a subgroup of $W(R)$ with $R = D_n$ or E_n, and the G–module $\operatorname{Pic}\bar{X}$ is isomorphic to $M(E_n)$ or $N(D_n)$. As in the case of maximal tori in adjoint groups, the problems arising here concern the study of the module $M(R)$. However, if two surfaces are k–birationally equivalent, the corresponding Picard modules are similar; therefore, any invariant of the similarity class $[\operatorname{Pic}\bar{X}]$ is also a birational invariant of X. This is the case for $H^1(G,\operatorname{Pic}\bar{X})$, $Ш^1_\omega(G, \operatorname{Pic}\bar{X})$, and $Ш^2_\omega(G, \operatorname{Pic}\bar{X})$. From Section 1 it is clear how to compute the latter invariants.

Note by the way that the tori dual to the $W(R)$–modules $M(R)$ and $Q(R)$ are stably equivalent. This follows from Voskresenskiĭ's theorem ([43], Ch. VI, § 2) and the exact sequence (1).

DEFINITION 2.15. *Let* X *be a rational surface defined over a field* k*. The algebraic torus dual to the* $\mathfrak{g}$*-module* $\operatorname{Pic}\bar{X}$ *is called the Néron-Severi torus of* X*.*

All these invariants play an important role in studying arithmetic properties of a surface. Let k be a global or a local field of characteristic 0, and let X be a rational variety (complete and smooth), then there exists an exact sequence

$$0 \to \operatorname{Pic} X \to (\operatorname{Pic}\bar{X})^{\mathfrak{g}} \to Br\, k \to Br\, X \to H^1(k, \operatorname{Pic}\bar{X}) \to 0,$$

where $Br\,k$ is the Brauer group of k (the group of classes of central simple algebras), and $Br\,X = H^2_{ét}(X, \mathbb{G}_{m,X})$ is the cohomological Brauer group of X (it coincides with the group of classes of Azumaya algebras on X) (cf. [28], [10]). Let k be a number field. We say that the Hasse principle holds for X if the condition $X(k_v) \neq \emptyset$ for every place v of the field k implies $X(k) \neq \emptyset$. Each class $A \in Br\,X$ determines the map "value at a point" : $X(k) \longrightarrow Br\,k$, and $X(k_v) \longrightarrow Br\,k_v$. Local class field theory yields an embedding $inv_v : Br\,k_v \hookrightarrow \mathbb{Q}/\mathbb{Z}$. Define the map $i_A : \prod_v X(k_v) \longrightarrow \mathbb{Q}/\mathbb{Z}$ by $i_A(x) = \sum_v inv_v(A_{x_v})$. The reciprocity law states that if $x \in X(k)$, then $\sum_v inv_v(A_x) = 0$. Therefore, if $X(k_v) \neq \emptyset$ for every v but $\bigcap_{A \in H^1(k, \mathrm{Pic}\,\bar{X})} \mathrm{Ker}\, i_A = \emptyset$, then the Hasse principle fails to hold for X. This obstruction is called the Manin obstruction to the Hasse principle.

PROPOSITION 2.16 ([10]). *Let* $Б(X)$ *denote the group of classes of locally constant Azumaya algebras modulo constant algebras* ($Б(X)$ *is the image in* $H^1(k, \mathrm{Pic}\,\bar{X})$ *of the kernel of* $Br\,X \longrightarrow \prod_v Br\,X_v/Br\,k_v$). *Then* $Б(X) \subseteq Ш^1_\omega(G, \mathrm{Pic}\,\bar{X})$.

Proof : We have an exact sequence

$$0 \longrightarrow Б(X) \longrightarrow H^1(G, \mathrm{Pic}\,\bar{X}) \longrightarrow \prod_v H^1(G_v, \mathrm{Pic}\,\bar{X}),$$

where G is the splitting group of X, and G_v is the splitting group of X_v. From Chebotarev's density theorem it follows that for every $g \in G$ there exists a valuation v such that $G_v = \langle g \rangle$.

Remark 2.17 ([10]). Let k be a number field, and assume $X(k_v) \neq \emptyset$ for every v. Then Manin's obstruction associated to $Б(X)$ vanishes if and only if the exact sequence of $\mathfrak{g}$–modules

$$1 \longrightarrow \bar{k}^* \longrightarrow \bar{k}(X)^* \longrightarrow \bar{k}(X)^*/\bar{k}^* \longrightarrow 1$$

splits.

Conjecturally, if we restrict ourselves to rational surfaces, then Manin's obstruction to the Hasse principle is the only one. Thus, if $H^1(k, \mathrm{Pic}\,\bar{X}) = 0$, then the Hasse principle holds for X (see [36] for discussion of some results in this direction).

On the other hand, let X be a rational surface, and let $CH_0(X)$ denote the Chow group of zero–cycles on X. Let $A_0(X) = \mathrm{Ker}\,[CH_0(X) \xrightarrow{\deg} \mathbb{Z}]$ be the group of zero–cycles of degree 0. Let S_X be the Néron–Severi torus of X. There exists Bloch's homomorphism $\phi : A_0(X) \to H^1(k,S_X)$, cf. [2]; if k is local or global, then ϕ is injective [4]. Let k be a number field. Then there exists the commutative diagram with exact rows, defining $Ш A_0(X)$ and $Ш^1(S_X)$:

$$\begin{array}{ccccccc} 0 & \longrightarrow & Ш A_0(X) & \longrightarrow & A_0(X) & \longrightarrow & \oplus_v A_0(X_v) \\ & & \downarrow & & \downarrow & & \downarrow \\ 0 & \longrightarrow & Ш^1(S_X) & \longrightarrow & H^1(k,S_X) & \longrightarrow & \oplus_v H^1(k_v,S_X) . \end{array}$$

Tate–Nakayama duality implies that $Ш^1(S_X)$ and $Ш^2(k,\hat{S}_X) = Ш^2(k, \operatorname{Pic} \bar{X})$ are dual to each other as finite abelian groups ($Ш^2(k, \operatorname{Pic} \bar{X})$ denotes the kernel of the map $H^2(k, \operatorname{Pic} \bar{X}) \to \prod_v H^2(k_v, \operatorname{Pic} \bar{X})$).

PROPOSITION 2.18. *There exists a natural embedding of groups*

$$Ш^2(k, \operatorname{Pic} \bar{X}) \hookrightarrow Ш^2_\omega(G, \operatorname{Pic} \bar{X}). \tag{14}$$

Proof: It is analogous to the proof of Proposition 2.16.

Conjecturally, there exists the following exact sequence

$$0 \to Ш^1(S_X) \to A_0(X) \to \oplus_v A_0(X_v) \to H^1(k, \operatorname{Pic} \bar{X})^*,$$

reducing the computation of $A_0(X)$ to the computation of $A_0(X_v)$ [9] (cf. the note at the end of Section 0).

To conclude this section, let us return to tori and recall Voskresenskiĭ's theorem ([43], Ch. VI, § 7) which computes the cokernel of the homomorphism (14). Let G be a linear algebraic group over a global field k, and let $\overline{G(k)}$ be the topological closure of the image of $G(k)$ under its diagonal embedding into $\prod_v G(k_v)$. The group $A(G) = \prod_v G(k_v)/\,\overline{G(k)}$ is called the *weak approximation defect*, it is a birational invariant of G. Let T be an algebraic torus, then the quotient $Ш^2_\omega(G,\hat{T})/Ш^2(G,\hat{T})$ is dual to $A(T)$.

3
INTERSECTIONS OF TWO QUADRICS IN $\mathbb{P}^4_k$ AND CONIC BUNDLES

Recall that for $d = (K^2_X) \geq 3$ the anticanonical class $-K_X$ embeds a Del Pezzo surface X into $\mathbb{P}^d_k$, and the image is a smooth surface of degree d. According to Theorem 2.12 the orthogonal complement to K_X in $\operatorname{Pic} \bar{X}$ is the lattice $Q(E_n)$ generated by roots, here $n = 9 - d$, $3 \leq n \leq 8$. For $d = 4$ this fact has a nice geometric interpretation.

It is well known that a Del Pezzo surface of degree 4 is defined in $\mathbb{P}^4_k$ as a geometrically integral smooth intersection of two quadrics $X = Q_0 \cap Q_\infty$. Let us consider the entire pencil of quadrics containing X: $Q_\lambda = Q_0 + \lambda\, Q_\infty$, $\lambda \in \bar{k}$. Assume Q_0 and Q_∞ to be nonsingular. Singular quadrics of the pencil correspond to the roots of the polynomial $P(\lambda) = \det(Q_\lambda)$. Since X is smooth all the roots λ_i of $P(\lambda)$ are distinct [31], therefore precisely 5 quadrics of the pencil Q_λ are singular. These are cones over smooth quadrics Q^b_i in $\mathbb{P}^3_{\bar{k}}$. By v_i we denote the vertices of these cones. Since $Q^b_i \simeq \mathbb{P}^1_{\bar{k}} \times \mathbb{P}^1_{\bar{k}}$, we have $\operatorname{Pic} Q^b_i \simeq \operatorname{Pic} Q_i \simeq \mathbb{Z} \oplus \mathbb{Z}$. This group is generated by classes of projective planes passing through the vertex v_i and a projective line belonging to one of the two pencils on $\mathbb{P}^1_{\bar{k}} \times \mathbb{P}^1_{\bar{k}}$. We denote these classes by h^+_i and h^-_i. Each of the planes intersects X in a conic $q^\pm_i \cap Q_0$ (by abuse of notation we denote by $q^\pm_i$ both the conic and its class in $\operatorname{Pic} \bar{X}$).

PROPOSITION 3.1. *In* $\operatorname{Pic} \bar{X}$ *the following relations hold:*

a) $-K_X = q^+_i + q^-_i$ *for any* i,

b) $(q^{+2}_i) = (q^{-2}_i) = 0$, $(q^+_i \,.\, q^-_i) = 2$ *for any* i,

c) $(q^\pm_i \,.\, q^\pm_j) = 1$ *for* $i \neq j$ *(for all combinations of signs* $+$ *and* $-$*).*

Proof:

a) Choose $L \simeq \mathbb{P}^3_{\bar{k}} \subset \mathbb{P}^4_{\bar{k}}$ in a special way. Namely, let L contain v_i, and let the projection of L from v_i be tangent to Q^b_i. Then $L \cap Q_i = h^+_i \cup h^-_i$, hence $L \cap \bar{X} = q^+_i \cup q^-_i$.

b) The projection from v_i represents X as a double covering of Q_i, $pr_i : \bar{X} \longrightarrow Q^b_i$. Hence if $\ell, m \in \operatorname{Pic} Q^b_i$, then $(pr^*_i\, \ell \,.\, pr^*_i\, m)_X = 2(\ell . m)_{Q^b_i}$, and b) follows.

c) By definition of $q^\pm_i$ we have $(q^\pm_i \,.\, q^\pm_j)_{\bar{X}} = (h^\pm_i \,.\, q^\pm_j)_{Q_i} = (h^\pm_i \,.\, h^\pm_j)_{\mathbb{P}^4} = 1$.

COROLLARY 3.2. *The orthogonal complement to* K_X *in* $\operatorname{Pic}\bar{X}$ *is generated by* $q_i^{\pm} - q_j^{\pm}$ *for all possible choices of* $+$ *and* $-$. *Equipped with a quadratic form (intersection pairing), this lattice is isomorphic to* $Q(D_5)$ *with the usual quadratic form given by the Cartan matrix.*

Proof : It is easy to check that the intersection matrix of $q_1^+ - q_2^+$, $q_1^- - q_2^+$, $q_2^+ - q_3^+$, $q_3^+ - q_4^+$, $q_4^+ - q_5^+$ is the Cartan matrix of the root system D_5. The discriminant of this lattice equals 4. The intersection pairing defines a unimodular quadratic form on $\operatorname{Pic}\bar{X}$, hence the discriminant of the orthogonal complement to K_X equals $(K_X^2) = 4$ since K_X is not divisible in $\operatorname{Pic}\bar{X}$. It follows that $K_X^{\perp}$ is generated by $q_i^{\pm} - q_j^{\pm}$.

Let us extend the $W(D_5)$–action on $Q(D_5) = \{\gamma \in \operatorname{Pic}\bar{X} \mid (\gamma.K_X) = 0\}$ to the entire group $\operatorname{Pic}\bar{X}$ so that K_X is $W(D_5)$–invariant. Then the set $\{q_1^+, q_1^-, \ldots, q_5^+, q_5^-\}$ with $W(D_5)$–action can be identified with the graph Δ (cf. § 0, Fig. 3).

Now let us consider the configuration of lines on X. Note that the projection of $Q_i^b \simeq \mathbb{P}^1_{\bar{k}} \times \mathbb{P}^1_{\bar{k}}$ onto any factor equips $\bar{X}$ with a conic bundle structure over $\mathbb{P}^1_{\bar{k}}$. Its fibres are conics of the pencil q_i^+ (respectively of q_i^-). In view of 3.1 a) it is natural to call q_i^+ and q_i^- *complementary* pencils. Degenerate conics in any of these pencils can be described for example in the following way.

Let us choose a coordinate system $(x_0, \ldots, x_4)$ on $\mathbb{P}^4_{\bar{k}}$ such that $v_i = (1,0,0,0,0)$. Then Q_i is given by a quadratic form in $x_1, \ldots, x_4$, and Q_0 after a suitable linear change of coordinates is given by $x_0^2 - Q_0'(x_1, \ldots, x_4)$. The projection pr_i from v_i, $pr_i : (x_0, x_1, \ldots, x_4) \longmapsto (x_1, \ldots, x_4)$, makes X a double covering of the quadric given by $Q_i^b = 0$ with the equation $x_0^2 = Q_0'(x_1, \ldots, x_4)$, ramified along a smooth elliptic curve $C = Q_0' \cap Q_i^b$. Let $\ell \simeq \mathbb{P}^1_{\bar{k}}$ lie on X. Then $pr_i(\ell) \simeq \mathbb{P}^1_{\bar{k}}$ lies on Q_i^b. However $pr_i^{-1}(\mathbb{P}^1_{\bar{k}}) = \ell \cup \ell'$ only if $\mathbb{P}^1_{\bar{k}}$ is tangent to C. The number of such lines belonging to one family on $\mathbb{P}^1_{\bar{k}} \times \mathbb{P}^1_{\bar{k}}$ equals the number of ramification points of the covering $C \longrightarrow \mathbb{P}^1_{\bar{k}}$, which is given by the projection of $\mathbb{P}^1_{\bar{k}} \times \mathbb{P}^1_{\bar{k}}$ to the corresponding factor, that is 4. Above these 4 lines there lie 4 pairs of lines on X, which constitute the degenerate fibres of one of the two complementary pencils of conics. The entire set of these 16 lines forms the configuration described by the graph Γ ([28], Ch. IV, § 4; cf. § 0, Fig. 1). Recall ([28], Ch. IV) that the classes of lines in $\operatorname{Pic}\bar{X}$ are precisely the classes ℓ such that $(\ell.K_X) = (\ell^2) = -1$. Each class contains only one line. This gives an injection $W(D_5) \hookrightarrow \operatorname{Aut}\Gamma$, which in fact is an isomorphism.

The Galois group $\mathfrak{g} = \mathrm{Gal}(\bar{k}/k)$ acts on Γ. It is often useful to know the partition of Γ into $\mathfrak{g}$–orbits. For example, $H^1(k, \mathrm{Pic}\, \bar{X})$ depends only on the partition ([28], Ch. IV, 9.3). Recall that X is k–minimal (section 2) iff there is no $\mathfrak{g}$–invariant set of pairwise nonintersecting lines on X, i.e. there is no $\mathfrak{g}$–invariant subgraph of Γ with the property that any two vertices of it are not joined.

DEFINITION 3.3. *A subgroup $G \subseteq W(D_5)$ is called minimal if Γ has no G-orbit such that any two vertices of it are not joined.*

This property holds for a splitting group G of X iff X is k–minimal.

The complete list of all possible partitions of Γ into G–orbits for minimal subgroups $G \subseteq W(D_5)$ is obtained in [28], Ch. IV, table 2 (see § 0, Figure 4). The types are always numerated as in Figure 4. A type A is called a subtype of a type B, if after the action of a suitable element of $W(D_5)$ the partition of type A becomes a subpartition of the partition of type B.

Starting with the list of all possible types V.A. Iskovskih [20] proved that if a Del Pezzo surface X of degree 4 is k–minimal then either $(\mathrm{Pic}\, \bar{X})^{\mathfrak{g}} = \mathbb{Z} \,.\, K_X$, or $(\mathrm{Pic}\, \bar{X})^{\mathfrak{g}} \simeq \mathbb{Z} \oplus \mathbb{Z}$. In the latter case $(\mathrm{Pic}\, \bar{X})^{\mathfrak{g}}$ is generated by q_i^+ and q_i^- for some i. Thus X has two different conic bundle structures, each defined over k. Since the property of being a component of a fibre of such a conic bundle is $\mathfrak{g}$–invariant, the partition of Γ into $\mathfrak{g}$–orbits is a subpartition of the following one : the 16 lines are divided into two orbits, each containing 8 lines, such that each orbit consists of lines which project to one family of lines on the quadric $Q_i^b \simeq C_1 \times C_2$, $C_1 \simeq C_2 \simeq \mathbb{P}^1_{\bar{k}}$. This partition is of type XV (see Figure 4). On the other hand, according to 2.13 the group of automorphisms of $\mathrm{Pic}\, \bar{X}$, preserving K_X and the intersection pairing, is isomorphic to $W(D_5)$. From 3.2 it follows that no element of $W(D_5)$ acts trivially on the set of rational equivalence classes of conics $q_i^{\pm}$. Considering the intersection matrix of $q_i^{\pm}$ obtained in 3.1 we see that this action is the "usual" action of $W(D_5)$ on graph Δ (see § 0). The stabilizer of q_1^+ in $W(D_5)$ is isomorphic to $W(D_4)$. This injection was called standard in § 1. Summing up we get the following statement.

COROLLARY 3.4. *Let X be a k-minimal Del Pezzo surface of degree 4. The following conditions are equivalent:*

i) $rk(\mathrm{Pic}\,\bar{X})^{\mathfrak{g}} = rk\,\mathrm{Pic}\,X = 2$;

ii) *the type of the partition of Γ into G-orbits is a subtype of type XV;*

iii) *the splitting group G of X is conjugate to a subgroup of $W(D_4) = <c_2c_3,(23),(2345)>$;*

iv) *there exists a k-morphism $f: X \longrightarrow C$ onto a rational curve C, representing X as a conic bundle with 4 degenerate fibres;*

v) *if $X = Q_0 \cap Q_\infty$, Q_0 and Q_∞ being nonsingular quadrics, then*

a) *the polynomial* $\det(Q_0 + \lambda\, Q_\infty)$ *has a root λ_1 in k,*

b) *for any hyperplane H such that the restriction of $Q_1 = Q_0 + \lambda_1 Q_\infty$ to H is nonsingular, we have* $\det\,(Q_1|_H) \in (k^*)^2$.

Proof : The equivalence of i), ii), iii) and iv) is proved above. Let us prove that v) is equivalent to iii). The pair $\{q_1^+, q_1^-\}$ is $\mathfrak{g}$–invariant, hence the quadric Q_1 is defined over k. The equivalence of v) b) to the property of q_1^+ and q_1^- to be individually $\mathfrak{g}$–invariant follows from the following fact.

LEMMA 3.5. *Let Q be a smooth quadric in $\mathbb{P}^3_k$, then* $\mathrm{Pic}\,Q \simeq \mathbb{Z} \oplus \mathbb{Z}$ *iff* $\det Q \in (k^*)^2$.

Proof : The quadratic form Q can be reduced to a diagonal form $(x_1^2 - ax_2^2) - b(x_3^2 - cx_4^2)$, $\det Q = acb^2$. The two families of lines on Q are then given by the equations :

$$\begin{cases} x_1 + \sqrt{a}\, x_2 & = t(x_3 + \sqrt{c}\, x_4) \\ x_1 - \sqrt{a}\, x_2 & = bt^{-1}(x_3 - \sqrt{c}\, x_4) \end{cases}$$

$$\begin{cases} x_1 + \sqrt{a}\, x_2 & = t(x_3 - \sqrt{c}\, x_4) \\ x_1 - \sqrt{a}\, x_2 & = bt^{-1}(x_3 + \sqrt{c}\, x_4) \end{cases}$$

Each family is $\mathfrak{g}$–invariant iff $k(\sqrt{a}) = k(\sqrt{c})$, i.e. $a = c\tau^2$ for some $\tau \in k$.

Remark 3.6. As we have already mentioned, the image of the standard embedding of $W(D_4)$ into $W(D_5)$ preserves q_1^+ and hence the fundamental weight $\omega_1 = e_1$ ([3], Ch. VI, 4.8). Proposition 1.14 states that the $W(D_4)$–module $M(D_5)$ is isomorphic to $N(D_4)$ (this also follows from the above geometric description of Del Pezzo surfaces of degree 4 with

Pic $X \simeq \mathbb{Z} + \mathbb{Z}$). Let us also note that a subgroup $G \subseteq W(D_5)$, which in fact belongs to $W(D_4)$, is minimal iff $rk\, N(D_4)^G = 2$.

The remaining part of this section is organized in the following way. First we recall a result of Iskovskih describing all relatively k–minimal conic bundles of degree 4. The study of anticanonical models of these surfaces leads us to the conclusion that any such surface is k–birationally equivalent to a Del Pezzo surface of degree 4. This is valid when k is large enough, for example, infinite; as we suppose it to be throughout this paper. If k is small, there exists a unique counterexample, for $k = \mathbb{F}_3$ [41]. This provides an answer to a question of Iskovskih ([22], § 2, remark 1). Then we try to obtain some necessary and sufficient conditions for a Del Pezzo surface of degree 4 to be k–birationally equivalent to an Iskovskih surface.

Here are some useful facts extracted from the proof of proposition 1 of [20]. The anticanonical class $-K_X$ of a relatively k–minimal conic bundle X defines a k–birational morphism $h : X \longrightarrow Y$, such that either h is an isomorphism, Y being a Del Pezzo surface, or it contracts a curve from the class $-K_X - 2\,\ell_0$ to a singular point (here $\ell_0 \in (\text{Pic}\,\bar{X})^{\mathfrak{g}}$ is the class of a geometric closed fibre of $f : X \longrightarrow C$). In the latter case this curve splits over a quadratic extension of k into two absolutely irreducible smooth rational curves, each of them having self–intersection -2. Hence Y has two singularities (double points of type A_1) conjugate over k. In the former case (when h is an isomorphism) the linear system $|-K_X - 2\,\ell_0|$ is empty.

Note that according to Proposition 2.11 any relatively k–minimal conic bundle of degree 4 is k–minimal. In this case $(\text{Pic}\,\bar{X})^{\mathfrak{g}} = \mathbb{Z}\,K_X + \mathbb{Z}\,\ell_0$.

DEFINITION 3.7 [14]. *An Iskovskih surface is a geometrically integral intersection of two quadrics* Y *in* $\mathbb{P}^4_k$, *having precisely two conjugate double points, with the property that the line joining them does not lie on* Y.

PROPOSITION 3.8. *The anticanonical model of a relatively k-minimal conic bundle of degree* 4, *such that* $|-K_X - 2\,\ell_0|$ *is not empty, is an Iskovskih surface. Conversely, the minimal desingularization of an Iskovskih surface has a conic bundle structure defined over* k, *such that if* ℓ_0 *is the class of a fibre, then* $|-K_X - 2\,\ell_0|$ *is not empty.*

Proof : The Riemann–Roch theorem gives

$$h^0(X, \mathcal{O}(-2K_X)) - h^1(X, \mathcal{O}(-2K_X)) + h^2(X, \mathcal{O}(-2K_X)) = 3(K_X^2) + 1 = 13.$$

By Serre duality $h^2(X, \mathcal{O}(-2K_X)) = h^0(X, \mathcal{O}(3K_X)) = 0$, since $(K_X.C) < 0$ for some curves C on X. Let us prove that $h^1(X, \mathcal{O}(-2K_X)) = h^1(X, \mathcal{O}(3K_X)) = 0$. Let C be a curve from $|-2K_X|$. There is an exact sequence of sheaves

$$0 \longrightarrow \mathcal{O}(K_X - C) \longrightarrow \mathcal{O}(K_X) \longrightarrow \mathcal{O}(K_X) \otimes \mathcal{O}_C \longrightarrow 0. \tag{15}$$

Since X is rational, Serre duality yields $h^1(X,\mathcal{O}(K_X)) = h^1(X,\mathcal{O}) = 0$. On the other hand $h^0(X,\mathcal{O}(K_X) \otimes \mathcal{O}_C) = h^0(C,\mathcal{O}(K_X)|_C) = 0$ since $(K_X. C) < 0$. Now the long cohomology exact sequence provided by (15) implies $h^1(X,\mathcal{O}(K_X - C)) = h^1(X,\mathcal{O}(3K_X)) = 0$. Therefore we have $h^0(X,\mathcal{O}(-2K_X)) = 13$. Let Y be the image of X in $\mathbb{P}^4_k$. Since $h^0(\mathbb{P}^4_k, \mathcal{O}(2)) = 15$, Y is contained in at least two quadrics Q_0 and Q_∞. The degree of Y equals 4, thus $Y = Q_0 \cap Q_\infty$. An effective curve from $|-K_X - 2\ell_0|$ is clearly contracted to a singular point on Y, since $(-K_X . -K_X - 2\ell_0) = 0$; the two singular $\bar{k}$–points correspond to the two components. The singularities are rational of type A_1, since the self–intersection of each component equals -2. Suppose that the line v joining the singular points lies on Y. Let b be its class in $\operatorname{Pic} \bar{X}$. We have $(-K_X . b) = 1$, $(-K_X - 2\ell_0 . b) = 2$, hence $(\ell_0.b)$ is not an integer. The contradiction proves our claim.

Let us prove the converse. In the pencil of quadrics defining Y there exists the unique (and hence defined over k) quadric containing the line v joining the singularities of Y. Indeed, the condition that a quadric Q contains a point $x \in v \setminus Y$ gives one linear restriction on the coefficients of Q. Therefore (since $x \notin Y$) such Q is unique, we denote it by Q_0. Three different points of v lie on Q_0, hence $v \subset Q_0$. The line v intersects Q_∞ transversally, and the intersection points are both singular on $Y = Q_0 \cap Q_\infty$. Thus $v \subset (Q_0)_{\text{sing}}$. Since $(Q_0)_{\text{sing}} \cap Q_\infty \subseteq Y_{\text{sing}}$ and $Y_{\text{sing}} \subset v$, we see that $v = (Q_0)_{\text{sing}}$.

The same method shows that there are no more quadrics of rank 3 in the pencil $Q_\lambda = Q_0 + \lambda Q_\infty$, and the polynomial $\lambda^{-2} \det Q_\lambda$ has no multiple roots. Assume Q_∞ to be smooth. Let M be the 2–dimensional k–vector space such that $\mathbb{P}(M) = v$. Since $Q_\infty|_M$ is nondegenerate, $Q_\infty|_{M^\perp}$ is also nondegenerate ($M^\perp$ is the orthogonal complement to M with respect to Q_∞). Let x_0 and x_1 be the orthogonal coordinates on M, and x_2, x_3 and x_4 be the coordinates on $M^\perp$. Then Q_∞ is of the form $x_0^2 - ax_1^2 + Q'_\infty(x_2,x_3,x_4) = 0$, and Q_0 is given by $Q_0^b(x_2,x_3,x_4) = 0$. Consider the projection pr from v, $pr : Y \setminus Y_{\text{sing}} \longrightarrow Q_0^b \subset \mathbb{P}(M^\perp)$.

Since $Y_{\text{sing}} \subset v$, a look on the Jacobian matrix of Y reveals that the set $Q_0^b = Q'_\infty = 0$ consists of 4 distinct points. The equations show that the fibres of pr are conics, and exactly 4 of them are singular. Let $\tilde{Y}$ be the proper transform of Y with respect to the blow–up centred at Y_{sing}. Clearly $\tilde{Y}$ is the minimal desingularization of Y, and pr can be lifted to a morphism $f : \tilde{Y} \longrightarrow Q_0^b$ defining a conic bundle structure of degree 4 on $\tilde{Y}$. The two components of the exceptional curve are conjugate over k. By the adjunction formula $-K_Y$ is the class of a hyperplane section on Y. Therefore $-K_{\tilde{Y}}$ is the proper transform on $\tilde{Y}$ of this class. By $e \in \operatorname{Pic} \tilde{Y}$ let us denote the class of the exceptional curve, then the equations $(e.K_{\tilde{Y}}) = 0$ and $(e.\ell_0) = 2$ imply $e \in |-K_{\tilde{Y}} - 2\,\ell_0|$.

Remark 3.9. The second part of the proof was influenced by the proof of lemma 2a of section 9 of [12].

Note also that theorem 7.2 of [14] states that if a singular intersection of two quadrics $Y \subset \mathbb{P}_k^4$ is k–minimal (in the sense that its minimal desingularization is k–minimal), then Y is an Iskovskih surface.

Remark 3.10. Assume that there is a k–rational point on the conic Q_0^b. Then up to a linear transformation Q_0^b is of the form $x_2\, x_3 - x_4^2 = 0$. Let $x_2 = x^2$, $x_3 = t^2$, $x_4 = xt$ be a rational parametrization, i.e. an isomorphism $Q_0^b \simeq \mathbb{P}_k^1$. Substituting these expressions into the equation of Q_0^b we get a surface

$$x_0^2 - a\, x_1^2 = P(x,t) \tag{16}$$

$P(x,t)$ being a homogeneous polynomial of degree 4 without multiple roots. The natural smooth compactification of the affine conic bundle (16) is called a (generalized) Châtelet surface (cf. [7]). This compactification can be constructed in the following way. Let

$$y^2 - a\, z^2 = f(x)\, t^2$$

and

$$Y^2 - a\, Z^2 = f(\lambda^{-1})\, T^2$$

be the equations defining surfaces in $\mathbb{P}_k^2 \times \mathbb{A}_k^1$. Let us glue them in $\mathbb{P}_k^2 \times (\mathbb{A}_k^1 \backslash 0)$, putting $y = Y$, $z = Z$, $t = \lambda^2 T$, $x = \lambda^{-1}$. The fibre over $t = \infty$ is smooth iff the leading coefficient of $f(x)$ is not zero.

The condition $Q_0^b(k) \neq \emptyset$ is equivalent to the condition that the class ℓ_0 of a closed geometric fibre of $f : \tilde{Y} \longrightarrow Q_0^b$ is defined over k, i.e. $\ell_0 \in \text{Pic } \tilde{Y}$. This is equivalent to $\text{Pic } \tilde{Y} = (\text{Pic } \bar{\tilde{Y}})^{\mathfrak{g}}$, i.e. to the condition that the natural homomorphism $Br\, k \longrightarrow Br\, \tilde{Y}$ is injective (cf. the exact sequence after 2.15). Both conditions are clearly satisfied if $Y(k) \neq \emptyset$. We reformulate Proposition 3.8 as follows.

COROLLARY 3.11. *The minimal desingularization X of an Iskovskih surface satisfying any of the equivalent properties:*

i) $\text{Pic } X = (\text{Pic } \bar{X})^{\mathfrak{g}}$;

ii) *the natural map $Br\, k \longrightarrow Br\, X$ is injective;*

is isomorphic to a (generalized) Châtelet surface.

COROLLARY 3.12. *Any relatively k-minimal (generalized) Châtelet surface is biregularly isomorphic to the minimal desingularization of some Iskovskih surface.*

Proof : Let us write the equation of the Châtelet surface X in the form

$$x_0^2 - ax_1^2 = P(x,t)x_2^2.$$

The curve on X given by $x_2 = 0$ is $\mathfrak{g}$–invariant. It is easy to check that its class is just $-K_X - 2\,\ell_0$.

Remark 3.13 : It is not difficult to obtain the equations of two quadrics defining the anticanonical model of X. Let $y^2 - az^2 = P(x)$ be the equation of the (generalized) Châtelet surface X. Set $u = x^2$. If $P(x) = \sum_{i=0}^{4} a_i x^i$, $a_4 \neq 0$, then the anticanonical model is the intersection of two quadrics Y:

$$\begin{cases} y^2 - az^2 = a_0 + a_2 u + a_4 u^2 + a_1 x + a_3 xu \\ x^2 = u \end{cases}$$

Remark 3.14 : Consider an Iskovskih surface Y, and let X be its minimal desingularization. The group of automorphisms of $\text{Pic } \bar{X}$ preserving K_X, ℓ_0 and the intersection pairing, is the Weyl group $W(D_4)$ (Corollary 2.13). The splitting group G of X acts on degenerate fibres of $f : X \longrightarrow C$, hence on the 4 singular points of these fibres. Here is a funny statement (the proof is left to the reader). Let $x_1,\ldots,x_4$ be the intersection points of the 4 pairs of lines on Y. Let

Q_1, Q_2 and Q_3 be the quadrics of rank 4 in the pencil $Q_\lambda = Q_0 + \lambda\, Q_\infty$, v_1, v_2, v_3 being their singular points (the vertices of the cones). Then the points v_i are the intersection points of the lines joining x_i and x_j (see Figure 5).

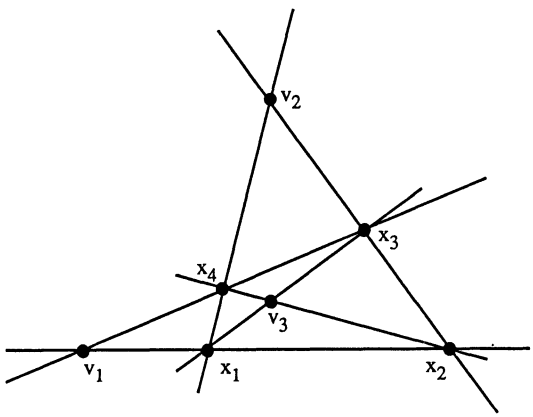

Figure 5

In particular it follows that G acts on $\{v_1, v_2, v_3\}$ via the map $G \hookrightarrow W(D_4) \xrightarrow{\alpha} S_4 \xrightarrow{\beta} S_3$, where α is the natural projection corresponding to the action on $\{x_1,\dots,x_4\}$, and β is the factorization by Klein's Vierergruppe.

The proper transform on X of a conic on an Iskovskih surface is also called a conic.

LEMMA 3.15. *Let Z be a Del Pezzo surface of degree 4 (respectively, an Iskovskih surface). Let $q_i^\pm$, $i = 1,\dots,5$, be the one-dimensional linear systems of conics on Z cut by the planes lying on the quadrics Q_i of rank 4 of the pencil $Q_\lambda = Q_0 + \lambda Q_\infty$ (respectively, let ℓ_0 and $q_i^\pm$, $i = 1,2,3$, be the one-dimensional linear systems of conics on Z cut by planes lying on the quadric Q_0 of rank 3 and on the quadrics Q_i of rank 4 in the pencil $Q_\lambda = Q_0 + \lambda Q_\infty$). Then any conic on Z belongs to one of these families.*

Proof : Let $L \simeq \mathbb{P}^2_{\bar{k}}$ be the plane passing through a conic $q \subset Z$. Since Z contains no plane, there is a unique quadric Q in the pencil Q_λ which contains $x \in L \setminus Z$. Then Q contains any line ℓ passing through x, since it contains 3 distinct points of ℓ. Thus Q contains L. This can happen only if $rk\,Q \leq 4$.

If Z is a Del Pezzo surface, $rk\,Q = 4$ and $Q = Q_i$ for some i, $i = 1,...,5$. If Z is an Iskovskih surface, then either $rk\,Q = 3$ and $Q = Q_0$, or $rk\,Q = 4$ and $Q = Q_i$ for some i, $i = 1,2,3$.

Let us fix some notation. Let $\pi : X \longrightarrow Y$ be the minimal desingularization of an Iskovskih surface. Let $f : X \longrightarrow Q_0^b$ be the natural conic bundle over the base of the cone defined by the unique quadric Q_0 of rank 3. By $R \subset Q_0^b$ we denote the minimal subscheme outside of which f is smooth, i.e. $\bar{R} \subset \bar{Q}_0^b$ consists of 4 points over which the fibres of f are singular. On $\bar{X}$ there are two irreducible curves S_1 and S_2 with $(S_1^2) = (S_2^2) = -2$ (the inverse images of singularities of $\bar{Y}$). Let S_X be the k–subscheme of X such that $\bar{S}_X = S_1 \cup S_2$.

We say that a curve C on $\bar{X}$ is a line (respectively, a conic) if it is a proper transform of a line (respectively, of a conic) on $\bar{Y}$.

LEMMA 3.16. *There exists a positive number N such that if* card $k > N$ *and X is a minimal desingularization of an Iskovskih surface Y defined over k, then the following properties hold.*

a) If $X(k) \neq \emptyset$, then there exist k-points on X which do not lie on lines of $\bar{X}$.

b) If $X(k) = \emptyset$ and $X(K) \neq \emptyset$ for some quadratic extension K/k, then there exists a $\mathfrak{g}$*-invariant pair of points $x_1,x_2 \in \bar{X}$ such that each of them does not lie on lines of $\bar{X}$, and they do not lie both on one conic of $\bar{X}$.*

c) Let $X(K) = \emptyset$ for all quadratic extensions K/k. Then there exists a $\mathfrak{g}$*–invariant 4-tuple of points $x_1,...,x_4 \in \bar{X}$, such that $f(x_i) \neq f(x_j)$ for $i \neq j$, none of these points is mapped to a singularity of Y, and there does not exist a* $\mathfrak{g}$*–invariant pair of irreducible curves W_1 and W_2 on $\bar{X}$ with $(W_i \cdot l_0) = 1$, $(W_i^2) = 2$, $(W_i \cdot K_X) = -4$, and $W_1 \cap W_2 = \{x_1,...,x_4\}$ (scheme-theoretically).*

Proof : a) If k is infinite, the statement follows from the k–unirationality of X (cf. [14], Lemma 7.1), therefore k–points on X are Zariski–dense. If k is finite it is enough to use the Lang–Weil theorem.

b) First of all note that if k is finite, $X(k)$ is always non–empty (cf. [28], IV.5.1, Corollary 1); therefore in what follows we suppose that k is infinite.

Let us first prove that X_K is K–unirational. If Y has a smooth K–point, we just use the same lemma of [14]. The only case left is when all the K–points of X are mapped to the singularities of Y. In this case each singularity is defined over K, the 4 lines passing through one of them are globally Gal $(\bar{K}/K)$–invariant. Contracting these lines on X we obtain a ruled surface. The only section having negative self–intersection is the inverse image of the other singular point of Y. This is a smooth K–rational curve with self–intersection -2. Contracting it we obtain a quadratic cone with a K–point on its base.

In what follows we use Weil's descent variety $R_{K/k}(X_K)$, cf. [45]. We need the following

SUBLEMMA 3.17. *Let L/k be a finite Galois extension of degree m.*

a) Let Z be an L-unirational L-variety. Then $R_{L/k}(Z)$ is a k-unirational k-variety.

b) Let X be a k-variety such that X_L is L-unirational. Then S^mX is k-unirational.

Proof : a) The dominant map $\mathbb{A}^n_L \longrightarrow Z$ yields a dominant map $\mathbb{A}^{mn}_k \simeq R_{L/k}(\mathbb{A}^n_L) \longrightarrow R_{L/k}(Z)$.

b) By definition, $\mathrm{Hom}_k(S,R_{L/k}(X_L)) \simeq \mathrm{Hom}_L(S_L,X_L)$ for any k–scheme S. There is a natural map $R_{L/k}(X_L) \longrightarrow S^mX$ which can be defined on $\bar{k}$–points as

$$\pi : \mathrm{Hom}(\mathrm{Spec}\ \bar{k},R_{L/k}(X_L)) \simeq \mathrm{Hom}(\mathrm{Spec}(L\otimes_k \bar{k}),X_L) \simeq X(\bar{k})^m \longrightarrow \mathrm{Hom}(\mathrm{Spec}\ \bar{k},\ S^mX),$$

sending a $(L\otimes_k \bar{k})$–point, which is an ordered m–tuple of $\bar{k}$–points, to this very m–tuple unordered. This morphism is dominant since $\dim_k(S^mX) = \dim_k(R_{L/k}(X_L))$ and it is finite. The sublemma is proved.

Now we can complete the proof of b). Consider *bad* k–points of $R_{K/k}(X_k)$, i.e. such that the corresponding pair $x_1,x_2 \in \bar{X}$ (we can also view x_1,x_2 as conjugate elements of $X(K)$) does not satisfy the conditions of the lemma. We are going to prove that *bad* k–points of $R_{K/k}(X_K)$ are contained in a Zariski–closed subset of smaller dimension. Then from Sublemma 3.17 and K–unirationality of X_K, it will follow that *good* k–points are dense on $R_{K/k}(X_K)$, which will prove b).

First of all it is clear that $R_{K/k}((S_X)_K)$ is closed in $R_{K/k}(X_K)$ and does not coincide with it.

Let σ be the non–trivial element of $\mathrm{Gal}(K/k)$, it becomes a k–automorphism of $R_{K/k}(X_K)$. Let us first look at the case when x and σx both belong to a conic from the pencil q_i^+, $i = 1,2$, or 3. If x and σx also belong to a conic from the pencil ℓ_0 ,q_j^+ or q_j^-, $j \neq i$, then since $(q_i^+ . \ell_0) = (q_i^+ . q_i^{\pm}) = 1$ (cf. Proposition 3.1) we have either $x = \sigma x$ which is not true, or these two conics have a component in common. But this component is a $\mathfrak{g}$–invariant line, which is impossible since X is k–minimal. Assume that there is no conic from the pencil q_i^-

containing x and σx. The Galois group $\mathfrak{g}$ acts on the set of conic pencils by permutations. Since q_i^+ is the class of the unique conic containing the $\mathfrak{g}$–invariant pair $(x, \sigma x)$, we have $q_i^+ \in (\mathrm{Pic}\,\bar{X})^{\mathfrak{g}}$. However this contradicts $(\mathrm{Pic}\,\bar{X})^{\mathfrak{g}} = \mathbb{Z}K_X + \mathbb{Z}l_0$ (Proposition 2.10).

We have proved that x and σx are necessarily the two intersection points of a conic from the pencil q_i^+ and a conic from the pencil q_i^-. It follows that the line joining the images of x and σx on Y passes through the vertex v_i of the cone Q_i. Let us prove that the set of points x satisfying this property is Zariski–closed in $R_{K/k}(X_K)$. Note that Q_i is defined over k. Let the coordinates $x_0,\dots,x_4$ be such that q_i is a quadratic form in $x_1,\dots,x_4$ and $v_i = (1,0,0,0,0)$. The anticanonical model Y of X is given by $Q_i(x_1,\dots,x_4) = Q_0(x_0,\dots,x_4) = 0$. We can assume that Q_0 is of the form $x_0^2 + Q_0'(x_1,\dots,x_4)$, where Q_0' is a quadratic form of rank 2. Since x, σx, and v_i are collinear, it follows that $\sigma(x) = \sigma(x_0,x_1,x_2,x_3,x_4) = (-x_0,x_1,x_2,x_3,x_4)$, where by abuse of notation we denote by x its image on Y. This defines a Zariski–closed set V_1. It is clear that $V_1 \neq R_{K/k}(X_K)$, which is enough to check over $\bar{k}$.

Likewise if x and σx both lie on a conic from the pencil l_0, then this conic is $\mathfrak{g}$–invariant. In particular, if the coordinates are chosen in the same way as in the proof of Proposition 3.8, then $\sigma(x_2) = x_2$, $\sigma(x_3) = x_3$, $\sigma(x_4) = x_4$. Let V_2 be the closed subset of $R_{K/k}(X_K)$ defined by these equations. Again $V_2 \neq R_{K/k}(X_K)$, which is clear if one checks it on $\bar{k}$–points.

c) We want to find a k–point $x = (x_1,\dots,x_4)$ on S^4X, outside the following closed subsets : $V_1 = \{x \mid f(x_i) = f(x_j)$ for some $i \neq j, 1 \leq i,j \leq 4\}$, $V_2 = \{x \mid f(x_i) \in \bar{R}$ for some $i, i = 1,\dots,4\}$, $V_3 = S^4S_X$, and V_4 which is the closure of the union of all $W_1 \cap W_2$, where W_i are curves satisfying the conditions of the lemma. It is clear that $V_1 \cup V_2 \cup V_3 \neq S^4X$ (just check it over $\bar{k}$). If we prove that $V_4 \neq S^4X$ then our claim will follow from Sublemma 3.17 and L–unirationality of X_L, where L is the residue field of a closed point of degree 4 on X, whose image on Y is smooth. Indeed, under our assumptions a k–point on $S^4X \setminus \bigcup_{i=1}^{4} V_i$ gives rise to such a point of degree 4.

First note that the conditions $(W_i.l_0) = 1$, $(W_i.K_X) = -4$ determine the class of the curve W_i in $\mathrm{Pic}\,\bar{X}$ up to a finite number of possibilities. In fact, since W_i is a section of f, $(W_i.l_j)$ is either 0, or 1 for any component l_j of a degenerate fibre. From Lemma 2.10 it follows that the class in $\mathrm{Pic}\,\bar{X}$ is uniquely determined by its intersection numbers with l_j, $j = 0,1,\dots,4$, and with K_X. Let us fix one possible class of W_1 in $\mathrm{Pic}\,\bar{X}$, call it D.

Let us proceed over $\bar{k}$. We see that $H^2(X,\mathcal{O}(D)) = H^0(X,\mathcal{O}(K_X-D)) = 0$ since $(K_X-D.D) = -6 < 0$, D being the class of a curve. We claim that $H^1(X,\mathcal{O}(D) = 0$. Consider the exact sequence

$$0 \longrightarrow \mathcal{O}(K_X-D) \longrightarrow \mathcal{O}(K_X) \longrightarrow \mathcal{O}(K_X)\otimes \mathcal{O}_{W_1} \longrightarrow 0.$$

We have $H^1(X,\mathcal{O}(K_X)) = 0$. Since $(K_X.D) = -4 < 0$,

$$H^0(X,\mathcal{O}(K_X)\otimes \mathcal{O}_{W_1}) = H^0(W_1,\mathcal{O}(K_X)) = 0,$$

and the long cohomology sequence gives $H^1(X,\mathcal{O}(D)) = 0$. The Riemann–Roch theorem now gives

$$\dim H^0(X,\mathcal{O}(D)) = (D.D-K_X)/2 + 1 = 4.$$

Hence $\dim |W_1| = 3$.

Let K/k be the quadratic extension over which D is defined. Consider the open subset T of $|W_1|$, whose K–points are curves C_1, such that $C_1 \cap C_2$ consists of 4 distinct $\bar{k}$–points (here $C_2 \in |W_2|$ is the conjugate curve). The morphism $h : R_{K/k}(T) \longrightarrow S^4X$ mapping C_1 to $C_1 \cap C_2$ goes from a 6–dimensional variety, hence it is not dominant. Now remark that V_4 is the union of a finite number of images of such maps, hence $V_4 \neq S^4X$. The lemma is proved.

Let $f : X \longrightarrow C$ be a conic bundle. Assume that we are given a $\mathfrak{g}$–invariant set P of points of $\bar{X}$, which do not lie on geometric degenerate fibres of $\bar{X} \longrightarrow \bar{C}$, and such that each fibre contains at most one point of P. Then we can define the elementary transformation centred at P, $elm_P : X \dashrightarrow X'$, as the composition of the blow–up centred at the points of P and the blow–down of the proper transforms of the fibres containing these points. Then X' is a conic bundle over the same base C, and the generic fibres of X and X' are isomorphic.

THEOREM 3.18. *Let X be the minimal desingularization of an Iskovshih surface Y over k.*

a) If $X(k) \neq \emptyset$, then an elementary transformation centred at a good k-point of X (i.e. at a point which does not lie on any line) transforms X into a Del Pezzo surface of degree 4 on which there lies a k-pair of intersecting lines.

b) If $X(k) = \emptyset$ and $X(K) \neq \emptyset$ for some quadratic extension K/k, then an elementary transformation centred at a good $\mathfrak{g}$–invariant pair of points of $\bar{X}$ (i.e. at a pair of points $x_1, x_2 \in \bar{X}$ such that each of them does not lie on any line and is not mapped to a singular point of $\bar{Y}$, and they do not both lie on any conic) transforms X into a Del Pezzo surface of degree 4.

c) If $X(K) = \emptyset$ for any quadratic extension K/k, then an elementary transformation centred at a good $\mathfrak{g}$–invariant 4-tuple of points of X (i.e. one satisfying the conditions described in part c) of Lemma 3.16) transforms X into a Del Pezzo surface of degree 4.

Therefore (if the cardinality of k is large enough) an Iskovskih surface is always birational to a del Pezzo surface of degree 4).

Note that the elementary transformations are well defined since by definition good points do not belong to degenerate fibers, a good pair of points does not lie in a fibre, and a good 4–tuple of points is mapped to 4 distinct $\bar{k}$–points on the base.

Proof : a) We give two different proofs. The first one is based on the Proposition 1 of [20]. It implies that a relatively k–minimal conic bundle $Z = elm_x X$ is a Del Pezzo surface iff the linear system $|-K_Z - 2\ell_0|$ is empty. Assume that it is not. Let s_2 be a curve from $|-K_Z - 2\ell_0|$, and s_1 be a curve from $|-K_Z - 2\ell_0|$ (the anticanonical map contracts s_1 to the singularities of Y). Let $\sigma_1 : X' \longrightarrow X$ be the blow–up of a good point $x \in X(k)$. Let $\sigma_2 : X' \longrightarrow Z$ be the blow–down to $z \in Z(k)$ of the proper transform of the fibre containing x. Then $\sigma_1^* s_1 = -K_{X'} + \sigma_1^{-1}(x) - 2\ell_0$, $\sigma_2^* s_2 = -K_{X'} + \sigma_2^{-1}(z) - 2\ell_0$, $(\sigma_1^* s_1 \cdot \sigma_2^* s_2) = -2$, which is impossible since $\sigma_1^* s_1$ and $\sigma_2^* s_2$ are curves. Thus $|-K_X - 2\ell_0|$ and $|-K_Z - 2\ell_0|$ cannot be both non–empty.

The other proof has the advantage that it does not require the relative k–minimality of $f : X \longrightarrow C$. The two components of a curve from $|-K_X - 2\ell_0|$ are conjugate and do not intersect each other, therefore a good point $x \in X(k)$ does not lie on this curve. The remaining part of the proof is valid over any extension L of k. Since x is a good point, the projection from x maps Y birationally onto a cubic surface Y', this map being the blow–up of x (isomorphic on $Y \setminus x$). The cubic Y' has precisely two singular points, which are the images of the two singular points on Y. The line l joining them lies on Y'. Clearly, l is the image of the conic from the pencil l_0 containing x and the singularities of Y. Let $X' \longrightarrow Y'$ be the minimal desingularization of Y', then the proper transform of l is an exceptional curve of the first kind. Let $X' \longrightarrow X''$ be the contraction of this curve, then X'' is smooth. In theorem 1.3b of [14] it is proved (using the Nakai–Moishezon criterion) that $-K_{X''}$ is ample, hence X'' is isomorphic to a Del Pezzo surface of degree 4.

b) Let $x = (x_1,x_2)$ be a good $\mathfrak{g}$–invariant pair of $\bar{k}$–points of $\bar{X}$ satisfying the assumptions of the theorem. Let $L = k(x_1) = k(x_2)$. Over L, elm_x decomposes into two elementary transformation elm_{x_1} and elm_{x_2}. Let $Z = elm_{x_1}(X)$. According to the second proof of a) Z is a Del Pezzo surface of degree 4. It remains to prove that $elm_{x_1}(x_2)$ is a good point on Z, and that an elementary transformation of a Del Pezzo surface centred at a good point is again a Del Pezzo surface. Assume that $elm_{x_1}(x_2)$ lies on a line $l_1 \subset Z$, which in this case is a section of the conic bundle $f' : Z \longrightarrow C$. Let $z \in Z(L)$ be the image under elm_{x_1} of the fibre of f containing x_1. There are two possibilities :

1) $z \in l_1$. Then the self–intersection of $l_2 = elm_z(l_1)$ on X equals -2 (note that elm_z is the inverse of elm_{x_1}). Hence the anticanonical map contracts l_2 to a singular point, thus contradicting the choice of $x = (x_1,x_2)$.

2) $z \notin l_1$. Then the self–intersection of $l_2 = elm_z(l_1)$ on X equals 0, hence by the genus formula we have $(-K_X \cdot l_2) = 2$. Thus l_2 is a conic, which again contradicts the choice of $x = (x_1,x_2)$. Therefore $elm_{x_1}(x_2)$ does not lie on any line of Z, i.e. is a good point.

Let us finally show that the elementary transformation centred at a good point maps Z onto a Del Pezzo surface. Indeed, blowing up a good point we get a smooth cubic surface, the proper transform of the fibre being a line. Contracting a line on a smooth cubic surface we obtain a del Pezzo surface of degree 4.

c) Let $x = (x_1,\dots,x_4)$ be a good $\mathfrak{g}$–invariant 4–tuple; set $L = k(x_1)$, then $[L:k] = 4$. Let $Z = elm_x(X)$. Since Z is a relatively minimal conic bundle with 4 degenerate geometric fibres, it is either a Del Pezzo surface, or the minimal desingularization of an Iskovskih surface. Let us exclude the second possibility. In fact, let $S_Z \in |-K_Z - 2l_0|$; we know that $\bar{S}_Z = S_1 \cup S_2$, S_i being irreducible, $(S_i^2) = -2$, $(S_i \cdot K_Z) = 0$, $(S_i \cdot l_0) = 1$, $(S_1 \cdot S_2) = 0$, S_1 and S_2 are conjugate over k. We know that $elm_x^{-1} = elm_y$ for some $\mathfrak{g}$–invariant 4–tuple $y = (y_1,\dots,y_4)$, $k(y_1) = L$.

Since $y_1,\dots,y_4$ are all conjugate and $S_1 \cap S_2 = \emptyset$, there are only two possibilities : either each S_i contains two points from $y_1,\dots,y_4$, or each S_i contains none. Now we proceed over $\bar{k}$. It is easy to see that $(S_i^2) = (W_i^2) + c_i - (4-c_i)$, where c_i is the number of y_i's lying on S_i; $(W_i \cdot K_X) = -2 - (W_i^2)$, $(W_i \cdot l_0) = 1$. If $c_1 = c_2 = 2$, then $(W_i^2) = -2$, $(W_i \cdot K_X) = 0$. This means that W_1 and W_2 are contracted into singularities under the anticanonical map. On the other hand, since two of the points $\{y_1,\dots,y_4\}$ do not lie on S_1, two of the points $\{x_1,\dots,x_4\}$ lie on W_1, which contradicts the assumption that none of them is mapped to a singularity of Y. If $c_1 = c_2 = 0$, then W_1 and W_2 satisfy the assumptions of Lemma 3.16, which also gives a contradiction. Thus Z is a Del Pezzo surface.

By Lemma 3.16 there always exists a good point, pair, or 4–tuple; and the theorem is proved.

LEMMA 3.19. *On a Del Pezzo surface X of degree 4 through a point there can pass at most two lines.*

Proof : Let $x \in X(\bar{k})$, and let $\Theta_{x,X}$ be the tangent space to X in x. If $x \in \ell$ and $\ell \subset X$, then clearly $\ell \subset X \cap \Theta_{x,X}$. If $X \cap \Theta_{x,X}$ contains more than two lines, then $\Theta_{x,X} \subset X$. Indeed in this case $\Theta_{x,X}$ belongs to any quadric of the pencil $Q_\lambda = Q_0 + \lambda\, Q_\infty$. Since X is irreducible, the lemma is proved.

THEOREM 3.20. a) *Let X be a k-minimal Del Pezzo surface of degree 4 containing two conjugate intersecting lines (in this case the intersection point is defined over k, hence $X(k) \neq \emptyset$). Then X is birationally equivalent to an Iskovskih surface.*

b) *Let X be a k-minimal Del Pezzo surface of degree 4, $X = Q_0 \cap Q_\infty$. Let the following conditions hold :*

1) *the polynomial* $\det(Q_0 + \lambda\, Q_\infty)$ *has at least two roots λ_1 and λ_2 in k,*

2) *the determinant of the restriction of the quadratic form $Q_1 = Q_0 + \lambda_1 Q_\infty$ to a hyperplane H (such that* $\mathrm{Ker}(Q_1) \not\subset H$) *is a square in k,*

3) *the quadric $Q_2 = Q_0 + \lambda_2\, Q_\infty$ has a smooth k-point. Then X is birationally equivalent to an Iskovskih surface ($X(k)$ can be empty).*

Proof : Let us prove that the birational map inverse to the map described in Theorem 3.18 a) gives the required birational equivalence. Namely, let s be a curve on X, splitting in $\bar{X}$ into two lines. Let q be the pencil of conics on X containing s as a degenerate conic. Let $f: X \longrightarrow C$ be the conic bundle structure on X, corresponding to the pencil of conics complementary to q. Let us blow up the intersection point x of the components of s. Consider the proper transform of the fibre of f containing x. It is nonsingular according to Lemma 3.19. Let us contract it to a point y. Denote the blow–up of x (respectively of y) by $\sigma_1 : X' \longrightarrow X$ (respectively by $\sigma_2 : X' \longrightarrow Y$). Let us prove that $|-K_Y - 2\ell_0|$ is not empty, which is enough to show that Y is an Iskovskih surface. By $\sigma_1^{-1}(s)$ denote the proper transform of s. Clearly $\sigma_2(\sigma_1^{-1}(s)) \in |-K_Y - 2\ell_0|$. Indeed, $s = -K_Y - \ell_0$ (here ℓ_0 is the class of a fibre of f), and the multiplicity of x on s equals two. Hence

$$\begin{aligned}\sigma_1^{-1}(S) = \sigma_1^*(S) - 2\,\sigma_1^{-1}(x) = -\sigma_1^*(K_X) - \ell_0 - 2\,\sigma_1^{-1}(x) &= -K_{X'} - \ell_0 - \sigma_1^{-1}(x)\\ &= -K_{X'} - 2\,\ell_0 + \sigma_2^{-1}(y),\end{aligned}$$

since $\ell_0 = \sigma_1^{-1}(x) + \sigma_2^{-1}(y) \in \operatorname{Pic}\bar{X}'$. It follows that $\sigma_2(\sigma_1^{-1}(s)) = -K_Y - 2\ell_0$, and we are done.

b) The quadric Q_2 of rank 4 is a cone with a vertex v_2, its base being a nonsingular quadric $Q_2^b \subset \mathbb{P}_k^3$. There is a smooth k–point on Q_2 if $Q_2^b(k) \neq \emptyset$, i.e. iff there exists a k–line ℓ such that $v_2 \in \ell \subset Q_2$. Since $Q_2^b(k) \neq \emptyset$, k–points are dense in Q_2^b, thus we can choose ℓ in such a way that $\ell \cap Q_1 = \{x_1, x_2\}$, $x_1 \neq x_2$. Since $\bar{Q}_2^b \simeq \mathbb{P}_{\bar{k}}^1 \times \mathbb{P}_{\bar{k}}^1$, there exists a $\mathfrak{g}$–invariant pair of projective planes (h_2^+, h_2^-) in Q_2 such that $h_2^+ \cap h_2^- = \ell$. Let $q_2^{\pm} = h_2^{\pm} \cap Q_2$ be the pair of conics on X cut by h_2^+ and h_2^-. Since $(\operatorname{Pic}\bar{X})^{\mathfrak{g}} \simeq \mathbb{Z} \oplus \mathbb{Z}$, the condition 2 of the theorem implies that $(\operatorname{Pic}\bar{X})^{\mathfrak{g}}$ is generated by the classes of q_1^+ and q_1^- (cf. comments before Proposition 3.5). Consider the conic bundle structure $f: X \longrightarrow C$ related to one of these pencils, say, to q_1^+. By Proposition 3.1 c) $(q_2^{\pm}, q_1^{\pm}) = 1$, hence q_2^+ and q_2^- are sections of f. The birational map inverse to that of theorem 3.18 blows up x_1 and x_2, and contracts the proper transform of the fibres passing through x_1 and x_2 to some points y_1 and y_2. Let $\sigma_1 : X' \longrightarrow X$ and $\sigma_2 : X' \longrightarrow Y$ be the corresponding monoidal transformations. Let us prove that $|-K_Y - 2\ell_0|$ is not empty.

The multiplicity of x_1 or x_2 on the curve $q_2^+ + q_2^-$ equals two, therefore $\sigma_1^{-1}(q_2^+ + q_2^-) = \sigma_1^*(-K_X) - 2(\sigma_1^{-1}(x_1) + \sigma_1^{-1}(x_2))$ (we have $-K_X = q_2^+ + q_2^-$ in $\mathrm{Pic}\,\bar{X}$). Since $\sigma_1^*(-K_X) = -K_{X'} + \sigma_1^{-1}(x_1) + \sigma_1^{-1}(x_2)$, we have

$$\sigma_1^{-1}(q_2^+ + q_2^-) = -K_{X'} - (\sigma_1^{-1}(x_1) + \sigma_1^{-1}(x_2)) = -K_{X'} - 2\ell_0 + \sigma_2^{-1}(y_1) + \sigma_2^{-1}(y_2).$$

Thus $\sigma_2(\sigma_1^{-1}(q_2^+ + q_2^-)) = -K_Y - 2\ell_0$, and we are done.

COROLLARY 3.21. a) *The birational transformation (inverse to that introduced in Theorem* 3.18 a) *maps a k-minimal Del Pezzo surface of degree* 4 *onto an Iskovskih surface iff the graph* Γ *contains a* $\mathfrak{g}$*-orbit of precisely two vertices. Equivalently, the type of the decomposition of* Γ *into* $\mathfrak{g}$*-orbits is a subtype of type* III.

b) *If the inverse of the birational transformation introduced in Theorem* 3.18 b *maps a k-minimal Del Pezzo surface of degree* 4 *onto an Iskovskih surface, then* $\mathfrak{g}$ *acts on the set of* 10 *conic pencils* $q_i^{\pm}$ *in such a way that* q_1^+ *and* q_1^- *are individually* $\mathfrak{g}$*-invariant (this is equivalent to the existence of a* $\mathfrak{g}$*-invariant conic pencil on* X), *and there exists a* $\mathfrak{g}$*-orbit* $\{q_j^+, q_j^-\}$ *for some* $j \neq 1$.

Remark 3.22 : The conclusions of a) and b) of Corollary 3.21 look somewhat similar. This can be explained as follows. Assume $X(k) \neq \emptyset$, then there is a good k–point x on X ([28], ch. IV, 8.1). Let $\sigma : Y \to X$ be the blow–up of this point; Y is a smooth cubic surface. The point x is contained in a unique conic from the pencil q_1^+ (respectively from q_1^-). Let ℓ_1 and ℓ_2 be the proper transforms of these conics on Y, naturally ℓ_1 and ℓ_2 are lines. If $(\mathrm{Pic}\,\bar{X})^{\mathfrak{g}} \simeq \mathbb{Z} \oplus \mathbb{Z}$ as it was assumed in Corollary 3.21, then ℓ_1 and ℓ_2 are defined over k. Therefore there is a triangle of k–lines on Y, namely ℓ_1, ℓ_2 and $\sigma^{-1}(x)$. Planes containing any of the three k–lines cut three pencils of conics on Y, having 5 degenerate fibres each. For example, planes containing ℓ_1 cut a pencil birationally equivalent to the pencil q_1^- on X, equipped with an additional degenerate fibre $\ell_2 \cup \sigma^{-1}(x)$. Planes containing ℓ_2 cut a pencil birationally equivalent to the pencil q_1^+ on X, equipped with an additional degenerate fibre $\ell_1 \cup \sigma^{-1}(x)$. Finally, planes passing through $\sigma^{-1}(x)$ cut a pencil of conics on Y with degenerate fibres $\sigma^{-1}(q_i^+) \cup \sigma^{-1}(q_i^+)$, $i = 2,\dots,5$, and $\ell_1 \cup \ell_2 = \sigma^{-1}(q_1^+) \cup \sigma^{-1}(q_1^-)$. Contracting any of the lines ℓ_1, ℓ_2 and $\sigma^{-1}(x)$ we get a Del Pezzo surface X'. If we contract ℓ_1 or ℓ_2, then X' is not necessarily isomorphic to X.

The splitting group $G_{X'}$ is obtained from G_X by the action of an automorphism of $W(D_4)$ induced by an automorphism of the root system D_4 [40]. Let us call such automorphisms of $W(D_4)$ *admissible*, they constitute a group denoted by $\mathrm{Aut}' W(D_4)$. The factor of $\mathrm{Aut}' W(D_4)$ over the group of inner automorphisms of $W(D_4)$ is isomorphic to S_3 ([40], 3.1). On the other hand, according to Proposition 4.3 of [40], if $G = h(G_{X'})$ for some $h \in \mathrm{Aut}' W(D_4)$, $X'(k) \neq \emptyset$, and card k is large enough, then there exists a Del Pezzo surface X of degree 4, which is k-birational to X', and $G_X = G$ (see [40] for precise statements).

COROLLARY 3.23. *A k-minimal Del Pezzo surface X of degree 4, $X(k) \neq \emptyset$, is birationally equivalent to an Iskovskih surface iff* $(\mathrm{Pic}\, \bar{X})^{\mathfrak{g}} \simeq \mathbb{Z} \oplus \mathbb{Z}$, *and the following equivalent conditions hold:*

i) *for some* $h \in \mathrm{Aut}' W(D_4)$ *there exists an* $h(G_X)$*–orbit of* Γ *consisting of two joined vertices,*

ii) *for some* $h \in \mathrm{Aut}' W(D_4)$ *there exists an* $h(G_X)$*–orbit of* Δ *consisting of two joined vertices, other than* q_1^+ *and* q_1^- *(i.e. generators of* $(\mathrm{Pic}\, \bar{X})^{\mathfrak{g}}$).

Proof: If X is birationally equivalent to an Iskovskih surface, then X is birationally equivalent to a Del Pezzo surface of degree 4 such that Γ has an orbit of two vertices (Corollary 3.21 a). Then $(\mathrm{Pic}\, \bar{X})^{\mathfrak{g}} \simeq \mathbb{Z} \oplus \mathbb{Z}$. According to [40], $rk\, (\mathrm{Pic}\, \bar{X})^{\mathfrak{g}}$ and the conjugacy class of the splitting group G_X in $W(D_4)$ modulo $\mathrm{Aut}' W(D_4)$ are birational invariants. Conversely, if Γ has an $h(G_X)$–orbit of two vertices, then assuming $X(k) \neq \emptyset$ we can transform X into a Del Pezzo surface Y of degree 4 such that $G_Y = h(G_X)$ ([40], 4.3). Now use Corollary 3.21 a). It remains to prove the equivalence of i) and ii). We do it by the following combinatorial argument. Let $(\mathrm{Pic}\, \bar{X})^G = \mathbb{Z}\, q_1^+ \oplus \mathbb{Z}\, q_1^-$, and let $\Gamma = \Sigma_1^+ \cup \Sigma_1^-$ be the decomposition of Γ introduced in section 0. Let $\Delta = \Sigma_1 \cup \{q_1^+, q_1^-\}$. The group of automorphisms of Λ (the graph also introduced in section 0; the set of its vertices is $\Sigma_1^+ \cup \Sigma_1^- \cup \Sigma_1 \cup \{e, q_1^+, q_1^-\}$), preserving the triangle $\{e, q_1^+, q_1^-\}$, is canonically isomorphic to the automorphism group $A(D_4)$ of the root system D_4 (cf. [30]); $A(D_4) = W(F_4) = W(D_4) \rtimes S_3$ ([3], Ch. VI, § 4). This group permutes e, q_1^+, q_1^-, and acts on $W(D_4)$ by admissible automorphisms. Since each vertex of Σ_1 (respectively of Σ_1^+, Σ_1^-) is joined with e (respectively with q_1^-, q_1^+), and is not joined with the two remaining elements of $\{e, q_1^+, q_1^-\}$, $A(D_4)$ permutes $\Sigma_1, \Sigma_1^+, \Sigma_1^-$. Therefore if one of these graphs has an $h(G)$–orbit of two joined vertices, then for a suitable $h' \in \mathrm{Aut}' W(D_4)$ any of the two other graphs also has an $h'(G_X)$–orbit of the same type.

COROLLARY 3.24. *Assume that a k-minimal Del Pezzo surface X of degree 4 is birationally equivalent to an Iskovskih surface, and $X(k) = \emptyset$. Then the graph Δ has a G_X-orbit of two joined vertices, other than q_1^+ and q_1^-.*

Proof : The surface X is k-birational to a surface Y satisfying the assumptions of Corollary 3.21 b). Since G_X modulo the action of $\text{Aut}' W(D_4)$ is a k-birational invariant, $G_X = h(G_Y)$ for some $h \in \text{Aut}' W(D_4)$. The graph $\Sigma_1 = \Delta \setminus \{q_1^+, q_1^-\}$ contains a G_Y-orbit of two joined vertices. Since $A(D_4)$ acts on Λ permuting $\Sigma_1^+, \Sigma_1^-, \Sigma_1$, in one of these graphs there is a G_X-orbit of two joined vertices. If this orbit belongs to $\Gamma = \Sigma_1^+ \cup \Sigma_1^-$, then the intersection point of the corresponding lines is defined over k, contradictory to $X(k) = \emptyset$; therefore it belongs to Σ_1, and we are done.

We see that Corollary 3.21 b) gives a necessary condition for a k-minimal Del Pezzo surface of degree 4 to be birationally equivalent to an Iskovskih surface. On the other hand, a sufficient condition is given by Theorem 3.20 b). Let us reformulate it in the following way. If a k-minimal Del Pezzo surface X of degree 4 is an intersection of two quadrics given in homogeneous coordinates (x, y, z, t, u) by

$$Q_1(x, y, z, t) = Q_2(x, y, z, u) = 0$$

where $\det Q_1 \in (k^*)^2$, and Q_2 represents 0, then X is birationally equivalent to an Iskovskih surface. If, in addition Q_1 also represents 0, then X is birationally equivalent to a Châtelet surface (Corollary 3.11).

4
SUBGROUPS OF $W(D_4)$.

In this section we show that the algorithms given by Theorems 1.22 and 1.26 are efficient enough to compute $Ш^2_\omega(G,Q(R))$ and $Ш^1_\omega(G,Q(R)^o)$; the root system $R = D_4$ serves us as a model. From the geometric point of view D_4 corresponds to Del Pezzo surfaces of degree 4 with $\mathrm{Pic}\, X \simeq \mathbb{Z} \oplus \mathbb{Z}$, since the splitting group G_X can be naturally represented as a subgroup of $W(D_4)$ (cf. § 0). Minimal subgroups (in the sense of Definition 3.3) correspond to k–minimal surfaces. We shall list all such subgroups, and compute $Ш^2_\omega(G,Q(D_4))$ and $Ш^1_\omega(G,Q(D_4)^o)$ (in the case where G is not minimal, both invariants vanish, see Corollary 4.3).

Let us recall several facts we need. The Weyl group $W(D_4)$ is given by its generators and relations using the Dynkin diagram D_4 ([3], Ch. VI, § 4). Embedding $W(D_4)$ into the group of automorphisms of the integral quadratic form $x_1^2 + x_2^2 + x_3^2 + x_4^2$, we shall write its elements as products of permutations from S_4 and reflections c_i associated to the coordinate hyperplanes. The generators of $W(D_4)$ corresponding to the vertices of the Dynkin diagram can be chosen as it is shown in Figure 6.

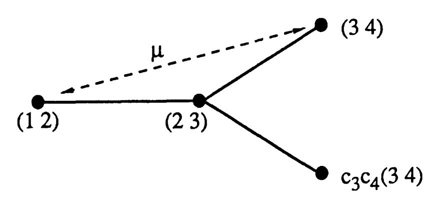

Figure 6

It is easy to check that the centre $Z(W(D_4))$ coincides with $\langle c_1\, c_2\, c_3\, c_4 \rangle \simeq \mathbb{Z}/2$. Let $\mathrm{Aut}'\, W(D_4)$ be the group of such automorphisms of $W(D_4)$ which are induced by automorphisms of the root system D_4 itself, i.e. by conjugations in

$A(D_4) = W(D_4) \rtimes S_3 = W(F_4)$. We call such automorphisms of $W(D_4)$ *admissible.* Let Inn be the subgroup of inner automorphisms of $W(D_4)$, clearly Inn $W(D_4) = W(D_4)/Z(W(D_4))$. One can prove (see, for example, [40], Proposition 3.1) that the factor group Aut$'$ $W(D_4)$/Inn $W(D_4) \simeq S_3$ coincides with the group of automorphisms of the Dynkin diagram D_4.

PROPOSITION 4.1. *Let* $G \subseteq W(D_4)$, $\sigma \in$ Aut$'$ $W(D_4)$, *then* $Ш^1_\omega(G, P(D_4)) = Ш^1_\omega(\sigma(G), P(D_4))$, *and* $Ш^2_\omega(G, Q(D_4)) = Ш^2_\omega(\sigma(G), Q(D_4))$.

Proof : We claim that the G–module $Q(D_4)^\sigma$ obtained from $Q(D_4)$ by twisting it by the automorphism σ, is isomorphic to $Q(D_4)$. Indeed, σ is induced by a conjugation in $A(D_4)$, which is the group of orthogonal automorphisms of $Q(D_4)$. The second assertion now follows. The first one is obtained by duality.

This result being granted we can consider subgroups $G \subseteq W(D_4)$ up to the action of the group of all automorphisms of $W(D_4)$ while computing $Ш^2_\omega(G, Q(D_4))$ and $Ш^1_\omega(G, Q(D_4)^o)$. In particular, since $W(D_4)$ is a normal subgroup of $W(C_4)$, it is sufficient to classify subgroups $G \subseteq W(D_4)$ up to a conjugation in $W(C_4)$. Let us now proceed with the classification.

PROPOSITION 4.2. *Let* G *be a non-minimal subgroup of* $W(D_5)$ *lying in* $W(D_4)$. *Then the algebraic torus dual to the* G*-module* $Q(D_4)$ *is stably* k*-rational.*

Proof : In view of [43], 4.16, it suffices to prove k–rationality of the torus dual to the G–module $M(C_4)$. This module can be represented by the extension (7) :

$$0 \longrightarrow \mathbb{Z} \longrightarrow M(C_4) \longrightarrow Q(B_4) \longrightarrow 0.$$

By 3.6 G is not minimal if and only if there exists a G–invariant element in $Q(B_4)$. Let $\{e_i\}$ be the standard basis of $Q(B_4)$. Since G permutes elements of the set $\{e_1,\dots,e_4, -e_1,\dots,-e_4\}$ and $Q(B_4)^G \neq 0$, it follows that $Q(B_4)$ can be decomposed into a direct sum of a permutation G–module P and a G–module $Q(B_m)$ where $m < 4$. Note that the image of the G–action on $Q(B_m)$ lies in $W(D_m)$. The extension (7) can be written as a pair of exact sequences

$$0 \longrightarrow \mathbb{Z} \longrightarrow X \longrightarrow Q(B_m) \longrightarrow 0, \tag{17}$$

$$0 \longrightarrow X \longrightarrow M(C_4) \longrightarrow P \longrightarrow 0, \tag{18}$$

the sequence (18) being the definition of X. One can easily show that $X \simeq M(C_m)$. We thus obtain a short exact sequence of G–modules

$$0 \longrightarrow Q(D_m) \longrightarrow X \longrightarrow \mathbb{Z} \longrightarrow 0$$

(it is just the exact sequence (1)). Applying [43], 4.16 again, we see from (17) and (18) that the torus dual to $M(C_4)$ is stably equivalent to the torus dual to $Q(D_m)$. All the tori of dimension 1 or 2 are k–rational ([43], Ch. IV, § 9), hence it is enough to treat the case $m = 3$. Since the root system D_3 is isomorphic to A_3 and $Q(A_3)$ is the kernel of the homomorphism $\mathbb{Z}[S_4/S_3] \longrightarrow \mathbb{Z}$ sending all base elements to 1, it follows that the dual torus is k–rational (cf. 1.9).

COROLLARY 4.3. *Let G be a non-minimal subgroup of $W(D_5)$ lying in $W(D_4)$. Then* $\text{Ш}^1_\omega(G,Q(D_4)^o) = \text{Ш}^2_\omega(G,Q(D_4)) = 0$.

Let us begin to list minimal subgroups of $W(D_4)$. Let $pr: W(D_4) \longrightarrow S_4$ denote the natural epimorphism as in § 1. Denote by $G^B \subseteq S_4$ the group $pr(G)$, and let $G^A \subseteq \langle c_1c_2,c_2c_3,c_3c_4\rangle$ be the kernel of $pr: G \longrightarrow G^B$. Certainly, G^A is G^B–invariant. Our algorithm is organized as follows. Given G^A and G^B we find all possible extensions

$$1 \longrightarrow G^A \longrightarrow G \longrightarrow G^B \longrightarrow 1. \tag{19}$$

Assume that G^B is generated by permutations α, β then $G = \langle \alpha c_I, \beta c_J, G^A\rangle$, where I and J are multiindices : $I = (i_1,\dots,i_k)$, $J = (j_1,\dots,j_\ell)$. Multiplying c_I and c_J by elements of G^A we exclude coinciding groups and groups which are conjugate in $W(C_4)$. Let us assign two indices to each subgroup $G \subseteq W(D_4)$ (to be more precise, to its conjugacy class in $W(C_4)$). The subscript i indicates that G^B is the i–th element in the following sequence $\{H_i\}$ of conjugacy classes of subgroups in S_4 : 1, <(12)>, <(12)(34)>, <(12),(34)>, $\mathbb{Z}/3$, S_3, $\mathbb{Z}/4$, V_4, D_4, A_4, S_4 (i = 0,1,...,10). In what follows by V_4, D_4, A_4 we denote Klein's Vierergruppe

$<(12)(34),(13)(24)>$, the dihedral group $<(1234),(13)>$, and the alternating group, respectively. The second index j is the number of $G_{i,j}$ in the list of subgroups G with given $G^B = H_i$.

PROPOSITION 4.4. *There are exactly* 50 *distinct minimal subgroups of* $W(D_4)$ *considered up to a conjugation in* $W(C_4)$. *They are given by the following list*:

$G_{i,0} = <c_1c_2,\ c_2c_3,\ c_3c_4> \times H_i$ $(i = 0,1,\dots,10)$; $G_{i,1} = <c_1c_2c_3c_4> \times H_i$ $(i = 0,\dots,10)$;
$G_{0,2} = <c_1c_2,\ c_3c_4>$; $G_{1,2} = <c_1c_2,\ c_3c_4> \times <(12)>$; $G_{1,3} = <c_3c_4,\ c_1c_3(12)>$;
$G_{1,4} = <c_1c_3,\ c_1c_4(12)>$; $G_{1,5} = <c_1c_2,\ c_3c_4(12)>$; $G_{2,2} = <c_1c_2,\ c_3c_4> \times <(12)(34)>$;
$G_{2,3} = <c_1c_2,\ c_1c_3(12)(34)>$; $G_{2,4} = <c_1c_3,(12)(34)>$; $G_{2,5} = <c_1c_3(12)(34)>$;
$G_{3,2} = <c_1c_2,\ c_3c_4> \times H_3$; $G_{3,3} = <c_1c_3(12),(34)>$; $G_{3,4} = <c_1c_3(12),\ c_1c_3(34)>$;
$G_{3,5} = <c_1c_2,\ c_3c_4(12),\ (34)>$; $G_{3,6} = <c_3c_4(12),\ c_1c_2(34)>$; *if* $i = 4$, *then* $j \leq 1$;
$G_{5,2} = <c_1c_4(12),\ c_1c_4(13)>$; $G_{5,3} = <c_3c_4(12),\ c_2c_4(13)>$; $G_{6,2} = <c_1c_3,(1234)>$;
$G_{7,2} = <c_1c_2,\ (12)(34),\ (13)(24)>$; $G_{7,3} = <c_1c_2,c_1c_3(12)(34),\ (13)(24)>$;
$G_{7,4} = <(12)(34),\ c_1c_3(13)(24)>$; $G_{7,5} = <(12)(34),\ c_1c_2(13)(24)>$;
$G_{7,6} = <c_1c_3(12)(34),\ c_2c_3(13)(24)>$; $G_{8,2} = <c_1c_3,\ (1234),\ (13)>$;
$G_{8,3} = <(1234),\ c_1c_2(12)(34)>$; $G_{8,4} = <(1234),\ c_1c_3(13)>$; $G_{8,5} = <(1234),\ c_1c_2c_3c_4(13)>$;
$G_{9,2} = <(123),\ c_1c_2(124)>$; $G_{10,2} = <c_3c_4(234),(1234)>$.

Proof : Let us exploit the following practical criterion for minimality of $G \subsetneq W(D_4)$: any G–orbit from the set $\{e_1, e_2, e_3, e_4, -e_1, -e_2, -e_3, -e_4\}$ is stable under multiplication by -1. The case $i = 0$ is obvious. Let $i = 1$, i.e. $G^B = H_1 = <(12)>$. If $G^A = <c_1c_2,\ c_3c_4>$, then $G = <c_I(12),\ G^A>$, where I is a multiindex. Multiplying by an element of G^A we reduce c_I to 1 or c_1c_3. If $G^A = <c_1c_2>$, then c_I can be reduced to c_3c_4 or $c_1c_2c_3c_4$, otherwise G would be non–minimal. The obtained groups are conjugate under c_1. For the remaining G^A no new minimal group appears, except for $G^A = <c_1c_2c_3c_4>$ and $G = G_{1,1}$. Let $i = 2$, i.e. $G^B = H_2 = <(12)(34)>$. This case can be treated in a similar way. Let $i = 3$, i.e. $G^B = H_3 = <(12),(34)>$. If $G^A = <c_1c_2,\ c_3c_4>$, then $G = <c_I(12),\ c_J(34),\ G^A>$, and multiplying by elements of G^A, we reduce c_I and c_J to 1 or c_1c_3. If $G^A = <c_1c_2>$, then

$c_J \in \langle c_1c_2, c_3c_4 \rangle$, otherwise $(c_I(34))^2 = c_3c_4 \notin G$. Therefore, conjugating by c_3 and multiplying by c_1c_2 (if necessary) we can reduce c_J to 1. Besides, $c_I = c_3c_4$ or c_1c_3 (by minimality). However in the second case, $c_1c_2c_3c_4 \in \langle c_1c_3(12), (34) \rangle$, which does not agree with the condition $G^A = \langle c_1c_2 \rangle$. If $G^A = \langle c_1c_2c_3c_4 \rangle$, then no new group appears apart from $G_{3,1}$, and if $G^A = 1$, then the only minimal group is $G_{3,6}$. All the other cases are treated along the same lines.

As above we say that the group $W(D_{n-1})$ is embedded into $W(D_n)$ in a standard way if it is realized as the subgroup of $W(D_n)$ fixing the fundamental weight ω_1 $(= e_1)$.

DEFINITION 4.5. *Let the inclusion* $\varphi : W(B_{n-1}) = W(C_{n-1}) \longrightarrow W(D_n)$ *be the extension of the standard inclusion of* $W(D_{n-1})$: *if an element* g *is such that* $\sigma(g) = -1$, *then set* $\varphi(g) = c_1 \cdot g$ *(here* $\sigma(c_{i_1} \dots c_{i_m} \rho) = (-1)^m, \rho \in S_n)$.

LEMMA 4.6. *If the group* $W(D_{n-1})$ *is embedded into* $W(D_n)$ *in a standard way, then its normalizer in* $W(D_n)$ *coincides with* $\varphi(W(C_{n-1}))$.

Proof : This is quite clear.

Let Γ be the incidence graph of lines on a Del Pezzo surface of degree 4, and let G be a minimal subgroup of $W(D_5)$. In [27] there is a description of 19 possible types of the decomposition of Γ into G–orbits (cf. § 0). The graph Γ is presented on Figure 1. Each vertex is meant to be joined with the vertex in the same row and on the same side of the vertical line, and with exactly one more vertex from each pair lying on the other side of the vertical line; moreover the left (right) vertex from each pair is joined with the left (right) vertex from the pair in the same row, and with the right (left) vertices from the pairs in other rows. The vertices of Γ are denoted by ℓ_i, ℓ_{ij}, q $(1 \leq i,j \leq 5,\ i \neq j)$. From the geometric point of view ℓ_i corresponds to one of 5 blown–up points on $\mathbb{P}^2$, ℓ_{ij} corresponds to one of 10 lines joining the points, and q corresponds to the conic through all the five points. The group $W(D_5)$ acts on Γ in the following way : S_5 permutes indices, and c_1c_5 is the reflection with respect to the vertical line. Note that having classified subgroups of $W(D_4)$ up to a conjugation in $W(C_4)$, we have also classified them up to a conjugation in $W(D_5)$ (see Lemma 4.6).

PROPOSITION 4.7. *Let G be a minimal subgroup of $W(D_5)$ lying in $W(D_4)$. Then the decomposition of Γ into G-orbits belongs to one of the 12 types*: II–IX *and* XII–XV *from Figure* 4.

Proof : It is enough to determine the decomposition type for each of 50 groups listed in Proposition 4.4. Here is the result : groups of type II : $G_{4,1}, G_{5,1}, G_{5,3}$; of type III : $G_{9,1}, G_{10,1}, G_{10,2}$; of type IV : $G_{0,1}$; of type V : $G_{2,5}, G_{7,4}$; of type VI : $G_{2,1}$; of type VII : $G_{1,1}$; of type VIII : $G_{3,1}, G_{3,6}$; of type IX : $G_{0,2}, G_{1,2}, G_{1,5}, G_{2,2}, G_{3,2}, G_{3,5}$; of type XII : $G_{7,1}$; of type XIII : $G_{6,1}, G_{8,1}, G_{8,5}$; of type XIV : $G_{2,4}, G_{6,2}, G_{7,2}, G_{7,3}, G_{8,2}$; of type XV : $G_{i,0}, G_{1,3}, G_{1,4}, G_{2,3}, G_{3,3}, G_{3,4}, G_{5,2}, G_{7,3}, G_{7,6}, G_{8,4}, G_{9,2}$ (to check this it is convenient to conjugate all the groups by the elements (15) before considering the action on Γ).

DEFINITION 4.8. *Given a decomposition type of the graph Γ, the maximal subgroup of $W(D_5)$ preserving the decomposition type is called its maximal group of automorphisms.*

COROLLARY 4.9. *The maximal groups of automorphisms of the types which are subtypes of type* XV *are listed below* :

Aut(II) $= G_{5,1}$,	Aut(III) $= G_{10,1}$,	Aut(IV) $= G_{0,1}$,
Aut(V) $= G_{7,4}$,	Aut(VI) $= G_{2,1}$,	Aut(VII) $= G_{1,1}$,
Aut(VIII) $= G_{3,1}$,	Aut(IX) $= G_{3,2}$,	Aut(XII) $= G_{7,1}$,
Aut(XIII) $= G_{8,1}$,	Aut(XIV) $= G_{8,2}$,	Aut(XV) $= G_{10,0}$.

Our next goal is to find out how admissible automorphisms of $W(D_4)$ act on the set of subgroups of $W(D_4)$. Let μ be the admissible outer automorphism of $W(D_4)$ marked by the dotted arrow in Figure 6, and let θ be the composition of μ and the conjugation by (14)(23). It is not difficult to see that θ acts on $S_4 = <(1234),(12)>$ trivially and sends c_1c_2 to $c_3c_4(12)(34)$. By ν we denote the admissible outer automorphism of $W(D_4)$ arising from the conjugation by c_i in $W(D_4)$ (the choice of the index i is of no importance since we are interested in outer automorphisms only up to inner ones). Both θ and ν are of order 2, together they generate $\mathrm{Aut}'\, W(D_4)/\mathrm{Inn}\, W(D_4) \simeq S_3$.

PROPOSITION 4.10. *In* $W(D_4)$ *there are exactly* 67 *conjugacy classes of minimal subgroups. The group* $\mathrm{Aut}'\,W(D_4)$ *acts on* 16 *classes trivially. The other classes are divided into* 17 *orbits containing* 3 *classes each.*

Proof : a) Let us list 16 subgroups representing the conjugacy classes with the trivial action of $\mathrm{Aut}'\,W(D_4)$. To begin with, such are $G_{i,0}$ for $i \geq 7$. Indeed, $\nu(G_{i,0}) = G_{i,0}$ since $G_{i,0}$ contains all c_ic_j; $\theta(G_{i,0}) = G_{i,0}$ since $G_{i,0}$ contains V_4 and $c_1c_2c_3c_4$. Next, such are also $G_{i,1}$ $(i = 0,1,4,5)$. Indeed, $\nu(G_{i,1}) = G_{i,1}$ since $pr(G_{i,1}) \subseteq S_4$ fixes some $m \in \{1,2,3,4\}$, thus c_m commutes with $G_{i,1}$; $\theta(G_{i,1}) = G_{i,1}$ since θ acts on $<c_1c_2c_3c_4> \times S_4$ trivially. One can easily check that $G_{2,2}, G_{2,5}, G_{3,2}, G_{3,5}, G_{5,3}$ are also invariant under $\mathrm{Aut}'\,W(D_4)$. The invariance of $G_{7,3}, G_{7,6}$ and $G_{9,2}$ can be verified in a bit more tedious way.

b) In order to verify the second statement of the proposition we break up the set of 34 subgroups remaining in the list of Proposition 4.4 into pairs, such that groups in each pair are transformed into each other by an admissible outer automorphism of $W(D_4)$. This being done, for one of the groups in each pair we shall point out the admissible outer automorphism of order 2 fixing the conjugacy class of the group. It will be thus shown that each orbit of $\mathrm{Aut}'\,W(D_4)/\mathrm{Inn}\,W(D_4)$ on the set of conjugacy classes of subgroups of $W(D_4)$ contains either one or three elements. In the following formulae ν_i denotes the conjugation by c_i.

$\nu_3 \circ \theta(G_{0,0}) = G_{7,1}$, $\nu_3 \circ \theta(G_{0,2}) = G_{2,1}$, $\nu_3 \circ \theta(G_{1,0}) = G_{8,1}$, $\nu_3 \circ \theta(G_{1,2}) = G_{3,1}$,
$\nu_3 \circ \theta(G_{2,0}) = G_{7,2}$, $\nu_3 \circ \theta(G_{3,0}) = G_{8,2}$, $\nu_3 \circ \theta(G_{4,0}) = G_{9,1}$, $\nu_3 \circ \theta(G_{5,0}) = G_{10,2}$,
$\nu_3 \circ \theta(G_{1,3}) = G_{6,1}$, $\nu_3 \circ \theta(G_{1,4}) = G_{8,5}$, $\nu_1 \circ \theta(G_{1,5}) = G_{3,6}$, $\nu_3 \circ \theta(G_{2,3}) = G_{7,5}$,
$\nu_2 \circ \theta(G_{5,2}) = G_{10,2}$, $\nu_3 \circ \theta(G_{3,3}) = G_{8,4}$, $\theta(G_{3,4}) = G_{6,2}$, $\theta(G_{7,4}) = G_{2,4}$, $\theta(G_{6,0}) = G_{8,3}$.

Now let us remark that the conjugacy classes of $G_{i,0}$ are invariant under ν since $G_{i,0}$ contains all the elements of the shape c_ic_j (the conjugation of a permutation by c_i coincides with multiplication by some c_ic_j). The conjugacy classes of $G_{0,j}, G_{1,j}, G_{4,j}, G_{5,j}$ are also ν–invariant since $pr(G) = G^B \subset S_4$ fixes some element of $\{1,2,3,4\}$. One can immediately see that the classes of $G_{2,3}, G_{3,3}, G_{3,4}, G_{7,4}$ are also ν–invariant.

As the following result shows, in fact one needs not require the minimality of a subgroup.

COROLLARY 4.11. *Each orbit of* $\mathrm{Aut}'\,W(D_4)$ *on the set of conjugacy classes of subgroups of* $W(D_4)$ *contains one or three elements.*

Proof : We have already proved the statement for minimal subgroups. Non–minimal subgroups can be divided into 4 types (after conjugating in $W(C_4)$, if necessary) : the subgroups fixing e_1, those permuting e_1 and e_2, those permuting e_1, e_2, e_3 and finally those permuting e_1, e_2, e_3, e_4. Let us show that in all cases there exists an admissible outer automorphism of order 2 fixing the conjugacy class of the subgroup. In the first case, one can take ν_1 (conjugation by c_1), in the third case, one can take ν_4; in the fourth case where $G \subsetneq S_4$, one can take θ. It remains to deal with the groups of the second type. The maximal group of this type which includes all the others is $<(12),(34),c_3c_4>$. Its conjugacy class is stable under θ. The other subgroups of this type are $<c_3c_4(12),(34)>$, $<(12),c_3c_4>$, $<c_3c_4(12)>$ (up to a conjugation in $W(C_4)$). The second and the third subgroups are ν_3–invariant, and the first subgroup can be transformed into the second one by θ.

Let us remark that we have no explanation of this phenomenon.

Now let us go over to the direct computation of $Ш^2_\omega(G,Q(D_4))$ for $G \subsetneq W(D_4)$.

LEMMA 4.12. *Let* G *be generated by some elements, each conjugate to an element of* S_4 *in* $W(C_4)$. *Then* $Ш^2_\omega(G,Q(D_4)) = 0$.

Proof : It follows immediately from Theorem 1.22.

THEOREM 4.13. *If* $G \subsetneq W(D_4)$ *is such that* $Ш^2_\omega(G,Q(D_4)) \neq 0$, *then it can be transformed by an admissible automorphism of* $W(D_4)$ *into one of the following groups:* $G_{3,6} = <c_3c_4(12),\ c_1c_2(34)>$, $G_{2,3} = <c_1c_2,\ c_1c_3(12)(34)>$, $G_{7,3} = <c_1c_2,\ c_1c_3(12)(34),(13)(24)>$ (*for these groups* $Ш^2_\omega(G,Q(D_4)) = \mathbb{Z}/2$) *or* $G_{7,6} = <c_1c_3(12)(34),c_2c_3(13)(24)>$ (*for this group* $Ш^2_\omega(G,Q(D_4)) = (\mathbb{Z}/2)^2$).

Proof : Let us exclude all the groups satisfying the condition of Lemma 4.12 as well as their transforms under $\mathrm{Aut}'\, W(D_4)$ from the list of Proposition 4.4. It is clear that the condition of Lemma 4.12 is satisfied for $G_{i,0}$ $(i \geq 6)$ and for $G_{i,1}$ $(i = 2,3$ or $i \geq 6)$ (it is enough to check that G is generated by permutations ρ and by elements of the form $c_I\rho$ with the index set I contained entirely in the union of orbits of ρ, such that each orbit contains an even number of elements of I. See Lemma 1.21). The groups $G_{10,2} = <c_3c_4(12),\ (1234)> = <c_3c_4(234),\ (1234)>$, $G_{7,4}$, $G_{7,2}$, $G_{6,2}$, $G_{8,4}$, $G_{8,5} = <(1234),\ c_1c_2c_3c_4(13)> = <(1234),\ c_1c_2c_3c_4(12)(34)>$ also satisfy the condition. Let us look at the list of conjugacy classes of subgroups of $W(D_4)$ with non–trivial action of $\mathrm{Aut}'\,W(D_4)$ (see the proof of Proposition 4.10), taking into account that the action of $\mathrm{Aut}'\,W(D_4)$ does not affect $Ш^2_\omega(G,Q(D_4))$ (Proposition 4.1).

We see that the only cases to examine are in fact $G_{3,6}$ and $G_{2,3}$.

Now consider conjugacy classes stable under $\mathrm{Aut}'\,W(D_4)$. Here the condition of Lemma 4.12 is satisfied for $G_{i,0}$ $(i \geq 7)$, $G_{2,2}$, $G_{3,2}$, $G_{3,5}$, $G_{9,2}$. The groups $G_{0,1}$ and $G_{2,5}$ are cyclic, and $Ш^2_\omega(G,Q(D_4)) = 0$ by definition. It remains to examine the groups $G_{i,1}$ $(i = 1,4,5)$, $G_{5,3}$, $G_{7,3}$, $G_{7,6}$. If G is one of the groups $G_{1,1}$, $G_{4,4}$, $G_{5,1}$, then G_0 (the normal closure of the elements conjugated to permutations) is isomorphic to $\mathbb{Z}/2$, $\mathbb{Z}/3$, S_3 respectively, and G/G_0 is generated by the image of $c_1c_2c_3c_4$. However, among the characters $\chi_{G,i}$ there is one defined by the action of $g \in G$ on e_4 which is stable under $pr(G)$. This character is non–trivial on $c_1c_2c_3c_4$, therefore it immediately follows from Theorem 1.22 that $Ш^2_\omega(G,Q(D_4)) = 0$ for $G_{i,1}$ $(i = 1,4,5)$. Likewise $G_{5,3} = \langle c_3c_4(12),\, c_2c_4(13)\rangle \simeq S_3$, $(G_{5,3})_0 = \langle c_1c_3(132)\rangle \simeq \mathbb{Z}/3$; the factor $G_{5,3}/(G_{5,3})_0 = \mathbb{Z}/2$ is generated by the image of $c_3c_4(12)$. Among $\chi_{G,i}$ there is the same character as above, and it is non–trivial on $c_3c_4(12)$. Therefore in this case we also have $Ш^2_\omega(G_{5,3},Q(D_4)) = 0$.

Now it remains to carry out the computations for the groups $G_{2,3}$, $G_{3,6}$, $G_{7,3}$, $G_{7,6}$; as above we use Theorem 1.22. We have $G_{2,3} = \langle c_1c_2,\, c_1c_3(12)(34)\rangle \simeq \mathbb{Z}/2 \times \mathbb{Z}/4$, $(G_{2,3})_0 = 1$, $\mathrm{Hom}(G_{2,3}/(G_{2,3})_0\,, \mathbb{Z}/2) \simeq (\mathbb{Z}/2)^2$. On the other hand, the orbits of $pr(G_{2,3})$ are $\{1,2\}$ and $\{3,4\}$. The corresponding characters $\chi_{G,1}$ and $\chi_{G,2}$ are trivial on c_1c_2 and non–trivial on $c_1c_3(12)(34)$. Since $\chi_{G,1} = \chi_{G,2}$, it follows that $Ш^2_\omega(G_{2,3},Q(D_4)) = \mathbb{Z}/2$. Consider $G_{3,6} = \langle c_3c_4(12),\, c_1c_2(34)\rangle \simeq (\mathbb{Z}/2)^2$; we have $(G_{3,6})_0 = \langle c_1c_2c_3c_4(12)(34)\rangle$, $\mathrm{Hom}(G_{3,6}/(G_{3,6})_0\,, \mathbb{Z}/2) = \mathbb{Z}/2$. The orbits of $pr(G_{3,6})$ are $\{1,2\}$ and $\{3,4\}$; the arising characters $\chi_{G,1}$ and $\chi_{G,2}$ are both trivial, hence $Ш^2_\omega(G_{3,6},Q(D_4)) = \mathbb{Z}/2$ (cf. Example 1.25). Next $G_{7,3} = \langle c_1c_2,\, c_1c_3(12)(34),(13)(24)\rangle = \langle c_1c_2,\, c_1c_3(14)(23),\, (13)(24)\rangle$, since $c_1c_2 \cdot c_1c_3(12)(34) \cdot (13)(24) = c_2c_3(14)(23)$. Therefore, $(G_{7,3})_0 \supseteq \langle c_2c_3(14)(23),\, (13)(24)\rangle$. Let us show that here in fact we have an equality. The group $\langle c_2c_3(14)(23),\, (13)(24)\rangle$ is dihedral, it contains 8 elements. One can easily write them down and verify that the product of any element by c_1c_2 is not conjugate to a permutation. Thus, $G_{7,3}/(G_{7,3})_0 = \mathbb{Z}/2$, and since the unique character $\chi_{G,1}(g) = \det(g)$ is trivial, we obtain $Ш^2_\omega(G_{7,3}, Q(D_4)) = \mathbb{Z}/2$. For $G_{7,6} = \langle c_1c_3(12)(34),\ c_2c_3(13)(24)\rangle$ we have $\mathrm{Hom}(G_{7,6}, \mathbb{Z}/2) \simeq \mathbb{Z}/2)^2$. Let us show that $G_{7,6}$ contains no element conjugate to a permutation. In fact $G_{7,6}$ is isomorphic to the quaternion group $\{\pm 1, \pm i, \pm j, \pm k\}$, and the elements 1, $c_1c_3(12)(34)$, $c_2c_3(13)(24)$, $c_3c_4(14)(23)$ and $c_1c_2c_3c_4$ exhaust (up to inversion) all the elements of $G_{7,6}$, hence $(G_{7,6})_0 = 1$. The unique character $\chi_{G,1}(g) = \det(g)$ is trivial, therefore $Ш^2_\omega(G_{7,6}, Q(D_4)) \simeq (\mathbb{Z}/2)^2$.

COROLLARY 4.14. *If* $G \subsetneq W(C_3)$ *is such that* $Ш^2_\omega(G, Q(C_3)) \neq 0$, *then* G *is conjugate to* $\langle c_1c_2,\ c_3(12)\rangle$.

Proof : Proposition 1.24 shows that we are in fact in the situation of Theorem 4.13. Since $pr(\varphi(G))$ has a fixed point in $\{1,2,3,4\}$, $\varphi(G)$ is conjugate to $G_{1,5}$ in $W(C_3)$.

Now let us compute $Ш^1_\omega(G, Q(D_4)^\circ) = Ш^1_\omega(G, P(D_4))$ for all $G \subsetneq W(D_4)$.

LEMMA 4.15. *If* $c_1c_2c_3c_4 \in G$, *then* $Ш^1_\omega(G, Q(D_4)^\circ) = 0$.

Proof : Recall that if $g \in G$, then $I_{\langle g\rangle, j}$ denote orbits of the group $\langle pr(g)\rangle$, and I_g denotes the union of all $I_{\langle g\rangle, j}$ such that $\chi_{\langle g\rangle, j}$ is a non–trivial character of the group $\langle g\rangle$. Therefore, $I_{\langle c_1c_2c_3c_4\rangle} = I$ is the entire set $\{1,2,3,4\}$. According to Theorem 1.26, in order to compute $Ш^1_\omega(G, Q(D_4)^\circ)$ we consider the group of all linear combinations of $a_{G,k}$ which satisfy the following condition (for any g) : the union of $I_{G,k}$ with the same indices k either contains I_g, or is disjoint from it. Then we factorize this group by $a_G = \Sigma\, a_{G,k}$. Taking $g = c_1c_2c_3c_4$ we see that the only linear combination satisfying the condition of Theorem 1.26 is an a_G itself, hence $Ш^1_\omega(G, Q(D_4)^\circ) = 0$.

LEMMA 4.16. *If* G *contains an element of the form* $c_ic_j(ik)(j\ell)$, $i,j,k,\ell \in \{1,2,3,4\}$ *being distinct, then* $Ш^1_\omega(G, Q(D_4)^\circ) = 0$.

Proof : Similar to the proof of Lemma 4.15.

LEMMA 4.17. *If* $pr(G)$ *is a transitive subgroup of* S_4, *then* $Ш^1_\omega(G, Q(D_4)^o) = 0$.

Proof : It trivially follows from Theorem 1.26, since $H^1(G, Q(B_n)) = \mathbb{Z}/2$ is generated by a_G.

THEOREM 4.18. *Let* $G \subseteq W(D_4)$ *be such that* $Ш^1_\omega(G, Q(D_4)^o) \neq 0$, *then* G *can be transformed by an admissible automorphism of* $W(D_4)$ *either into* $G_{3,6} = \langle c_3c_4(12), c_1c_2(34)\rangle$, *or into* $G_{3,5} = \langle c_1c_2, (34), c_3c_4(12)\rangle$. *In both cases,* $Ш^1_\omega(G, Q(D_4)^o) = \mathbb{Z}/2$.

Proof : Among all the conjugacy classes stable under $\mathrm{Aut}' W(D_4)$ only $G_{3,5}$ and $G_{5,3}$ do not satisfy the conditions of Lemmas 4.15–4.17. Next, one can also immediately see that all the $\mathrm{Aut}' W(D_4)$–orbits consisting of three elements contain at least one group satisfying the condition of one of Lemmas 4.15–4.17, except for the orbit containing $G_{3,6}$.

Let $G = G_{3,5} = \langle c_1c_2, c_3c_4(12), (34)\rangle$, then $F = H^1(G_{3,5}, Q(B_4)) \simeq (\mathbb{Z}/2)^2$, $Ш^1_\omega(G_{3,5}, Q(D_4)^o) = F/\langle a_G\rangle = \mathbb{Z}/2$.

For $G_{5,3} = \langle c_3c_4(12), c_2c_4(13)\rangle$ we have $H^1(G_{5,3}, Q(B_4)) \simeq (\mathbb{Z}/2)^2$, however, $F = \langle a_G\rangle$ because $I_{c_3c_4(12)} = \{3,4\}$ intersects both $\{1,2,3\}$ and $\{4\}$ which are orbits of $pr(G_{5,3})$. Hence $Ш^1_\omega(G_{5,3}, Q(D_4)^o) = 0$.

For $G_{3,6} = \langle c_3c_4(12), c_1c_2(34)\rangle$ we have $H^1(G_{3,6}, Q(B_4)) = (\mathbb{Z}/2)^2 = F$, hence $Ш^1_\omega(G_{3,6}, Q(D_4)^o) = \mathbb{Z}/2$.

Let us conclude this section with one more application of the classification of subgroups of $W(D_4)$.

THEOREM 4.19. *If a Del Pezzo surface* X *of degree* 4 *with* $\mathrm{Pic}\, X \simeq \mathbb{Z} \oplus \mathbb{Z}$ *is stably* k*-rational but is not* k*-rational, then it is birationally equivalent to a cubic surface given by*

$$y^2 - az^2 = P(x), \tag{20}$$

$P(x)$ *being an irreducible polynomial of degree* 3, *whose discriminant is equal to* a, $a \notin (k^*)^2$.

Proof : Suppose X is stably k–rational. Let L/k be the splitting field of X, and let $G_X = \mathrm{Gal}(L/k)$ be its splitting group. Then the G_X–module $\mathrm{Pic}\, \bar{X}$ is a direct summand of a permutation G_X–module (see the beginning of section 2, or [43], 4.35). Therefore for any subgroup $G \subsetneq G_X$ we have $H^1(G, \mathrm{Pic}\,\bar{X}) = 0$. Assume that X is k–minimal, i.e. G_X is a minimal subgroup of $W(D_4)$ (then X is k–irrational [20]). According to Manin, $H^1(G, \mathrm{Pic}\,\bar{X})$ depends only on the decomposition type of the graph Γ into G–orbits. He computed this invariant for all the types ([28], Tab. 3). Note, that $G_{0,1} = <c_1c_2c_3c_4>$ is of type IV, hence $H^1(G_{0,1}, \mathrm{Pic}\,\bar{X}) \simeq (\mathbb{Z}/2)^2$. Next, $G_{3,6} = <c_3c_4(12), c_1c_2(34)>$ is of type VIII, hence $H^1(G_{3,6}, \mathrm{Pic}\,\bar{X}) = \mathbb{Z}/2$. Finally, $G_{1,5} = <c_1c_2, c_3c_4(12)>$ is of type IX, hence $H^1(G_{1,5}, \mathrm{Pic}\,\bar{X}) = \mathbb{Z}/2$. Now observe that $c_1c_2c_3c_4$ is contained in $G_{i,0}$ and $G_{i,1}$ for all i, and also in $G_{0,2}$, $G_{1,2}$, $G_{1,3}$, $G_{2,2}$, $G_{2,3}$, $G_{2,4}$, $G_{2,5}$, $G_{3,2}$, $G_{3,3}$, $G_{3,4}$, $G_{6,2}$, $G_{7,2}$, $G_{7,3}$, $G_{7,4}$, $G_{7,5}$, $G_{7,6}$, $G_{8,2}$, $G_{8,3}$, $G_{8,4}$, $G_{9,2}$. The group $G_{3,5}$ is of type IX. Therefore it only remains to consider $G_{1,4}$, $G_{5,2}$, $G_{5,3}$, $G_{8,5}$, $G_{10,2}$. The group $G_{1,4} = <c_1c_3, c_1c_4(12)>$ contains $<c_1c_2, c_3c_4(12)> = G_{1,5}$. The group $G_{5,2} = <c_1c_4(12), c_1c_3(13)>$ also contains $G_{1,5}$. The group $G_{8,5} = <(1234), c_1c_2c_3c_4(13)>$ contains $<(13)(24), c_1c_2c_3c_4(13)>$ which is conjugate in $W(D_4)$ to the group $<c_1c_2c_3c_4(13)(24), c_2c_4(13)> = G_{3,6}$. The group $G_{10,2} = <c_3c_4(12), (1234)>$ contains $G_{3,6}$. Thus, any stably k–rational k–minimal Del Pezzo surface X of degree 4 with $\mathrm{Pic}\, X \simeq \mathbb{Z} \oplus \mathbb{Z}$ has the splitting group $G_X = G_{5,3} \simeq S_3$. This group always has the decomposition of Γ of type II, in particular, there exists a G_X–orbit consisting of two joined vertices (see Figure 4). To them there correspond two lines on X meeting in a point x which is thus defined over k. Let us fix a conic bundle structure $f : X \longrightarrow C$. Arguing as in the proof of Theorem 3.20 a), let us blow–up x and blow–down the transform of the fibre of f passing through x. We obtain an Iskovskih surface which (by Corollary 3.11) is isomorphic to a generalized Châtelet surface given by

$$y^2 - az^2 = R(x,t).$$

Note that G_X acts on the components of singular fibres in the same way as it acts on singular fibres of f, i.e. as on the 8 vertices of Γ all lying on the same side of the vertical line. One immediately checks that this action is conjugate to the G_X–action on the graph $\Sigma_5 = \Delta \backslash \{q_5^+, q_5^-\}$ (it can be deduced from the fact that the conjugacy class of $G_X = G_{5,3}$ is $\mathrm{Aut}'\, W(D_4)$–invariant). Since $pr(G_X) = pr(G_{5,3}) \simeq S_3$, the polynomial $R(x,t)$ has a linear factor. A linear change of x and t yields the equation

$$y'^2 - az'^2 = t\,.\,P(x,t)$$

where $\deg P(x,t) = 3$. From the explicit action of G_X one can easily deduce that $a \equiv \operatorname{discr} P(x,t) \pmod{(k^*)^2}$. The birational transformation $y' = yt$, $z' = zt$ provides the equation $t(y^2 - az^2) = P(x,t)$; it is just the equation of a cubic surface written in homogeneous coordinates. This proves the theorem under the assumption of k–minimality of X.

Assume that X is not k–minimal. Then X contains a G_X–invariant set of mutually skew lines. Blowing them down we get a Del Pezzo surface of degree at least 5. For such surfaces the following alternative holds : either Z is k–rational, or $Z(k) = \emptyset$. By the assumption X is not k–rational; on the other hand, any stably k–rational variety has a k–point. This proves the theorem.

Remark 4.20. In the remarkable paper [1] it is proved that the surface (20) is actually stably k–rational.

5
SUBGROUPS OF $W(D_5)$

This section is devoted to the computation of the invariant $Ш^2_\omega(G, Q(D_5))$ for all the subgroups $G \subseteq W(D_5)$ that are not contained in $W(D_4)$. From the geometric point of view such groups correspond to Del Pezzo surfaces of degree 4 with $\mathrm{Pic}\, X = \mathbb{Z}$. Together with the results of the preceding section it completes the computation of $Ш^2_\omega(G_X, \mathrm{Pic}\, \bar{X})$ for all Del Pezzo surface of degree 4.

However, the group $W(D_5)$ has no admissible outer automorphisms (see, for example, [40], Proposition 3.1), i.e. this case is characterized by lower degree of symmetry than that of subgroups of $W(D_4)$. The methods used in this section are quite different from those employed in section 4.

Let us return for a while to an arbitrary torsion–free G–module N of finite type, and let G be a finite group.

LEMMA 5.1. *Let U be a normal subgroup of G, and suppose that $H^1(U,N) = 0$. Then* $|Ш^2_\omega(G,N)| \leq |Ш^2_\omega(U,N)| + |H^2(G/U, N^U)|$.

Proof : The statement follows from the commutative diagram

$$
(21) \qquad \begin{array}{ccccccc}
 & & & & 0 & & 0 \\
 & & & & \downarrow & & \downarrow \\
 & & & & Ш^2_\omega(G,N) & \longrightarrow & Ш^2_\omega(U,N) \\
 & & & & \downarrow & & \downarrow \\
0 & \longrightarrow & H^2(G/U,N^U) & \longrightarrow & H^2(G,N) & \longrightarrow & H^2(U,N) \\
 & & & & \downarrow & & \downarrow \\
 & & & & \prod\limits_{g \in G} H^2(<g>,N) & \longrightarrow & \prod\limits_{g \in U} H^2(<g>,N),
\end{array}
$$

the vertical sequences being exact by definition, and the horizontal one arising from the Hochschild–Serre spectral sequence on account of the condition $H^1(U,N) = 0$.

LEMMA 5.2. *Let $G = U \rtimes V$, and suppose that $H^1(U,N) = 0$. Then* $|Ш^2_\omega(G,N)| \leq |Ш^2_\omega(U,N)| + |Ш^2_\omega(V,N)| + |H^1(V,N/N^U)|$.

Proof : If follows from (21) that $|\text{Ш}^2_\omega(G,N)| \le |\text{Ш}^2_\omega(U,N)| + |A|$, where A is the kernel of composite homomorphism $H^2(V,N^U) \to H^2(G,N) \to \prod_{g\in G} H^2(<g>,N)$. The exact sequence of V–modules $0 \to N^U \to N \to N/N^U \to 0$ implies that $|A| \le |H^1(V,N/N^U)| + |B|$, B being the image of A in $H^2(V,N)$. From the commutative diagram

$$\begin{array}{ccccccc} 0 & \longrightarrow & A & \longrightarrow & H^2(V,N^U) & \longrightarrow & \prod_{g\in G} H^2(<g>,N) \\ & & \downarrow & & \downarrow & & \downarrow \\ 0 & \longrightarrow & \text{Ш}^2_\omega(V,N) & \longrightarrow & H^2(V,N) & \longrightarrow & \prod_{g\in V} H^2(<g>,N) \end{array}$$

it follows that $B \subseteq \text{Ш}^2_\omega(V,N)$.

Recall that the G_X–module $\operatorname{Pic}\bar{X}$ is isomorphic to $M(D_5)$ because of 2.12 a), and that $\text{Ш}^2_\omega(G,M(D_5)) = \text{Ш}^2_\omega(G,Q(D_5))$.

COROLLARY 5.3. *Let $G \subseteq W(D_5)$, and suppose that $M(D_5)^G \simeq \mathbb{Z}$ (this property characterizes minimal subgroups of $W(D_5)$ not lying in $W(D_4)$). Assume that U is a subgroup of G of index 2, such that U is a minimal subgroup of $W(D_4)$ and $H^1(U,M(D_5)) = 0$. Then $|\text{Ш}^2_\omega(G,M(D_5))| \le |\text{Ш}^2_\omega(U,M(D_5))|$.*

Proof : In view of 5.1, it is enough to show that $M(D_5)^U = M(D_5)^{W(D_4)} \simeq \mathbb{Z} \oplus \mathbb{Z}$ is a permutation $W(D_5)$–module. It can be done in the following way : $(\operatorname{Pic}\bar{X})^{G_X \cap W(D_4)}$ is generated by K_X and ℓ_0, K_X being the canonical class of X, and ℓ_0 being the class of a fibre of the conic bundle, therefore $-K_X - \ell_0$ is the class of the fibre of the complementary conic bundle, and the group $W(D_5)$ interchanges ℓ_0 and $-K_X - \ell_0$. We can express the same thing in another way : the $W(D_4)$–invariants are generated by sums of lines corresponding to the joined pairs of vertices of the graph Γ lying on the left and on the right of the vertical line, and the group $W(D_5)$ interchanges these pairs.

COROLLARY 5.4. *Let $G = U \rtimes V \subseteq W(D_5)$, and let both U and V be non-minimal subgroups of $W(D_5)$. If $H^1(V,M(D_5)/M(D_5)^U) = 0$, then $\text{Ш}^2_\omega(G,M(D_5)) = 0$.*

Proof : In view of [28] (Ch. IV, 7.7) we have $H^1(U,M(D_5)) = 0$ (U being a non-minimal subgroup of $W(D_5)$).

COROLLARY 5.5. *Let* $G = U \rtimes V \subseteq W(D_5)$, *and let* $M(D_5)^U \simeq \mathbb{Z}$. *Suppose that* $H^1(U,M(D_5)) = 0$. *Let* V *be a subgroup of* S_4 *which is embedded into* $S_5 \subset W(D_5)$. *Then* $|\text{Ш}^2_\omega(G,M(D_5))| \leq |\text{Ш}^2_\omega(U,M(D_5))|$.

Proof : Let us employ Lemma 5.2. We have $\text{Ш}^2_\omega(V,M(D_5)) = 0$ since $Q(D_5)$ is a permutation V-module. Besides, $M(D_5)/M(D_5)^{W(D_5)} = P(D_5)$ is the weight lattice (see the exact sequence (11)). As a submodule of $Q(B_5) \otimes \mathbb{Q}$ this lattice is generated by $Q(B_5)$ with the element $\frac{1}{2}(e_1+...+e_5)$ added to it. One can take $\{e_1,e_2,e_3,e_4,\frac{1}{2}(e_1+...+e_5)\}$ for a basis of $P(D_5)$. Since S_4 permutes the first four elements and fixes the last one, it follows that $P(D_5)$ is a permutation V-module, in particular, $H^1(V,M(D_5)/M(D_5)^U) = 0$.

Let G be a minimal group, $G \not\subseteq W(D_4)$. According to Figure 4 the decomposition of Γ into G-orbits belongs to one of the following types : I, X–XI, XVI–XIX. Any more or less complete list of such groups would be, probably, too lengthy, therefore in this section we act in the following way. We compute the maximal groups of automorphisms of all the types, then we establish k-rationality of the torus dual to G-module $Q(D_5)$ whenever the decomposition type of G is XVI (or its subdecomposition). For the remaining types (except for type I) we use enumeration of cases excluding some series of groups with $\text{Ш}^2_\omega(G,M(D_5)) = 0$.

PROPOSITION 5.6. *The maximal groups of automorphisms of the types, which are not subtypes of type* XV, *are* :

$\text{Aut(I)} = \langle c_2c_3c_4c_5,(15)\rangle \times \langle(23),(34)\rangle$, $\text{Aut(X)} = \langle c_2c_3c_4c_5,(15)\rangle$,
$\text{Aut(XI)} = \langle c_1c_2c_3c_4, c_1c_5(23)\rangle$, $\text{Aut(XVI)} = \langle c_2c_3c_4c_5,(15)\rangle \times \langle(34)\rangle$,
$\text{Aut(XVII)} = \langle c_3c_4,(34)\rangle \times \langle c_2c_5,(25),(12)\rangle$, $\text{Aut(XVIII)} = \langle(12)(34), c_2c_5(1243)\rangle$,
$\text{Aut(XIX)} = W(D_5)$.

Proof : One can immediately check that the above generators preserve corresponding types. Note that a vertex in any horizontal row of Γ is joined to a unique vertex in the first row. It follows that the action of any element g preserving all the rows (type X) is determined by its action on an arbitrary row. However, a row viewed as a graph is nothing but a square, hence Aut(X) is at most the dihedral group.

Any element of Aut(I) is equal to a product of an element of Aut(X) by an element of $S_3 = <(23),(34)>$, since Aut(X) (respectively S_3) determines the law of permutation for the columns (respectively for the rows).

The group Aut(XVI) is a proper subgroup of Aut(I), and $<c_2c_3c_4c_5,(15),(34)>$ is a Sylow 2–subgroup of Aut(I) whose index in Aut(I) is equal to 3, hence these two subgroups coincide.

To compute Aut(XVII) we note that any permutation from S_5 fixes the vertex q of Γ. Therefore, if $c_I\rho \in$ Aut(XVII), then $c_I \in \{1,c_3c_4,c_1c_2,c_2c_5\}$. All these elements belong to Aut(XVII), hence ρ also belongs to Aut(XVII), i.e. $\rho \in \{1,(34),(12),(25)\}$.

We begin the computation of Aut(XVIII) by observing that this group contains no element of order 5 or 3. Indeed, an element of order 5 is conjugate to (12345), and its orbits on Γ are of lengths 5,5,5,1, hence such a decomposition cannot be a subdecomposition of type XVIII. Elements of order 3 are conjugate to (125). Γ should have been decomposed into two orbits of 8 elements each in such a way that (125) preserves this decomposition, and each vertex is joined with three vertices in its own orbit and two vertices in the other. One easily checks, that there is no way to do so, other than the decomposition of type XVII. The latter is not equivalent to the decomposition of type XVIII. Thus, Aut(XVIII) is a 2–group. We have Aut(XVIII) $\cap <c_1c_2,c_2c_3,c_3c_4,c_4c_5> = <c_1c_2c_3c_4>$, hence the index of Aut(XVIII) in a Sylow 2–subgroup of $W(D_5)$ is at least 2^3. The order of a Sylow 2–subgroup of $W(D_5)$ is equal to 2^7, and the group $<(12)(34), c_2c_5(1243)>$ is isomorphic to the dihedral group of order 2^4. It follows that Aut(XVIII) $= <(12)(34), c_2c_5(1243)>$.

Finally, since the decomposition of type XI is a subdecomposition of type XVIII, in order to prove that Aut(XI) $= <c_1c_2c_3c_4, c_1c_5(23)> \simeq (\mathbb{Z}/2)^2$ it is enough to show that Aut(XI) contains no element of order 4. There exists only one element of order 4 in Aut(XVIII) (up to an inversion), and it does not preserve the decomposition of type XI.

PROPOSITION 5.7. *Let* $G =$ Aut(XVI) $= <c_2c_3c_4c_5,(15),(34)>$, ***then the algebraic torus dual to the*** *G-module* $Q(D_5)$ ***is rational over the ground field.***

Proof : Recall that $Q(D_5)$ is isomorphic to $Q(C_5)$ as a $W(D_5)$–module; it is a submodule of index 2 in $Q(B_5)$. Let $\{e_1,\dots,e_5\}$ be the standard basis of $Q(B_5)$, then $Q(C_5)$ is defined by the condition that the sum of all coordinates is even. Let $\epsilon_1 = e_2 - e_5$, $\epsilon_2 = e_1 - e_5$, $\epsilon_3 = e_1 + e_2$, $\epsilon_4 = e_3 - e_4$, $\epsilon_5 = e_2 - e_3$ be a basis of $Q(C_5)$. We immediately observe that the G–module

$Q(D_5)$ can be decomposed into the direct sum of the G–module generated by ϵ_1, ϵ_2, ϵ_3 and the G–module generated by ϵ_4, ϵ_5. Respectively, the dual torus is a product of a two–dimensional torus by a three–dimensional one. According to [43], all the two–dimensional tori are rational over the ground field. Consider the G–module $<\epsilon_1,\epsilon_2,\epsilon_3>$. Let us remark that $(34) \in G$ acts trivially. We claim that this G–module is isomorphic to the kernel of the homomorphism $\mathbb{Z}[<c_2c_3c_4c_5,(15)>/<(15)>] \to \mathbb{Z}$ mapping elements of the canonical basis to 1. Let us define this isomorphism explicitly :

$$\begin{aligned} f(\epsilon_1) &= c_2c_3c_4c_5 \,.\, <(15)> - <(15)>, \\ f(\epsilon_2) &= c_2c_3c_4c_5 \,.\, <(15)> - c_1c_2c_3c_4 \,.\, <(15)>, \\ f(\epsilon_3) &= c_2c_3c_4c_5 \,.\, <(15)> - c_1c_5 \,.\, <(15)>. \end{aligned}$$

It follows that the torus T dual to $<\epsilon_1, \epsilon_2, \epsilon_3>$ is rational : indeed, let $k \subset F \subset L$ be the extensions corresponding (in the sense of Galois theory) to the subgroups $<c_2c_3c_4c_5,(15)> \supset <(15)> \supset \{1\}$, then $T = R_{F/k}(G_{m,F})/G_{m,k}$, where $R_{F/k}$ denotes Weil's descent [45]. T admits an open embedding into $\mathbb{P}^3_k$, thus it is k–rational (see also [43], 4.8).

COROLLARY 5.8. *Let* $G \subset W(D_5)$ *provide the orbit decomposition of* Γ *of type* X, XI *or* XVI. *Then the algebraic torus dual to* $Q(D_5)$ *is rational over the ground field. In particular,* $Ш^2_\omega(G,Q(D_5)) = 0$. *If the decomposition of* Γ *into* G*-orbits is of type* I, *then* $Ш^2_\omega(G,Q(D_5)) = 0$.

Proof : If G provides the orbit decomposition of type X, XI or XVI, then the statement is quite clear, G being a subgroup of Aut(XVI). Let $G \subsetneq \text{Aut(I)} = <c_2c_3c_4c_5,(15)> \times <(23),(34)>$. As it has been already noted, Aut(XVI) is a Sylow 2–subgroup of Aut(I). Therefore, if $G(2)$ is a Sylow 2–subgroup of G, then $Ш^2_\omega(G(2),Q(D_5)) = 0$. Obviously, $H^2(G,Q(D_5))$ can be embedded into $\prod_p H^2(G(p),Q(D_5))$, and $\text{Ker}[H^2(G,Q(D_5)) \to \prod_{g\in G} H^2(<g>,Q(D_5))]$ can be embedded into $\prod_p \text{Ker}[H^2(G(p),Q(D_5)) \to \prod_{g\in G(p)} H^2(<g>,Q(D_5))]$. For every $p \neq 2$ the group $G(p)$ is cyclic (this follows from the fact that $|W(D_5)| = 2^7.3.5$), hence $Ш^2_\omega(G(p),Q(D_5)) = 0$.

It remains to investigate groups of types XVII, XVIII, XIX. We begin with type XVIII.

PROPOSITION 5.9. *Let the orbit decomposition of* Γ *defined by* G *be of type* XVIII, *then* G *is conjugate to one of the following groups* : $G_1 = \mathrm{Aut(XVIII)} = <(12)(34), c_2c_5(1243)>$, $G_2 = <c_2c_5(1243)>$ *or* $G_3 = <c_2c_4(14)(23), c_2c_5(14)>$.

Proof : $\mathrm{Aut(XVIII)} \simeq D_8$ is the dihedral group of degree 8, i.e. $<a,b|\ a^8 = baba = b^2 = 1>$. We are interested in its subgroups of order 8 because such is the orbit length for the decomposition of type XVIII. If G does not coincide with $<a>$, then $G \cap <a> = <a^2>$, hence G is either $<a^2,b>$, or $<a^2,ab>$. However, $<a^2,b> = <c_2c_4(14)(23),(12)(34)>$ coincides with the group Aut(V), i.e. does not belong to type XVIII. The group $G_3 = <a^2,ab>$ is of type XVIII.

COROLLARY 5.10. *Let the orbit decomposition for* G *be of type* XVIII, *then* $\coprod^2_\omega(G,Q(D_5)) = 0$.

Proof : The statement is trivial for the cyclic group G_2; G_1 contains the subgroup $<a^2,b> = \mathrm{Aut(V)}$ of index 2, and G_3 contains the subgroup $<c_2c_4(14)(23)> = <a^2>$ also of index 2. The latter is conjugate to $G_{2,5}$ and thus belongs to type V (Proposition 4.7). Since minimal subgroups of type V have the property $H^1(G,M(D_5)) = 0$, we can apply Corollary 5.3. It remains to use Theorem 4.13.

Let us go over to the groups of type XVII. Let $A = <c_3c_4,(34)>$, and let $B = <(12),(25),c_1c_5>$, then $\mathrm{Aut(XVII)} = A \times B$. Looking at the graph Γ we immediately see that neither A, nor B is a minimal subgroup of $W(D_5)$. Now we need the explicit action of $W(D_5)$ on $M(D_5)$. We have a decomposition of $\mathbb{Z}$–modules : $M(D_5) \simeq Q(D_5) \oplus \mathbb{Z}$, let β be a generator of $\mathbb{Z}$.

If $\alpha \in Q(D_5)$, then g acts on α in the usual way, and $g\beta = \beta + \omega_5 - g\,\omega_5$, where ω_5 is the fundamental weight corresponding to the vertex at the short end of the Dynkin diagram (cf. section 1). In the standard basis $\{e_1,...,e_5\}$ we have $\omega_5 = \frac{1}{2}(e_1+...+e_5)$, therefore, if $g \in S_5$, then $g\beta = \beta$, and if $g = c_ic_j$, then $g\beta = \beta + e_i + e_j$.

THEOREM 5.11. *Let the orbit decomposition for* G *be of type* XVII, *and let* $\coprod^2_\omega(G,Q(D_5)) \neq 0$, *then* G *is conjugate to one of the following groups* : $H_1 = <c_1c_5(12)(34), c_2c_3c_4c_5(34)>$, $H_2 = <c_3c_4, c_1c_5(12)(34)>$, $H_3 = <c_3c_4, c_1c_2, c_1c_5(34)>$, *or* $H_4 = <c_3c_4,(12),c_2c_5(34)>$; *for these groups* $\coprod^2_\omega(G,Q(D_5)) = \mathbb{Z}/2$.

Proof : Set $G_A = G \cap A$, and denote the group $GA/A \subseteq B$ by G_B, then G can be represented as an extension $1 \to G_A \to G \to G_B \to 1$.

Let $G = G_A \times G_B$, then $M(D_5)^{G_B} \supseteq <e_3,e_4>$, hence $M(D_5)/M(D_5)^{G_B}$ is a trivial G_A–module. Corollary 5.4 now states that for such groups $Ш^2_\omega(G,M(D_5)) = 0$. Note that if $G_A = A$, then $G = G_A \times G_B$.

Let $G_A = <(34)>$ or $<c_3c_4(34)>$. One may assume that $G_A = <(34)>$, otherwise one can conjugate in $W(D_5)$, for example by c_1c_3. In this case, $G = <(34)> \times G'$, where $G' \subseteq <c_3c_4(34)> \times B$; we have $M(D_5)^{G'} \supseteq <e_3 - e_4>$, hence $M(D_5)/M(D_5)^{G'}$ is a trivial $<(34)>$–module. Since neither $<(34)>$, nor G' is minimal, one can apply Corollary 5.4 again.

Let $G_A = 1$, then $G \simeq G_B \subset B$. The orbit length for the decomposition of type XVII is equal to 8, and B is isomorphic to S_4, hence G_B is isomorphic either to S_4 or to D_4. Let $G_B = B = <(125),(15),c_1c_5>$, and let $G = <(125), \mu(15),\nu c_1c_5>$. Since $(\nu c_1c_5(125))^3 = \nu$, it follows that $\nu = 1$. The only μ for which G is minimal is $\mu = c_3c_4$, i.e. $G = <(125),c_1c_5,c_3c_5(15)>$. Let us compute $Ш^2_\omega(G,Q(D_5))$ according to the recipe of Theorem 1.22. Since $G_0 = <(125),c_1c_5>$ is isomorphic to A_4, we have $\mathrm{Hom}(G/G_0,\mathbb{Z}/2) = \mathbb{Z}/2$; the character $\chi_{G,1}$ given by the G–action on $\{e_3, -e_3\}$ is non–trivial on G, hence $Ш^2_\omega(G,Q(D_5)) = 0$. Let $G_B = <(15),c_2c_5>$ be the dihedral group, then $G = <\mu(15),\nu c_2c_5>$ with $\mu,\nu \in A$. Now one has to analyse all the possible cases. If $\mu = 1$, (34) or $c_3c_4(34)$, then G cannot belong to type XVII. Consequently, $\mu = c_3c_4$, and there remain three possibilities : $<c_3c_4(15),c_2c_5>$, $<c_3c_4(12),c_2c_3c_4c_5>$ and $<c_3c_4(15),c_2c_3c_4c_5(34)>$. For each of these groups we have $G_0 = 1$. In the first case, $\chi_{G,1}$ corresponding to the G–action on $\{e_3, -e_3\}$ is non trivial on $c_3c_4(15)$ and trivial on c_2c_5, and $\chi_{G,2}$ corresponding to the action on $\{e_2, -e_2\}$ is trivial on $c_3c_4(15)$ and non–trivial on c_2c_5, i.e. $Ш^2_\omega(G,Q(D_5)) = 0$. In the second case, one can use similar arguments. In the third case, the orbits of $pr(G)$ are $\{3,4\}$, $\{1,5\}$ and $\{2\}$. The character corresponding to the first of these is trivial on G, and two others coincide, hence $Ш^2_\omega(G,Q(D_5)) = \mathrm{Hom}(G,\mathbb{Z}/2)/(\mathbb{Z}/2) = \mathbb{Z}/2$. Thus we obtain H_1.

Now consider the case where $G_A = <c_3c_4>$, then $G = G_A \times G'$ with $G' \subseteq <(34)> \times B$. Since the orbit length for the decomposition of type XVII is equal to 8, the order of G' is divisible by 4, i.e. $G' \simeq G_B$ is isomorphic to one of the following subgroups of $B \simeq S_4$: $<c_1c_2,(12)>$, $<c_1c_5(12)> \simeq \mathbb{Z}/4$, $<c_1c_2,c_1c_5> \simeq V_4$, $<c_1c_2,(15)> \simeq D_4$, A_4, S_4. In the first case, G is contained in $<c_3c_4,(34),c_1c_2,(12)>$ and cannot belong to type XVII. Let $G_B \simeq A_4$. The group A_4 is generated by its elements of order 3, let γ be one of them. Suppose for a moment that $\mu \in A$ is such that $\mu\gamma \in G$, and let $\mu \neq 1$; then $(\mu\gamma)^3 = \mu \in A$ which contradicts the assumption $G_A = <c_3c_4>$, therefore $G = <c_3c_4> \times A_4 = G_A \times G_B$, and we are in the situation considered at the very beginning of the proof. Assuming that $G_B \simeq \mathbb{Z}/4$ we get $H_2 = <c_3c_4,c_1c_5(12)(34)>$. For this group we have : $(H_2)_0 = 1$, $\mathrm{Hom}(H_2/(H_2)_0,\mathbb{Z}/2) \simeq (\mathbb{Z}/2)^2$. The orbits of $pr(H_2)$ are $\{1,2\}$, $\{3,4\}$, $\{5\}$; let us denote the corresponding characters by $\chi_{G,1}, \chi_{G,2}, \chi_{G,3}$. They are trivial on c_3c_4, and $\chi_{G,1}, \chi_{G,3}$ are non–trivial on $c_1c_5(12)(34)$, hence

$$Ш^2_\omega(H_2,Q(D_5)) = \mathrm{Hom}(H_2/(H_2)_0,\mathbb{Z}/2)/<\chi_{G,1}, \chi_{G,2}, \chi_{G,3}> = \mathbb{Z}/2.$$

Assuming that $G_B \simeq V_4 = <c_1c_2,c_1c_5>$ and excluding the case $G = G_A \times G_B$ we get (up to a conjugation) $H_3 = <c_3c_4,c_1c_2,c_1c_5(34)>$. Here $(H_3)_0 = 1$, and $\mathrm{Hom}(H_3/(H_3)_0,\mathbb{Z}/2) \simeq (\mathbb{Z}/2)^3$. The character corresponding to the action of the group on $\{e_2, -e_2\}$ is trivial on all the generators except the second one, and the one, corresponding to the action on $\{e_5, -e_5\}$, is trivial on all the generators except the third one. Finally, each $\chi_{G,i}$ is trivial on c_3c_4, hence $Ш^2_\omega(H_3,Q(D_5)) \simeq (\mathbb{Z}/2)^3/(\mathbb{Z}/2)^2 = \mathbb{Z}/2$. Assuming that $G_B = D_4$ we have several possibilities :

$$H_4 = <c_3c_4,c_1c_2(34),(15)>, <c_3c_4,c_1c_2,(15)(34)>, \text{ and } <c_3c_4,c_1c_2(34),(15)(34)>.$$

In the last two cases, $G_0 = <c_3c_4,c_1c_5,(15)(34)>$, and the character corresponding to $\{1,5\}$ is non–trivial on c_1c_2 and $c_1c_2(34)$, hence $Ш^2_\omega(G,Q(D_5)) = 0$. On the other hand, $(H_4)_0 = <(15),c_1c_5>$, the character corresponding to $\{2\}$ is non–trivial on $c_1c_2(34)$, and all the characters are trivial on c_3c_4 and (15), hence $Ш^2_\omega(H_4,Q(D_5)) = \mathbb{Z}/2$. Consider the very last case, where $G_B = B$ and $G = <c_3c_4,(125),\nu c_1c_2,\mu(12)>$ with ν and μ being equal either to (34) or to 1. Since $(\nu c_1c_2(125))^3 = \nu$, it follows that $\nu = 1$. Finally, $G = <c_3c_4,(125),c_1c_2,(12)(34)>$, and one can see that $G = G_0$, hence $Ш^2_\omega(G,Q(D_5)) = 0$. The theorem is proved.

THEOREM 5.12. *Let a group* $G \subsetneq W(D_5)$ *act on the graph* Γ *transitively (i.e. its orbit decomposition is of type* XIX*). If* $\text{Ш}^2_{\omega}(G,Q(D_5)) \neq 0$, *then* G *is conjugate to one of the following groups:* $J_1 = \langle c_3c_5(12), c_1c_5(1324)\rangle$, $J_2 = \langle c_1c_2, c_1c_5(1324)\rangle$, $J_3 = \langle c_1c_2, c_3c_5(12), (13)(24)\rangle$, $J_4 = \langle c_1c_2, c_3c_5(12), c_1c_5(34)\rangle$; *for these groups* $\text{Ш}^2_{\omega}(G,M(D_5)) = \mathbb{Z}/2$.

This class of subgroups of $W(D_5)$ is especially numerous, and the proof will be divided into several steps. As in section 4, let G^A denote the kernel of the homomorphism $pr: W(D_5) \longrightarrow S_5$ restricted to G, and let G^B denote its image. We shall list all the subgroups $G^A \subsetneq \langle c_1c_2, c_2c_3, c_3c_4, c_4c_5\rangle$, and look through all possible G's for each G^A. The transitivity of the G–action on Γ implies that $|G|$ is divisible by 16. Furthermore, the order of a Sylow 2–subgroup of S_5 is equal to 8, hence $G^A \neq 1$. Note, by the way, that a Sylow 2–subgroup of S_5 is conjugate to the dihedral group D_4, therefore it lies in S_4.

LEMMA 5.13. *Let* $G^A \simeq (\mathbb{Z}/2)^4$, *then* $\text{Ш}^2_{\omega}(G,Q(D_5)) = 0$.

Proof : In view of the argument used in the proof of Corollary 5.8, it is enough to show that $\text{Ш}^2_{\omega}(G(2),Q(D_5)) = 0$, where $G(2)$ is a Sylow 2–subgroup of G. Since $G(2)^B \subset S_4$, one can apply Corollary 5.5, taking $U = G(2)^A$ and $V = G(2)^B$. One easily checks that $G^A \simeq (\mathbb{Z}/2)^4$ is of type XIX, in particular, $H^1(G^A, M(D_5)) = 0$. The immediate application of Theorem 1.22 yields $\text{Ш}^2_{\omega}(G^A, Q(D_5)) = 0$, and Corollary 5.5 gives the statement of the lemma.

LEMMA 5.14. *Let* $G^A = \langle c_1c_2, c_2c_3, c_3c_4\rangle$, *then* $\text{Ш}^2_{\omega}(G,Q(D_5)) = 0$.

Proof : The group G^B is contained in S_4 realized as a permutation group on $\{1,2,3,4\}$, since G^A is stable under conjugations by elements of G^B. Let us assume that G^B is generated by permutations α and β, then we have to consider three cases : $\langle G^A, c_1c_5\alpha, \beta\rangle$, $\langle G^A, \alpha, c_1c_5\beta\rangle$, and $\langle G^A, c_1c_5\alpha, c_1c_5\beta\rangle = \langle G^A, c_1c_5\alpha, \alpha\beta\rangle$ (multiplying by elements of G^A, one can always write the generators in such a form). Note that according to Theorem 4.13 we have $\text{Ш}^2_{\omega}(G,Q(D_4)) = 0$ for subgroups $G \subsetneq W(D_4)$ containing G^A. Our group G contains a subgroup U of such a form : $U = \langle G^A, \beta\rangle$, $\langle G^A, \alpha\rangle$ or $\langle G^A, \alpha\beta\rangle$. It is not difficult to see that U is a minimal subgroup of $W(D_4)$ with the orbit decomposition of type XV, in particular, $H^1(U, M(D_5)) = 0$. By Corollary 5.3 we conclude that $\text{Ш}^2_{\omega}(G,Q(D_5)) = 0$.

LEMMA 5.15. *Let* $G^A = <c_1c_2, c_2c_3> \times <c_4c_5>$, *then* $Ш^2_\omega(G, Q(D_5)) = 0$.

Proof : In this case, $G^B \subseteq <(12),(23)> \times <(45)>$. Indeed, if $\sigma \in S_5$ normalizes G^A and $\sigma \notin <(12),(23),(45)>$, then there exist $i,j \in \{1,2,3\}$ such that $\sigma(i) \in \{1,2,3\}$ and $\sigma(j) \in \{4,5\}$. Consequently, $\sigma\, c_ic_j\, \sigma^{-1} \notin G$ which is absurd. Thus, $G = <G^A, c_1c_4\, \alpha,\beta,\gamma>$, where α, β, γ generate G^B (otherwise, $G \subseteq <G^A,(12),(23),(45)> = \mathrm{Aut(XVII)}$).

Suppose first that G is a 2–group. If $G^B = <(12)>$, then $G = <G^A, c_1c_4(12)>$. Let $U = <c_1c_4(12), c_1c_3>$, it is the group $G_{1,4}$ from Proposition 4.4. It belongs to type XV (Proposition 4.7), in particular, $H^1(U, M(D_5)) = 0$. According to Theorem 4.13 we have $Ш^2_\omega(U, M(D_5)) = 0$. Corollary 5.3 now yields $Ш^2_\omega(G, M(D_5)) = 0$. If $G^B = <(12)(45)>$, then $G = <G^A, c_1c_4(12)(45)> = <c_2c_3, c_1c_4(12)(45)>$, since

$$c_2c_3.c_1c_4(12)(45).c_2c_3.(c_1c_4(12)(45))^{-1} = c_1c_2.$$

Let us compute $Ш^2_\omega(G, Q(D_5))$ using Theorem 1.22. We have $G_0 = 1$, $\mathrm{Hom}(G, \mathbb{Z}/2) \simeq (\mathbb{Z}/2)^2$; the character associated to the G–action on $\{e_3, -e_3\}$ is non–trivial on c_2c_3 and trivial on $c_1c_4(12)(45)$, and the character associated to the orbit $\{4,5\}$ is, on the contrary, trivial on c_2c_3 and non–trivial on $c_1c_4(12)(45)$. Consequently, $Ш^2_\omega(G, Q(D_5)) = 0$. If $G^B = <(45)>$, then $G = <G^A, c_1c_4(45)>$. Set $U = <c_1c_2, c_1c_4(45)>$; this group is conjugate to $G_{1,3}$ and thus belongs to type XV. Applying Corollary 5.3 we get $Ш^2_\omega(G, M(D_5)) = 0$. If $G^B = <(12),(45)>$, then $G = <G^A, c_1c_4(12),(45)>$, $<G^A,(12), c_1c_4(45)>$ or $<G^A, c_1c_4(12),(12)(45)>$. Let $U = <G^A, c_1c_4(12)>$, $<G^A, c_1c_4(45)>$ or $<G^A, c_1c_4(12)>$, respectively. As we have just proved, $Ш^2_\omega(U, M(D_5)) = 0$. Since these groups belong to type XIX, we have $H^1(U, M(D_5)) = 0$. Set $V = <(45)>$, $<(12)>$ or $<(12)(45)>$, then $G = U \rtimes V$. By Corollary 5.5 we conclude that $Ш^2_\omega(G, M(D_5)) = 0$.

If G is not a 2–group, a Sylow 2–subgroup $G(2)$ is either of type XIX (and has been considered above), or of type XVII (because $G(2)$ contains G^A belonging to type XVII). In the latter case, we can write $G(2) = G(2)_A \times G(2)_B$ (in the notation of the proof of Theorem 5.11), hence $Ш^2_\omega(G(2), M(D_5)) = 0$ (Theorem 5.11).

LEMMA 5.16. *There are no groups* $G \subsetneq W(D_5)$ *with transitive action on* Γ, *and* G^A *conjugate to* $<c_1c_2,c_2c_3>$, $<c_1c_2,c_2c_3c_4c_5>$ or $<c_1c_2>$.

Proof : Arguing as in the proof of the previous lemma we see that in the first case, G^B is contained in $<(12),(23),(45)>$. However, $G \not\subseteq \mathrm{Aut(XVII)}$, hence G contains an element of the form $c_1c_4(12)$. The order of G^B is divisible by 4, therefore G also contains either (45) or $c_4c_5(45)$. The squared product of this element and the preceding one does not lie in G, which leads to a contradiction. In the second case, $G^B \subsetneq <(12),(34),(45)>$. If $\mu(34) \in G$, then multiplying by an element of G^A, we reduce μ to 1 or to c_3c_4. If $\nu(12) \in G$, then we can assume that $\nu = c_3c_4, c_4c_5$ or c_3c_5. Multiplying $c_ic_j(12)$ by (ik), where $\{i,j,k\} = \{3,4,5\}$, we get $(c_ic_j(12)(ik))^2 = c_ic_k$, which contradicts the assumption on G^A. Finally, if $G^A = <c_1c_2>$, then $G^B \subsetneq <(12),(34),(45)>$, and the order of G^B is not divisible by 8.

We have investigated all the subgroups of $<c_1c_2,c_2c_3,c_3c_4,c_4c_5>$ except for $<c_1c_2,c_3c_4,>$ and $<c_1c_2c_3c_4>$.

LEMMA 5.17. *Let* $G^A = <c_1c_2c_3c_4>$, *and suppose that* $Ш^2_\omega(G,M(D_5)) \neq 0$, *then* G *is conjugate to* $J_1 = <c_3c_5(12),c_1c_5(1324)>$, *and* $Ш^2_\omega(G,M(D_5)) = \mathbb{Z}/2$.

Proof : Obviously, $G^B \subsetneq <(12),(1234)>$; G^B is either D_4 or S_4, since $|G^B|$ is divisible by 8. Let $G^B = S_4$. If $\gamma \in G^B$ is a 3–cycle and $c_ic_j\gamma \in G$, then both i and j are from $\{1,2,3,4\}$ (indeed, $c_ic_5\gamma \in G$ implies $(c_ic_5)^3 \in G$, which is impossible). Therefore, denoting the natural projection $G \longrightarrow S_4$ by pr, we see that $U = pr^{-1}(A_4)$ is an index 2 subgroup of G, lying in $W(D_4)$. Besides, U is conjugate in $W(D_5)$ either to $G_{9,1}$ or to $G_{9,2}$, belonging respectively to type III or XV; in particular, $H^1(U,M(D_5)) = 0$. According to 4.13 we have $Ш^2_\omega(U,M(D_5)) = 0$, hence (by 5.3) $Ш^2_\omega(G,M(D_5)) = 0$. Now suppose that $G^B = D_4 = <(12),(1324)>$ and $G = <c_1c_2c_3c_4, \mu(12), \nu(1324)>$. If ν does not contain c_5, conjugating by an element of $<c_1c_2,c_2c_3,c_3c_4,c_4c_5>$ we obtain $\nu = 1$, then $\mu = c_3c_5$ or c_4c_5. These elements are conjugate under $(12)(34) \in G$, i.e. $G = <c_3c_5(12),(1324)>$; G contains $c_1c_3 = (c_3c_5(12)(1324))^2$, which contradicts the choice of G^A. Thus $\nu = c_ic_5$. If $\mu = 1$ or c_1c_2, then multiplying $(c_ic_5(1324))^2 = c_ic_j(12)(34)$ by $\mu(12)$ on the right and looking at the square of this product, we see that $c_3c_4 \in G$, which again contradicts the choice of G^A. Consequently, $\mu = c_3c_5$ or c_4c_5. These elements are conjugate under (34) which maps $<c_5c_i(1324)>$ to $<c_5c_j(1324)>$. The conjugation by (12) fixes $c_3c_5(12)$, maps $c_1c_5(1324)$ to $<c_4c_5(1324)>$, and $<c_2c_5(1324)>$ to $<c_3c_5(1324)>$. Now we must analyse two groups. The first one is $J_1 = <c_3c_5(12),c_1c_5(1324)>$. Let us apply Theorem 1.22. We have $(J_1)_0 = 1$; orbits of $pr(J_1)$ provide a unique non–trivial character, hence $Ш^2_\omega(G,M(D_5)) = (\mathbb{Z}/2)^2/(\mathbb{Z}/2) = \mathbb{Z}/2$. The other group is $G = <c_3c_5(12),c_2c_5(1324)>$. One can check that G is conjugate to Aut(XVIII).

LEMMA 5.18. *Let* $G^A = <c_1c_2,c_3c_4>$, *and suppose that* $Ш^2_\omega(G,M(D_5)) \neq 0$, *then* G *is conjugate either to* $J_2 = <c_1c_2,c_1c_5(1324)>$, *or to* $J_3 = <c_1c_2,c_3c_5(12),(13)(24)>$, *or to* $J_4 = <c_1c_2,c_3c_5(12),c_1c_5(34)>$, *and* $Ш^2_\omega(G,M(D_5)) = \mathbb{Z}/2$.

Proof : We have $G^B \subseteq <(12),(1324)>$ since G^B is contained in $S_4 = <(12),(1234)>$ but contains no 3–cycle. Since $|G^A| = 4$, it follows that $G^B = \mathbb{Z}/4$, V_4, D_4, or $<(12),(34)>$. If $G^B = <(1324)>$, then $G = J_2 = <c_1c_2,c_1c_5(1324)>$. We have $(J_2)_0 = 1$, $\mathrm{Hom}(J_2/(J_2)_0,\mathbb{Z}/2) \simeq (\mathbb{Z}/2)^2$; the characters of J_2 arising from the orbits $\{1,2,3,4\}$ and $\{5\}$ coincide, therefore $Ш^2_\omega(J_2,M(D_5)) = \mathbb{Z}/2$. If $G^B = <(12)(34),(13)(24)>$, then $G = <c_1c_2,\mu(13)(24),\nu(14)(23)>$ with μ and ν ranging over the set $\{1,c_1c_5,c_3c_5,c_1c_3\}$. If $\mu = c_1c_5$, then $(c_1c_5(13)(24))^2 = c_1c_3$, which is impossible. Therefore, μ and ν can only be equal to 1 or c_1c_3, but in these cases, $G \subset W(D_4)$. If $G^B = <(12),(1324)> = <(12),(13)(24)>$, then $G = <c_1c_2,\mu(12),\nu(13)(24)>$, where $\mu \in \{1,c_1c_5,c_3c_5,c_1c_3\}$ and $\nu \in \{1,c_1c_3\}$. Conjugating by c_3 we reduce ν to 1. Since $G \not\subset W(D_4)$, it follows that $\mu = c_1c_5$ or c_3c_5. Consider $G = <c_1c_5(12),(13)(24)>$. We have $G_0 = <(13)(24),c_2c_3(14)(23)> \simeq D_4$; $G/G_0 \simeq \mathbb{Z}/4$ is generated by the image of $c_1c_5(1324)$, $\mathrm{Hom}(G/G_0,\mathbb{Z}/2) \simeq \mathbb{Z}/2$, and the character associated with $\{5\}$ is non–trivial, hence $Ш^2_\omega(G,Q(D_5)) = 0$.

Consider $G = J_3 = <c_3c_5(12),(13)(24),c_1c_2>$. We have

$$(J_3)_0 = <(13)(24),c_2c_3(14)(23)> \simeq D_4;$$

$J_3/(J_3)_0 \simeq (\mathbb{Z}/2)^2$ is generated by the images of c_1c_2 and $c_3c_5(12)$, however the characters given by the orbits $\{1,2,3,4\}$ and $\{5\}$ of $pr(J_3) = D_4$ coincide, hence $\text{Ш}^2_\omega(J_3, M(D_5)) = \mathbb{Z}/2$. It remains to treat the case $G^B = <(12),(34)>$. Here $G = <G^A, \mu(12), \nu(34)>$ with μ and ν from $\{1, c_1c_5, c_3c_5, c_1c_3\}$. If $\mu = 1$, then (for any ν) G does not belong to type XIX. The case $\nu = 1$ is similar. If $\mu = c_1c_3$, then $U = <G^A, c_1c_3(12)> = G_{1,3}$ belongs to type XV, and $\text{Ш}^2_\omega(U, M(D_5)) = 0$. By Corollary 5.3 we see that $\text{Ш}^2_\omega(G, M(D_5)) = 0$. The case $\nu = c_1c_3$ is similar. The remaining groups are $G = <c_1c_5(12), c_3c_5(34)>$, $J_4 = <c_1c_5(34), c_3c_5(12), c_1c_2>$, $<c_1c_5(12), c_1c_5(34), c_3c_4>$, and $<c_3c_5(12), c_3c_5(34), c_1c_2>$. The last two groups do not belong to type XIX. Further, $G_0 = 1$, $\mathrm{Hom}(G/G_0, \mathbb{Z}/2) \simeq (\mathbb{Z}/2)^2$; the character corresponding to $\{1,2\}$ is non–trivial on $c_1c_5(12)$ and trivial on $c_3c_5(34)$, and vice versa for the character corresponding to $\{3,4\}$, hence $\text{Ш}^2_\omega(G, M(D_5)) = 0$. We have $(J_4)_0 = 1$, $\mathrm{Hom}(J_4/(J_4)_0, \mathbb{Z}/2) \simeq (\mathbb{Z}/2)^3$; the character corresponding to $\{1,2\}$ is non–trivial on the first generator and trivial on the others, the character corresponding to $\{3,4\}$ is non–trivial on the second generator and trivial on the others, and the character corresponding to $\{5\}$ is the product of two preceding ones. Thus $\text{Ш}^2_\omega(J_4, M(D_5)) = \mathbb{Z}/2$.

The theorem now follows from Lemmas 5.13–5.18.

On account of Proposition 1.24 we can restate Theorems 5.11 and 5.12 as follows.

COROLLARY 5.19. a) *Let* $G \subseteq W(C_4)$, $G \not\subseteq W(C_3) = <(12),(123),c_1>$, $G \not\subseteq W(D_4) = <(12),(1234),c_1c_2>$, *and let* $\text{Ш}^2_\omega(G, Q(C_4)) \neq 0$, *then* G *is conjugate to one of the following groups*:

$<c_1(12)(34), c_2c_3c_4(34)>$, $<c_3c_4, c_1(12)(34)>$, $<c_3c_4, c_1c_2, c_1(34)>$, $<c_3c_4, (12), c_2(34)>$, $<c_1c_2, c_3(12), c_1(34)>$, $<c_3(12), c_1(1324)>$, $<c_1c_2, c_1(1324)>$, *or* $<c_1c_2, c_3(12), (13)(24)>$,

for which $\text{Ш}^2_\omega(G, Q(C_4)) = \mathbb{Z}/2$.

b) *If* $G \subseteq W(D_5)$ *is not conjugate to* $G' \subseteq \varphi(W(C_4))$, *then* $\text{Ш}^2_\omega(G, Q(D_5)) = 0$.

We end this section with an analog of Theorem 4.19 (unfortunately, a less precise one) :

THEOREM 5.20. *If a Del Pezzo surface* X *of degree* 4 *with* $\mathrm{Pic}\, X \simeq \mathbb{Z}$ *is stably* k*-rational but not* k*-rational, then its splitting group* G_X *is conjugate to one of the following groups* : $I_1 = <(234), c_2c_3c_4c_5(23), c_1c_5>$, $I_2 = <(234), c_2c_3c_4c_5(15)(23)>$ *or* $I_3 = <(234), (15), c_2c_3c_4c_5(23)>$.

Proof : Our arguments are similar to those used in the proof of Theorem 4.19. Stable rationality of X implies that the G_X–module $\mathrm{Pic}\,\bar{X}$ is a direct summand of a permutation G_X–module, hence for any subgroup $G \subseteq G_X$ we have $H^1(G, \mathrm{Pic}\,\bar{X}) = 0$. Therefore, a minimal group G_X can only belong to one of the types I, XVIII, XIX (see Table 3 in [28], Ch. IV), and the case of non–minimal groups can be treated in the same manner as in Theorem 4.19. Assume that G_X is of type XIX. In the proof of Theorem 5.12 we have listed all the subgroups $G_X^A \subseteq <c_1c_2, c_2c_3, c_3c_4, c_4c_5>$ which can appear as $\mathrm{Ker}(pr)$, where $pr: G_X \longrightarrow S_5$ is the natural projection. Any subgroup from that list contains $<c_1c_2c_3c_4> = \mathrm{Aut(IV)}$ for which $H^1(\mathrm{Aut(IV)}, M(D_5)) \simeq (\mathbb{Z}/2)^2$ and thus does not satisfy the necessary condition. If G_X is of type XVIII, then, according to Proposition 5.9, G_X is conjugate to G_1, G_2 or G_3. All these groups contain $c_1c_2c_3c_4$. Hence G_X is of type I. Recall that $\mathrm{Aut(I)} = <c_2c_3c_4c_5, (23), (234), (15)>$. Since G_X is of type I, $|G_X|$ is divisible by 12 (the length of an orbit). Since $c_2c_3c_4c_5 \notin G_X$, there remain two possibilities for G_X^A: 1 or $<c_1c_5>$. In the first case, $G = <\mu(23), \eta(234), \nu(15)>$. Since $\mu, \eta, \nu \in <c_2c_3c_4c_5, c_1c_5>$, it follows that $\eta = 1$, and $\nu = 1$ or c_1c_5. If $\mu = 1$ or c_1c_5, then G_X is non–minimal. If $\mu = c_2c_3c_4c_5$ or $c_1c_2c_3c_4$, then $(\mu(23).\nu(15))^2 = c_1c_5 \in G_X$, which contradicts the assumption $G_X^A = 1$. Therefore, there are no groups $G_X \subseteq \mathrm{Aut(I)}$ with $G_X^A = 1$. In the second case $G_X^A = <c_1c_5>$, G_X^B can be one of the following groups : $<(23),(234)>$, $<(234),(15)>$, $<(23),(234),(15)>$ or $<(234),(23)(15)>$. We get respectively : $I_1 = <c_2c_3c_4c_5(23), (234), c_1c_5>$, $G = <(234), c_2c_3c_4c_5(15)>$, $G' = <(23),(234), c_2c_3c_4c_5(15)>$, $I_3 = <c_2c_3c_4c_5(23), (234), (15)>$, $G'' = <c_2c_3c_4c_5(15), (234), (23)(15)>$, $I_2 = <(234), c_2c_3c_4c_5(23)(15)>$. Note that G, G' and G'' contain the group $<c_2c_3c_4c_5(15)>$, which belongs to type X, and $H^1(<c_2c_3c_4c_5(15)>, M(D_5)) \simeq (\mathbb{Z}/2)^2$.

Remark 5.21. All the Sylow subgroups of I_1, I_2 and I_3 are non–minimal, therefore for any group $U \subseteq I_j$ its Sylow subgroups are non–minimal too. Hence $H^1(U(p), M(D_5)) = 0$, therefore $H^1(U, M(D_5)) = 0$.

According to [17] if all $G(p)$ are cyclic for $p > 2$ and $G(2)$ is either cyclic or dihedral, then the following statement holds : if $H^1(U,N) = H^{-1}(U,N) = 0$ for any subgroup $U \subseteq G$, then the G–module N is a direct summand of a permutation module. In our case, we have $M(D_5) \simeq M(D_5)^\circ$, therefore, if G_X is conjugate to I_1, I_2 or I_3, then the G_X–module $\mathrm{Pic}\,\bar{X}$ is a direct summand of a permutation G_X–module.

6
CONIC BUNDLES AND DEL PEZZO SURFACES OF DEGREE 4 WITH PRESCRIBED SPLITTING GROUPS.

In this section we are interested in the following problem : given a subgroup $G \subsetneq W(D_n)$ does there exist a conic bundle with n geometric degenerate fibres having G for its splitting group ? (and in the same problem for Del Pezzo surfaces of degree 4). The answer is provided by Theorem 6.3 (respectively, by Corollary 6.10). The developed machinery is then applied to various problems. We prove the existence of a k–minimal conic bundle which is not split over any extension of degree 2^r, thus answering a question of Colliot–Thélène and Coray [5]. We also construct (more or less explicitly) Del Pezzo surfaces of degree 4 whose splitting groups are either the maximal groups of automorphisms of the 19 types (see subsection 2), or the groups with non–vanishing invariants (see subsection 3).

1. General method.

LEMMA 6.1. *Let Z be a generalized Châtelet surface, i.e. a natural smooth compactification of the affine surface given by*

$$y^2 - az^2 = f(x) \,, \tag{22}$$

where $\deg f(x) = 2n$, *and* $f(x)$ *has no multiple roots* (*cf. Remark* 3.10, *or* [7]). *Let* L_f *denote the splitting field of the polynomial* $f(x)$. *If* $\sqrt{a} \notin L_f$, *then* $G_Z \simeq \mathbb{Z}/2 \times G_f$.

Proof. Obviously, the minimal field over which all the components of singular fibres are defined is the compositum of L_f and $k(\sqrt{a})$; and the statement of the lemma follows.

Let $f(x) = a_{2m} x^{2m} + \ldots + a_0$ be a polynomial without multiple roots, and let $a_{2m} \neq 0$, $a_0 \neq 0$. We define surfaces W_1 and W_2 by the equations

$$\begin{aligned} y^2 - xz^2 &= f(x)t^2 \quad , \\ Y^2 - \lambda Z^2 &= \lambda^{2m} f(\lambda^{-1}) T^2 \end{aligned}$$

in $\mathbb{P}^2_k \times \mathbb{A}^1_k$. We patch them together in $\mathbb{P}^2_k \times (\mathbb{A}^1_k \setminus 0)$ by the map : $y = Y$, $z = \lambda Z$, $t = \lambda^m T$, $x = \lambda^{-1}$, and get a smooth compact surface W. Its projection onto the x–axis equips W with a structure of a conic bundle over $\mathbb{P}^1_k$. The fibres at $x=0$ and $x=\infty$ are degenerate, their components are defined over $k(\sqrt{a_0})$ and $k(\sqrt{a_{2m}})$, respectively.

Quite similarly, let $f(x) = a_{2m+1}\, x^{2m+1} + ... + a_0$ be a polynomial without multiple roots with $a_{2m+1} \neq 0$, $a_0 \neq 0$. Let us define surfaces $V_1, V_2 \subset \mathbb{P}^2_k \times \mathbb{A}^1_k$ by the equations

$$\begin{aligned} y^2 - xz^2 &= f(x)t^2 , \\ \lambda Y^2 - Z^2 &= \lambda^{2m+1} f(\lambda^{-1}) T^2 , \end{aligned}$$

and patch them together in $\mathbb{P}^2_k \times (\mathbb{A}^1_k \setminus 0)$ by the map $y = Y$, $z = Z$, $t = \lambda^m T$, $x = \lambda^{-1}$. We get a smooth compact surface V; the projection onto the x–axis defines on V a structure of a conic bundle over $\mathbb{P}^1_k$. The components of the degenerate fibres at $x=0$ and $x=\infty$ are defined over $k(\sqrt{a_0})$ and $k(\sqrt{-a_{2m+1}})$, respectively.

LEMMA 6.2. *The splitting group of the conic bundle W (respectively of V) is just the Galois group of the compositum of the fields $k(\sqrt{a_0})$, $k(\sqrt{a_{2m}})$ (respectively, $k(\sqrt{-a_{2m+1}})$), and the splitting field of the polynomial $f(x^2)$.*

Proof. The components of the degenerate fibres at the points $x \neq 0, \infty$ are defined over the field obtained from k by adjoining $\sqrt{\alpha_i}$, α_i being the roots of $f(x)$.

In what follows in the case when $f(x) = a_n x^n + ... + a_0$ is such that $a_0 \in (k^*)^2$ and (or) $(-1)^n a_n \in (k^*)^2$ we consider the conic bundle Z obtained by contracting rational components in the fibres at $x=0$ and (or) $x=\infty$ (Z being a relatively k–minimal model of the conic bundle W or V).

THEOREM 6.3. *Let G be a subgroup of $W(D_n)$, and let k be an infinite field,* char. $k \nmid 2\#G$. *Then there exists a conic bundle X/k of degree $8-n$ with splitting group G iff there exists a Galois extension L/k with* $\mathrm{Gal}(L/k) \simeq G$.

Proof. Recall that since char. $k \nmid \#G$, any k–representation is a direct sum of irreducible k–representations (Maschke's theorem), and each irreducible k–representation injects into the regular k–representation ([42], §§ 105, 108). We have $G \subsetneq W(D_n) \subset W(B_n)$. Consider the $W(B_n)$–module $Q(B_n)$ with the standard basis $e_1, ..., e_n$; recall, that the set

$\{e_1,\ldots,e_n,-e_1,\ldots,-e_n\}$ is $W(B_n)$–invariant (see §1). Let V be a k–representation of G obtained by tensoring the G–module $Q(B_n)$ by k, $V = Q(B_n)\otimes_{\mathbb{Z}} k$. Let $V = \bigoplus_{i=1}^{s}\bigoplus_{j=1}^{m_i} W_{ij}$ be the decomposition into irreducible k–representations of G such that W_{ij} and $W_{i'j'}$ are isomorphic iff $i = i'$. Set $W_{ij} = V_i$, and let $\varphi_i : V_i \hookrightarrow k[G]$ be an injection. Let $e_m = \sum_{i,j} \epsilon_m^{ij}$, $\epsilon_m^{ij} \in W_{ij}$.

Now let L/k be a Galois extension with $\mathrm{Gal}(L/k) \simeq G$. By the existence of a normal basis ([42], § 67) the G–modules $k[G]$ and L are isomorphic. Let $\alpha : k[G] \longrightarrow L$ be any G–isomorphism. Set

$$\alpha_m = \sum_{i=1}^{s}\sum_{j=1}^{m_i} b_{ij}\,\alpha(\varphi_i(\epsilon_m^{ij}))$$

where $b_{ij} \in k$ will be chosen further on. We claim that the set $\{\alpha_1,\ldots,\alpha_n, -\alpha_1,\ldots,-\alpha_n\} \subset L$ is G–invariant, and the map $e_i \longmapsto \alpha_i$ commutes with the action of G. Indeed, $g\alpha_m = \sum_{i,j} b_{ij}\,\alpha(\varphi_i(g\epsilon_m^{ij}))$. Since $ge_m = \pm e_{g(m)}$, we have $g\epsilon_m^{ij} = \pm\epsilon_{g(m)}^{ij}$, therefore $g\alpha_m = \pm\alpha_{g(m)}$. Now let us show that for sufficiently general b_{ij} the sets $\{e_1,\ldots,e_n,-e_1,\ldots,-e_n\}$ and $\{\alpha_1,\ldots,\alpha_n,-\alpha_1,\ldots,-\alpha_n\}$ contain the same number of elements. If it is not so, then either $\alpha_m = -\alpha_m$ for some m, i.e. $\alpha_m = 0$, or $\alpha_\ell = \pm\alpha_r$ for some $\ell \neq r$. In the first case we have $\sum_{j=1}^{m_i} b_{ij}\,\alpha(\varphi_i(\epsilon_m^{ij})) = 0$ for all i, because $\alpha(\varphi_i(V_i))$ are different direct summands of the G–module L. Thus to ensure that $\alpha_m \neq 0$ for all m it suffices to choose b_{ij} so that $\sum_{j=1}^{m_i} b_{ij}\,\alpha(\varphi_i(\epsilon_m^{ij})) \neq 0$ for all m and i such that the vector $(\epsilon_m^{i,1},\ldots,\epsilon_m^{i,m_i})$ is non–zero. Likewise to ensure that $\alpha_r \neq \pm\alpha_\ell$ for all $r \neq \ell$ it suffices to choose b_{ij} so that $\sum_{j=1}^{m_i} b_{ij}\,\alpha(\varphi_i(\epsilon_r^{ij} \pm \epsilon_\ell^{ij})) = 0$ for all $r \neq \ell$ and i such that the vector $(\epsilon_r^{i,1} \pm \epsilon_\ell^{i,1},\ldots,\epsilon_r^{i,m_i} \pm \epsilon_\ell^{i,m_i})$ is non–zero. Therefore, for each i we have to choose $(b_{i,1},\ldots,b_{i,m_i}) \in k^{m_i}$ outside a finite number of hyperplanes. This is clearly possible for any infinite k.

Now let Z/k be a conic bundle, obtained from the conic bundle given by

$$y^2 - xz^2 = (-1)^n \prod_{i=1}^{n} (x-\alpha_i^2) \tag{23}$$

by contracting k-lines in the fibres at $x=0$ and $x=\infty$. This is possible since $\prod_{i=1}^{n} \alpha_i^2 \in (k^*)^2$, because $\prod_{i=1}^{n} \alpha_i \in L^G = k$ (recall that $G \subseteq W(D_n)$). To complete the proof with the help of Lemma 6.2 we remark that the polynomial $\prod_{i=1}^{n} (x-\alpha_i)(x+\alpha_i)$ is split in L, and is not split in any subfield of L. Indeed, if it were so, then there would exist $g \in G$, $g \neq 1$, such that $g\alpha_i = \alpha_i$ for all i, but in this case $ge_i = e_i$, therefore $g=1$. The theorem is proved.

In fact this theorem holds for arbitrary k of char $k \neq 2$.

PROPOSITION 6.4. *Let* $f_1,\dots,f_n$ *be independent variables,* $K = k(f_1,\dots,f_n)$, $f(x) = (-1)^n(x^n+f_1x^{n-1}+\dots+f_{n-1}x+(-1)^nf_n^2)$, *and let a conic bundle* Z *over a field* K *be a relatively* K*-minimal model of the natural smooth compactification of the affine surface* $y^2-xz^2 = f(x)$. *Then the splitting group of* Z *is the Weyl group* $W(D_n)$.

Proof. Let $\mathfrak{g}$ be a semisimple Lie algebra of the type D_n. We identify $\mathfrak{g}$ with the algebra of square matrices g of order $2n$ satisfying $g = -s^t.g.s$, where (in the basis $\epsilon_1,\dots,\epsilon_n,-\epsilon_n,\dots,-\epsilon_1$) all entries of the matrix s are equal to zero except for $s_{1,2n} = s_{2,2n-1} = \dots = s_{2n,1} = 1$ (see [3], VIII.13.4). Let σ denote the tautological representation of $\mathfrak{g}$, let $y = \xi_1H_1+\dots+\xi_nH_n$ be a general element of the Cartan subalgebra $\mathfrak{h}$ (here H_i denotes the difference of elementary matrices $E_{i,i} - E_{-i,-i}$). Then the characteristic polynomial of the endomorphism $\sigma(y)$ is of the form $h(t) = t^{2n} + F_1(y)t^{2n-1} +\dots+ F_{2n-1}(y)t + F_{2n}(y)$, where (up to a sign) $F_i(y)$ is the i-th elementary symmetric function in $\xi_1,\dots,\xi_n,-\xi_1,\dots,-\xi_n$, in particular, $F_{2i+1}(y) = 0$. Moreover, $F_{2n}(y) = \det(y) = (-1)^n\det(sy) = (-1)^n(Pf(sy))^2$. Set $\tilde{F}(y) = Pf(sy)$, then $F_2(y),\dots,F_{2n-2}(y)$ and $\tilde{F}(y)$ are algebraically independent and form a basis of the space of $W(D_n)$-invariant polynomial functions on $\mathfrak{h}$ ([3], *loc. cit.*).It means that the Galois group of the polynomial $h(t) = t^{2n} + F_2t^{2n-2} +\dots+ F_{2n-2}t^2 + (-1)^n\tilde{F}^2$ over $k(F_2,\dots,F_{2n-2},\tilde{F})$ is isomorphic to $W(D_n)$. It remains to apply Lemma 6.2.

COROLLARY 6.5. *Let* k *be a number field, then for any* n *there exists a conic bundle over* k *with* n *degenerate fibres and with the Weyl group* $W(D_n)$ *for its splitting group. Moreover, there exists a conic bundle over a suitable finite extension with any given subgroup* $G \subseteq W(D_n)$ *for its splitting group.*

Proof. One applies Hilbert's irreducibility theorem.

Remark 6.6. Let $f(x) = (-1)^n(x^n+f_1x^{n-1}+...+f_{n-1}x+f_n)$ and consider the associated conic bundle Z over $k(f_1,...,f_n)$ with $(n+1)$ degenerate fibres. From analogous results on the invariant polynomial functions on the Lie algebra of type C_n it follows that $G_Z = \varphi(W(C_n))$, φ being the inclusion $W(C_n) \longrightarrow W(D_{n+1})$ defined in 1.24.

PROPOSITION 6.7. *There exists a conic bundle X defined over an algebraic number field k such that for any extension L/k of degree 2^n the surface $X_L = X \times_k L$ is not L-rational.*

Proof. Let $G \simeq A_5$ be the subgroup of S_5 of index 2. It is simple, and it contains neither a subgroup of index 2 (since any such subgroup would have been normal), nor a subgroup of index 4 (since $|A_5| = 60$, and any subgroup of order 15 is cyclic). Consider an icosahedron I whose centre is placed at the point $(0,0,0)$ of $\mathbb{R}^3$. It is well known that its group of rotations is A_5. Let $\{v_1,...,v_6,-v_1,...,-v_6\}$ be the set of 12 vertices of I. The action of A_5 on this set defines an injection $\varphi : A_5 \hookrightarrow W(B_6)$, whose image in fact belongs to $W(D_6)$. Indeed, the character $\sigma : W(B_6) \longrightarrow \{\pm 1\}$, Ker $\sigma = W(D_6)$, is trivial on $\varphi(A_5)$ since any character of a simple group is trivial. We set $G = \varphi(A_5)$. According to Corollary 6.5 over a suitable number field k there exists a conic bundle X with 6 geometric degenerate fibres such that $G_X = G$. Let us prove that X enjoys the required property. Indeed, since G_X is transitive on the set of components of degenerate fibres, X is relatively k-minimal, hence k-irrational [21]. Assume that X_L is L-rational for some extension L/k of degree 2^m. Then the splitting group of X_L is a subgroup of G of index 2^ℓ, but according to the beginning of the proof $\ell=0$. Therefore X_L is also relatively L-minimal, hence L-irrational. This contradiction proves the proposition.

Remark 6.8. The proposition just proved answers in the negative a question of Colliot–Thélène and Coray [5].

We end this subsection with an analog of Theorem 6.3 for Del Pezzo surfaces of degree 4.

Let Z be a conic bundle of degree $(8-n)$, where $n=4$ or 5, let $\ell_0 \in \text{Pic}\,\bar{Z}$ be the class of a geometrically closed fibre. We choose one component in each singular fibre and denote these components by $\ell_1,...,\ell_n$. The action of the splitting group G_Z on the set of components $\{\ell_1,\ell_0-\ell_1,...,\ell_n,\ell_0-\ell_n\}$ induces an inclusion $\theta : G_Z \hookrightarrow W(D_n)$ defined up to a conjugation in

$W(B_n)$. Let Ψ be a standard embedding of $W(D_4)$ into $W(D_5)$ as the stabilizer of ℓ_1. If $n=5$, any conjugation in $W(B_5)$ is an inner automorphisme of $W(D_5)$, therefore the conjugacy class $\theta(G_Z) \subseteq W(D_5)$ is well–defined. If $n=4$, a conjugation in $W(B_4)$ might be an outer automorphism of $W(D_4)$, but according to Lemma 4.6 it is reduced to a conjugation in the normalizer of $\Psi(W(D_4))$ in $W(D_5)$, hence the conjugacy class $\Psi\circ\theta(G_Z)$ is well–defined again.

PROPOSITION 6.9. *Let G be a minimal subgroup of $W(D_5)$. If G is conjugate to a subgroup of $\Psi(W(D_4))$, i.e. $M(D_5)^G \simeq \mathbb{Z}\oplus\mathbb{Z}$, then in order to construct a Del Pezzo surface of degree 4 with the splitting group G it suffices to construct a conic bundle Z with 4 degenerate fibres, such that $\Psi\circ\theta(G_Z)$ is conjugate to G and $Z(k)\neq\emptyset$. If G is not conjugate to a subgroup of $\Psi(W(D_4))$, i.e. $M(D_5)^G \simeq \mathbb{Z}$, then in order to construct a Del Pezzo surface of degree 4 with the splitting group G it suffices to construct a conic bundle Z with 5 degenerate fibres, such that $\theta(G_Z)$ is conjugate to G.*

Proof. According to [21] (Theorems 4 and 5) a relatively minimal conic bundle Z with 5 degenerate fibres is isomorphic to a smooth cubic surface with a k–line ℓ. Its blow–down X is a Del Pezzo surface of degree 4. The conic bundle structure on Z is related to a pencil of conics cut by planes passing through ℓ. Singular conics of this pencil are exactly the pairs of intersecting lines on Z each of them meeting ℓ. They are the proper transforms of the conics on X passing through x. There are 10 pencils of conics $q_i^{\pm}$ on X (see 3.1, 3.15), and exactly one conic of each pencil passes through x. It follows that the splitting group $G_X = G_Z \subseteq W(D_5)$ acts on the components of the singular conics on Z in the same way as it acts on the set $\{q_1^+,q_1^-,\dots,q_5^+,q_5^-\}$, i.e. on the graph Δ (cf. §0). We have proved the proposition in the case when $M(D_5)^G \simeq \mathbb{Z}$.

We now assume that a relatively k–minimal conic bundle Z has 4 degenerate fibres, and $Z(k)\neq\emptyset$. Then according to [20] (Proposition 1) Z is either isomorphic to a Del Pezzo surface of degree 4 or to an Iskovskih surface. In the first case, the blow–up of a good (i.e. not lying on the lines) point is a cubic surface equipped with a conic bundle structure, and we are in the situation considered above. In the second case, we perform an elementary transformation and apply Theorem 3.18 a) ; after that everything again reduces to the case of Del Pezzo surfaces of degree 4 considered above. As before it is easy to check that contracting a k–line on Y which is a 2–section of the conic bundle under consideration, we obtain a Del Pezzo surface X of degree 4 with G_X conjugate to G.

COROLLARY 6.10. *Let* G *be a minimal subgroup of* $W(D_5)$, *and let* k *be an infinite field,* char. $k \nmid 2\#G$. *Then there exists a Del Pezzo surface* X/k *of degree* 4 *with the splitting group* G *iff there exists a Galois extension* L/k, *such that* $\mathrm{Gal}(L/k) \simeq G$.

Proof. Apply Theorem 6.3 for $n=5$. Clearly the conic bundle (23) has a k-point (for example in the fibre at $x=0$). Now apply Proposition 6.9.

2. Maximal groups of the 19 types.

PROPOSITION 6.11. *Let* G_i *denote the maximal group of automorphisms of type* i, $G \subseteq \varphi(W(D_4))$. *A Del Pezzo surface of degree* 4 *over* $\mathbb{Q}$ *with the splitting group* G_i *is obtained by a birational transformation from a generalized Châtelet surface* $y^2-az^2 = f_i(x)$ *for the following polynomials:* $f_2(x) = x(x^3+x-1)$, $f_3(x) = x^4-x+1$, $f_4(x) = x(x-b_1)(x-b_2)(x-b_3)$, $f_6(x) = (x^2-b)(x^2-bc^2)$, $f_7(x) = x(x-c)(x^2-b)$, $f_8(x) = (x^2-b)(x^2-c)$, $f_{12}(x) = x^4+b^2$, $f_{13}(x) = x^4+b$, *where* $\sqrt{a}$ *does not lie in the splitting field of* $f_i(x)$, b *and* c *are sufficiently general rationals,* $b_i \neq b_j$ *for* $i \neq j$.

Proof. In view of Corollary 4.9, the maximal groups of automorphisms of types II, III, IV, VI, VII, VIII, XII and XIII are conjugate to subgroups of $W(D_4)$ of the form $\langle c_1c_2c_3c_4, G^B\rangle$, G^B being a subgroup of S_4. Hence by Lemma 6.1 the only thing to do is to construct prescribed polynomials of degree 4 (which is standard) and to choose a, such that $\sqrt{a}$ does not lie in the splitting field of the polynomial. For types IV, VI–VIII the construction of $f(x)$ is quite elementary. For type II one should take $f(x) = (x-b)h(x)$, where $h(x)$ is an irreducible polynomial of degree 3 whose discriminant is not a square. For type XII one can take an irreducible polynomial $f(x)$ of degree 4 with its resolvent cubic splitting over k into linear factors, for example, $f(x) = x^4+b^2$. For type XIII take an irreducible polynomial $f(x)$ such that the Galois group of its resolvent cubic is isomorphic to $\mathbb{Z}/2$, and $f(x)$ itself is irreducible over $\mathbb{Q}(\sqrt{\delta})$, δ denoting the discriminant of $f(x)$. One can take $f(x) = x^4+b$, i.e. a "general" binomial of degree 4 (see, for example, [23], Ch. II). Finally, for type III we need a polynomial $f(x)$ with $G_f \simeq S_4$, i.e. a "general" polynomial of degree 4, for example, $f(x) = x^4-x+1$ (its discriminant is not a square, and its resolvent cubic is irreducible).

Since all our polynomials $f(x)$ are monic, the point $x=\infty$, $y=1$, $z=0$ is a rational point of Z.

THEOREM 6.12. *Let Z_i be a conic bundle over $\mathbb{Q}$ which is defined as a relatively minimal model of the natural smooth compactification of the surface $y^2-xz^2 = f_i(x)$, where*

$$f_1(x) = (a-x^2)(x^3+a),\ f_5(x) = x^4-2b(1-6a^2+a^4)x^2+b^2(1+a^2)^4,$$
$$f_9(x) = (x^2-2(a+b)x+(a-b)^2)(x^2-2(c+d)x+(c-d)^2),\ f_{10}(x) = (x^2-a)(x+ab^2)(x+ac^2),$$
$$f_{11}(x) = a(b-x)(x^2-2b(a+1)x+b^2(a-1)^2),\ f_{14}(x) = ((x-a)^2-\tfrac{1}{2}b^2)^2 - b^2(a^2-\tfrac{3}{4}b^2),$$
$$f_{15}(x) = x^4-x+1\ ,\ f_{16}(x) = (x^2-a)(x^2+2a(1+b)x+a^2(1-b)^2),\ f_{17}(x) = (x^2+b^2)(-x^3-x+1),$$
$$f_{18}(x) = x^4-2\ ,\ f_{19}(x) = -x^5+x+1,$$

for general enough $a,b,c \in \mathbb{Q}$. Then Z_i is birationally equivalent to a Del Pezzo surface of degree 4, whose splitting group coincides with the group of automorphisms of type i.

Proof. Denote by G_i the Galois group of the polynomial $f_i(x^2)$, and let L_i be its splitting field. Then $L_1 = \mathbb{Q}(\sqrt{-1},\sqrt{-3},\sqrt[12]{a})$, $G_1 = <\alpha,\beta,\gamma>$, α multiplies $\sqrt[12]{a}$ by $\frac{1}{2}(\sqrt{3}+\sqrt{-1})$, $\beta:\sqrt{-1}\longmapsto-\sqrt{-1}$, $\gamma:\sqrt{-3}\longmapsto-\sqrt{-3}$. Straightforward computation shows that G_1 acts on the components of singular fibres of Z_1 as follows: $\alpha = c_1c_2c_3c_4(15)(234)$, $\beta = c_2c_3c_4c_5(24)$, $\gamma = (24)$. These elements generate Aut(I).

Further, the roots of $f_5(x^2)$ are $\pm\sqrt[4]{b}(1\pm ai)$, $\pm\sqrt[4]{b}(a\pm i)$, $L_5 = \mathbb{Q}(\sqrt[4]{b},i)$, G_5 is isomorphic to the dihedral group. Complex conjugation acts on the components of the fibres as (12)(34), and the element $\sigma:\sqrt[4]{b}\longmapsto i\sqrt[4]{b}$ – as $c_2c_4(14)(23)$. Together they generate $\mathrm{Aut(V)} = G_{7,4}$.

The roots of $f_9(x^2)$ are $\pm(\sqrt{a}\pm\sqrt{b})$, $\pm(\sqrt{c}\pm\sqrt{d})$. Clearly its Galois group is isomorphic to $(\mathbb{Z}/2)^4$. It can be immediately identified with $\mathrm{Aut(IX)} = G_{3,2}$.

The conic bundle Z_{10} has 5 singular fibres : 4 fibres at the roots of $f_{10}(x)$ and the fifth fibre at $x=0$. The group G_{10} is isomorphic to the Galois group of the polynomial x^4-a, i.e. to the dihedral group ; it is generated by complex conjugation and the transformation $\sigma:\sqrt[4]{a}\longmapsto i\sqrt[4]{a}$. The splitting field of the fibre at $x=0$ is $\mathbb{Q}(\sqrt{-a})$. It is contained in $L_{10} = \mathbb{Q}(\sqrt[4]{a},i)$, hence $G_Z = G_{10}$. Complex conjugation acts on the components of the fibres as $c_2c_3c_4c_5$, and σ as $c_2c_3c_4c_5(15)$, these elements generate Aut(X).

The conic bundle Z_{11} has 5 singular fibres : at the roots of f_{11}, at $x=0$, and $x=\infty$. $L_{11} = \mathbb{Q}(\sqrt{a},\sqrt{b})$ contains the splitting fields of the fibres at $x=0$ and $x=\infty$, hence $G_Z = G_{11} \simeq (\mathbb{Z}/2)^2$. The generators of G_Z act on the components of the fibres as $c_1c_2c_3c_4$ and $c_4c_5(23)$, generating Aut(XI).

Denote $c = b^2(a^2 - \frac{3}{4}b^2)$, $d = \frac{1}{2}b^2$, then the roots of $f_{14}(x^2)$ are $\pm\sqrt{a\pm\sqrt{d\pm\sqrt{c}}}$. Let us find G_{14}. Denote by $K = \mathbb{Q}(\sqrt{c},\sqrt{d+\sqrt{c}},\sqrt{d-\sqrt{c}})$ the normal closure of the extension $\mathbb{Q}(\sqrt{c},\sqrt{d+\sqrt{c}})$. The group $\mathrm{Gal}(K/\mathbb{Q})$ is isomorphic to the dihedral group ; it is generated by the transformations $\alpha : \sqrt{c} \longmapsto -\sqrt{c}$, $\beta : \sqrt{d+\sqrt{c}} \longmapsto -\sqrt{d+\sqrt{c}}$. Since $(a+\sqrt{d+\sqrt{c}})(a-\sqrt{d+\sqrt{c}}) \in (K^*)^2$ and $(a+\sqrt{d-\sqrt{c}})(a-\sqrt{d-\sqrt{c}}) \in (K^*)^2$, $L_{14} = K(\sqrt{a\pm\sqrt{d\pm\sqrt{c}}})$ is biquadratic over K. Thus, G_{14} can be represented by an extension $1 \rightarrow (\mathbb{Z}/2)^2 \rightarrow G_{14} \rightarrow D_4 \rightarrow 1$; it is generated by α, β and $\gamma : \sqrt{a+\sqrt{d+\sqrt{c}}} \longmapsto -\sqrt{a+\sqrt{d+\sqrt{c}}}$. These elements act on singular fibres as (23)(45), (25), and c_2c_5, generating $\mathrm{Aut(XIV)} = G_{8,2}$. Note that the components of the singular fibres at $x = 0$ and $x = \infty$ are defined over $\mathbb{Q}$, hence $G_Z = G_{14} = \mathrm{Aut(XIV)}$.

Now let us show that $G_{15} = W(D_4) = \mathrm{Aut(XV)}$. The polynomial $f_{15}(x^2)$ modulo 2 is a square of an irreducible polynomial of degree 4, and splits modulo 3 into the product of irreducible polynomials x^2+1 and $x^6-x^4+x^2+1$ of degrees 2 and 6. It shows that G_{15} contains the elements (1234) and $c_1c_4(123)$ generating $W(D_4)$.

The surface Z_{16} has 5 singular fibres : 4 fibres at the roots of $f_{16}(x)$ and the fifth fibre at $x=0$: $L_{16} = \mathbb{Q}(\sqrt[4]{a},i,\sqrt{b})$, $G_{16} \simeq \mathbb{Z}/2 \times D_4$. The splitting field of the fibre at $x=0$ is contained in L_{16}. The action of generators of G_{16} on the components of the fibres can be easily written.

The leading coefficient of $f_{17}(x)$ equals -1, $f_{17}(0) = b^2$, hence Z_{17} has 5 singular fibres. If $b \in \mathbb{Q}$ is sufficiently general, G_{17} is the direct product of splitting groups of polynomials x^4+b^2 and $-x^6-x^2+1$, which (as we shall prove below) are isomorphic to $(\mathbb{Z}/2)^2$ and $W(D_3) \simeq S_4$, respectively. Indeed, x^3+x-1 is irreducible over $\mathbb{Q}$ (being irreducible modulo 2), its discriminant is not a square in $\mathbb{Q}$, hence its Galois group is S_3. On the other hand, x^3+x-1 splits modulo 3 into the product of a linear factor by an irreducible factor of degree 2 : $x^3+x-1 \equiv (x+1)(x^2-x-1) \pmod 3$, hence $x^6+x-1 \equiv (x^2+1)(x^4-x^2-1) \pmod 3$. But x^2+1 and x^4-x^2-1 are irreducible mod 3 ; it means that the Galois group of the polynomial x^6+x^2-1 contains an element acting on the components of the fibres as $c_1c_2(23)$. Since its map onto S_3 is an epimorphism, this group contains $W(D_3)$. It coincides with $W(D_3)$, since the product of the roots of x^3+x-1 is a square in $\mathbb{Q}$.

The group G_{18} is isomorphic to the dihedral group of order 16, $L_{18} = \mathbb{Q}(\sqrt[8]{2},i)$, $G_{18} = <\alpha,\beta>$, where $\alpha : \sqrt[8]{2} \longmapsto i\sqrt[8]{2}$, and β denotes complex conjugation. α acts on the components of singular fibres over $x \neq 0$ as $c_2(1243)$, and β – as $c_1c_3c_4(14)$. The splitting field of the fibre at $x=0$ is $\mathbb{Q}(\sqrt{2}) \subset L_{18}$, hence $\alpha = c_2c_5(1243)$, $\beta = c_1c_3c_4c_5(14)$. These elements generate Aut(XVIII) while $(12)(34) = \alpha^4\beta\alpha$.

Finally, in order to prove that $G_{19} = \mathrm{Aut(XIX)}$ it is enough to show that : 1) the Galois group of x^5-x-1 is S_5 ([42], Ch. 8, § 66); 2) $x^{10}-x^2-1$ does not split into a product of two irreducible polynomials of degree 5. Indeed, it follows from the first statement that the natural map $pr: G_{19} \longrightarrow S_5$ is an epimorphism. Suppose that its kernel G^A_{19} is non–trivial. Since the product of the roots of x^5-x-1 is a square in $\mathbb{Q}$, G_{19} is contained in $W(D_5)$. Conjugating by permutations from S_5 we conclude that $G^A_{19} = \langle c_1c_2, c_2c_3, c_3c_4, c_4c_5 \rangle$, i.e. $G_{19} = W(D_5)$. If $G^A_{19} = 1$, then $G^A_{19} \simeq S_5$. Let us show that in this case the set $\{e_1, -e_1, \dots, e_5, -e_5\}$ decomposes into two G–orbits (that corresponds to splitting of $x^{10}-x^2-1$ into a product of two irreducible polynomials of degree 5). $S_5 = \langle (12345),(12) \rangle$. All the elements of $W(D_5)$ of order 5 are conjugate to each other, hence $pr^{-1}((12345))$ is conjugate to (12345). Let $pr^{-1}((12)) = c_I(12)$. Since $(c_I(12))^2 = 1$, it follows that $c_I = \mu\nu$, where $\mu \in \{1, c_1c_2\}$, $\nu \in \{1, c_3c_4; c_3c_5, c_4c_5\}$. $\mu=1$ because $(c_I(12)(12345))^4 = 1$. Let $pr^{-1}((345)) = c_I(345)$. However, $pr^{-1}((12))$ commutes with $pr^{-1}((345))$, therefore $\nu=1$, hence $G_{19} = \langle (12),(12345) \rangle$. It remains to prove the second statement. In view of Gauss' lemma it is enough to do it in $\mathbb{Z}[x]$. One can employ the method of indeterminate coefficients, taking into account that the values of both factors at $x=1$ should be equal to ± 1.

3. "Interesting" groups.

In this subsection we obtain examples of Del Pezzo surfaces of degree 4 with prescribed "interesting" splitting groups. We call a group G_X *interesting* in one of the following cases : either $Ш^2_\omega(G_X, \mathrm{Pic}\,\bar{X}) \neq 0$, or $Ш^1_\omega(G_X, \mathrm{Pic}\,\bar{X}) \neq 0$, or the G_X–module $\mathrm{Pic}\,\bar{X}$ is a direct summand of a permutation G_X–module. In the latter case, the surface X can happen to be a counterexample to the Zariski conjecture. In order to construct a Del Pezzo surface with a prescribed splitting group we (due to 6.9) write down an equation of a corresponding conic bundle with 4 or 5 degenerate fibres (cf. §0).

We begin with interesting groups corresponding to Del Pezzo surfaces X of degree 4 with $\mathrm{Pic}\,X \simeq \mathbb{Z} \oplus \mathbb{Z}$.

THEOREM 6.13. *Any subgroup* $G \subset W(D_4)$ *with* $Ш^2_\omega(G,Q(D_4)) \neq 0$ *or* $Ш^1_\omega(G,Q(D_4)^o) \neq 0$ *can be realized as a splitting group of a Del Pezzo surface of degree* 4 *over* $\mathbb{Q}$.

Proof. Let the conic bundles $X_1,...,X_4$ be given by the following equations :

$(X_1)\quad y^2-abz^2 = (x^2-a)(x^2-b),$

$(X_2)\quad y^2-xz^2 = x^4-12x^2+4\ ,$

$(X_3)\quad y^2-xz^2 = x^4-12ax^2+4a^2\ ,$

$(X_4)\quad y^2-xz^2 = (x^2-2ax+c^2)[x^2-2(a^2-c^2+b)x+(a^2-c^2-b)^2]\ .$

(Till the end of this section $a,b,c,...$ denote sufficiently general rationals).

Let us show that the splitting groups of these surfaces are respectively :

$$<(12)c_1c_2c_3c_4,(34)c_1c_2c_3c_4> \simeq G_{3,6},\quad <c_1c_2,c_1c_3(12)(34)> \simeq G_{2,3},$$
$$<c_1c_2,c_1c_3(12)(34),(13)(24)> \simeq G_{7,3},\quad <c_1c_2,(34),c_3c_4(12)> \simeq G_{3,5}.$$

The right hand side of the equation of X_i is denoted by $f_i(x)$.

The surface X_1 has been investigated in [26].

Consider X_2 . Let us denote by μ a primitive root of unity, $\mu^{16} = 1$. The Galois group of $\mathbb{Q}(\mu)/\mathbb{Q}$ is isomorphic to $\mathbb{Z}/2 \times \mathbb{Z}/4$, its generators being $\rho : \mu \longmapsto \mu^7$ and $\sigma : \mu \longmapsto \mu^5$.

The roots of $f_2(x^2) = x^8-12x^4+4$ are $\pm\sqrt{2\pm\sqrt{2}}\ , \pm i\sqrt{2\pm\sqrt{2}}$.

On account of the equality $\mu = (\sqrt{2+\sqrt{2}}+i\sqrt{2-\sqrt{2}})/2$, the degenerate fibres of X_2 at $x = 2+\sqrt{2}\ , 2-\sqrt{2}\ , -2+\sqrt{2}$, and $-2-\sqrt{2}$ are split over $\mathbb{Q}(\mu-\mu^7)$, $\mathbb{Q}(\mu^5-\mu^3)$, $\mathbb{Q}(\mu+\mu^7)$, and $\mathbb{Q}(\mu^5+\mu^3)$, respectively.

We see that ρ acts on the components of degenerate fibres as c_1c_2 , and σ acts as $c_1c_3(12)(34)$.

The case of X_3 is similar. The roots of $f_3(x^2)$ are $\pm\sqrt[4]{a}.\sqrt{2\pm\sqrt{2}}\ , \pm i\sqrt[4]{a}.\sqrt{2\pm\sqrt{2}}$; the generators of G_X are ρ , σ , and τ with ρ and σ as above, and $\tau : \sqrt[4]{a} \longmapsto i\sqrt[4]{a}$ acting as $c_1c_2(14)(23)$.

The roots of $f_4(x^2)$ are $\pm\sqrt{(a\pm\sqrt{a^2-c^2})}, \pm(\sqrt{a^2-c^2}\pm\sqrt{b})$. The generators of G_X are $(\sqrt{b} \longmapsto -\sqrt{b}) = (34)$, $(\sqrt{a^2-c^2} \longmapsto -\sqrt{a^2-c^2}) = (12)(34)c_3c_4$, $\sqrt{(a+\sqrt{a^2-c^2})} \longmapsto -\sqrt{a+\sqrt{a^2-c^2}}) = c_1c_2$ (since $(a+\sqrt{a^2-c^2})(a-\sqrt{a^2-c^2})$ is a square, $G_X \simeq (\mathbb{Z}/2)^3$).

The only remaining group is $G_{7,6}$. As was remarked in the proof of Theorem 4.13, $G_{7,6}$ is isomorphic to the group of quaternions of order 8. Let $L/\mathbb{Q}$ be a Galois extension, $\mathrm{Gal}(L/\mathbb{Q}) \simeq G_{7,6}$ (such extensions are described, for example, in [16]). Now use Theorem 6.3 to complete the proof.

Now let us go over to interesting subgroups $G \subseteq W(D_5)$, $G \not\subseteq W(D_4)$.

THEOREM 6.14. *Any subgroup* $G \subset W(D_5)$ *with* $\text{Ш}^2_\omega(G, Q(D_5)) \neq 0$ *can be realized as a splitting group of a Del Pezzo surface of degree* 4 *over* $\mathbb{Q}$.

Proof. According to Theorems 5.11 and 5.12 there are (up to a conjugation) 8 subgroups $G \subseteq W(D_5)$, $G \not\subseteq W(D_4)$ with $\text{Ш}^2_\omega(G, Q(D_5)) \neq 0$. Let us write down the equations of the corresponding conic bundles :

(Y_1) $y^2 - xz^2 = -[x^2 - 2(a-1)x + (a+1)^2](x^2 - a)(x + a)$,

(Y_2) $y^2 - xz^2 = -[x^2 + 4(a^2 + 2b^2)x + 4(a^2 - 2b^2)^2](x^2 + 4x + 2)(x - 2)$,

(Y_3) $y^2 - xz^2 = -(x - a)(x - b)(x - ab)[x^2 - 2(a+1)cx + c^2(a-1)^2]$,

(Y_4) $y^2 - xz^2 = -[x^2 - 2b(a-1)x + b^2(a+1)^2](x^2 - a)(x + a)$,

(V_1) $y^2 - xz^2 = x^4 - 32$,

(V_3) $y^2 - xz^2 = x^4 - 16a$,

(V_4) $y^2 - xz^2 = (x^2 - a)(x^2 + ab^2)$.

In order to check that their splitting groups are H_1–H_4 , J_1, J_3, J_4 (in the notation of 5.11 and 5.12) one has to find the roots of $f_i(x^2)$ and to show that G_X acts on the components of degenerate fibres in the required way. It can be done quite as in Theorem 6.13. The only remaining group in the lists of Theorems 5.11, 5.12 is $J_2 = \langle c_1c_2, c_1c_5(1324)\rangle$. Since J_2 is an extension of the abelian group $\langle(1324)\rangle$ by the abelian group $\langle c_1c_2, c_3c_4\rangle$, J_2 is solvable (cf. Lemma 5.18). Therefore there exists a Galois extension $L/\mathbb{Q}$, $\mathrm{Gal}(L/\mathbb{Q}) \simeq J_2$ (see [39]). Now we complete the proof using Theorem 6.3.

Let us go over to interesting groups which may correspond to stably $\mathbb{Q}$–rational but not $\mathbb{Q}$–rational Del Pezzo surfaces of degree 4 (these groups are listed in Theorem 5.20).

THEOREM 6.15. *Let* Z_1, Z_2, Z_3 *be conic bundles given over* $\mathbb{Q}$ *by the following affine equations :*

$(Z_1)\quad y^2-xz^2 = -(x^2-9)(x^3+9),$

$(Z_2)\quad y^2-xz^2 = -(x^3+2apx^2+a^2p^2x-a^3q^2)(x^2-2rx+s),$

$(Z_3)\quad y^2-xz^2 = -(x^2-3)(x^3+3).$

In the second case, let the following conditions hold : 1) a *is not a square* ; 2) *the polynomial* x^3+px+q *is irreducible over* $\mathbb{Q}$; 3) $\mathcal{D}_1/\mathcal{D}_2$ *is a square* ($\mathcal{D}_1 = discr(x^3+px+q) = -4p^3-27q^2$, $\mathcal{D}_2 = discr(x^2-2rx+s) = 4(r^2-s)$); 4) $s/\mathcal{D}_2$ *is a square* ; 5) $a/\mathcal{D}_1$ *is a square.*

Then the splitting group of Z_i *coincides with* I_i (*in the notation of Theorem* 5.20).

Proof. For Z_1 and Z_3 the verification is immediate. Let us consider Z_2. Denote the roots of x^3+px+q by ϵ_i $(i = 1,2,3)$. Applying the Viète formulae we see that the roots of $x^3+2apx^2+a^2p^2x-a^3q^2$ are $a\epsilon_i^2$. Let $K/\mathbb{Q}$ be the splitting field of x^4-2rx^2+s, then the splitting field of the surface is $L = K(\epsilon_1,\epsilon_2,\epsilon_3)$ (since conditions 3 and 5 yield $\sqrt{a} \in K$). The conditions imposed on the coefficients ensure that $\mathrm{Gal}(K/\mathbb{Q}) \simeq \mathbb{Z}/4$ and $\mathrm{Gal}(L/K) \simeq \mathbb{Z}/3$. Now let us show that $G = \mathrm{Gal}(L/\mathbb{Q})$ acts on the singular fibres in the required way. Let us numerate the singular fibres in such a way that the splitting field of the i-th fibre is $\mathbb{Q}(\sqrt{r+\sqrt{s}})$, $\mathbb{Q}(\epsilon_1\sqrt{a})$, $\mathbb{Q}(\epsilon_2\sqrt{a})$, $\mathbb{Q}(\epsilon_3\sqrt{a})$, and $\mathbb{Q}(\sqrt{r+\sqrt{s}})$, respectively. According to the Cardano formulae $\epsilon_1 = u+v$, $\epsilon_2 = \alpha u+\alpha^2 v$, $\epsilon_3 = \alpha^2 u+\alpha v$, where $\alpha^3=1$, $\alpha\neq 1$, $u = \sqrt[3]{-\frac{q}{2}+\frac{\sqrt{-3}}{18}\sqrt{\mathcal{D}_1}}$, $v = \sqrt[3]{-\frac{q}{2}+\frac{\sqrt{-3}}{18}\sqrt{\mathcal{D}_1}}$. Let ρ (respectively, σ) be the generator of $\mathrm{Gal}(K/\mathbb{Q})$ (respectively, of $\mathrm{Gal}(L/K)$), then G can be represented as a semidirect product $<\sigma>\rtimes<\rho>$. In fact σ acts via cyclic permutation of ϵ_i's, i.e. $\sigma = (234)$. Further, ρ acts in the splitting field K via the cyclic permutation $\sqrt{r+\sqrt{s}} \longmapsto \sqrt{r-\sqrt{s}} \longmapsto -\sqrt{r+\sqrt{s}} \longmapsto -\sqrt{r-\sqrt{s}}$. In view of conditions 3–5 the product $a.s$ is a square, hence ρ maps $\sqrt{a}$ to $-\sqrt{a}$ and therefore interchanges the components of the second, third and fourth fibres. On the other hand, ρ sends $\sqrt{\mathcal{D}_2}$ to $-\sqrt{\mathcal{D}_2}$ because of condition 4, hence ρ transposes u and v, and therefore also ϵ_2 and ϵ_3, i.e. the third and the fourth fibres. Summing up, we conclude that $\rho = c_2c_3c_4c_5(15)(34)$, as required.

It is worth remarking that conditions 1–5 can be easily fulfilled, one can take, for example, $p=-4$, $q=1$, $a=r=229$, $s=4.229$; we get the surface

$$y^2-xz^2 = -(x^3-1832x^2+839056x-12008989)(x^2-458x+916)$$

(probably, the example is not the simplest one). It is an interesting question whether this surface is actually stably $\mathbb{Q}$–rational.

7
RATIONALITY PROBLEM FOR NERON–SEVERI TORI.

Let T be an algebraic torus defined over a field k, and let L be its splitting field, $G = \mathrm{Gal}(L/k)$. The character lattice $\hat{T}$ admits the following resolution of G–modules (a canonical sequence [43, 4.53], or a flasque resolution [6]) :

$$0 \longrightarrow \hat{T} \longrightarrow S \longrightarrow F \longrightarrow 0, \tag{24}$$

where S is a permutation module, and F is a flasque module, i.e. $H^{-1}(U,F) = 0$ for any $U \subseteq G$ (cf. the discussion at the beginning of section 2). According to the basic result of Voskresenskiĭ [43, 4.60] the torus T is stably k–rational if and only if the G–module F is similar to a permutation module. The goal of this section is to show that in some simple cases this theorem provides quite a practical method to establish the k–irrationality of some tori.

The exact sequence (24) can be constructed explicitly in terms of torus embeddings (cf. for example, [44]). The crucial point is the proof of the non–triviality of the similarity class $[F]$. We establish this by going over from F to $F \otimes_{\mathbb{Z}} \mathbb{Z}_p$ and applying the Krull–Schmidt unique decomposition theorem which holds in the category of $\mathbb{Z}_p[G]$–modules, $\mathbb{Z}_p$ denoting the ring of p–adic integers, G being a p–group [15, 76.26].

THEOREM 7.1. *Let $G \subseteq W(D_4)$ and suppose that the decomposition of the graph Γ into G-orbits is of type* III, VIII, XII, XIII *or* XV. *Then the algebraic k-torus dual to the G-module $Q(D_4)$ is not stably k-rational.*

Proof : Consider $G_{6,1} = \langle c_1c_2c_3c_4,(1234)\rangle$ or $G_{7,1} = \langle c_1c_2c_3c_4,(12)(34),(13)(24)\rangle$. These groups are contained in $G = G_{10,1} = \langle c_1c_2c_3c_4,(12),(1234)\rangle$. Let us construct the resolution (24) for the latter group following [44]. We have to choose the G–invariant fan Σ in the weight lattice

$$\hat{T}^0 = \mathrm{Hom}(\hat{T},\mathbb{Z}) = \mathrm{Hom}(Q(D_4),\mathbb{Z}) = P(D_4).$$

This lattice is volume–centered, i.e. it is the cubic lattice with centres of all the cubes added to it. Consider the vectors e_i, $\bar{e}_i = -e_i$ $(i = 1,\dots,4)$, and all the vectors $f_I = \frac{1}{2}(\sum_{i\in I} e_i + \sum_{j\in \bar{I}} \bar{e}_j)$, $I \subseteq \{1,2,3,4\}$, $\bar{I} = \{1,2,3,4\}\backslash I$. We construct the fan Σ as follows : let the vectors e_i, $\bar{e}_i$, f_I form the 1–skeleton of Σ; the vector f_I divides the simplex with the edges $e_i, i \in I$ and $\bar{e}_j$,

$j \in \bar{I}$ into four cones. This procedure yields a G–invariant partition of $\mathbb{R}^4$ into 64 cones; the corresponding toric variety X_Σ is $(\mathbb{P}^1)^4$ with 16 points blown up. The permutation module S generated by the edges of Σ is of rank 24, the rank of $F = \operatorname{Pic} \bar{X}_\Sigma$ is equal to 20. The resolution (24) is thus constructed.

Consider the exact sequence of $W(C_n)$–modules :

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathbb{Z} & \longrightarrow & M(C_n) & \xrightarrow{\ b\ } & P(C_n) & \longrightarrow & 0 \\ & & & & {}_a\nwarrow & & \nearrow_c & & \\ & & & & & Q(C_n) & & & \end{array}$$

Let us recall that the restriction of the pairing $M(R) \times M(R^\vee) \longrightarrow \mathbb{Z}$ to $Q(R) \times Q(R^\vee)$ coincides with the usual Cartan pairing. It follows that $b \circ a = c$, where a is the natural embedding $Q(R) \longrightarrow M(R)$, $(c(\alpha))(\beta^\vee) = <\alpha,\beta^\vee>$. This provides a commutative diagram of $W(C_n)$–modules :

$$\begin{array}{ccccccccc} & & 0 & & 0 & & 0 & & \\ & & \uparrow & & \uparrow & & \uparrow & & \\ 0 & \longrightarrow & Q(C_n) & \xrightarrow{\ c\ } & Q(B_n) & \longrightarrow & \mathbb{Z}/2 & \longrightarrow & 0 \\ & & \uparrow\wr & & \uparrow b & & \uparrow & & \\ 0 & \longrightarrow & Q(C_n) & \xrightarrow{\ a\ } & M(C_n) & \longrightarrow & \mathbb{Z} & \longrightarrow & 0 \\ & & \uparrow & & \uparrow & & \uparrow & & \\ & & 0 & \longrightarrow & \mathbb{Z} & \xrightarrow{\ \sim\ } & \mathbb{Z} & \longrightarrow & 0 \\ & & & & \uparrow & & \uparrow & & \\ & & & & 0 & & 0 & & \end{array}$$

and a commutative diagram of the cohomology groups of U–modules (U being either $G_{7,1}$ or $G_{6,1}$) :

$$\begin{array}{ccccccccc} & & & & & & \operatorname{Hom}(U,\mathbb{Q}/\mathbb{Z}) & & \\ & & & & & & \uparrow \delta_1 & & \\ 0 & \longrightarrow & \mathbb{Z}/2 & \longrightarrow & H^1(U,Q(C_4)) & \longrightarrow & H^1(U,Q(B_4)) & \xrightarrow{\ \delta_2\ } & \operatorname{Hom}(U,\mathbb{Z}/2) \\ & & & & \uparrow\wr & & \uparrow & & \uparrow \\ & & & & H^1(U,Q(C_4)) & \longrightarrow & H^1(U,M(C_4)) & \longrightarrow & 0. \end{array}$$

Here $H^1(U,Q(B_4)) = \mathbb{Z}/2$, and Lemma 1.23 yields $\delta_1 = 0$. Thus $\delta_2 = 0$, therefore $|H^1(U,Q(C_4))| = 4$.

Now let us compute $\hat{H}^0(U,S)$. It is easy since we know the permutation basis of S. Here is the result. If $U = G_{7,1}$, then $S = \mathbb{Z}[U] \oplus \mathbb{Z}[U] \oplus \bigoplus_{i=1}^{4} \mathbb{Z}[U/H_i]$, where $|H_i| = 4$, therefore $\hat{H}^0(U,S) \simeq (\mathbb{Z}/4)^4$ and $rk\, S^U = 6$. If $U = G_{6,1}$, then $S = \mathbb{Z}[U] \oplus \mathbb{Z}[U] \oplus \mathbb{Z}[U/H] \oplus \bigoplus_{i=1}^{2} \mathbb{Z}[U/H_i]$, where $|H| = 2$, $|H_i| = 4$, therefore $\hat{H}^0(U,S) \simeq \mathbb{Z}/2 \times (\mathbb{Z}/4)^2$ and $rk\, S^U = 5$.

Let us tensor the resolution (24) by the ring of 2–adic integers $\mathbb{Z}_2$, and apply the unique decomposition theorem, which holds in the category of $\mathbb{Z}_2[U]$–modules, U being a 2–group [15, 76.26]. Suppose for a moment that T is stably rational, then the U–module F is a direct summand of a permutation U–module. Then $F \otimes \mathbb{Z}_2$ is a permutation $\mathbb{Z}_2[U]$–module (cf. [17]). We know the structure of U, and the ranks of F and F^U, therefore we can list all the possible decomposition types of $F \otimes \mathbb{Z}_2$ into indecomposable modules. Since $\hat{H}^0(U,\mathbb{Z}[U/H]) = \hat{H}^0(H,\mathbb{Z}) \simeq \mathbb{Z}/|H|$, to compute $\hat{H}^0(U,F) = \hat{H}^0(U,F \otimes \mathbb{Z}_2)$ it suffices to know the ranks of indecomposable components of $F \otimes \mathbb{Z}_2$ (tensoring by $\mathbb{Z}_2$ does not change the cohomology of a 2–group). If $U = G_{7,1}$, then the following cases are a priori possible :

a) (8,8,1,1,1,1), then $\hat{H}^0(U,F) \simeq (\mathbb{Z}/8)^4$;
b) (8,4,4,2,1,1), then $\hat{H}^0(U,F) \simeq (\mathbb{Z}/8)^2 \times \mathbb{Z}/4 \times (\mathbb{Z}/2)^2$;
c) (4,4,4,4,2,2), then $\hat{H}^0(U,F) \simeq (\mathbb{Z}/4)^2 \times (\mathbb{Z}/2)^4$;
d) (8,4,2,2,2,2), then $\hat{H}^0(U,F) \simeq (\mathbb{Z}/4)^4 \times \mathbb{Z}/2$.

Comparing the orders of the groups in the exact sequence

$$(25) \qquad 0 \longrightarrow \hat{H}^0(U,S \otimes \mathbb{Z}_2) \longrightarrow \hat{H}^0(U,F \otimes \mathbb{Z}_2) \longrightarrow H^1(U,\hat{T} \otimes \mathbb{Z}_2) \longrightarrow 0,$$

we notice that cases a), c), d) are impossible. Case b) is also impossible, since $(\mathbb{Z}/4)^4$ cannot be embedded into $(\mathbb{Z}/8)^2 \times \mathbb{Z}/4 \times (\mathbb{Z}/2)^2$.

Now let $U = G_{6,1}$. Then the following ranks of indecomposable components of $F \otimes \mathbb{Z}_2$ are a priori possible :

a) (8,4,4,2,2), then $\hat{H}^0(U,F) \simeq (\mathbb{Z}/4)^2 \times (\mathbb{Z}/2)^2$;
b) (8,8,2,1,1), then $\hat{H}^0(U,F) \simeq (\mathbb{Z}/8)^2 \times \mathbb{Z}/4$;
c) (4,4,4,4,4), then $\hat{H}^0(U,F) \simeq (\mathbb{Z}/2)^5$.

Now, (25) leads to a contradiction in all cases.

Summing up, we have proved that tori corresponding to $G_{6,1}$ and $G_{7,1}$ are not stably k–rational. The same is obviously true for tori with splitting groups $G_{1,5}$, $G_{3,6}$, $G_{2,3}$, $G_{7,5}$, $G_{7,6}$, $G_{7,3}$ because in these cases the birational invariant $Ш^2_\omega(G,\hat{T})$ is non–zero. In a similar way, the torus with the group $G_{3,5}$ is not stably k–rational because $Ш^1_\omega(G_{3,5},\hat{T}^o) = \mathbb{Z}/2$ (Theorems 4.13 and 4.18). Let us show that all the other groups of types III, VIII, XII, XIII and XV from the list of Proposition 4.4 contain one of the above groups; this is sufficient to prove Theorem 7.1. We can consider subgroups of $W(D_4)$ up to the action of $\mathrm{Aut}' W(D_4)$ since the dual tori are stably equivalent (Proposition 4.1). In fact $G_{i,0}$ contains $G_{0,0}$ which is transformed into $G_{7,1}$ by an automorphism of $W(D_4)$. The group $G_{9,2} = <(123),c_1c_2(124)>$ contains $(123)c_1c_2(124) = c_2c_3(13)(24)$ and $(123)c_2c_3(13)(24)(132) = c_1c_3(12)(34)$, these two elements generate $G_{7,6}$; $G_{1,2}$ and $G_{1,4}$ contain $G_{1,5}$; $G_{10,2} \supset G_{3,6}$; $G_{3,3}$ and $G_{3,4}$ contain $G_{1,3}$ which is transformed into $G_{6,1}$ by an outer automorphism from $\mathrm{Aut}' W(D_4)$. Other groups of types considered are transformed by an outer automorphism from $\mathrm{Aut}' W(D_4)$ into one of the above groups. The theorem is proved.

To complete the picture let us present the following result.

THEOREM 7.2. *Let* $G \subseteq W(D_4)$ *and suppose that the decomposition of the graph* Γ *into G-orbits is of type* II *or* IV–VII. *Then the algebraic torus dual to the* G*-module* $Q(D_4)$ *is rational over the ground field.*

Proof : The k–rationality of tori of types IV and VII follows from Proposition 5.7. The k–rationality of the torus corresponding to the unique group $G_{2,1}$ of type VI is proved in [26]. Let $G = \mathrm{Aut(II)} = G_{5,1} = <(12),(23),c_1c_2c_3c_4>$. Let us choose the following basis in $Q(D_4)$: $\epsilon_i = e_4 - e_i$ $(i = 1,2,3)$, $\epsilon_4 = e_4 - e_1 - e_2 - e_3$ (e_i being the standard basis of $Q(B_4)$). The group $S_3 = <(12),(23)>$ permutes ϵ_i's, and the element $c_1c_2c_3c_4$ changes their signs. It follows that the dual torus admits an open embedding into a k–form of $(\mathbb{P}^1_k)^4$, therefore it is k–rational [44]. Let $G = \mathrm{Aut(V)} = G_{7,4} = <(12)(34),c_1c_3(13)(24)>$. Let us choose the basis $\epsilon_1 = e_4 - e_3$, $\epsilon_2 = e_1 + e_2$, $\epsilon_3 = e_4 - e_2$, $\epsilon_4 = e_1 - e_3$ in $Q(D_4)$, then the G–module $Q(D_4)$ is a direct sum of G–modules of rank 2 : $Q(D_4) = <\epsilon_1,\epsilon_2> \oplus <\epsilon_3,\epsilon_4>$, and all the two dimensional tori are k–rational [43, 4.74]. The theorem is proved.

Remark 7.3. Both k–rational and k–irrational tori occur among the tori corresponding to the groups of types IX and XIV.

In fact, $\mathrm{Aut(XIV)} \supset \mathrm{Aut(IX)} \supset G_{3,6}$. On the other hand, the torus corresponding to $G_{0,2}$ is k–rational [26]. For the torus corresponding to the group $G_{2,4}$ of type XIV the situation is quite similar.

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Résumé

Les intersections de deux quadriques dans $\mathbb{P}^4_k$ (c'est–à–dire les surfaces de Del Pezzo de degré 4, soit lisses, soit "singulières") constituent la première classe de surfaces rationnelles dont l'arithmétique est non–triviale. L'arithmétique de telles surfaces X dépend de leurs propriétés algébriques (combinatoires) et géométriques, propriétés que l'on peut lire sur l'action du groupe de Galois $\mathrm{Gal}(\bar{k}/k)$ sur le groupe de Picard $\mathrm{Pic}\,\bar{X}$ (ici $\bar{k}$ est une clôture séparable de k et $\bar{X} = X \times_k \bar{k}$). Pour étudier ces propriétés, nous donnons des formules générales pour certains invariants cohomologiques importants. Ces formules nous permettent d'établir la liste des cas "intéressants", c'est–à–dire des cas où ces invariants sont non–triviaux. Nous étudions les équivalences birationnelles entre divers types de surfaces rationnelles de degré 4, tant en termes géométriques que combinatoires. Puis nous exhibons de nombreux exemples explicites (y compris tous les cas "intéressants") et nous donnons une méthode générale de construction de tels exemples. Nous étudions aussi les propriétés de rationalité du tore de Néron–Severi.