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S. HALPERIN

Lectures on minimal models

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LECTURES ON MINIMAL MODELS

S. Halperin

INTRODUCTION

Algebraic topology has, classically, meant the study of algebraic invariants associated with topological spaces. These invariants (homology, homotopy,...) are normally not "geometric" in the sense that one cannot recover a space from them.

Although there is still no satisfactory algebraic description of homotopy theory (over $\mathbb{Z}$), the rational homotopy theory of Quillen and Sullivan is a practical and complete solution - if one is willing to forget torsion. Here one models the homotopy category by the category of commutative graded differential algebras (c.g.d.a.'s). Then to each c.g.d.a. one associates a "minimal model" with the property that if two c.g.d.a.'s are connected by a homomorphism which is an isomorphism of cohomology then the minimal models are isomorphic.

The process space $\to$ c.g.d.a. $\to$ minimal model gives the minimal model of a space. Its isomorphism class is an invariant of the weak homotopy type of the space, S. Moreover, if S is a 1-connected CW complex of finite type then from the model one can recover a space $S_{\mathbb{Q}}$ and a continuous map $S \to S_{\mathbb{Q}}$ which induces isomorphisms $\pi_i(S) \otimes \mathbb{Q} \cong \pi_i(S_{\mathbb{Q}})$.

Minimal models have proved to be a powerful tool in the solution of geometric problems. While the fact that one can indeed recover $S_{\mathbb{Q}}$ from the model is undoubtedly the philosophic reason for the power of the machine, this fact plays little direct role in the applications. Rather the two key ingredients turn out to be:

(a) A detailed understanding of the algebraic behaviour of the models, and

(b) A dictionary from classical topological invariants to invariants of the models.

My aim in these notes has been to provide a self-contained reference for many of the basic theorems needed for (a) and (b), in which complete, formal proofs were given down to the last technical detail. I have also tried to make the hypotheses as weak as possible and the conclusions as strong as possible. While this approach tends to make for difficult reading, it does (or so I hope!) result in a safely quoteable source for those whose main interest is the applications.

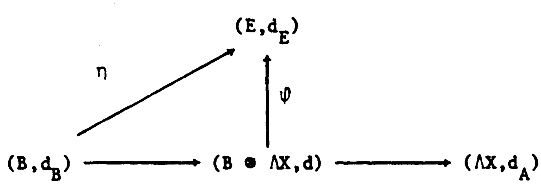
For the sake of completeness I have also included (with proofs!) many well known results and definitions (eg. simplicial sets in chap. 12, $\pi_1(M)$-modules in chap. 16 and Serre fibrations in chap. 19). In fact, the only prerequisite is some multilinear algebra and a little basic topology.

The material presented here divides naturally into three parts. The first (chaps. 1 to 11) is pure differential algebra: suppose

$$\eta : (B,d_B) \to (E,d_E)$$

is a homomorphism of c.g.d.a.'s (over a field k of characteristic zero). Assume $H^0(B) = H^0(E) = k$, and B is augmented.

Then there is a commutative diagram of c.g.d.a. homomorphisms



in which:

i) ψ^* is an isomorphism.

ii) ΛX is the free commutative graded algebra over the graded space X

iii) A certain "nilpotence-type" condition and a certain minimality condition (cf. chap. 1) are satisfied by d.

Moreover the bottom row is uniquely determined (up to isomorphism).

The diagram above is called the <u>minimal model</u> for η (cf. chap. 6). When B = k we have simply

$$\psi : (\Lambda X, d_A) \to (E, d_E) \quad ;$$

it is called the minimal model for (E, d_E).

The second part of the theory is a functor $M \rightsquigarrow (A(M), d)$ from topological spaces to c.g.d.a.'s (over k) such that H(A(M)) is naturally isomorphic with the singular cohomology H(M ; k). This is described in chaps. 13 to 15. The minimal model of (A(M),d) is called the <u>minimal model</u> for M.

The third part is the study of fibrations (chaps. 16 to 20). Suppose $F \xrightarrow{j} E \xrightarrow{\pi} B$ is a Serre fibration in which F, E, B are path connected. Then we can form the model of $A(\pi) : A(B) \to A(E)$, obtaining the commutative diagram:

$$\begin{array}{ccccc}
A(B) & \xrightarrow{A(\pi)} & A(E) & \xrightarrow{A(j)} & A(F) \\
\| & & \uparrow{\scriptstyle \psi} & & \uparrow{\scriptstyle \alpha} \\
A(B) & \longrightarrow & A(B) \otimes \Lambda X & \longrightarrow & \Lambda X
\end{array}$$

in which ψ^* is an isomorphism. The fundamental theorem of this part reads

20.3. - <u>Theorem</u>. Assume that

i) Either H(B ; k) or H(F ; k) has finite type.

ii) $\pi_1(B)$ acts nilpotently in each $H^p(F ; k)$.

Then α^* is an isomorphism, and so $\alpha : \Lambda X \to A(F)$ is the minimal model for F.

This theorem was proved first by P. Grivel [G] in the case B is 1-connected. Another proof was given independently a little later by J.C. Thomas (unpublished), again for the case B is 1-connected. The proof given in these notes follows the general idea of Grivel's proof, but the technicalities are substantially more complex. In particular, heavy use is made of the notion of "local system over a simplicial set" (chap. 12) which is a simplicial analogue of a sheaf.

Let $\Lambda X \to A(M)$ be the minimal model (over $\mathbb{Q}$) of a path connected space M. There are obvious linear maps

$$X^p \to \mathrm{Hom}_{\mathbb{Z}}(\pi_p(M)\ ;\ \mathbb{Q}), \qquad p \geq 2.$$

Using theorem 20.3 it is easy to deduce the

Theorem. - Assume that

i) Each $\pi_p(M) \otimes \mathbb{Q}$ is a nilpotent finite dimensional $\pi_1(M)$ module for $p \geq 2$).

ii) The minimal model for $K(\pi_1(M)\ ;\ 1)$ has generators only in degree 1.

Then the linear maps $X^p \to \mathrm{Hom}_Z(\pi_p(M)\ ;\ \mathbb{Q})$, $p \geq 2$, are isomorphisms.

I had originally planned to include this and other applications, but ran out of time. They will appear elsewhere.

The theory of minimal models is due to Dennis Sullivan, and his paper "Infinitesimal Computations in Topology" [S] is the fundamental work on the subject. Indeed the first two parts of these notes (chaps. 1 to 11 and 13 to 15) follow [S] very closely.

The reader who makes this comparison will discover that aside from the occasional modification in the assertions of [S] I have frequently merely expanded the ideas there into formal proofs. (One exception is "de Rham's theorem" in chap. 14 whose proof is based on that of Chris Watkiss [W]; another version of this proof is given by Cartan [C]. Other proofs abound in the literature.) Of course the overlap of these notes with [S] covers only part of [S]. I haven't touched solvable models, let alone the latter half of [S].

Another approach to minimal models is via localizations and Postnikov towers. If M is a nilpotent space it can be localized to produce a rational space $M_{\mathbb{Q}}$. The data which define the Postnikov decomposition of $M_{\mathbb{Q}}$ are exactly the data which define the minimal model of M, and so it follows that the minimal model of M determines its rational homotopy type. The theorem above on homotopy groups follows at once, at least for nilpotent spaces. This approach is that of Friedlander et al. [F] and Lehmann [L_2]. The resumé by Lehmann [L_1] is particularly elegant and readable.

A different approach is taken by Bousfield and Gugenheim [B-G] who provide a complete exposition in the context of the closed model categories of Quillen. Other expositions (eg. [W-T]) are also available.

At least two other algebraic categories have been successfully used to model rational homotopy theory: the iterated integrals of Chen [Ch] and the category of graded differential Lie algebras. In the latter category the notion of minimal model was introduced by Baues and Lemaire [B-L].

The recent book of Tanré [Ta] provides a clear description of the relation between these categories and goes very much further than the present notes in describing topological invariants in terms of the model.

These notes are a greatly expanded version of lectures I gave at Lille in 1976 and 1977 in the seminar on algebraic topology and differential geometry. They first appeared in 1977 in the Publications Internes of the U.E.R. de Mathématiques, Université de Lille I and are presented here unchanged, except for changes to the introduction.

The seminar discussions were, naturally, enormously helpful - I want particularly to mention Daniel Lehmann and Chris Watkiss. My thanks also go to Mmes Tatti and Bérat for their careful typing of the manuscript, and to the Université de Lille I, whose hospitality made the whole thing possible. It is a great pleasure to be able, now, to say thank you to my Lille colleagues not only for their hospitality that year, but for all the subsequent years as well.

Finally, I should like to take this opportunity to express my warm gratitude to my teacher, friend and colleague Werner Greub from whom I first learned about commutative graded differential algebras and Koszul complexes.

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Chapter 0

Notation and conventions

All vector spaces, algebras, multilinear operations, ... in these notes are defined over a fixed field k of characteristic zero. Occasionally we specify $k = \mathbb{Q}, \mathbb{R}$ or $\mathbb{C}$.

All algebras are associative, and have an identity, 1, which is preserved by homomorphisms.

By a graded vector space we mean a direct sum $V = \Sigma_{p\geq 0} V^p$ (note that the sum is over the non negative integers). We write $V^+ = \Sigma_{p>0} V^p$. Elements of V^p are homogeneous of degree p. V has finite type if each V^p has finite dimension. If W is a second graded space then a linear map $\varphi : V \to W$ has degree r if $\varphi(V^p) \subset W^{p+r}$, $p \geq 0$.

A graded algebra $A = \Sigma_{p\geq 0} A^p$ is one which satisfies $A^p.A^q \subset A^{p+q}$. An augmentation of A is a homomorphism $\varepsilon_A : A \to k$ such that $\varepsilon_A(A^+) = 0$. A derivation of degree p in A is a linear map, θ, of degree p such that $\theta(ab) = \theta(a).b+(-1)^{pq}a.\theta(b)$, $a \in A^q$, $b \in A$. A is called n-connected if $A^0 = k$ and $A^p = 0$, $1 \leq p \leq n$. By a homomorphism of graded algebras we mean a homomorphism of degree zero. A homomorphism $\varphi : (A,\varepsilon_A) \to (B,\varepsilon_B)$ of augmented graded algebras satisfies $\varepsilon_A = \varepsilon_B\varphi$.

A homomorphism of graded algebras is called n-regular if it is an isomorphism in degrees $\leq n$ and injective in degree n. The tensor product of graded algebras A and B is the graded algebra $A \otimes B$ with product

$$(a \otimes b).(a' \otimes b') = (-1)^{pq}\, aa' \otimes bb' \quad , \quad \begin{matrix} a \in A \\ b \in B^p \\ a' \in A^q \\ b' \in B \,. \end{matrix}$$

A commutative graded algebra (c.g.a.), A, is one which satisfies $ab = (-1)^{pq}ba$, $a \in A^p$, $b \in A^q$. If $\varphi : C \to A$ and $\psi : C \to B$ are homomorphisms of c.g.a.'s the elements $a\varphi(x) \otimes b - a \otimes \psi(x)b$ $(a \in A, x \in C, b \in B)$ span a graded subspace $I \subset A \otimes B$. In fact I is an ideal and we write

$$A \otimes_C B = (A \otimes B)/I \quad ;$$

it is again a c.g.a.

A graded differential algebra is a graded algebra A together with a derivation d_A of degree 1 such that $d_A^2 = 0$. The spaces $\ker d_A$, $\operatorname{Im} d_A$ are called the <u>cocycle</u> and <u>coboundary</u> spaces and the graded algebra

$$H(A,d_A) = \ker d_A/\operatorname{Im} d_A$$

is called the cohomology algebra. It is sometimes written $H(A)$. A homomorphism $\varphi : (A,d_A) \to (B,d_B)$ of g.d.a.'s is a homomorphism of graded algebras which satisfies $\varphi d_A = d_B \varphi$. It induces a homomorphism of cohomology algebras, written

$$\varphi^* : H(A) \to H(B).$$

If φ^* is an isomorphism we sometimes write $\varphi : (A,d_A) \xrightarrow{\cong} (B,d_B)$. (Note that $\varphi \rightsquigarrow \varphi^*$ is a covariant functor !).

A is called <u>acyclic</u> if $H(A) = k$. An ideal $J \subset A$ (which is d_A stable) is called <u>acyclic</u> if $H(J) = 0$.

The tensor product of g.d.a.'s A and B is again a g.d.a. with

$$d_{A\otimes B}(a \otimes b) = d_A a \otimes b + (-1)^p a \otimes d_B b \quad , \quad a \in A^p, \ b \in B.$$

Multiplication defines an isomorphism $H(A \otimes B) = H(A) \otimes H(B)$.

If (A,d_A) is a g.d.a. and A is a c.g.a. then (A,d_A) is called a commutative graded differential algebra (c.g.d.a.). If $C \to A$ and $C \to B$ are c.g.d.a. homomorphisms then $A \otimes_C B$ is naturally a c.g.d.a..

Chapter 1

KS-extensions

1.1.- Definitions.- Let X be a graded space. ΛX will denote the free graded commutative algebra over X :

$$\Lambda X = \text{Exterior algebra } (X^{odd}) \otimes \text{Symmetric algebra } (X^{even})$$

$(\Lambda^k X)^p$ is the subspace generated by $x_1 \wedge \ldots \wedge x_k$ with $\sum_i \deg x_i = p$; we say the elements in $(\Lambda X)^p$ have degree p.

Let $A \xrightarrow{\varepsilon} k$ be an augmented graded algebra. We define a graded space $Q(A) = \sum_{p \geq 0} Q^p(A)$ (also written Q_A) by

$$Q(A) = \ker \varepsilon / \ker \varepsilon . \ker \varepsilon \quad ;$$

it is the space of indecomposables of A. We denote the canonical projection (of graded spaces) by $\zeta_A : \ker \varepsilon \to Q(A)$.

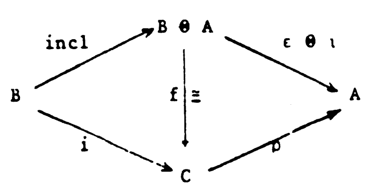
An extension is a sequence of augmented c.g.d.a.'s

$$\mathcal{E} : (B, d_B) \xrightarrow{i} (C, d_C) \xrightarrow{\rho} (A, d_A)$$

such that i and ρ preserve the augmentations and

i) $A = \Lambda X$ for some graded subspace $X \subset A$, and $\Lambda^+ X$ is the augmentation ideal.

ii) There is a commutative diagram of algebra homomorphisms

(1.1') 

,

where f is an isomorphism and ε is the augmentation for B. (We make no requirement about how f behaves with the differentials). Note that $f(\ker \varepsilon \otimes A + B \otimes \Lambda^+ X)$ is the augmentation ideal for C.

We call the c.g.d.a.'s (B,d_B), (C,d_C) and (A,d_A) <u>the base</u>; <u>total space</u>, and <u>fibre</u> of the extension.

The above definition involves the existence of two "non-canonical objects", namely X and f. The extension $\mathcal{E}$, together with (X,f) will be called a <u>structured</u> <u>extension</u>. We often use the structure to identify C with $B \otimes A$. In this case the elements $b \otimes 1$ (= i(b)) and $1 \otimes a$ (=f(1 $\otimes$ a)) will often be denoted simply by b and by a.

An extension is called <u>elementary</u> if there is a structure (X,f) such that

(1.2) $$d_C(X) \subset B.$$

Suppose $\{x_\alpha\}_{\alpha\in I}$ is an ordered homogeneous basis for X. Then we will write $A_{<\alpha}$ and $A_{\leq\alpha}$ for the subalgebras generated by the x_β with $\beta < \alpha$ (resp. $\beta \leq \alpha$). Note that

(1.3) $$A_{\leq\alpha} \cong A_{<\alpha} \otimes \Lambda x_\alpha \quad ,$$

the isomorphism being multiplication in $A_{\leq\alpha}$.

An extension $\mathcal{E}$ is called a <u>Koszul-Sullivan (KS) extension</u> if it admits a structure (X,f) and a homogeneous basis $\{x_\alpha\}_{\alpha\in I}$ of X, indexed by a <u>well-ordered</u> set I such that

(1.4.) $$d_C(x_\alpha) \in B \otimes A_{<\alpha}, \quad \alpha \in I.$$

An extension $\mathcal{E}$ is called <u>positive</u> if A is connected : $(X=\sum_{p\geq 1} X^p)$.

A KS-extension $\mathcal{E}$ is called <u>minimal</u> if there is a structure (X,f) and a well ordered homogeneous basis $\{x_\alpha\}_{\alpha\in I}$ for X such that

$\deg x_\beta < \deg x_\alpha \Rightarrow \beta < \alpha$, **and such that (1.4) holds.**

If $B = k$ we replace "extension" by "complex" in the definitions, obtaining KS-complex, minimal KS-complex.

1.5.- *Remarks*.-

1) If $B \to C \to A$ is a **KS extension (resp. minimal KS) then A is a KS complex (resp. minimal KS).**

2) A KS-extension is a generalized sequence of elementary extensions.

Finally, a <u>morphism</u> between two **extensions is a commutative diagram**

$$\begin{array}{ccccc} B & \longrightarrow & C & \longrightarrow & A \\ \downarrow{\scriptstyle \psi} & & \downarrow{\scriptstyle \varphi} & & \downarrow{\scriptstyle \alpha} \\ B' & \longrightarrow & C' & \longrightarrow & A' \end{array}$$

of homomorphisms of augmented c.g.d.a.'s ; it is written $(\psi, \varphi, \alpha) : E \to E'$.

1.6.- *Example*.- *contractible extensions*.

A contractible KS extension $B \to C \to A$ is one which admits a structure (X, f) and a decomposition $X = X_1 \oplus X_2$ such that

$$d_C : X_1 \xrightarrow{\cong} X_2 .$$

Thus we can write

$$C = B \otimes (\Lambda X_1 \otimes \Lambda X_2)$$

$$d_C = d_B \otimes \iota + \omega_B \otimes d_A$$

$(\omega_B b = (-1)^p b,\ b \in B^p)$ and (A, d_A) is the free c.g.d.a. generated by X_1.

The ideal $J \subset C$ generated by X_1 and X_2 is acyclic :

$$H^\bullet(J) = 0$$

and this shows that the projection $C \to B$ with kernel J induces an isomorphism of cohomology inverse to i^*.

If $B = k$ we call $C(=A)$ a <u>contractible</u> KS <u>complex</u>.

Suppose next that (X,f) is a structure for an extension $\mathcal{E} : B \to C \to A$, and that $\{x_\alpha\}_{\alpha\in I}$ is a well ordered homogeneous basis for X.

1.7.- <u>Lemma</u>.- Let $y_\alpha \in C$ be homogeneous elements such that

i) $\rho y_\alpha = x_\alpha$

ii) $y_\alpha - f(1 \otimes x_\alpha) \in f(B \otimes A_{<\alpha})$, $\alpha \in I$.

Then a second structure (X,g) is defined by

$$g(1 \otimes x_\alpha) = y_\alpha .$$

<u>Proof</u> : A unique homomorphism $g : B \otimes A \to C$ is defined by $g(b \otimes 1) = i(b)$ and $g(1 \otimes x_\alpha) = y_\alpha$. Because of i) we have $\rho \circ g = \varepsilon \otimes \iota$; it remains to show that g is an isomorphism.

Set $\varphi = f^{-1} \circ g$. Then $\varphi(b \otimes 1) = b \otimes 1$ and ii) implies that

$$\varphi(1 \otimes x_\alpha) - 1 \otimes x_\alpha \in B \otimes A_{<\alpha} \tag{1.8}$$

We show now that $\varphi : B \otimes A_{\leq\alpha} \xrightarrow{\cong} B \otimes A_{\leq\alpha}$ for all α.

If not there is a least α for which it fails ; since $B \otimes A_{<\alpha} = \varinjlim_{\beta<\alpha} B \otimes A_{\leq\beta}$, we have

$$\varphi : B \otimes A_{<\alpha} \xrightarrow{\cong} B \otimes A_{<\alpha} .$$

Write $B \otimes A_{\leq\alpha} = (B \otimes A_{<\alpha}) \otimes \Lambda x_\alpha$ and use (1.8) to complete the proof.

Q.E.D.

Consider next a morphism

$$(\psi,\varphi,\iota) : (B' \to C' \to A) \to (B \to C \to A)$$

between Λ-extensions E' and E. Assume that E' and E admit structures (X,f') and $(X,\tilde{f})$ and that $\{x_\alpha\}_{\alpha\in I}$ is a well ordered homogeneous basis for X such that :

i) $d_{C'}f'(1 \otimes x_\alpha) \in B' \otimes A_{<\alpha}$

and

ii) $\phi f'(1 \otimes x_\alpha) - \tilde{f}(1 \otimes x_\alpha) \in \tilde{f}(B \otimes A_{<\alpha})$.

1.9.- *Lemma*.- With the hypotheses above :

i) There is a second structure (X,f) for E such that $\phi f'(1 \otimes x_\alpha) = f(1 \otimes x_\alpha)$.

ii) With respect to this second structure E satisfies (1.4) and ϕ is given by $\phi = \psi \otimes \iota$.

iii) If ψ (resp. ψ^*) is an isomorphism then ϕ (resp. ϕ^*) is an isomorphism .

Proof :

i) Define f by $f(1 \otimes x_\alpha) = \phi f'(1 \otimes x_\alpha)$. Then

$$\rho f(1 \otimes x_\alpha) = \rho\phi f'(1 \otimes x_\alpha) = \rho' f'(1 \otimes x_\alpha) = x_\alpha$$

and

$$f(1 \otimes x_\alpha) - \tilde{f}(1 \otimes x_\alpha) \in \tilde{f}(B \otimes A_{<\alpha}).$$

It follows that (X,f) is a structure on E (lemma 1.7).

ii) Note that

$$d_C f(1 \otimes x_\alpha) = d_C \phi f'(1 \otimes x_\alpha)$$
$$= \phi d_{C'} f'(1 \otimes x_\alpha) \in \phi(B' \otimes A_{<\alpha}) \subset f(B \otimes A_{<\alpha}).$$

iii) We may assume by i) and ii) that both E' and E satisfy equation (1.4) with respect to (X,f') and (X,f), and that $\phi = \psi \otimes \iota$.

Thus if ψ is an isomorphism, so is ϑ. Now assume ψ^* is an isomorphism. It is enough to show that $(\psi \otimes \iota) : B' \otimes A_{\leq\alpha} \to B \otimes A_{\leq\alpha}$ induces a cohomology isomorphism for each α. If not, let α be the least α for which it fails. Since ψ^* is an isomorphism and

$$B' \otimes A_{<\alpha} = \varinjlim_{\beta<\alpha} B' \otimes A_{\leq\beta} \quad ; \quad B \otimes A_{<\alpha} = \varinjlim_{\beta<\alpha} B \otimes A_{\leq\beta} ,$$

a direct limit argument shows that $(\psi \otimes \iota)^* : H(B' \otimes A_{<\alpha}) \xrightarrow{\cong} H(B \otimes A_{<\alpha})$.

Set $E' = B' \otimes A_{<\alpha}$, $E = B \otimes A_{<\alpha}$, $\gamma = \psi \otimes \iota : E' \to E$.

Then $\psi \otimes \iota : B' \otimes A_{\leq\alpha} \to B \otimes A_{\leq\alpha}$ can be identified with

$$\gamma \otimes \iota : E' \otimes \Lambda x_\alpha \to E \otimes \Lambda x_\alpha .$$

Moreover $d'x_\alpha \in E'$, $dx_\alpha \in E$.

Set $F'_k = \sum_{j\leq k} E' \otimes \Lambda^j x_\alpha$ and $F_k = \sum_{j\leq k} E \otimes \Lambda^j x_\alpha$.

Then $E' \otimes \Lambda x_\alpha = \varinjlim F'_k$, $E \otimes \Lambda x_\alpha = \varinjlim F_k$ and so we need only prove that $(\gamma \otimes \iota)^* : H(F'_k) \xrightarrow{\cong} H(F_k)$.

Define (for $k = 0,1$ when $\deg x_\alpha$ is odd and for all k when $\deg x_\alpha$ is even) projections $F'_k \to E'$ (and $F_k \to E$) by

$$\sum_{j=0}^{k} \Phi_j \otimes x_\alpha^j \longrightarrow \Phi_k .$$

Then we have the commutative row exact diagrams of differential spaces

$$\begin{array}{ccccccccc} 0 & \longrightarrow & F_{k-1} & \longrightarrow & F'_k & \longrightarrow & E' & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle \gamma\otimes\iota} & & \downarrow{\scriptstyle \gamma\otimes\iota} & & \downarrow{\scriptstyle \gamma} & & \\ 0 & \longrightarrow & F_{k-1} & \longrightarrow & F_k & \longrightarrow & E & \longrightarrow & 0 \end{array} .$$

Now the 5-lemma, plus induction complete the proof. Q.E.D.

1.10.- Pullbacks.- We consider now a KS-extension $B \to C \to A$ which satisfies (1.4) for a structure (X,f) and a well ordered homogeneous basis $\{x_\alpha\}_{\alpha\in I}$ for X. Suppose in addition that

$$\psi : (B',d_{B'}) \longrightarrow (B,d_B)$$

is a homomorphism of augmented c.g.d.a.'s such that ψ^* is an isomorphism.

1.11.- Proposition.- There is a KS-extension $B' \to C' \to A$ and a morphism $(\psi,\varphi,\iota) : E' \to E$ such that

i) E' admits a structure (X,f') such that

$$d_{C'}(1 \otimes x_\alpha) \in B' \otimes A_{<\alpha}$$

and

$$\varphi(1 \otimes x_\alpha) - 1 \otimes x_\alpha \in B \otimes A_{<\alpha} , \qquad \alpha \in I.$$

ii) φ^* is an isomorphism.

Proof.- We set $C' = B' \otimes A$, $f' = \iota$. We have to construct $d_{C'}$ and φ so that $B' \to C' \xrightarrow{\varepsilon'\otimes\iota} A'$ is a sequence of c.g.d.a.'s and so that φ is a morphism of extensions with i) holding.

We induct over I, as usual, starting off by setting $\varphi = \psi$ in B' and $d_{C'} = d_{B'}$ in B'. Now assume φ and $d_{C'}$ are constructed in $B' \otimes A_{<\alpha}$ so that φ and $d_{C'}$ satisfy i) for all β, $\beta < \alpha$. Then by lemma 1.9,

$$\varphi^* : H(B' \otimes A_{<\alpha}) \xrightarrow{\cong} H(B \otimes A_{<\alpha}).$$

But $d_C(1 \otimes x_\alpha)$ is a d_C-cocycle in $B \otimes A_{<\alpha}$. Thus there is a $d_{C'}$-cocycle $\Phi \in B' \otimes A_{<\alpha}$ and an element $\Omega \in B \otimes A_{<\alpha}$ such that

$$d_C(1 \otimes x_\alpha) = \hat{\psi}(\Phi) - d_C\Omega.$$

In particular, $\rho\,\Omega \in A_{<\alpha}$.

Extend $d_{C'}$ and $\hat{\psi}$ to $B' \otimes A_{\leq\alpha}$ by setting

$$d_{C'}(1 \otimes x_\alpha) = \Phi - d_{C'}(1 \otimes \rho\Omega)$$

and

$$\hat{\psi}(1 \otimes x_\alpha) = 1 \otimes x_\alpha + (\Omega - \hat{\psi}(1 \otimes \rho\Omega)).$$

Then i) holds by definition. Straightfoward calculations show that

$$\rho\hat{\psi} = \varepsilon' \otimes \iota, \quad \hat{\psi}\, d_{C'} = d_C\hat{\psi}, \quad (\varepsilon' \otimes \iota) \circ d_{C'} = d_A \circ (\varepsilon' \otimes \iota).$$

The existence of $d_{C'}$ and $\hat{\psi}'$ in $B' \otimes A$ now follows by induction.

Finally, lemma 1.9. yields ii). Q.E.D.

Corollary.- E admits a structure $(X,\tilde{f})$ such that with respect to this structure and (X,f'), $\hat{\psi} = \psi \otimes \iota$ and

$$d_C(1 \otimes x_\alpha) \in B \otimes A_{<\alpha} \quad ; d_{C'}(1 \otimes x_\alpha) \in B' \otimes A_{<\alpha} .$$

Proof : Apply lemma 1.9. Q.E.D.

1.12. - *Example*.- Suppose $E : B \to C \to A$ is a KS-extension, and $H^0(B) = k$ and $H^q(B) = 0$, $0 < q < m$ (where we allow $m = \infty$). Let $\bar{B}^m \subset B^m$ satisfy $\bar{B}^m \oplus d_B(B^{m-1}) = B^m$ and define a sub. c.g.d.a. $\bar{B} \subset B$ by

$$\bar{B}^0 = k, \quad \bar{B}^q = 0, \quad 0 < q < m, \quad \bar{B}^m = \text{space just chosen}$$

$$\bar{B}^q = B^q,\ q > m.$$

Then $H^*(\bar{B}) = H^*(B)$.

Now we can apply the corollary to prop. 1.11 to obtain a structure (X,f) on $\mathcal{E}$ such that

i) $\bar{B} \otimes A$ is d_C-stable

ii) $d_C(1 \otimes x_\alpha) \in \bar{B} \otimes A_{<\alpha}$

and

iii) $H^*(\bar{B} \otimes A) \xrightarrow{\cong} H^*(B \otimes A)$.

This will be called a normalized structure for $\mathcal{E}$.

1.13. - *Corollary*.- Assume $H(B)$ is connected and let $\bar{B}$, (X,f) be as above. Then

(1.14) $$d_C(1 \otimes a) - 1 \otimes d_A a \in B^+ \otimes A, \qquad a \in A.$$

Proof.- The left hand side is in $(\bar{B} \otimes A) \cap \ker \rho = B^+ \otimes A$.

Q.E.D.

1.15. - *Corollary*.- Assume $H(B)$ is 1-connected and let $\bar{B}$, (X,f) be as above. Then

(1.16) $$d_C(1 \otimes a) - 1 \otimes d_A a \in \sum_{j\geq 2} B^j \otimes A, \quad a \in A.$$

Chapter 2

Reduction to a minimal extension

2.1.- Introduction.- In this chapter we consider a KS extension

$$\mathcal{E} : B \xrightarrow{i} C \xrightarrow{\rho} A$$

with augmentations ε_B, ε_C and ε_A. We always assume $H^0(C) = k$; then $H^0(B) = k$ as well. <u>However, we do not suppose $H^0(A) = k$.</u>

Recall from (1.1) the projection $\zeta_A : \ker \varepsilon_A \to Q(A)$. If (X,f) is any structure for $\mathcal{E}$ then ζ_A restricts to an isomorphism

$$\zeta_A : X \xrightarrow{\cong} Q(A).$$

Hence if $\Phi \in \ker \varepsilon_A$ $(=\Lambda^+X)$ we may regard Φ as a polynomial with no constant term in the elements of X, and then $\zeta_A\Phi$ can be interpreted as the "linear part" of Φ.

Observe that a differential $Q(d_A)$ is defined in $Q(A)$ by

$$Q(d_A)\zeta_A = \zeta_A d_A .$$

The object of this chapter is to prove

2.2.- Theorem.- There is a minimal KS extension

$$B \xrightarrow{\tilde{i}} \tilde{C} \xrightarrow{\tilde{\rho}} \tilde{A}$$

and a contractible KS-complex $R = \Lambda T \otimes \Lambda dT$ such that : if $\tilde{C} \otimes R$ and $\tilde{A} \otimes R$ denote the tensor products (as augmented c.g.d.a.'s) then there is a commutative diagram of homomorphisms of augmented c.g.d.a.'s

$$\begin{array}{ccccc} B & \xrightarrow{\tilde{i} \otimes 1} & \tilde{C} \otimes R & \xrightarrow{\tilde{\rho} \otimes \iota} & \tilde{A} \otimes R \\ \| & & \cong \downarrow \phi & & \cong \downarrow \psi \\ B & \xrightarrow[i]{} & C & \xrightarrow[\rho]{} & A \end{array}$$

in which the vertical arrows are isomorphisms.

2.3.- *Corollary*.- ψ induces an isomorphism

$$Q(\psi) : Q(\tilde{A}) \otimes Q(R) \xrightarrow{\cong} Q(A).$$

2.4.- *Corollary*.- E is minimal if and only if $Q(d_A) = 0$; i.e., if and only if $\operatorname{Im} d_A$ consists of polynomials with no linear term.

In particular the isomorphism of cor. 2.3 induces an isomorphism

$$Q(\tilde{A}) \xrightarrow{\cong} H(Q(A), Q(d_A)).$$

Proof.- If E is minimal it follows directly from the definitions that $Q(d_A) = 0$. Conversely, assume $Q(d_A) = 0$. Then because $Q(\psi)$ is an isomorphism and R is a contractible complex we conclude that $R = k$ and so E is minimal.

Q.E.D.

The rest of the chapter is devoted to the proof of theorem 2.2.

2.5.- *Conventions*.- We fix a connected c.g.d.a. $\bar{B} \subset B$ such that $H(\bar{B}) = H(B)$ (cf. example 1.12). By a normalized structure we shall always mean normalized with respect to $\bar{B}$.

If $U = \sum_{j\geq 0} U^j$ is a graded space, then we write

$$U^{\geq p} = \sum_{j\geq p} U^j .$$

The spaces $U^{>p}$, $U^{<p}$, $U^{\leq p}$ are defined analogously. Note that $\Lambda U^{\leq p}$ is the free c.g.a. on $U^{\leq p}$, while $(\Lambda U)^{\leq p}$ is the subspace of ΛU of elements of degree $\leq p$!

If K is contained in an ordered set I then for $\alpha \in I$ $K_{<\alpha} = \{\gamma \in K | \gamma < \alpha\}$. $K_{\leq\alpha}$, $K_{>\alpha}$, $K_{\geq\alpha}$ are defined analogously.

2.6.- *Lemma*.- Suppose $Q(d_A) = 0$. Let (X,f) be a normalized structure for E satisfying (1.4) with respect to a well ordered homogeneous basis $\{x_\alpha\}_{\alpha\in I}$. Then if $q = \deg x_\alpha$,

(2.7) $$d_A\, x_\alpha \in (\Lambda X^{\leq q})_{<\alpha} \quad , \qquad \alpha \in I.$$

Moreover if $\Phi \in C^{p+1}$ $(p \geq 0)$ is a d_C-cocycle and

$$\rho\Phi = \Phi_1 x_{\beta_1} + \dots + \Phi_n x_{\beta_n} + \Psi ,$$

where $x_{\beta_i} \in X^{p+1}$, $\beta_1 > \dots > \beta_n$, and $\Psi \in \Lambda X^{\leq p}$, then

$$\Phi_1 \text{ is a scalar : } \Phi_1 \in k.$$

Proof.- We show first that (2.7) implies the rest of the lemma. In view of (1.14) it does imply that

(2.8) $$d_C(1 \otimes x_\alpha) \in \bar{B} \otimes (\Lambda X^{\leq q})_{<\alpha} \quad , \qquad \alpha \in I,$$

where $q = \deg x_\alpha$. Hence $\bar{B} \otimes \Lambda X^{\leq q}$ is d_C-stable, $q = 0,1,\dots$.

Now we recall that $\bar{B} \otimes A \to B \otimes A$ induces a cohomology isomorphism, and so $\Phi = \bar{\Phi} + d_C\Omega$, with $\bar{\Phi} \in \bar{B} \otimes A$ and $\Omega \in B \otimes A$. Then $\rho\Omega \in A^p \subset \Lambda X^{\leq p}$, which is d_A-stable by (2.7). Thus $d_A \rho\, \Omega \in \Lambda X^{\leq p}$, and so

$$\rho\bar{\Phi} = \sum_{i=1}^{n} \Phi_i x_{\beta_i} + \bar{\Psi} \quad , \qquad \bar{\Psi} \in \Lambda X^{\leq p} .$$

We may thus, without loss of generality, assume that $\Phi \in \bar{B} \otimes A$, and we do.

In view of (2.8) this implies that

$$\begin{aligned} d_C(1 \otimes \rho\Phi) &= d_C(1 \otimes \rho\Phi - \Phi) \\ &\in d_C(\bar{B} \otimes \Lambda X^{\leq p}) \\ &\subset \bar{B} \otimes \Lambda X^{\leq p}. \end{aligned}$$

Again because of (2.8) we conclude from this that

$$d_C(1 \otimes \sum_{i=1}^{n} \Phi_i x_{\beta_i}) \in \bar{B} \otimes \Lambda X^{\leq p}.$$

A degree calculation shows that each $\Phi_i \in \Lambda X^o$, and so because of (2.8)

$$d_C\Phi_i \in \bar{B}^1 \otimes \Lambda X^o \quad , \qquad i = 1,\ldots,n.$$

Let $Y \subset X$ be the subspace spanned by the x_γ, $\gamma \neq \beta_1$. Since $\beta_i < \beta_1$ $(i > 1)$ we have $d_C(1 \otimes x_{\beta_i}) \in \bar{B} \otimes \Lambda Y$. Together with the relations above, this yields

$$d_C(1 \otimes \Phi_1) \cdot (1 \otimes x_{\beta_1}) \in \bar{B} \otimes \Lambda Y.$$

In view of the isomorphism

$$\bar{B} \otimes \Lambda X = \bar{B} \otimes \Lambda Y \otimes \Lambda x_{\beta_1}$$

we obtain $d_C(1 \otimes \Phi_1) = 0$, and so $\Phi_1 \in k$ as desired.

It remains to prove (2.7). Assume it holds for all $\alpha < \gamma$, some γ with $\deg x_\gamma = q$. Then $d_C(1 \otimes x_\gamma)$ is a cocycle in $B \otimes A_{<\gamma}$. Since (2.7) holds for $\alpha < \gamma$ we can apply the second half of the lemma to this cocycle. This gives

$$d_A x_\gamma = \rho d_C(1 \otimes x_\gamma) = \lambda x_{\beta_1} + \sum_{\beta_i < \beta_1} \Phi_i x_{\beta_i} + \Psi ,$$

where $\lambda \in k$, $\Phi_i \in (\Lambda X^o)_{<\gamma}, \Psi \in (\Lambda X^{\leq q})_{<\gamma}$.

Hence (since $Q(d_A) = 0$)

$$0 = Q(d_A)\zeta_A x_\gamma = \lambda\zeta_A x_{\beta_1} + \sum_{\beta_i<\beta_1} (\varepsilon_A \Phi_i)\ \zeta_A x_{\beta_i} .$$

It follows that $\lambda = 0$. Hence Φ_2 is a scalar and so zero. In this way we find all the Φ_i are zero :

$$d_A x_\gamma = \Psi \in (\Lambda X^{\leq q})_{<\gamma} .$$

Q.E.D.

Again consider the KS extension E. Choose a normalized structure (X,f) satisfying (1.4) with respect to a well ordered homogeneous basis $\{x_\alpha\}_{\alpha\in I}$. It is easy to arrange that the following condition hold as well :

There is a disjoint decomposition $I = J \cup J' \cup H$ and a bijection $' : J \to J'$ such that

$$Q(d_A)\zeta_A x_\alpha = \zeta_A x_{\alpha'} \qquad , \qquad \alpha \in J$$

(2.9) and

$$Q(d_A)\zeta_A x_\alpha = 0 \qquad , \qquad \alpha \in J' \cup H$$

Note that necessarily $\alpha' < \alpha$, $\alpha \in J$.

We shall assume henceforth that (2.9) holds.

2.10.- Lemma.- There are elements $z_\alpha \in \ker \varepsilon_C$ $(\alpha \in I)$ such that

i) $z_\alpha - 1 \otimes x_\alpha \in \bar{B} \otimes A_{<\alpha}$.

ii) $\zeta_A \rho z_\alpha = \zeta_A x_\alpha$.

iii) If $\alpha \notin J$ then $d_C z_\alpha$ is in the subalgebra generated by $\bar{B} \otimes A_{<\alpha}^{\leq p_\alpha}$ $(\deg z_\alpha = p_\alpha)$.

(iv) If $\alpha \in J$ then $d_C z_\alpha - z_{\alpha'}$ is in the subalgebra E_α defined as follows : E_α is generated by $\bar{B} \otimes A_{<\alpha}^{\leq p_\alpha}$ $(p_\alpha = \deg z_\alpha)$ and by the elements z_β such that $\deg z_\beta = p_\alpha + 1$ and $\beta < \alpha'$, and by the elements z_β, $d_C z_\beta$ with $\beta \in J_{<\alpha}$.

Proof.- We assume z_γ has been constructed for $\gamma < \alpha$ and construct z_α. First note that if we change the definition of x_β to ρz_β and if we change the definition of $f(1 \otimes x_\beta)$ to $f(1 \otimes x_\beta) = z_\beta$ $(\beta < \alpha)$ then we obtain a new structure (X,f) and basis $\{x_\gamma\}_{\gamma\in I}$ for X which still satisfy (1.4.) cf. lemma 1.7.

Moreover, the algebras $A_{<\gamma}$ are unaffected by this change, as are the algebras generated by $A_{<\gamma}^{\leq q}$. Finally, it follows from ii) that the elements $\zeta_A x_\gamma$ are also unchanged, and so (2.9) remains valid, as does the statement of the lemma for $\gamma < \alpha$.

Thus without loss of generality we may assume that

(2.11) $$z_\beta = 1 \otimes x_\beta \quad , \qquad \beta < \alpha \quad ,$$

and we make this assumption henceforth.

Now write $(J_{<\alpha})' = \{\beta' \in J'/\beta < \alpha\}$. Then $(J_{<\alpha})' \subset J'_{<\alpha}$, but equality may fail. Let K be the complement of $J_{<\alpha} \cup (J_{<\alpha})'$ in $I_{<\alpha}$:

$$I_{<\alpha} = J_{<\alpha} \cup (J_{<\alpha})' \cup K.$$

Let Y and U be graded spaces with bases $\{y_\gamma\}_{\gamma\in K}$ and $\{u_\gamma\}_{\gamma\in J_{<\alpha}}$ such that $\deg y_\gamma = \deg x_\gamma$ and $\deg u_\gamma = \deg x_\gamma$.

Let $\Lambda(U \oplus dU)$ be the free c.g.d.a. over U (= contractible KS-complex) -cf. example 1.6. Set

$$W = U \oplus dU \oplus Y \quad \text{and} \quad S = U \oplus dU.$$

Then an algebra homomorphism

$$g : B \otimes W \to B \otimes A_{<\alpha}$$

is given by $g(b) = b$, $g(u_\gamma) = 1 \otimes x_\gamma$, $g(du_\gamma) = d_C(1 \otimes x_\gamma)$ and $g(y_\gamma) = 1 \otimes x_\gamma$.

2.12.- Lemma. g is an isomorphism.

Proof.- g is surjective. We need only show $1 \otimes x_\gamma \in \operatorname{Im} g$, $\gamma < \alpha$. Suppose for some $\lambda < \alpha$ with $\deg x_\lambda = p$ we know that $1 \otimes x_\gamma \in \operatorname{Im} g$ whenever $\deg x_\gamma < p$ or $\deg x_\gamma = p$ and $\gamma < \lambda$. Then we show $1 \otimes x_\lambda \in \operatorname{Im} g$, and the result follows by induction.

But $1 \otimes x_\lambda \in \operatorname{Im} g$ by definition, unless $\lambda = \mu'$, some $\mu \in J_{<\alpha}$. But then lemma 2.10 (iv) shows that $1 \otimes x_\lambda - g(du_\mu) \in \operatorname{Im} g$; i.e. ; $1 \otimes x_\lambda \in \operatorname{Im} g$.

<u>g is injective</u>. Define $\sigma : \Lambda W \xrightarrow{\cong} A_{<\alpha}$ by $\sigma u_\gamma = x_\gamma$, $\sigma(du_\gamma) = x_\gamma$, $\sigma(y_\gamma) = x_\gamma$. Then

$$\iota \otimes \sigma : B \otimes \Lambda W \xrightarrow{\cong} B \otimes A_{<\alpha} .$$

Moreover $(g-\iota\otimes\sigma)\,\Phi = 0$ if $\Phi \in B$ or U or Y. Now because $g(\Lambda W) \subset \bar{B} \otimes \Lambda W$ we can write

$$(g-\iota\otimes\sigma)\, du_\gamma = 1 \otimes \Phi + \Psi \quad , \quad \Psi \in B^+ \otimes A_{<\alpha} .$$

Moreover, $\Phi \in \ker \varepsilon_A$ and by (2.9)

$$\begin{aligned} \zeta_A\Phi &= \zeta_A\rho(g-\iota\otimes\sigma)du_\gamma \\ &= Q(d_A)\zeta_A x_\gamma - \zeta_A x_\gamma, \\ &= 0. \end{aligned}$$

It follows that

(2.13) $$\mathrm{Im}(g-\iota\otimes\sigma) \subset B^+ \otimes A_{<\alpha} + B \otimes \ker \zeta_A .$$

Now suppose g fails to be injective, and let $\Omega(\neq 0)$ be in $\ker g$. Write

$$\Omega = \sum b_i \otimes \Phi_i + \Psi ,$$

where $b_i \in B^p$ are linearly independent, $\Phi_i \in \Lambda^q W$ are non zero, and $\Psi \in \sum_{j>p} B^j \otimes \Lambda W + B^p \otimes \sum_{j>q} \Lambda^j W$.

In view of (2.13), and the fact that $(\iota\otimes\sigma)\Omega = (\iota\otimes\sigma-g)\Omega$, we have

$$(\iota\otimes\sigma)\Omega \in \sum_{j>p} B^j \otimes \Lambda X + B^p \otimes \sum_{j>q} \Lambda^j X .$$

Since this relation is also satisfied by $(\iota\otimes\sigma)\Psi$ it is satisfied by $\sum b_i \otimes \sigma\Phi_i$; hence for each i

$$\sigma\Phi_i \in \Lambda^q X \cap \sum_{j>q} \Lambda^j X = 0,$$

a contradiction.

Q.E.D.

2.14.- Corollary.- A differential, d, is induced in $B \otimes \Lambda W$ by $d = g^{-1} d_C g$. It coincides with d_B in B and with the originally defined d in $\Lambda(U \oplus dU)$.

2.15.- Remark.-Since $\bar{B} \otimes A_{<\alpha}$ is d_C-stable, g restricts to a homomorphism

$$\bar{g} : \bar{B} \otimes \Lambda W \to \bar{B} \otimes A_{<\alpha} ,$$

which, by the proof of lemma 2.12, is an isomorphism. In particular, $\bar{B} \otimes \Lambda W$ is d-stable, and $\bar{g}d = d_C\bar{g}$.

Now reorder I by putting $\beta << \gamma$ if $\deg x_\beta < \deg x_\gamma$ or if $\deg x_\beta = \deg x_\gamma$ and $\beta < \gamma$. This is a new well ordering.

2.16. Lemma.- $dy_\gamma \in B \otimes \Lambda S \otimes (\Lambda Y)_{<<\gamma}$. In particular

$$B \otimes \Lambda S \xrightarrow{i} B \otimes \Lambda S \otimes \Lambda Y \xrightarrow{\pi} \Lambda Y$$

is a minimal KS extension (with Y having the differential $\bar{d}$ given by $\bar{d}\pi = \pi d$.).

Proof.- We show first that for $\gamma \in I_{<\alpha}$

$$g^{-1}(1 \otimes x_\gamma) \in B \otimes \Lambda S \otimes (\Lambda Y)_{\underline{<<}\gamma}. \tag{2.17}$$

Suppose this is proved for all $\beta<\gamma$. If $\gamma \in J$ then $g^{-1}(1 \otimes x_\gamma) = u_\gamma$, and (2.17) is clear. If $\gamma \in K$, $g^{-1}(1 \otimes x_\gamma) = y_\gamma$ and (2.17) is clear. Suppose $\gamma = \mu'$, $\mu \in J_{<\alpha}$. Suppose $\deg x_\gamma = p+1$.

Then by lemma 2.10 (iv) $g^{-1}(1 \otimes x_\gamma)$ is in the subalgebra generated by du_μ, $\bar{B}$, elements of degree $\leq p$, elements of the form $g^{-1}(1 \otimes x_\lambda)$ with $\lambda << \gamma$, and elements in ΛS. All these elements are in $B \otimes \Lambda S \otimes (\Lambda Y)_{\underline{<<}\gamma}$.

Thus (2.17) is proved.

But now if $\gamma \in K$ then by lemma 2.10 (iii) $d_C(1 \otimes x_\gamma)$ is in the subalgebra generated by $\bar{B} \otimes A^{\leq p}_{<\gamma}$. Hence by (2.17),

$$dy_\gamma = g^{-1} d_C(1 \otimes x_\gamma)$$
$$\in B \otimes \Lambda S \otimes (\Lambda Y)_{<<\gamma}.$$

Q.E.D.

2.17. *Proof of lemma 2.10 cont'd.* Consider the extension of lemma 2.16. Since S is contractible, the inclusion $\bar{B} \hookrightarrow \bar{B} \otimes \Lambda S$ induces an isomorphism of cohomology. Thus we can apply prop. 1.11 to obtain a KS extension

(2.18) $$\bar{B} \to \bar{B} \otimes \Lambda Y \to \Lambda Y$$

and a homomorphism $h : \bar{B} \otimes \Lambda Y \to \bar{B} \otimes \Lambda S \otimes \Lambda Y$ (of. c.g.d.a.'s) such that $h(b) = b$, and

$$h(1 \otimes y_\gamma) - 1 \otimes y_\gamma \in \bar{B} \otimes \Lambda S \otimes (\Lambda Y)_{<<\gamma}$$

and

$$D(1 \otimes y_\gamma) \in \bar{B} \otimes (\Lambda Y)_{<<\gamma}.$$

(D is the differential in $\bar{B} \otimes \Lambda Y$). Moreover, h^* is an isomorphism.

In particular we can write

(2.19) $$d_C(1 \otimes x_\alpha) = (gh)\Phi + d_C\Omega ,$$

for some $\Omega \in \bar{B} \otimes A_{<\alpha}$ and $\Phi \in \bar{B} \otimes \Lambda Y$, with $D\Phi = 0$.

Now the extension 2.18 is minimal by the definition of $<<$. Hence obviously $Q(\bar{d}) = 0$, so we can apply lemma 2.6 to Φ. We obtain

$$\pi\Phi = \Phi_1 y_{\beta_1} + \dots + \Phi_n y_{\beta_n} + \Psi ,$$

where $y_{\beta_i} \in Y^{p+1}$ ($p = \deg x_\alpha$), $\beta_1 > \dots > \beta_n$, $\Psi \in \Lambda Y^{\leq p}$, and Φ_1 is a scalar.

We choose Φ and Ω so that either all the Φ_i are zero, or so that $\Phi_1 \neq 0$ and β_1 is as small as possible.

Next recall that $\zeta_A \rho(\Psi_1 \cdot \Psi_2) = 0$ for $\Psi_i \in \ker \varepsilon_C$.

This yields

$$\zeta_A \rho g\, h(1 \otimes y_\gamma) = \zeta_A x_\gamma + \sum_{\beta \in K_{<\gamma}} \lambda_\beta \zeta_A x_\beta + \sum_{\beta \in J_{<\alpha} \cup (J_{<\alpha})'} \lambda_\beta \zeta_A x_\beta \ .$$

Hence

$$(\zeta_A \rho\, g\, h)\Phi = \sum_{i=1}^{n} \varepsilon(\Phi_i)\, \zeta_A \rho\, g\, h(y_{\beta_i})$$

(2.20)
$$= \Phi_1 \cdot \zeta_A x_{\beta_1} + \sum_{\beta \in K_{<\beta_1}} \mu_\beta \zeta\, x_\beta + \sum_{\beta \in J_{<\alpha} \cup (J_{<\alpha})'} \mu_\beta \zeta\, x_\beta \ ,$$

if $\Phi_1 \neq 0$. (ε is the augmentation for ΛY.).

On the other hand, if all the Φ_i are zero then

(2.21)
$$(\zeta_A \rho\, g\, h)\Phi = 0.$$

Now define scalars σ_β $(\beta \in I_{<\alpha})$ by

$$\zeta_A \rho\ \Omega = \sum_{\beta \in I_{<\alpha}} \sigma_\beta\, \zeta_A x_\beta \ .$$

We define (finally) the element z_α by

$$z_\alpha = 1 \otimes x_\alpha - \Omega + \sum_{\beta \in I_{<\alpha}} 1 \otimes \sigma_\beta x_\beta \ .$$

Then i) and ii) are satisfied by definition. It remains to verify iii) if $\alpha \notin J$ and iv) if $\alpha \in J$.

<u>*Case 1 :* $\alpha \notin J$.</u> In this case by (2.9)

$$\zeta_A\ \rho\ d_C(1 \otimes x_\alpha) = Q(d_A)\zeta_A\ x_\alpha = 0.$$

Were $\Phi_1 \neq 0$ we would combine (2.20) and (2.19) to obtain

$$\Phi_1 \cdot \zeta_A x_{\beta_1} + \sum_{\beta \in K_{<\beta_1} \cup J_{<\alpha} \cup (J_{<\alpha})'} \mu_\beta \zeta_A x_\beta$$

$$= - Q(d_A)\zeta_A \rho \Omega$$

$$= - \sum_{\beta \in J_{<\alpha}} \sigma_\beta \zeta_A x_{\beta'} ,$$

which implies $\Phi_1 = 0$. Hence all the Φ_i are zero (since $\beta_1 \in K$ by definition)

This implies that $\Phi \in \bar{B} \otimes \Lambda Y^{\leq p}$ and so $(g\ h)\Phi$ is in the subalgebra generated by $\bar{B} \otimes A^{\leq p}_{<\alpha}$. On the other hand, since all the $\Phi_i = 0$, (2.21) and (2.19) imply that $Q(d_A)\zeta_A \rho \Omega = 0$, whence

$$\sigma_\beta = 0 \quad , \quad \beta \in J_{<\alpha}$$

Thus

$$d_C z_\alpha = d_C(1 \otimes x_\alpha - \Omega) + \sum_{\beta \notin J, \beta < \alpha} d_C(1 \otimes \sigma_\beta x_\beta).$$

The first term on the right is $gh(\Phi)$, while the second is also in the subalgebra generated by $\bar{B} \otimes A^{\leq p}_{<\alpha}$, by the induction hypothesis.

It follows that iii) holds for z_α.

<u>*Case 2 :* $\alpha \in J$.</u> In this case

$$\zeta_A \rho\, d_C(1 \otimes x_\alpha) = \zeta_A x_{\alpha'} \quad .$$

Thus were the Φ_i all to vanish we would obtain via (2.19) and (2.21) that $\zeta_A x_{\alpha'} = Q(d_A) \zeta_A \rho \Omega = \sum_{\beta \in J_{<\alpha}} \sigma_\beta \zeta_A x_{\beta'}$, which is impossible. It follows that $\Phi_1 \neq 0$.

Now (2.19) and (2.20) yield

$$\zeta_A x_{\alpha'} = \Phi_1 \zeta_A x_{\beta_1} + \sum_{\beta \in K_{<\beta_1} \cup J_{<\alpha} \cup (J_{<\alpha})'} \mu_\beta \zeta_A x_\beta + \sum_{\beta \in J_{<\alpha}} \sigma_\beta \zeta_A x_{\beta'} .$$

Since $\Phi_1 \neq 0$ we can conclude that $\Phi_1 = 1$ and $\beta_1 = \alpha'$.

Next observe that

$$d_C z_\alpha = (g\ h)\Phi + \sum_{\beta \in I_{<\alpha}} d_C(1 \otimes \sigma_\beta\ x_\beta).$$

Since $\Phi_1 = 1$, and $\beta_1 = \alpha'$,

$$\Phi - 1 \otimes y_{\alpha'} \in \bar{B} \otimes \Lambda Y^{\leq p} + 1 \otimes Y^{p+1}_{<\alpha'} \otimes \Lambda Y^o .$$

Hence

$$h\Phi - 1 \otimes y_{\alpha'} \in \bar{B} \otimes \Lambda S \otimes \left[\Lambda Y^{\leq p} + \Lambda Y^o \otimes Y^{p+1}_{<\alpha'}\right].$$

Hence

$$gh\ \Phi - 1 \otimes x_{\alpha'} \in E_\alpha .$$

Let $\beta \in I_{<\alpha}$. Then $d_C(1 \otimes x_\beta) \in E_\alpha$ by definition if $\beta \in J_{<\alpha}$, and $d_C(1 \otimes x_\beta) \in E_\alpha$ by lemma 2.10 iii). Otherwise (by induction). Hence $d_C z_\alpha - 1 \otimes x_{\alpha'} \in E_\alpha$, and iv) is proved. Q.E.D.

2.22. Proof of theorem 2.2.

Let z_α ($\alpha \in I$) be the elements of lemma 2.10. As at the start of the proof of lemma 2.10 we can change the definition of x_α ($\alpha \in I$) and of f so that

$$f(1 \otimes x_\alpha) = z_\alpha \qquad , \qquad \alpha \in I \quad ,$$

while retaining the conditions of the lemma and formula (2.9). We do this.

Let V and T be graded spaces with bases $\{v_\gamma\}_{\gamma \in H}$ and $\{e_\gamma\}_{\gamma \in J}$ respectively, with $\deg v_\gamma = \deg x_\gamma$ and $\deg e_\gamma = \deg x_\gamma$.

Let $\Lambda T \otimes \Lambda dT$ be the free c.g.d.a. over T and set

$$W = T \oplus dT \oplus V \text{ and } R = T \oplus dT.$$

Define an algebra homomorphism $g : B \otimes \Lambda W \to C$ by $g(b) = b$, $g(e_\gamma) = 1 \otimes x_\gamma$, $g(de_\gamma) = d_C(1 \otimes x_\gamma)$ and $g(v_\gamma) = 1 \otimes x_\gamma$.

Then lemma 2.12 shows that g is an isomorphism, while corollary 2.14 shows that the differentials d_B in B and d in ΛR extend to a differential d in $B \otimes \Lambda W$ such that $gd = d_C g$.

Reorder I by putting $\beta \ll \alpha$ if $\deg x_\beta < \deg x_\alpha$ or if $\deg x_\beta = \deg x_\alpha$ and $\beta < \alpha$. This is a second well ordering. It follows from lemma 2.16 that

$$dv_\gamma \in B \otimes \Lambda R \otimes (\Lambda V)_{\ll\gamma}, \qquad \gamma \in H.$$

Hence if we endow ΛV with the differential $\bar{d}$ given by $\pi d = \bar{d}\pi$, where $\pi : B \otimes \Lambda R \otimes \Lambda V \to \Lambda V$ is the projection, then

$$B \otimes \Lambda R \xrightarrow{j} B \otimes \Lambda R \otimes \Lambda V \xrightarrow{\pi} \Lambda V$$

is a minimal KS extension.

Since the inclusion $B \to B \otimes \Lambda R$ induces a cohomology isomorphism, prop. 1.11 yields a minimal extension

$$B \xrightarrow{\tilde{i}} B \otimes \Lambda V \xrightarrow{\tilde{\rho}} \Lambda V.$$

where, (if D is the differential in $B \otimes \Lambda V$)

$$D(1 \otimes v_\gamma) \in B \otimes (\Lambda V)_{\ll\gamma}, \qquad \gamma \in H.$$

We also obtain a homomorphism of c.g.d.a.'s

$$h : B \otimes \Lambda V \to B \otimes \Lambda R \otimes \Lambda V$$

such that

$$h(b) = b$$

and

$$h(1 \otimes v_\gamma) - 1 \otimes v_\gamma \in B \otimes \Lambda R \otimes (\Lambda V)_{\ll\gamma}, \qquad \gamma \in H.$$

Put $(B \otimes \Lambda V, D) = (\tilde{C}, d_{\tilde{C}})$ and $(\Lambda V, \bar{d}) = (\tilde{A}, d_{\tilde{A}})$.

Then the above minimal extension becomes

$$B \xrightarrow{\tilde{i}} \tilde{C} \xrightarrow{\tilde{\rho}} \tilde{A}.$$

Extend h to a homomorphism of c.g.d.a.'s

$$h : \tilde{C} \otimes \Lambda R \longrightarrow B \otimes \Lambda R \otimes \Lambda V$$

by putting $h(e_\gamma) = e_\gamma$, $e_\gamma \in T$. Lemma 1.7 implies that h is an isomorphism.

Finally, let $\varphi = g \circ h : \tilde{C} \otimes \Lambda R \xrightarrow{\cong} C$.

Since φ is the identity in B it carries the ideal generated by $\tilde{i}$ (ker ε_B) isomorphically to the ideal generated by i (ker ε_B). These ideals are respectively ker $\tilde{\rho} \otimes \Lambda R$ and ker ρ. Thus φ induces an isomorphism of c.g.d.a.'s $\psi : \tilde{A} \otimes \Lambda R \xrightarrow{\cong} A$ such that the diagram of the theorem commutes.

Q.E.D.

Chapter 3

The structure of a minimal extension

3.1.- *Introduction*.- In this chapter we consider a minimal KS extension

$$\mathcal{E} : B \xrightarrow{i} C \xrightarrow{\rho} A$$

with augmentations ε_B, ε_C, ε_A. We assume $H^0(C) = k$, and it follows that $H^0(B) = k$. We do not assume $H^0(A) = k$, and indeed this may fail to be the case.

We shall show how to decompose $\mathcal{E}$ into a countable family of elementary extensions natural with respect to morphisms of extensions.

3.2.- *The canonical filtrations*.- Define c.g.d.a.'s $C_{p,n}$ $(p \geq -1,\ n \geq 0)$ contained in C, and subspaces $Z_p^n \subset C^n$ $(p \geq 0,\ n \geq 0)$ inductively as follows :

i) $C_{-1,0} = B$

ii) $Z_p^n = d_C^{-1}(C_{p-1,n}) \cap (\ker \varepsilon_C)^n$

iii) $C_{p,n}$ = subalgebra generated by $C_{p-1,n}$ and Z_p^n, $p \geq 0$, $n \geq 0$.

iv) $C_{-1,n} = \bigcup_p C_{p,n-1}$, $n > 0$.

Thus for each n,

$$C_{-1,n} \subset C_{0,n} \subset \dots \subset C_{p,n} \subset \dots \subset C ,$$

and

$$Z_0^n \subset Z_1^n \subset \dots \subset Z_p^n \subset \dots \subset (\ker \varepsilon_C)^n .$$

We set

$$Z^n = \bigcup_p Z_p^n , \qquad n \geq 0.$$

Now recall the projection $\zeta_A : \ker \varepsilon_A \to Q(A)$ (from 1.1) and define subspaces

$Q^n_p(A) \subset Q^n(A)$ by

$$Q^n_p(A) = \zeta_A \, \rho(Z^n_p) \quad , \; p \geq 0, \; n \geq 0.$$

Thus

$$Q^n_o(A) \subset \ldots \subset Q^n_p(A) \subset \ldots \quad Q^n(A).$$

Next, for $p \geq 0$, $n \geq 0$ let $I_{p,n} \subset C_{p,n}$ be the ideal generated by $C_{p-1,n} \cap \ker \varepsilon_C$. Since $C_{p,n}$ is generated by Z^n_p and $C_{p-1,n}$, and since $d_C(Z^n_p) \subset C_{p-1,n}$, it follows that

$$d_C(C_{p,n}) \subset I_{p,n} .$$

Set

(3.3) $$A_{p,n} = C_{p,n}/I_{p,n} .$$

Then (giving $A_{p,n}$ the zero differential) we obtain a sequence of augmented c.g.d.a.'s

$$\mathcal{E}_{p,n} : C_{p-1,n} \xrightarrow{j} C_{p,n} \xrightarrow{\pi} A_{p,n} \quad , \; p \geq 0, \; n \geq 0.$$

The main goal of this chapter is

3.4.- *Theorem*.- The extension $\mathcal{E}$ admits a structure (X,f) in which each X^n is decomposed as a direct sum $X^n = \sum_{p\geq 0} X^n_p$, such that with respect to (X,f) :

i) $Z^n_p = C^n_{p-1,n} \cap \ker \varepsilon_C \oplus (1 \otimes X^n_p)$, $n \geq 0$, $p \geq 0$.

ii) $C_{p,n} = B \otimes \Lambda(X^{<n} \oplus X^n_{\leq p})$, $n \geq 0$, $p \geq -1$.

iii) $\zeta_A : X^n_{\leq p} \xrightarrow{\cong} Q^n_p(A)$, $n \geq 0$, $p \geq 0$.

Before proving the theorem we establish some consequences.

3.5.- *Corollary*.- The sequences $\mathcal{E}_{p,n}$ are elementary extensions.

Proof.- Use the structure of theorem 3.4 to write $C_{p,n} = C_{p-1,n} \otimes \Lambda X_p^n$, $p \geq 0$, $n \geq 0$. Since $1 \otimes X_p^n \subset Z_p^n$ we have

$$d_C(1 \otimes X_p^n) \subset C_{p-1,n}.$$

Moreover $I_{p,n} = (C_{p-1,n} \cap \ker \varepsilon_C) \otimes \Lambda X_p^n$. This yields the commutative diagrams

$$(3.6) \qquad \begin{array}{ccccc} & \nearrow & C_{p-1,n} \otimes \Lambda X_p^n & \longrightarrow & \Lambda X_p^n \\ C_{p-1,n} & & \downarrow \cong & & \downarrow \cong \\ & \underset{j}{\searrow} & C_{p,n} & \xrightarrow[\pi]{} & A_{p,n} \end{array}$$

Q.E.D.

3.7.- *Corollary*.- Fix $n \geq 0$ and $p \geq -1$. Let $Q^n_{-1}(A) = 0$. Then the following are equivalent :

i) $Q^n_{p+1}(A) = Q^n_p(A)$.

ii) $Q^n(A) = Q^n_p(A)$.

iii) $Z^n_{p+1} \subset C_{p,n}$.

iv) $C_{-1,n+1} = C_{p,n}$.

Proof.- In view of theorem 3.4 we have iv) <=> ii) and iii) <=> i). Clearly ii) => i). On the other hand, if iii) holds then by definition, $C_{p+1,n} = C_{p,n}$, and so $Z^n_{p+2} = Z^n_{p+1}$. Continuing this way we obtain iv).

Q.E.D.

Now observe that d_C restricts to a linear map

$$\bar{d}_C : Z_p^n \longrightarrow C^{n+1}_{p-1,n} \cap \ker d_C.$$

This map induces (in the obvious way) a linear map

$$\alpha_{p,n} : Z_p^n \Big/ C^n_{p-1,n} \cap \ker \varepsilon_C \longrightarrow \mathrm{Ker}(H^{n+1}(C_{p-1,n}) \to H^{n+1}(C)).$$

On the other hand the inclusion $C^n \cap \ker d_C \to Z_o^n$ $(n > 0)$ induces a linear map

$$\beta_n : \mathrm{Coker}(H^n(C_{-1,n}) \to H^n(C)) \to Z_o^n \Big/ C_{-1,n}^n$$

To simplify notations write

$$\mathrm{Ker}_p^{n+1} = \mathrm{Ker}(H^{n+1}(C_{p-1,n}) \to H^{n+1}(C)) \qquad , \quad n \geq 0, \quad p \geq 0$$

and

$$\mathrm{Coker}^n = \mathrm{Coker}(H^n(C_{-1,n}) \to H^n(C)) \qquad , \quad n > 0.$$

3.8.- *Proposition.*-

i) The homomorphism $H(C_{-1,n}) \to H(C)$ is an isomorphism in degrees less than n and injective in degree n. If $n = 0$ it is an isomorphism in degree n.

ii) The homomorphism $H(C_{p,n}) \to H(C)$ is an isomorphism in degrees $\leq n$, if $p > 0$.

iii) The sequences

$$0 \to \mathrm{Coker}^n \xrightarrow{\beta} Z_o^n \Big/ C_{-1,n}^n \xrightarrow{\alpha} \mathrm{Ker}_0^{n+1} \longrightarrow 0 \qquad , \quad n > 0$$

are short exact.

iv) The linear maps

$$\alpha_{p,n} : Z_p^n \Big/ C_{p-1,n}^n \cap \ker \varepsilon_C \xrightarrow{\cong} \mathrm{Ker}_p^{n+1}$$

are isomorphisms if $p > 0$ or if $p = 0$ and $n = 0$.

Proof.- Theorem 3.4 ii) shows that $C_{p,n}^q = C^q$ if $q < n$. It follows that $H(C_{p,n}) \to H(C)$ is an isomorphism in degrees $< n$ and injective in degree n. Moreover, by definition if $n > 0$

$$C^n \cap \ker d_C \subset Z_o^n \subset C_{p,n} \quad , \quad p \geq 0 .$$

Thus for $p > 0$, $H^n(C_{p,n}) \xrightarrow{\cong} H^n(C)$. This proves i) and ii).

Finally, iii) and iv) follow at once from the definitions.

Q.E.D.

3.9.- *Corollary*.- The homomorphism $i^* : H(B) \to H(C)$ is n regular (i.e., an isomorphism in degrees $\leq n$ and injective in degree $n+1$) if and only if A is n-connected.

Proof.- If A is n-connected then $B^p = C^p$ $(p \leq n)$, whence i^* is n-regular.

Conversely, suppose i^* is n-regular. We show that $B = C_{-1,n+1}$. In view of theorem 3.4 ii) this implies A is n-connected.

Indeed if we know $B = C_{-1,m}$ (some $m \leq n$) then our hypothesis implies $\mathrm{Ker}_o^{m+1} = 0$ and $\mathrm{Coker}^m = 0$. Thus by prop. 3.8 iii) and iv)

$$Z_0^m \subset C_{-1,m} .$$

Hence by cor. 3.7 $B = C_{-1,m} = C_{-1,m+1}$. Q.E.D.

3.10.- *Corollary*.- The structure (X,f) of theorem 3.4 can be chosen so that $\wedge X^+$ is d_A stable. In particular, since $d_A(X^o) = 0$ we obtain

$$H(A) \cong \wedge X^o \otimes H(\wedge X^+, d_A) .$$

Proof.- Let (Y,g) be a structure satisfying the conclusions of the theorem. Let $E \subset B \otimes \wedge Y^o$ be a sub c.g.d.a. such that E is connected and $H(E) = H(B \otimes \wedge Y^o)$. Use the procedure of prop. 1.11 to construct a new structure (X,f) such that :

i) $X^o = Y^o$ and $f(1 \otimes x) = g(1 \otimes x)$, $x \in X^o$.

ii) There are isomorphisms of bigraded spaces,

$\sigma : X^n_p \xrightarrow{\cong} Y^n_p$ $(n > 0)$ such that

$$x - \sigma(x) \in Y^{<n} \oplus Y^n_{<p} \quad , \quad x \in X^n_p \quad ,$$

and

$$f(1 \otimes x) - g(1 \otimes \sigma x) \in g[B \otimes \Lambda(Y^{\leq n} \oplus Y^n_{<p})] \quad , \quad x \in X^n_p \; .$$

iii) If we use f to write $C = B \otimes \Lambda X^o \otimes \Lambda X^+$, then $E \otimes \Lambda X^+$ is d_C-stable.

Then $f(B \otimes \Lambda X^{<n} \otimes \Lambda X^n_{\leq p}) = g(B \otimes \Lambda Y^n \otimes \Lambda Y^n_{\leq p}) = C_{p,n}$ $(p \geq -1,\ n \geq 0)$,

and

$$f(1 \otimes X^n_p) \subset g(1 \otimes Y^n_p) \oplus C^n_{p-1,n} = Z^n_p \; , \quad n > 0.$$

It follows that i) and ii) of theorem 3.4 are satisfied by (X,f). Part iii) follows at once from i).

Moreover, if $x \in X^+$ then

$$d_C\, f(1 \otimes x) \in f(E \otimes \Lambda X^+) = f(1 \otimes \Lambda X^+) + f(E^+ \otimes \Lambda X^+)$$

$$\subset f(1 \otimes \Lambda X^+) + f(B^+ \otimes \Lambda X^o \otimes \Lambda X^+).$$

It follows that $d_A x \in \rho f(1 \otimes \Lambda X^+) = \Lambda X^+$. Q.E.D.

3.11.- <u>*Corollary*</u>.- Assume $H(B)$ is 1-connected, and that i^* is 1-regular. Then

$$Q^n(A) = Q^n_o(A) \quad , \quad n \geq 0.$$

<u>*Proof*</u>.- First note that A is 1-connected by cor.3.9. Let $\bar{B} \subset B$ be a 1-connected sub c.g.d.a. such that $H(\bar{B}) = H(B)$. Using the method of prop. 1.11, choose a structure (X,f) satisfying the conclusions of theorem 3.4 so that $\bar{B} \otimes \Lambda X$ is d_C-stable.

Then $\bar{B} \otimes \Lambda X^{\leq n}$ is 1-connected. Hence for $x \in X^n$ $d_C(1 \otimes x)$ is a polynomial in elements x_i with $2 \leq \deg x_i \leq n$ and elements $b_i \in \bar{B}$ with $2 \leq \deg b_i$. Since $\deg d_C(1 \otimes x) = n+1$ this yields

$$d_C(1 \otimes x) \in \bar{B} \otimes \Lambda X^{\leq n-1} .$$

Hence $1 \otimes X^n \subset Z_o^n$ and so

$$Q^n(A) = \zeta_A X^n = \zeta_A \rho\, Z_o^n = Q_o^n(A).$$

Q.E.D.

3.12.- *Remark*.- The extension E is called nilpotent if $\dim Q^n(A) < \infty$, $n = 0,1,2,\dots$. In view of cor. 3.7 this is equivalent to

(3.13) $\dim Q_p^n(A) < \infty$ all $n \geq 0$, $p \geq 0$

and

(3.14) For each $n \geq 0$ there is some $p \geq 0$ such that

$$Q_{p+1}^n(A) = Q_p^n(A).$$

If the hypotheses of cor. 3.11 hold then (3.14) is automatic and nilpotence is equivalent to (3.13).

3.15.- *Corollary*.- Suppose $H(B)$ has finite type. Assume A is connected and (3.14) holds. Then $H(C)$ has finite type if and only if E is nilpotent.

Thus if the hypotheses of cor. 3.11 hold (and $H(B)$ has finite type) then $H(C)$ has finite type if and only if E is nilpotent.

Proof.- Consider the elementary extensions $E_{p,n}$. Diagram (3.6) shows that $Q(A_{p,n}) \cong Q_p^n(A) \,/\, Q_{p-1}^n(A)$.

Now suppose $\dim Q_p^n(A) < \infty$ for all n and p. Then so does $Q(A_{p,n})$, and hence $A_{p,n}$ has finite type. It follows that if $H(C_{p-1,n})$ has finite type, so does $H(C_{p,n})$. Since (3.14) holds, cor 3.7 implies that $C_{-1,n+1} = C_{p,n}$ some p. Now by induction we obtain that $H(C_{p,n})$ always has finite type. But by prop. 3.8,

$$H^n(C_{-1,n+1}) = H^n(C).$$

Hence $H(C)$ has finite type.

Conversely, assume $H(C)$ has finite type. If $H(C_{p-1,n})$ has finite type then prop. 3.8 (iii) and iv)) shows that

(3.16) $$\dim Z_p^n \Big/ C_{p-1,n}^n \cap \ker \varepsilon_C < \infty .$$

Hence (cf. Theorem 3.4) $\dim Q_p^n(A)/Q_{p-1}^n(A) < \infty$, and so $\dim Q(A_{p,n}) < \infty$. It follows in this way that $H(C_{p,n})$ has finite type. Since $C_{-1,n+1} = C_{p,n}$, some p, $H(C_{-1,n+1})$ also has finite type. Thus each $H(C_{p,n})$ has finite type, and so each $Q_p^n(A)/Q_{p-1}^n(A)$ has finite dimension.

Q.E.D.

We turn now to the proof of theorem 3.4. It proceeds via several lemmas.

3.17.- *Lemma*.- Let (X,f) be a structure for E and let $\{x_\alpha\}_{\alpha\in I}$ be a well ordered homogeneous basis for X such that (1.4) holds and $\deg x_\alpha < \deg x_\beta \Rightarrow \alpha < \beta$. Then

i) $1 \otimes X^n \subset Z^n$, $n \geq 0$

ii) $B \otimes \Lambda X^{\leq n} = C_{-1,n+1}$, $n \geq -1$.

iii) $\bigcup_p Q_p^n(A) = Q^n(A)$, $n \geq 0$.

Proof.-

i) Assume $1 \otimes x_\beta \in Z^{n_\beta}$, $\beta < \alpha$, where $n_\beta = \deg x_\beta$. Let $n = \deg x_\alpha$. Then $d_C(1 \otimes x_\alpha)$ is a polynomial in elements from B and elements $1 \otimes x_{\beta_i}$ $(i= 1,\dots,m)$ with $\deg x_{\beta_i} = n_i \leq n$. By hypothesis for some p_i $(i=1,\dots,m)$ and $p > p_i$,

$$1 \otimes x_{\beta_i} \in Z_{p_i}^{n_i} \subset C_{p,n}$$

Hence $d_C(1 \otimes x_\alpha) \in C_{p,n}$, and so $1 \otimes x_\alpha \in Z_{p+1}^n \subset Z^n$.

ii) $C_{-1,n+1}$ is generated by B and $\sum_{m\leq n} Z^m$.
Since $Z^m \subset C^m \subset B \otimes \Lambda X^{\leq m} \subset B \otimes \Lambda X^{\leq n}$, we have

$$C_{-1,n+1} \subset B \otimes \Lambda X^{\leq n}.$$

The reverse inclusion follows from i).

iii) Apply i).

Q.E.D.

Now suppose (Y,g) is a structure for E, and $\{y_\alpha\}_{\alpha\in I}$ is a basis for Y such that the hypotheses of lemma 3.17 hold.
If $\deg y_\alpha = n$ we will say $\alpha \in I_{p,n}$ if p is the least integer such that for some scalars λ_β $(\beta < \alpha)$

$$\zeta_A\, y_\alpha + \sum_\beta \lambda_\beta\, \zeta_A\, y_\beta \in Q_p^n(A).$$

Lemma 3.17 iii) shows that

$$I = \bigcup_{p,n} I_{p,n} \qquad ; \tag{3.18}$$

this union is disjoint by definition. We define Y_p^n to be the span of the y_α with $\alpha \in I_p^n$.

3.19.- *Lemma*.- For each $p \geq -1$ and $n \geq 0$ an isomorphism

$$g_{p,n} : C_{p,n} \otimes \Lambda(Y^n_{>p} \oplus Y^{>n}) \xrightarrow{\cong} C$$

is given by $g_{p,n}(\Phi \otimes \Psi) = \Phi \cdot g(1 \otimes \Psi)$. Moreover, with respect to this isomorphism

$$d_C(1 \otimes y_\alpha) \in C_{p,n} \otimes (\Lambda(Y^n_{>p} \oplus Y^{>n}))_{<\alpha} \quad ,$$

if $y_\alpha \in Y^n_{>p} \oplus Y^{>n}$.

Proof.- By induction. When $p = -1$, the statement follows at once from lemma 3.17 ii). Now we assume it holds for some pair $(p-1,n)$, and prove it for (p,n).

To simplify notation denote $Y^n_{\geq p} \oplus Y^{>n}$ by U, and write

$$\bigcup_{q\geq p} I_{q,n} \cup \bigcup_{\substack{m>n \\ \text{all } q}} I_{q,m} = J.$$

Then $\{y_\alpha\}_{\alpha\in J}$ is a basis for U. Use $g_{p-1,n}$ to identify

(3.20) $$C_{p-1,n} \otimes \Lambda U = C.$$

Let $(E_{p-1,n}, d)$ be the c.g.d.a. defined by

$$E_{p-1,n} = C \Big/ \text{ideal generated by } C_{p-1,n} \cap \ker \varepsilon_C.$$

Then by hypothesis

(3.21) $$d_C(1 \otimes y_\alpha) \in C_{p-1,n} \otimes (\Lambda U)_{<\alpha} \quad , \quad \alpha \in J$$

and so

(3.22) $$C_{p-1,n} \xrightarrow{\ell} C \xrightarrow{\eta} E_{p-1,n}$$

is a minimal extension.

3.23.- Sublemma.- Let $\Phi \in Z^n_p$ and assume $\Phi \notin C^n_{p-1,n}$. With the notation above, for some $\alpha \in J$,

$$\Phi = \lambda(1 \otimes y_\alpha) + \Psi,$$

where λ is a non-zero scalar and $\Psi \in C_{p-1,n} \otimes (\Lambda U)_{<\alpha}$.

Proof.- We distinguish two cases.

Case 1 : n=0. Choose the least α such that $\Phi \in C_{p-1,0} \otimes (\Lambda U)_{\leq\alpha}$. Write

$$C_{p-1,0} \otimes (\Lambda U)_{\leq\alpha} = C_{p-1,0} \otimes (\Lambda U)_{<\alpha} \otimes \Lambda y_\alpha$$

and note that by (3.21) $C_{p-1,0} \otimes (\Lambda U)_{<\alpha}$ is d_C-stable and contains $d_C(1 \otimes y_\alpha)$.

Now Φ has the form

$$\Phi = \sum_{i=0}^{m} \Phi_i \otimes y_\alpha^i \quad , \qquad \Phi_i \in C_{p-1,0}^{o} \otimes (\Lambda U^o)_{<\alpha}$$

where $\Phi_m \neq 0$ and $m \geq 1$. (Note that each Φ_i has degree zero).

Applying d_C we find that

$$d_C \Phi_m \otimes y_\alpha^m + \sum_{i=0}^{m-1} \left[d_C \Phi_i + (i+1)\Phi_i \; d_C(1 \otimes y_\alpha) \right] \otimes y_\alpha^i \in C_{p-1,0} .$$

This implies that $d_C \Phi_m = 0$ and so (because $H^o(C) = k$) Φ_m is a non-zero scalar, λ. Were $m > 1$ we would also have

$$d_C(\Phi_{m-1} + m \; \lambda \otimes y_\alpha) = 0$$

which would imply $\Phi_{m-1} + m \; \lambda \; \otimes y_\alpha \in k$; i.e., $\lambda = 0$. Thus $m = 1$ and so $\Phi = \lambda(1 \otimes y_\alpha) + \Phi_0$ with $\Phi_0 \in C_{p-1,0} \otimes (\Lambda U)_{<\alpha}$.

<u>*Case 2* : $n > 0$</u>. Again choose the least α such that $\Phi \in C_{p-1,n} \otimes (\Lambda U)_{\leq\alpha}$. Then (for degree reasons)

$$\Phi = \Phi_o \otimes y_\alpha + \Psi \quad ,$$

where Φ_o and Ψ belong to $C_{p-1,n} \otimes (\Lambda U)_{<\alpha}$, and Φ_o is a non-zero element of degree zero.

As in case 1 this yields

$$(d_C \Phi_o) \otimes y_\alpha + \Phi_o \cdot d_C(1 \otimes y_\alpha) + \Psi \in C_{p-1,n} \, ,$$

whence $d_C \Phi_o = 0$ and so Φ_o is a non zero scalar.

Q.E.D.

3.24.- <u>*Proof of Lemma 3.19 cont'd.*</u>- Let K be the set of indices $\alpha \in J$ such that for some $\Psi_\alpha \in C_{p-1,n} \otimes (\Lambda U)_{<\alpha}$,

$$1 \otimes y_\alpha + \Psi_\alpha \in Z_p^n .$$

Denote $1 \otimes y_\alpha + \Psi_\alpha$ by z_α and set $w_\alpha = \eta\, z_\alpha$. We extend the definition by setting $w_\alpha = y_\alpha$ ($\alpha \in J$ but $\alpha \notin K$).

By lemma 1.7 a new structure (W,h) for the extension (3.22) is given as follows : W has as basis the w_α, $\alpha \in J$ and

$$h(1 \otimes w_\alpha) = \begin{cases} z_\alpha & , \alpha \in K \\ 1 \otimes y_\alpha & , \text{otherwise.} \end{cases}$$

Moreover, we have

$$(3.25) \qquad d_C h(1 \otimes w_\alpha) = d_C z_\alpha \in C_{p-1,n} \quad , \quad \alpha \in K,$$

and

$$(3.26) \qquad d_C h(1 \otimes w_\alpha) = d_C(1 \otimes y_\alpha) \in C_{p-1,n} \otimes (\Lambda U)_{<\alpha}, \text{ otherwise.}$$

Let W_p^n be the span of the w_α ($\alpha \in K$) and let $\tilde{U}$ be the span of the y_α ($\alpha \in J$ but $\alpha \notin K$). Then

$$W = W_p^n \oplus \tilde{U}.$$

Since $h : C_{p-1,n} \otimes (\Lambda W)_{<\alpha} \xrightarrow{\cong} C_{p-1,n} \otimes (\Lambda U)_{<\alpha}$, $\alpha \in J$; and since

$$(\Lambda W)_{<\alpha} \subset \Lambda W_p^n \otimes (\Lambda \tilde{U})_{<\alpha} \quad , \quad \alpha \in J ,$$

it follows from (3.26) that

$$(3.27) \qquad d_C\, h(1 \otimes y_\alpha) \in h(C_{p-1,n} \otimes \Lambda W_p^n \otimes (\Lambda \tilde{U})_{<\alpha}), \quad \alpha \in J \text{ but } \alpha \notin K.$$

Next observe that sublemma 3.23 yields

$$(3.28) \qquad Z_p^n = C_{p-1,n}^n \cap \ker \varepsilon_C \oplus h(1 \otimes W_p^n).$$

Since $C_{p-1,n}$ is generated by B and $Z_{p-1}^n \oplus \sum_{m<n} Z^m$, we obtain

(3.29) $$\zeta_A\, \rho(C^n_{p-1,n} \cap \ker \varepsilon_C) = Q^n_{p-1}(A) .$$

We show now that

$$K = I_{p,n}$$

First suppose $\alpha \in K$. Then $z_\alpha = 1 \otimes y_\alpha + \Psi_\alpha \in Z^n_p$, where $\Psi_\alpha \in C_{p-1,n} \otimes (\Lambda U)_{<\alpha}$. Since, clearly, $\Psi_\alpha \in \ker \varepsilon_C$ (3.29) implies that

$$\zeta_A\, y_\alpha + \sum_{\beta<\alpha} \lambda_\beta\, \zeta_A\, y_\beta \in Q^n_p(A) .$$

Hence $\alpha \in \bigcup_{q \leq p} I_{q,n}$. But also $\alpha \in J$, whence $\alpha \in I_{p,n}$:

$$K \subset I_{p,n}.$$

On the other hand, suppose $\alpha \in I_{p,n}$. Then

$$\zeta_A\, y_\alpha + \sum_{\beta<\alpha} \lambda_\beta\, \zeta_A\, y_\beta \in Q^n_p(A).$$

It follows from (3.28) and (3.29) that

$$Q^n_p(A) = \zeta_A\, \rho\, h(1 \otimes W^n_p) + Q^n_{p-1}(A).$$

Moreover, for $\gamma \in K$,

$$\begin{aligned}\zeta_A\, \rho\, h(1 \otimes w_\gamma) &= \zeta_A\, \rho(1 \otimes y_\gamma) + \zeta_A \rho\, \Psi_\gamma \\ &= \zeta_A\, y_\gamma + \sum_{\mu<\gamma} \sigma_{\gamma\mu}\, \zeta_A\, y_\mu + \zeta_A\, \Omega_\gamma ,\end{aligned}$$

where $\zeta_A\, \Omega_\gamma \in Q^n_{p-1}(A)$.

This yields

$$\zeta_A\, y_\alpha + \sum_{\beta<\alpha} \lambda_\beta\, \zeta_A\, y_\beta + \sum_{\gamma\in K} \tau_\gamma \{\zeta_A y_\gamma + \sum_{\mu<\gamma} \sigma_{\gamma\mu}\, \zeta_A y_\mu\} \in Q^n_{p-1}(A).$$

Suppose $\alpha \notin K$. If $\alpha > \gamma$ for each $\gamma \in K$ such that $\tau_\gamma \neq 0$, then this equation shows that

$$\zeta_A y_\alpha + \sum_{\beta<\alpha} \lambda'_\beta \zeta_A y_\beta \in Q^n_{p-1}(A).$$

This would give $\alpha \in \bigcup_{q<p} I_{q,n}$, which is impossible.

But if $\gamma > \alpha$ for some $\gamma \in K$ for which $\tau_\gamma \neq 0$ we conclude in the same way that the largest such γ is in $\bigcup_{q<p} I_{q,n}$, which is equally impossible. Hence $\alpha = \gamma$, some $\gamma \in K$; i.e.

$$I_{p,n} \subset K.$$

Since $K = I_{p,n}$,

$$\tilde{U} = Y^n_{>p} \oplus Y^{>n}.$$

Moreover, (3.28) shows that h carries $C_{p-1,n} \otimes \Lambda W^n_p$ (isomorphically) onto $C_{p,n}$. If we denote this restricted isomorphism by $\bar{h}$ then

$$g_{p,n} = h \circ (\bar{h} \otimes \iota)^{-1} : C_{p,n} \otimes \Lambda(Y^n_{>p} \oplus Y^{>n}) \to C.$$

Hence $g_{p,n}$ is an isomorphism.

Finally, if $y_\alpha \in Y^n_{>p} \oplus Y^{>n}$ then

$$g_{p,n}(1 \otimes y_\alpha) = h(1 \otimes y_\alpha),$$

and so (3.27) reads

$$d_C\, g_{p,n}(1 \otimes y_\alpha) \in g_{p,n}(C_{p,n} \otimes [\Lambda(Y^n{}_p \oplus Y^{>n})]_{<\alpha}).$$

3.30.- *Proof of theorem 3.4.*- Consider the spaces W^n_p constructed in sec. 3.24 above and set

$$L^n_p = h(1 \otimes W^n_p) \subset C, \qquad n \geq 0, \quad p \geq 0,$$

(where h is the isomorphism depending on p and n defined in sec. 3.24).

Then formula (3.28) reads

$$Z_p^n = C^n_{p-1,n} \cap \ker \varepsilon_C \oplus L_p^n \,. \tag{3.31}$$

It follows that the spaces L_p^n are linearly independent ; denote their direct sum by

$$L^n = \sum_{p\geq 0} L_p^n \subset C^n \cap \ker \varepsilon_C .$$

Let $L = \sum_{n\geq 0, p\geq 0} L_p^n \subset C$ (direct sum). The inclusion $L \hookrightarrow C$ induces an obvious homomorphism

$$\varphi : B \otimes \Lambda L \to C$$

We show that φ restricts to isomorphisms

$$\varphi_{p,n} : B \otimes \Lambda(L^{<n} \oplus L^n_{\leq p}) \xrightarrow{\cong} C_{p,n}, \qquad \begin{array}{l} p\geq -1, \\ n\geq 0. \end{array}$$

When $p = -1$ and $n = 0$ this is true by definition.
If it holds for some n and all p then by direct limits it holds for $\varphi_{-1,n+1}$. Thus we may assume $p \geq 0$ and that $\varphi_{p-1,n}$ is an isomorphism, and have only to prove that $\varphi_{p,n}$ is.

But in the notation of sec 3.24 we have an isomorphism

$$\bar{h} : C_{p-1,n} \otimes \Lambda W_p^n \xrightarrow{\cong} C_{p,n} \,.$$

It restricts to an isomorphism

$$\psi : W_p^n \xrightarrow{\cong} L_p^n .$$

Moreover, the diagram

$$\begin{array}{ccc} B \otimes \Lambda(L^{<n} \oplus L^n_{<p}) \otimes \Lambda L_p^n & & \\ \Big\downarrow \varphi_{p-1,n} \otimes \Lambda\psi^{-1} & \searrow^{\varphi_{p,n}} & \\ C_{p-1,n} \otimes \Lambda W_p^n & \xrightarrow[\bar{h}]{\hspace{3em}} & C_{p,n} \end{array}$$

commutes. Hence $\psi_{p,n}$ is an isomorphism.

Since each $\psi_{p,n}$ is an isomorphism, so is ψ. Thus ρ restricts to an isomorphism $\bar{\rho} : \Lambda L \xrightarrow{\cong} A$, which carries $\Lambda^+ L$ to $\ker \varepsilon_A$. Define $X = \bar{\rho}(L)$ and $X_p^n = \bar{\rho}(L_p^n)$.

Define f by

$$f(1 \otimes x) = \psi(1 \otimes \bar{\rho}^{-1}x) \quad , \quad x \in X.$$

Then f restricts to isomorphisms

$$f : B \otimes \Lambda(X^{<n} \oplus X^n_{\leq p}) \xrightarrow{\cong} C_{p,n} \quad , \quad \begin{array}{l} p \geq -1, \\ n \geq 0. \end{array}$$

Moreover formula (3.31) reads

$$Z_p^n = (\ker \varepsilon_C \cap C^n_{p-1,n}) \oplus f(1 \otimes X_p^n).$$

Thus parts i) and ii) of theorem 3.4 are proved. Part iii) follows at once from i).

Q.E.D.

Chapter 4

Morphisms of extensions

4.1.- *Introduction*.- In this chapter we consider a morphism $(\psi,\varphi,\alpha) : \check{E} \to E$ between KS extensions :

$$\check{E} : \check{B} \xrightarrow{\check{i}} \check{C} \xrightarrow{\check{\rho}} \check{A} \quad \text{and} \quad E : B \xrightarrow{i} C \xrightarrow{\rho} A.$$

We assume $H^o(\check{C}) = k = H^o(C)$. All augmentations are denoted by ε.

Note that a linear map $Q(\alpha) : Q(\check{A}) \to Q(A)$ is defined by

$$Q(\alpha) \circ \zeta_{\check{A}} = \zeta_A \circ \alpha.$$

It satisfies $Q(\alpha) \circ Q(d_{\check{A}}) = Q(d_A) \circ \alpha$.

Henceforth we assume that $\check{E}$ and E are minimal, and we use the notation developed in chapter 3. Then our morphism induces morphisms $\check{E}_{p,n} \to E_{p,n}$ $(p \geq 0,\ n \geq 0)$, written

$$\begin{array}{ccccc} \check{C}_{p-1,n} & \longrightarrow & \check{C}_{p,n} & \longrightarrow & \check{A}_{p,n} \\ \downarrow \varphi_{p-1,n} & & \downarrow \varphi_{p,n} & & \downarrow \alpha_{p,n} \\ C_{p-1,n} & \longrightarrow & C_{p,n} & \longrightarrow & A_{p,n} \end{array},$$

where $\varphi_{p,n}$ is the restriction of φ.

Moreover φ restricts to linear maps $\check{Z}_p^n \to Z_p^n$, and hence induces linear maps

$$\varphi_Z : Z_p^n / \check{C}^n_{p-1,n} \cap \ker \varepsilon \longrightarrow Z_p^n / C^n_{p-1,n} \cap \ker \varepsilon, \quad n \geq 0, \quad p \geq 0.$$

Further note that $\psi^*_{p-1,n}$ restricts to a linear map

$$\psi_{Ker} : \check{Ker}^{n+1}_p \longrightarrow Ker^{n+1}_p \,, \quad n \geq 0, \quad p \geq 0,$$

while ψ^* factors to give a linear map

$$\psi_{Coker} : \check{Coker}^n \longrightarrow Coker^n \,, \quad n > 0.$$

Clearly the diagrams

$$(4.2) \qquad \begin{array}{ccccccccc} 0 & \longrightarrow & \check{Coker}^n & \longrightarrow & \check{Z}^n_o / \check{C}^n_{-1,n} & \longrightarrow & \check{Ker}^{n+1}_o & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle \psi_{Coker}} & & \downarrow{\scriptstyle \psi_Z} & & \downarrow{\scriptstyle \psi_{Ker}} & & \\ 0 & \longrightarrow & Coker^n & \longrightarrow & Z^n_o / C^n_{-1,n} & \longrightarrow & Ker^{n+1}_o & \longrightarrow & 0, \quad n > 0, \end{array}$$

commute, as do the diagrams

$$(4.3) \qquad \begin{array}{ccc} \check{Z}^n_p / \check{C}^n_{p-1,n} \cap Ker\,\varepsilon & \xrightarrow{\cong} & \check{Ker}^{n+1}_p \\ \downarrow{\scriptstyle \psi_Z} & & \downarrow{\scriptstyle \psi_{ker}} \\ Z^n_p / C^n_{p-1,n} \cap Ker\,\varepsilon & \xrightarrow{\cong} & Ker^{n+1}_p \,, \end{array} \qquad \begin{array}{l} p > 0 \text{ or} \\ p=0,\ n=0. \end{array}$$

Finally, observe that the linear maps

$$\zeta_A \rho : Z^n_p \longrightarrow Q^n_p(A)$$

factor to yield isomorphisms

$$\eta_A : Z^n_p / C^n_{p-1,n} \cap Ker\,\varepsilon \xrightarrow{\cong} Q^n_p(A) / Q^n_{p-1}(A)$$

Clearly $Q(\alpha)$ restricts to linear maps

$$Q_p^n(\alpha) : Q_p^n(\check{A}) \to Q_p^n(A).$$

Hence it induces maps

$$\overline{Q(\alpha)} : Q_p^n(\check{A}) \Big/ Q_{p-1}^n(\check{A}) \longrightarrow Q_p^n(A) \Big/ Q_{p-1}^n(A) ,$$

$n \geq 0$, $p \geq 0$. (Note : we set $Q_{-1}^n(A) = 0 = Q_{-1}^n(\check{A})$!) the diagrams

$$(4.4) \qquad \begin{array}{ccc} \check{Z}_p^n \Big/ \check{C}_{p-1,n}^n \cap \ker \varepsilon & \xrightarrow{\cong} & Q_p^n(\check{A}) \Big/ Q_{p-1}^n(\check{A}) \\ \downarrow \varphi_Z & & \downarrow \overline{Q(\alpha)} \\ Z_p^n \Big/ C_{p-1,n}^n \cap \ker \varepsilon & \xrightarrow[\cong]{} & Q_p^n(A) \Big/ Q_{p-1}^n(A) \end{array} , \quad \begin{array}{l} n \geq 0 \\ p \geq 0 \end{array}$$

commute.

Our main aim is to prove

4.5.- Theorem.- Assume that ψ^* and φ^* are isomorphisms. Then α is an isomorphism and each $Q_p^n(\alpha)$ $(n \geq 0,\ p \geq 0)$ is an isomorphism.

We also prove :

4.6.- Theorem.- Assume ψ is an isomorphism, and φ^* is an isomorphism. Then φ is an isomorphism.

4.7.- Remark.-

1) We remind the reader we are dealing with minimal extensions.

2) Further isomorphism theorems are established in chapter 7.

4.8.- Lemma.- Assume that the morphism (ψ,φ,α) satisfies that each $Q_p^n(\alpha)$ is an isomorphism. (We do not assume φ^* or ψ^* is an isomorphism.)

Then α is an isomorphism. Moreover $\check{E}$ and E have structures $(\check{X},\check{f})$ and (X,f) which satisfy the conclusions of theorem 3.4 and with respect to which

$$\check{\psi} = \psi \otimes \alpha.$$

Proof.-

It follows from (4.4) that each ψ_Z is an isomorphism. Now let $(\check{X},\check{f})$ and (Y,g) be structures for $\check{E}$ and E which satisfy the conclusions of theorem 3.4. Then

$$\check{Z}^n_p = \left(\check{C}^n_{p-1,n} \cap \ker \varepsilon\right) \oplus \left(1 \otimes \check{X}^n_p\right)$$

and

$$Z^n_p = \left(C^n_{p-1,n} \cap \ker \varepsilon\right) \oplus \left(1 \otimes Y^n_p\right) .$$

Hence an isomorphism $\sigma : \check{X}^n_p \xrightarrow{\cong} Y^n_p$ $(n \geq 0,\ p \geq 0)$ is defined by

$$\psi(1 \otimes \check{x}) - 1 \otimes \sigma\check{x} \in C^n_{p-1,n} \cap \ker \varepsilon , \qquad \check{x} \in \check{X}^n_p .$$

Clearly (apply ρ)

$$\alpha(\check{x}) - \sigma\check{x} \in \Lambda^+(Y^{<n} \otimes Y^n_{<p}) \quad , \qquad \check{x} \in \check{X}^n_p.$$

This implies (same argument as in lemma 1.7) that α is an isomorphism.

Now set

$$X^n_p = \alpha(\check{X}^n_p) \quad \text{and} \quad X = \Sigma_{n,p}\, X^n_p .$$

Then $A = \Lambda X$. Define

$$f : B \otimes \Lambda X \to C$$

by $f(b) = b$ and

$$f(1 \otimes x) = \varphi(1 \otimes \alpha^{-1}x) \quad , \qquad x \in X.$$

It follows that

$$f(1 \otimes x) - 1 \otimes \sigma\,\alpha^{-1}(x) \in C^n_{p-1,n} \cap \ker \varepsilon, \qquad x \in X^n_p.$$

Because $\sigma\,\alpha^{-1} : X^n_p \xrightarrow{\cong} Y^n_p$ this implies (again as in lemma 1.7) that f is an isomorphism. It also implies that (X,f) satisfies the conditions of theorem 3.4. It is clear from the definitions that f converts φ to $\psi \otimes \alpha$.

Q.E.D.

4.9.- Lemma. Suppose $\check{E}$ and E are elementary extensions, and ψ^* and $Q(\alpha)$ are isomorphisms. Then φ^* is an isomorphism.

Proof.-

Since $\check{E}$ and E are elementary,

$$Q^n(\check{A}) = Q^n_o(\check{A}) \qquad \text{and} \qquad Q^n(A) = Q^n_o(A), \qquad n \geq 0.$$

It follows trivially that each $Q^n_p(\alpha)$ is an isomorphism. Thus by lemma 4.8 we can choose structures $(\check{X},\check{f})$ and (X,f) such that $d_C(1 \otimes \check{X}^n) \subset \check{B} \otimes \Lambda\check{X}^{<n}$, $d_C(1 \otimes X^n) \subset B \otimes \Lambda X^{<n}$, and $\varphi = \psi \otimes \alpha$.

Now the method of proof of lemma 1.9 shows that φ^* is an isomorphism.

Q.E.D.

4.10.- Proof of theorem 4.5.

We shall show by induction that each $\varphi^*_{p,n}$ and each $Q^n_p(\alpha)$ is an isomorphism. Indeed if $\varphi^*_{p,n}$ is an isomorphism for all p, then $\varphi^*_{-1,n+1} = \varinjlim \varphi^*_{p,n}$ is an isomorphism. Thus we may assume that for some $p \geq 0$, $n \geq 0$, $\varphi^*_{p-1,n}$ and $Q^n_{p-1}(\alpha)$ are isomorphisms, and we have to show that $\varphi^*_{p,n}$ and $Q^n_p(\alpha)$ are isomorphisms.

Since $\psi^*_{p-1,n}$ and ψ^* are isomorphisms it follows that

$$\psi_{Ker} : \check{Ker}^{n+1}_p \xrightarrow{\cong} Ker^{n+1}_p$$

and (if $p = 0$, $n > 0$)

$$\psi_{Coker} : \check{Coker}^n \xrightarrow{\cong} Coker^n.$$

Thus (4.2) and (4.3) show that

$$\psi_Z : \check{Z}^n_p \big/ \check{C}^n_{p-1,n} \cap \ker \varepsilon \xrightarrow{\cong} Z^n_p \big/ C^n_{p-1,n} \cap \ker \varepsilon .$$

Hence (4.4) implies that

$$\overline{Q(\alpha)} : Q^n_p(\check{A}) \big/ Q^n_{p-1}(A) \xrightarrow{\cong} Q^n_p(A) \big/ Q^n_{p-1}(A) .$$

We have assumed $Q^n_{p-1}(\alpha)$ is an isomorphism. It follows from the equation above and this that $Q^n_p(\alpha)$ is an isomorphism.

Next use structures (X,f) and $(\check{X},\check{f})$ satisfying the conclusions of theorem 3.4 to write

$$C_{p,n} = C_{p-1,n} \otimes \Lambda X^n_p \quad \text{and} \quad \check{C}_{p,n} = \check{C}_{p-1,n} \otimes \Lambda \check{X}^n_p .$$

It follows that $\rho_{p,n}$ and $\hat{\rho}_{p,n}$ induce isomorphisms

$$Z^n_p \big/ C^n_{p-1,n} \cap \ker \varepsilon \xrightarrow{\cong} X^n_p$$

(and similarly for $\check{E}$), where X^n_p is considered as a subspace of $A_{p,n}$.

If we compose these with the projections onto $Q(\check{A}_{p,n})$ and $Q(A_{p,n})$ we obtain isomorphisms

$$\check{Z}^n_p \big/ \check{C}^n_{p-1,n} \cap \ker \varepsilon \xrightarrow{\cong} Q(\check{A}_{p,n}) \quad \text{and} \quad Z^n_p \big/ C^n_{p-1,n} \cap \ker \varepsilon \xrightarrow{\cong} Q(A_{p,n})$$

These convert ψ_Z to $Q(\alpha_{p,n})$, and so $Q(\alpha_{p,n})$ is an isomorphism. Since $\check{E}_{p,n}$ and $E_{p,n}$ are elementary extensions, and since $\psi^*_{p-1,n}$ is assumed to be an isomorphism, lemma 4.9 shows that $\psi^*_{p,n}$ is an isomorphism.

It follows by induction that each $\psi^*_{p,n}$ and each $Q^n_p(\alpha)$ is an isomorphism. Now lemma 4.8 applies and shows that α is an isomorphism.

Q.E.D.

4.11.- *Proof of theorem 4.6.*

Since ψ and ψ^* are isomorphisms theorem 4.5 shows that α and each $Q^n_p(\alpha)$ are isomorphisms. But now lemma 4.8 applies and allows us to write $\psi = \psi \otimes \alpha$. Hence ψ is an isomorphism.

Q.E.D.

Chapter 5

Homotopies and liftings

5.1.- *The c.g.d.a.* $((C,B)^I,D)$.- Suppose $U = \Sigma_{p\geq 0} U^p$ is a graded space. The suspension of U, ΣU, is the graded space which coincides with U^+ as a vector space, but with degrees shifted down by 1 :

$$\Sigma U = \Sigma_{p\geq 0}(\Sigma U)^p \quad ,$$

and

$$(\Sigma U)^p = U^{p+1}.$$

The identity automorphism from U^{p+1} to $(\Sigma U)^p$ is written Σ and called the suspension map ; we extend it to U^o by setting $\Sigma(U^o) = 0$.

Now consider a KS extension

$$E : B \xrightarrow{i} C \xrightarrow{\rho} A,$$

with augmentations ε_B, ε_C and ε_A. To simplify notation denote ΣQ_A by $\bar{Q}_A$ and let $\Lambda\bar{Q}_A \otimes \Lambda D\bar{Q}_A$ be the contractible complex generated by $\bar{Q}_A$. Denote its augmentation by ε_Q.

Tensoring this with (C,d_C) we obtain the c.g.d.a.

$$(C,B)^I = C \otimes \Lambda\bar{Q}_A \otimes \Lambda D\bar{Q}_A$$

whose differential is denoted by D, and which is augmented by $\varepsilon = \varepsilon_C \otimes \varepsilon_Q$. Because $(\Lambda(\bar{Q}_A \oplus D\bar{Q}_A),D)$ is acyclic, the projection

$$\pi : ((C,B)^I,D) \to (C,d_C)$$

defined by $\pi(1\otimes\bar{Q}_A) = 0$ and $\pi(z\otimes 1) = z$, $z \in C$, and the inclusion

$$\lambda_o : (C,d_C) \to ((C,B)^I,D)$$

($\lambda_o z = z \otimes 1$) induce inverse cohomology isomorphisms. In particular $\ker \pi$ is an acyclic ideal :

(5.2) $$H(\ker \pi) = 0.$$

If $B = k$ we denote $(C,B)^I$ simply by C^I.

For general KS extensions $\mathcal{E}$, the projection ρ extends to a projection

$$\rho \otimes \iota : (C,B)^I \to A^I$$

with kernel the ideal generated by $\lambda_o\ i(\ker \varepsilon_B)$.

Next, suppose (X,f) is a structure for $\mathcal{E}$ and $(x_\alpha)_{\alpha\in I}$ is a wellordered homogeneous basis for X such that (1.4) holds. Then $\zeta_A : X \xrightarrow{\cong} Q_A$. The composite $\Sigma \circ \zeta_A : X \longrightarrow \bar{Q}_A$ will be denoted by

$$x \longmapsto \bar{x} \qquad\qquad x \in X.$$

Write $C = B \otimes A$ (via f). For each $\alpha \in I$ we have the KS extension $B \to B \otimes A_{<\alpha} \to A_{<\alpha}$. Since $B \otimes A_{<\alpha} \subset C$ and $Q(A_{<\alpha}) \subset Q(A)$ we can form

$$((B \otimes A_{<\alpha},B)^I,D) \subset ((C,B)^I,D) .$$

Now note that

(5.3) $$(C,B)^I = B \otimes \Lambda X \otimes \Lambda\bar{Q}_A \otimes \Lambda D\bar{Q}_A .$$

Thus a degree -1 derivation, i, is defined in $(C,B)^I$ by

$$i(B) = i(\bar{Q}_A) = i(D\bar{Q}_A) = 0$$

and

$$i(x) = \bar{x} \quad , \qquad x \in X.$$

We define a degree zero derivation, θ, in $(C,B)^I$ by

$$\theta = Di + iD.$$

It satisfies $D\theta = \theta D$.

Note that

(5.4) $$\pi i = 0 \qquad \text{and} \qquad \pi\theta = 0,$$

and so i and θ preserve $\ker \varepsilon$. Moreover, i and θ restrict to derivations in each $(B \otimes A_{<\alpha}, B)^I$. We use (5.3) to regard X, $\bar{Q}_A$ and $D\bar{Q}_A$ as subspaces of $(C,B)^I$. Since, $d_C x_\alpha \in B \otimes A_{<\alpha}$ we obtain

$$\theta x_\alpha - D\bar{x}_\alpha \in (B \otimes A_{<\alpha}, B)^I \quad , \quad \alpha \in I.$$

Clearly $\theta D\bar{x}_\alpha = i D^2\bar{x}_\alpha + D i D\bar{x}_\alpha = 0$, and so

$$\theta^2 x_\alpha \in (B \otimes A_{<\alpha}, B)^I \quad , \quad \alpha \in I.$$

Because $\theta(B) = 0$ an induction on I now shows that for each α there is an integer n_α with

$$\theta^{n_\alpha} x_\alpha = 0.$$

Since $\theta\bar{x} = \theta D\bar{x} = 0$, $\bar{x} \in \bar{Q}_A$, this implies that for each $\Phi \in (C,B)^I$ there is some N (depending on Φ) such that

$$\theta^N \Phi = 0.$$

Hence an automorphism e^θ of the augmented c.g.d.a. $((C,B)^I, D)$ (with inverse $e^{-\theta}$) is defined by

$$e^\theta = \sum_{n=0}^{\infty} \frac{\theta^n}{n!}$$

We define an inclusion of augmented c.g.d.a.'s

$$\lambda_1 : (C, d_C) \to ((C,B)^I, D)$$

by setting

$$\lambda_1 = e^{\theta} \circ \lambda_o .$$

(Recall $\lambda_o z = z \otimes 1,\ z \in C$.)

We now give an inductive formula for $\lambda_1 - \lambda_o$. Since i is a derivation of degree -1, i^2 is a derivation of degree -2.

But $i^2(b) = i^2(x) = i^2(\bar{x}) = i^2(D\bar{x}) = 0$ $(b \in B,\ x \in X)$ and hence

$$i^2 = 0.$$

It follows that

$$\theta^n = (Di+iD)^n = (Di)^n + (iD)^n.$$

In particular, since $iDix_\alpha = iD\bar{x}_\alpha = 0,\ \alpha \in I$,

(5.5) $$\lambda_1 x_\alpha = \sum_0^\infty \frac{1}{n!}\,\theta^n(x_\alpha) = x_\alpha + D\bar{x}_\alpha + \sum_{n=1}^\infty \frac{1}{n!}\,(iD)^n(x_\alpha)$$

Set $\Omega_\alpha = \sum_{n=1}^\infty \frac{1}{n!}\,(iD)^n(x_\alpha)$. Because of (1.4), $\Omega_\alpha \in (B \otimes A_{<\alpha},B)^I$.

Because $\Omega_\alpha \in \operatorname{Im} i$, $\pi\Omega_\alpha = 0$. Thus (5.5) reads.

(5.6) $$\lambda_1 x_\alpha - \lambda_o x_\alpha = D\bar{x}_\alpha + \Omega_\alpha \ ;\ \Omega_\alpha \in (B \otimes A_{<\alpha},B)^I \cap \operatorname{Ker} \pi,\quad \alpha \in I.$$

5.7.- *Lemma*.- The inclusions λ_1 and λ_o coincide in B. Moreover

$$\operatorname{Im}(\lambda_1 - \lambda_o) \subset \operatorname{Ker} \pi ,$$

and so $\lambda_1^* = \lambda_o^*$.

Proof.- Apply (5.4) .

Q.E.D.

Next, suppose that (E,d_E) and $(\check{E},d_{\check{E}})$ are c.g.d.a.'s. If E and $\check{E}$ are augmented by ε_E and $\varepsilon_{\check{E}}$, then by a homomorphism

$$\varphi : (\check{E},d_{\check{E}},\varepsilon_{\check{E}}) \to (E,d_E,\varepsilon_E)$$

we mean an augmentation preserving homomorphism of c.g.d.a.'s.

Assume

$$\psi_o, \psi_1 : (C, d_C) \to (E, d_E)$$

are homomorphisms restricting to the same $\psi : B \to E$.

5.8.- *Definition*.- ψ_o and ψ_1 are called homotopic (rel B) if there is a homomorphism

$$\Phi : ((C,B)^I, D) \to (E, d_E)$$

such that $\Phi \circ \lambda_i = \psi_i$, $i = 0,1$. We write $\psi_o \sim \psi_1$ (rel B). Φ is called a homotopy (rel B) from ψ_o to ψ_1. If $B = k$ we say simply that ψ_o and ψ_1 are homotopic.

Next assume E is augmented by ε_E, and

$$\psi_o, \psi_1 : (C, d_C, \varepsilon_C) \to (E, d_E, \varepsilon_E)$$

are homomorphisms.

5.9.- *Definition*.- ψ_o and ψ_1 are called based homotopic (rel B) if there is a homotopy, Φ, from ψ_o to ψ_1 such that Φ preserves augmentations. Φ is then called a based homotopy (rel B) from ψ_o to ψ_1. We write $\psi_o \overset{\sim}{\sim} \psi_1$ (rel B).

5.10.- *Remarks*.-

1. The definition above of $\lambda_1 : C \to (C,B)^I$ depends on the choice of structure. We shall see in prop. 5.14 that our definitions of "homotopic (rel B)" and "based homotopic (rel B)" are independent of the choice of structure.

2. Fix a homomorphism $\psi_o : (C, d_C) \to (E, d_E)$. By restriction to $\bar{Q}_A$ we obtain a bijection between homotopies (rel B) starting at ψ_o and linear

maps of degree zero from $\bar{Q}_A$ to E.

If E is augmented and ψ_o preserves augmentations, restriction to $\bar{Q}_A$ defines a bijection from based homotopies (rel B) starting at ψ_o to linear maps of degree zero from $\bar{Q}_A$ to $\operatorname{Ker} \varepsilon_E$.

3. If $\psi_o \sim \psi_1$ (rel B) then lemma 5.7 shows that $\psi_o^* = \psi_1^*$.

5.11.- *Lemma*.- Assume that $\psi_o, \psi_1 : (C, d_C) \to (E, d_E)$ restrict to the same ψ in B. Suppose that

$$\operatorname{Im}(\psi_1 - \psi_o) \subset I,$$

where $I \subset E$ is a d_E-stable ideal with $H(I) = 0$. Then

$$\psi_o \sim \psi_1 (\text{rel } B)$$

and the homotopy Φ can be chosen so that $\Phi(\bar{Q}_A) \subset I$.

5.12.- *Corollary*.- Suppose ε_E augments E and $I \subset \operatorname{Ker} \varepsilon_E$. If ψ_o and ψ_1 preserve augmentations then Φ is a based homotopy (rel B).

5.13.- *Proof of lemma* 5.11.- Set $\Phi = \psi_o$ in C. We have to construct elements $\Phi(\bar{x}_\alpha)$ in I ($\alpha \in \mathcal{I}$) such that the unique extension of Φ to a homomorphism $\Phi : ((C,B)^I, D) \to (E, d_E)$ satisfies $\Phi \circ \lambda_1 = \psi_1$.

Assume $\Phi(\bar{x}_\beta) \in I$ is constructed for $\beta < \alpha$. Then, since $d_C(x_\alpha) \in B \otimes A_{<\alpha}$, and Φ is defined in $(B \otimes A_{<\alpha}, B)^I$, we can apply (5.6) to find

$$\begin{aligned} d_E(\psi_1 x_\alpha - \psi_o x_\alpha - \Phi(\Omega_\alpha)) &= \psi_1 d_C x_\alpha - \psi_o d_C x_\alpha - \Phi D \Omega_\alpha \\ &= \Phi \lambda_1 d_C x_\alpha - \Phi \lambda_o d_C x_\alpha - \Phi D \Omega_\alpha \\ &= \Phi D(\lambda_1 x_\alpha - \lambda_o x_\alpha - \Omega_\alpha) \\ &= 0. \end{aligned}$$

Moreover, because $\Phi(\bar{x}_\beta) \in I$ it follows that $\Phi(D\bar{x}_\beta) \in I$ $(\beta < \alpha)$. Since the $\bar{x}_\beta$ and $D\bar{x}_\beta$ generate $\operatorname{Ker}\pi$ as an ideal, we find $\Phi(\operatorname{Ker}\pi) \subset I$. Hence $\psi_1 x_\alpha - \psi_o x_\alpha - \Phi(\Omega_\alpha)$ is a d_E-cocycle in I. By hypothesis we may write

$$\psi_1 x_\alpha - \psi_o x_\alpha - \Phi(\Omega_\alpha) = d_E y_\alpha \quad ,$$

some $y_\alpha \in I$. Note that if $\deg x_\alpha = 0$, then $\deg y_\alpha = -1$; i.e. $y_\alpha = 0$.

Now extend Φ to $\bar{x}_\alpha$ by setting

$$\Phi\, \bar{x}_\alpha = y_\alpha .$$

(If $\deg x_\alpha = 0$ then $\bar{x}_\alpha = 0 = y_\alpha$, so we are all right.) It follows from (5.6) and our choice of y_α that

$$\begin{aligned}(\Phi \circ \lambda_1)x_\alpha &= \Phi(\lambda_o x_\alpha + D\bar{x}_\alpha + \Omega_\alpha)\\ &= \psi_o x_\alpha + d_E y_\alpha + \Phi(\Omega_\alpha)\\ &= \psi_1 x_\alpha .\end{aligned}$$

Hence $\Phi \circ \lambda_1 = \psi_1$ in $B \otimes A_{\leq\alpha}$, and the induction on I is complete.

Q.E.D.

5.14.- *Proposition*.-

i) The definition (5.8) of "homotopic (rel B)" and the definition (5.9) of "based homotopic (rel B)" are independent of the choice of structure for E.

ii) Homotopy (rel B) is an equivalence relation on the set of homomorphisms $(C,d_C) \to (E,d_E)$ which restrict to a given ψ in B.

iii) If E is augmented, based homotopy (rel B) is an equivalence relation on the homomorphisms $(C,d_C,\varepsilon_C) \to (E,d_E,\varepsilon_E)$ which restrict to a given ψ in B.

Proof.-

i) Suppose a second homomorphism

$$\lambda_1' : (C,d_C,\varepsilon_C) \to ((C,B)^I,D,\varepsilon)$$

is defined via a second structure on E. Lemma 5.7 shows that

$$\mathrm{Im}(\lambda_1'-\lambda_o) \subset \mathrm{Ker}\ \pi \subset \mathrm{Ker}\ \varepsilon .$$

Hence by cor. 5.12, $\lambda_o \overset{\sim}{\sim} \lambda_1'$ (rel B) ; i.e. there is a homomorphism

$$\Psi : ((C,B)^I,D,\varepsilon) \to ((C,B)^I,D,\varepsilon)$$

such that $\Psi \circ \lambda_o = \lambda_o$ and $\Psi \circ \lambda_1 = \lambda_1'$.

Hence if $\Phi : ((C,B)^I,D) \to (E,d_E)$ satisfies $\Phi \circ \lambda_o = \psi_o$ and $\Phi \circ \lambda_1' = \psi_1$ for homomorphisms $\psi_o,\psi_1 : (C,d_C) \to (E,d_E)$ it follows that $\Phi \circ \Psi \circ \lambda_o = \psi_o$ and $\Phi \circ \Psi \circ \lambda_1 = \psi_1$. Moreover, if Φ is augmentation preserving so is $\Phi \circ \Psi$.

ii) Reflexivity. Suppose $\psi : (C,d_C) \to (E,d_E)$. Then $\psi \circ \pi : ((C,B)^I,D) \to (E,d_E)$ is a homotopy (rel B) from ψ to ψ.

Symmetry. Suppose Φ is a homotopy (rel B) from ψ_o to ψ_1 ($\psi_o,\psi_1 : (C,d_C) \to (E,d_E)$.) Lemmas 5.7 and 5.11 show that $\lambda_1 \sim \lambda_o$ (rel B) ; let Ψ be a homotopy (rel B) from λ_1 to λ_o. Then $\Phi \circ \Psi$ is a homotopy (rel B) from ψ_1 to ψ_o.

Transitivity. Suppose $\psi_o,\psi_1,\psi_2 : (C,d_C) \to (E,d_E)$ restrict to the same ψ in B. Assume

$$\psi_2 \sim \psi_o \text{ (rel B)} \quad \text{and} \quad \psi_o \sim \psi_1 \text{ (rel B)}.$$

Then by symmetry, $\psi_o \sim \psi_2$ (rel B).

Let Φ_i be a homotopy (rel B) from ψ_o to ψ_i $(i = 1,2)$. Let Y be an isomorphic copy of $\bar{Q}_A$ and fix an isomorphism $\alpha : \bar{Q}_A \xrightarrow{\cong} Y$. Let $\Lambda Y \otimes \Lambda DY$ be the contractible complex generated by Y, and tensor it with $((C,B)^I,D)$ to obtain a c.g.d.a. augmented by $\bar{\varepsilon}$ (the tensor product of the augmentations). Define a homomorphism of c.g.d.a.'s.

$$\Phi : C \otimes \Lambda\bar{Q}_A \otimes \Lambda D\bar{Q}_A \otimes \Lambda Y \otimes \Lambda DY \to E$$

by

$$\Phi(z) = \psi_o z, \qquad z \in C \quad ,$$

$$\Phi(\bar{x}) = \Phi_1\bar{x}, \qquad \bar{x} \in \bar{Q}_A \quad ,$$

and

$$\Phi(\alpha\bar{x}) = \Phi_2\bar{x}, \qquad \bar{x} \in \bar{Q}_A .$$

We may use α to identify $(C,B)^I$ with $C \otimes \Lambda Y \otimes \Lambda DY$; then λ_1 is identified with a homomorphism

$$\lambda_2 : C \to C \otimes \Lambda Y \otimes \Lambda DY$$

of augmented c.g.d.a.'s. Use the obvious inclusions to regard λ_1 and λ_2 as homomorphisms

$$\lambda_i : (C,d_C,\varepsilon_C) \to (C \otimes \Lambda\bar{Q}_A \otimes \Lambda D\bar{Q}_A \otimes \Lambda Y \otimes \Lambda DY, D, \bar{\varepsilon}) .$$

Observe that for $z \in C$,

$$\lambda_2 z - \lambda_1 z = (\lambda_2 z - z) - (\lambda_1 z - z)$$

$$\in C \otimes \Lambda^+(\bar{Q}_A \oplus D\bar{Q}_A \oplus Y \oplus DY),$$

and this is an acyclic ideal in $\operatorname{Ker} \bar{\varepsilon}$. Hence cor. 5.12 yields a based homotopy (rel B), Ψ, from λ_1 to λ_2.

On the other hand the homomorphism Φ defined above satisfies $\Phi \circ \lambda_i = \psi_i$, $i = 1,2$. Hence $\Phi \circ \Psi$ is a homotopy (rel B) from ψ_1 to ψ_2.

iii) Note (in the proof of ii) that if Φ_i (i = 1,2) are based homotopies (rel B) then Φ is augmentation preserving, and so $\Phi \circ \Psi$ is a based homotopy (rel B). This proves transitivity.

Refexivity and symmetry are proved as in the proof of ii).

Q.E.D.

Next, suppose $\check{E} : \check{B} \to \check{C} \to \check{A}$ is a second KS extension, and $(\check{E}, d_{\check{E}})$ is a second c.g.d.a. Assume

$$\chi_C : (\check{C}, d_{\check{C}}) \to (C, d_C) \quad \text{and} \quad \gamma : (E, d_E) \to (\check{E}, d_{\check{E}})$$

are homomorphisms and that χ_C restricts to $\chi_B : \check{B} \to B$.

Finally, assume

$$\psi_o, \psi_1 : (C, d_C) \to (E, d_E)$$

are homomorphisms restricting to the same ψ in B.

5.15. *Proposition*.-

i) If $\psi_o \sim \psi_1$ (rel B) then

$\gamma \circ \psi_o \sim \gamma \circ \psi_1$ (rel B) and $\psi_o \circ \chi_C \sim \psi_1 \circ \chi_C$ (rel $\check{B}$).

ii) If γ^* is an isomorphism then $\psi_o \sim \psi_1$ (rel B) if and only if $\gamma \circ \psi_o \sim \gamma \circ \psi_1$ (rel B).

5.16.- *Proposition*.- Assume that E and $\check{E}$ are augmented by ϵ_E and $\epsilon_{\check{E}}$, and that γ is augmentation preserving and that so are χ_C, ψ_o and ψ_1.

i) If $\psi_o \tilde{\sim} \psi_1$(rel B) then $\gamma \circ \psi_o \tilde{\sim} \gamma \circ \psi_1$ (rel B) and $\psi_o \circ \chi_C \tilde{\sim} \psi_1 \circ \chi_C$ (rel B).

ii) If γ^* is an isomorphism then $\psi_0 \simeq \psi_1$ (rel B) if and only if $\gamma \circ \psi_0 \simeq \gamma \circ \psi_1$ (rel B).

5.17.- *Proof of prop. 5.15.*

i) If Φ is a homotopy (rel B) from ψ_0 to ψ_1, then $\gamma \circ \Phi$ is a homotopy (rel B) from $\gamma \circ \psi_0$ to $\gamma \circ \psi_1$.

On the other hand, $\lambda_0 \circ \chi_C$ and $\lambda_1 \circ \chi_C$ restrict to $\lambda_0 \circ \chi_B$ in $\check{B}$, and

$$\operatorname{Im}(\lambda_1 \circ \chi_C - \lambda_0 \circ \chi_C) \subset \operatorname{Im}(\lambda_1 - \lambda_0) \subset \operatorname{Ker} \pi.$$

Since $H(\operatorname{Ker} \pi) = 0$, lemma 5.11 shows that $\lambda_0 \circ \chi_C \sim \lambda_1 \circ \chi_C$ (rel $\check{B}$). Hence $\Phi \circ \lambda_0 \circ \chi_C \sim \Phi \circ \lambda_1 \circ \chi_C$ (rel $\check{B}$) ; i.e.,

$$\psi_0 \circ \chi_C \sim \psi_1 \circ \chi_C \quad (\text{rel } \check{B}).$$

ii) We have only to show that if $\gamma \circ \psi_0 \sim \gamma \circ \psi_1$ (rel B) then $\psi_0 \sim \psi_1$ (rel B).

Denote by F the graded space $\check{E}$, regarded simply as a graded space (i.e., $F = \check{E}$ without the algebra structure). It generates a contractible complex, $F \oplus \Lambda \bar{D}F$. Tensor this with (E, d_E) to obtain a c.g.d.a. $(\bar{E}, \bar{D})$:

$$\bar{E} = E \otimes \Lambda F \otimes \Lambda \bar{D}F.$$

Denote by

$$j : (E, d_E) \to (\bar{E}, \bar{D}) \quad \text{and} \quad p : (\bar{E}, \bar{D}) \to (E, d_E)$$

the obvious inclusion and projection. Then $p \circ j = \iota$, and $\operatorname{Ker} p$ is an acyclic ideal in $\bar{E}$.

Extend γ to a homomorphism $\bar{\gamma} : (\bar{E}, \bar{D}) \to (E, d_E)$ by setting $\bar{\gamma}\bar{z} = \bar{z}$ and $\bar{\gamma}\bar{D}\bar{z} = d_E\bar{z}$ $(\bar{z} \in F)$. Then $\bar{\gamma}$ is surjective and $\bar{\gamma}^*$ is an isomorphism, so $\operatorname{Ker}\bar\gamma$ is an acyclic ideal.

Now suppose Ψ is a homotopy (rel B) from $\gamma\psi_0$ to $\gamma\psi_1$. Let $\Phi : \bar{Q}_A \to \bar{E}$ be a linear map of degree zero such that

$$\bar\gamma \circ \Phi = \Psi.$$

(This is possible because $\bar\gamma$ is surjective.) Extend Φ to a (unique) homotopy (rel B), Φ, starting at $j \circ \psi_0$. Then

$$\bar\gamma \circ \Phi = \Psi : (C,B)^I \to E.$$

In particular it follows that

$$\bar\gamma \circ \Phi \circ \lambda_1 = \gamma \circ \psi_1 = \bar\gamma \circ j \circ \psi_1.$$

Hence $\bar\gamma(\Phi \circ \lambda_1 - j \circ \psi_1) = 0$; by lemma 5.11 $\Phi \circ \lambda_1 \sim j \circ \psi_1$ (rel B). On the other hand, $j \circ \psi_0 \sim \Phi \circ \lambda_1$ (rel B) with homotopy Φ. Thus we obtain (cf. Prop. 5.14 ii))

$$j \circ \psi_0 \sim j \circ \psi_1 \text{ (rel B)}.$$

Finally apply p to find (since $p \circ j = \iota$)

$$\psi_0 \sim \psi_1 \text{ (rel B)}.$$

Q.E.D.

5.16.- *Proof of prop. 5.16.*

i) The proof is word for word the same as the proof of prop. 5.15, i), except that we rely on cor. 5.12.

ii) The proof is the same as that of prop. 5.15, ii) except for the following changes.

Let $F = \ker \varepsilon_E$ and tensor the augmentations of E and $\Lambda F \otimes \Lambda \bar{D}F$ to augment $\bar{E}$. Note that j, p and $\bar\gamma$ preserve augmentations. In particular

$$\ker p \subset \ker \bar{\varepsilon} \quad \text{and} \quad \ker \bar{\gamma} \subset \ker \bar{\varepsilon}.$$

Moreover $\bar{\gamma} : \ker \bar{\varepsilon} \to \ker \varepsilon_{\check{E}}$ is surjective, and so Φ can be chosen so that $\Phi(\bar{Q}_A) \subset \ker \bar{\varepsilon}$.

Then Φ extends to a based homotopy (rel B) from $j \circ \psi_0$ to $\Phi \circ \lambda_1$. Moreover $\operatorname{Im}(\Phi \circ \lambda_1 - j \circ \psi_1) \subset \ker \bar{\gamma}$, and so cor. 5.12 implies that $\Phi \circ \lambda_1$ is based homotopic to $j \circ \psi_1$ (rel B). By prop. 5.14 iii) $j \circ \psi_0 \simeq j \circ \psi_1$ (rel B). Since p is augmentation preserving we apply it to obtain

$$\psi_0 \simeq \psi_1 \text{ (rel B)}.$$

Q.E.D.

We come now to the lifting theorems, which are the main results of the chapter.

5.19.- *Theorem*. Assume

$$\begin{array}{ccccc} E & \xrightarrow{\ \gamma\ } & \check{E} & & \\ \uparrow{\scriptstyle \psi} & & \uparrow{\scriptstyle \eta} & & \\ B & \xrightarrow[\ i\ ]{} & C & \xrightarrow[\ \rho\ ]{} & A \end{array}$$

is a commutative diagram of homomorphisms of c.g.d.a.'s. Suppose that

i) γ^* is an isomorphism.

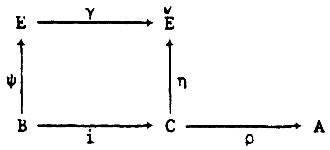
ii) The bottom row is a KS extension, $\mathcal{E}$.

Then there is a homomorphism $\varphi : (C, d_C) \to (E, d_E)$ such that

$$\varphi \circ i = \psi \quad \text{and} \quad \gamma \circ \varphi \sim \eta \text{ (rel B)}.$$

If $\bar{\psi} : (C,d_C) \to (E,d_E)$ also satisfies these conditions, then $\bar{\psi} \sim \psi$ (rel B).

5.20.- *Theorem*. Assume



is a commutative diagram of homomorphisms of augmented c.g.d.a.'s. Suppose that

i) γ^* is an isomorphism.

ii) The bottom row is a KS extension, $\mathcal{E}$.

Then there is a homomorphism $\psi : (C,d_C,\varepsilon_C) \to (E,d_E,\varepsilon_E)$ such that

$$\psi \circ i = \psi \quad \text{and} \quad \gamma \circ \psi \simeq \eta (\text{rel } B).$$

If $\bar{\psi} : (C,d_C,\varepsilon_C) \to (E,d_E,\varepsilon_E)$ also satisfies these conditions, then $\bar{\psi} \simeq \psi$ (rel B).

5.21.- *Proof of theorem 5.19.*

Construct $(\bar{E},\bar{D})$, j,p and $\bar{\gamma}$ exactly as in the proof of prop. 5.15 ii). We first construct a homomorphism

$$\psi_o : (C,d_C) \to (\bar{E},\bar{D})$$

suchthat $\psi_o \circ i = j \circ \psi$, and $\bar{\gamma} \circ \psi_o = \eta$.

Fix a structure (X,f) and a basis $\{x_\alpha\}_{\alpha \in I}$ for X so (1.4) holds. We define $\psi_o x_\alpha$ by induction over I.

Indeed if $\psi_o x_\beta$ is defined $(\beta < \alpha)$, then $\psi_o d_C x_\alpha$ is defined, and

$$\bar{\gamma}\, \psi_o\, d_C\, x_\alpha = \eta\, d_C\, x_\alpha = d_{\check{E}}\, \eta\, x_\alpha.$$

Choose $y_\alpha \in \bar{E}$ so that $\bar{\gamma}\, y_\alpha = \eta\, x_\alpha$. Then

$$\bar{\gamma}(\psi_o d_C x_\alpha - \bar{D} y_\alpha) = d_E \eta x_\alpha - d_E \bar{\gamma}\, y_\alpha = 0,$$

and

$$\bar{D}(\psi_o d_C x_\alpha - \bar{D} y_\alpha) = \psi_o\, d_C^2\, x_\alpha = 0.$$

Since $H(\ker \bar{\gamma}) = 0$ we can write (for some $w_\alpha \in \ker \bar{\gamma}$)

$$\psi_o\, d_C\, x_\alpha - \bar{D} y_\alpha = \bar{D} w_\alpha.$$

Extend ψ_o to $B \otimes A_{\leq \alpha}$ by setting

$$\psi_o\, x_\alpha = y_\alpha + w_\alpha.$$

By definition $\bar{D}\psi_o\, x_\alpha = \psi_o d_C\, x_\alpha$, while

$$\bar{\gamma}\, \psi_o\, x_\alpha = \bar{\gamma}\, y_\alpha = \eta\, x_\alpha.$$

This completes the inductive construction of ψ_o.

Now we define ψ by setting

$$\psi = p \circ \psi_o : C \to E.$$

Then $\psi \circ i = p \circ \psi_o \circ i = p \circ j \circ \psi = \psi$. Moreover, $\mathrm{Im}(j \circ \psi - \psi_o) \subset \ker p$, and so by lemma 5.11, $j \circ \psi \sim \psi_o$ (rel B). Since $\bar{\gamma} \circ j = \gamma$ while $\bar{\gamma} \circ \psi_o = \eta$ we can apply $\bar{\gamma}$ to this relation to obtain

$$\gamma \circ \psi \sim \eta \quad (\text{rel } B).$$

Finally, suppose $\bar{\psi} : (C, d_C) \to (E, d_E)$ satisfies $\bar{\psi} \circ i = \psi$ and $\gamma \circ \bar{\psi} \sim \eta$ (rel B). Then by prop 5.14 ii),

$$\gamma \circ \bar{\psi} \sim \gamma \circ \psi \ (\text{rel } B).$$

Thus prop. 5.15 ii) implies that $\bar{\psi} \sim \psi$ (rel B).

Q.E.D.

5.22.- *Proof of theorem 5.20.*

Construct $(\bar{E},\bar{D})$, $p,j,\bar{\gamma}$ and the augmentation $\bar{\varepsilon}:\bar{E}\to k$ exactly as in the proof of prop 5.16 ii).

Then repeat the construction of $\psi_o:(C,d_C)\to(\bar{E},\bar{D})$ given above, being careful at each step to choose $y_\alpha\in\ker\bar{\varepsilon}$.

Since $\ker\bar{\gamma}\subset\ker\bar{\varepsilon}$, ψ_o will preserve augmentations.

Hence so will $\psi = p\circ\psi_o$.

Use cor. 5.12 to conclude that $j\circ\psi \simeq \psi_o$ (rel B). Since $\bar{\gamma}$ preserves augmentations this implies

$$\gamma\circ\psi \simeq \eta \text{ (rel B)}.$$

Complete the proof using prop 5.14 iii) and prop 5.16 ii), just as above.

Q.E.D.

For the rest of this chapter we shall consider KS extensions $\mathcal{E}: B\to C\to A$ admitting a structure (X,f) and a well ordered homogeneous basis $\{x_\alpha\}_{\alpha\in I}$ such that (1.4) holds and

(5.23) $$\deg x_\beta = 0,\quad \deg x_\alpha > 0 \Rightarrow \beta < \alpha.$$

A KS extension satisfying this conditions will be called <u>0-minimal</u>. Clearly minimal extensions are 0-minimal. Moreover any KS extension with connected fibre is vacuously 0-minimal. It is easy to see that a KS extension is 0-minimal if and only if the subalgebra $C_o\subset C$ generated by B and C^o is d_C-stable.

5.24.- <u>Lemma</u>. Suppose $\mathcal{E}$ is 0-minimal and $\psi_o,\psi_1:(C,d_C)\to(E,d_E)$ are homotopic (rel B). Then ψ_o and ψ_1 coincide in C_o.

Proof.- Observe that the derivation i in $(C,B)^I$ has degree -1, and so $i(C_o) = 0$. Since E is 0-minimal, C_o is d_C-stable, and it follows that $\theta(C_o) = 0$.

This implies that $\lambda_1 = \lambda_o$ in C_o, and so $\psi_1 = \psi_o$ in C_o.

Q.E.D.

If E is 0-minimal we denote by $(C,B)^{\partial I}$ the augmented subalgebra of $(C,B)^I$ generated by $\operatorname{Im}\lambda_o$ $(=C)$ and $\operatorname{Im}\lambda_1$. $(C,B)^{\partial I}$ is stable under D.

Recall from 5.1. that the linear map $x \mapsto \bar{x}$ of X to $\bar{Q}_A$ is defined by $\bar{x} = \Sigma\zeta_A x$. This extends to a linear map

$$\Sigma\zeta_A : \Lambda^+ X \to \bar{Q}_A.$$

If E is 0-minimal then $d_A(X^o) = 0$ and it follows that a differential $\bar{Q}(d_A)$ is defined in $\bar{Q}_A$ by

$$\bar{Q}(d_A)\bar{x} = \Sigma\zeta_A\, d_A(x), \qquad x \in X.$$

Extend $\bar{Q}(d_A)$ to a derivation in $\Lambda\bar{Q}_A$; then $(\Lambda Q_A, \bar{Q}(d_A))$ is a a c.g.d.a. and

$$\bar{Q}(d_A) = 0$$

if and only if E is minimal.

Next define a projection of augmented c.g.d.a.'s

$$p : ((C,B)^I, D) \to (\Lambda\bar{Q}_A, -\bar{Q}(d_A))$$

by setting

$$p(z) = \varepsilon_C z \quad , \qquad z \in C$$
$$p(\bar{x}) = \bar{x} \quad , \qquad \bar{x} \in \bar{Q}_A$$

and

$$p(D\bar{x}) = -\bar{Q}(d_A)\bar{x} \ , \qquad \bar{x} \in \bar{Q}_A.$$

Let $j : (C,B)^{\partial I} \to (C,B)^I$ be the inclusion.

5.25.- *Proposition.*

If E is a O-minimal extension then

$$(5.26) \qquad (C,B)^{\partial I} \xrightarrow{j} (C,B)^I \xrightarrow{p} (\Lambda\bar{Q}_A, -\bar{Q}(d_A))$$

is a KS extension. If E is minimal so is (5.26).

Proof.- Let J be the subset of I of indices α with $\deg x_\alpha > 0$. Tnen $\{x_\alpha\}_{\alpha\in J}$ is a well ordered homogeneous basis of X^+, and $\{\bar{x}_\alpha\}_{\alpha\in J}$ is a well ordered homogeneous basis of $\bar{Q}_A$. Now let $\check{X}$ be an isomorphic copy of X^+ with isomorphism $x \to \check{x}$. Then $\{\check{x}_\alpha\}_{\alpha\in J}$ is a well ordered homogeneous basis of $\check{X}$.

In particular we have the subalgebras

$$(\Lambda\bar{Q}_A)_{<\alpha} \quad , \quad (\Lambda D\bar{Q}_A)_{<\alpha} \quad \text{and} \quad (\Lambda\check{X})_{<\alpha} \quad , \quad \alpha \in J.$$

Now define an algebra homomorphism

$$g : B \otimes \Lambda X \otimes \Lambda\check{X} \otimes \Lambda\bar{Q}_A \longrightarrow (C,B)^I$$

by

$$g(b) = b, \quad b \in B \quad , \quad g(\check{x}) = \lambda_1 x, \quad x \in X^+$$

$$g(x) = x, \quad x \in X \quad , \quad g(\bar{x}) = \bar{x} \; , \quad x \in X^+.$$

It follows from lemma 5.24 that if we set $\check{x} = x \quad (x \in X^0)$ then

$$(5.27) \qquad g(\check{x}) = \lambda_1 x \quad , \quad x \in X.$$

5.28.- *Lemma.* g is an isomorphism. Moreover

$$g^{-1} D\bar{x}_\alpha \in B \otimes \Lambda X \otimes \Lambda\check{X} \otimes (\Lambda\bar{Q}_A)_{<\alpha} \quad , \quad \alpha \in J.$$

Proof.- Define an isomorphism

$$h : B \otimes \Lambda X \otimes \Lambda \check{X} \otimes \Lambda \bar{Q}_A \xrightarrow{\cong} (C,B)^I$$

by $h(b) = b$, $h(x) = x$ ($x \in X$), $h(\check{x}) = D\bar{x}$ ($x \in X^+$) and $h(\bar{x}) = \bar{x}$ ($x \in X^+$). Then $g-h$ is zero in B, X and $\bar{Q}_A$.

Moreover, we obtain from (5.6) that

$$g\check{x}_\alpha - h\check{x}_\alpha = x_\alpha + \Omega_\alpha \quad , \quad \alpha \in J,$$

whence

$$(g-h)\check{x}_\alpha \in B \otimes \Lambda X \otimes (\Lambda\bar{Q}_A)_{<\alpha} \otimes (\Lambda D\bar{Q}_A)_{<\alpha}.$$

It follows as in lemma 1.7 that g is an isomorphism.

Finally, (5.6) implies that

$$g^{-1}(D\bar{x}_\alpha) = \check{x}_\alpha - x_\alpha - g^{-1}(\Omega_\alpha), \qquad \alpha \in J.$$

Thus induction on J shows that for $\alpha \in J$

$$g^{-1}(B \otimes A_{\leq\alpha},B)^I \subset B \otimes \Lambda X \otimes \Lambda\check{X} \otimes (\Lambda\bar{Q}_A)_{\leq\alpha}$$

It follows that $g^{-1}(\Omega_\alpha) \in B \otimes \Lambda X \otimes \Lambda\check{X} \otimes (\Lambda\bar{Q}_A)_{<\alpha}$, and the rest of the lemma is proved.

Q.E.D.

5.29.- *Corollary*.- g restricts to an isomorphism of $B \otimes \Lambda X \otimes \Lambda\check{X}$ onto $(C,B)^{\partial I}$.

Proof.- Clearly $g(B \otimes \Lambda X \otimes \Lambda\check{X}) \subset (C,B)^{\partial I}$. Formula (5.27) allows us to reverse the inclusion.

Q.E.D.

5.30.- *Proof of prop. 5.25 cont'd.*

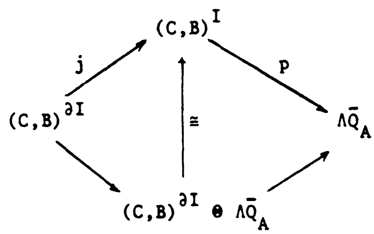
It follows from lemma 5.28 and cor 5.29 that multiplication in $(C,B)^I$ defines an isomorphism (of augmented graded algebras)

$$(C,B)^{\partial I} \otimes \Lambda\bar{Q}_A \xrightarrow{\cong} (C,B)^I$$

with respect to which

$$D(1 \otimes \bar{x}) \in (C,B)^I \otimes (\Lambda\bar{Q}_A)_{<\alpha}, \qquad \alpha \in J$$

To show that (5.26) is a KS extension we need only show that the diagram



commutes.

That the left triangle commutes is clear. To verify that the right triangle commutes we show that

$$p \circ \theta = 0 \tag{5.31}$$

where $\theta = Di + iD$ is the derivation in $(C,B)^I$ used to define λ_1.

In fact, since $\theta\bar{x} = \theta D\bar{x} = 0$, $x \in X$ we need only prove that $p \circ \theta(x) = 0$, $x \in X$. Now in $C = B \otimes \Lambda X$

$$d_C(1 \otimes x) - 1 \otimes d_A x \in \ker \varepsilon_B \otimes \Lambda X,$$

and so

$$i\, d_C(1 \otimes x) - i(1 \otimes d_A x) \in \ker \varepsilon_B \otimes \Lambda X \otimes \Lambda\bar{Q}_A.$$

Since the restriction of p to $B \otimes \Lambda X \otimes \Lambda \bar{Q}_A$ is just $\varepsilon_C \otimes \iota$, we find

$$p\ i\ d_C(1 \otimes x) = (\varepsilon_C \otimes \iota)\ i\ (1 \otimes d_A x)$$

$$= \bar{Q}(d_A)\bar{x}\ , \qquad x \in X.$$

On the other hand

$$p\ Di(1 \otimes x) = p\ D\bar{x} = -\bar{Q}(d_A)\bar{x}, \qquad x \in X$$

and adding these relations we obtain (5.31).

But this yields $p \circ \lambda_1 = p \circ \lambda_o = \varepsilon_C$; since $p\bar{x} = \bar{x}$ the commutativity of the right triangle is proved.

Finally, if E is minimal so is (5.26), almost by definition.

Q.E.D.

5.32.- *Corollary*.- An isomorphism

$$C \otimes_{C_o} C \xrightarrow{\cong} (C,B)^{\partial I}$$

is defined by

$$z \otimes w \longmapsto \lambda_o z \cdot \lambda_1 w.$$

5.33.- *Corollary*.-

i) Suppose $\psi_o, \psi_1 : (C,d_C) \to (E,d_E)$ coincide in C_o. Then a unique homomorphism

$$\psi : ((C,B)^{\partial I},D) \to (E,d_E)$$

is defined by $\psi_o \lambda_i = \psi_i$, $i = 0,1$. The homomorphisms ψ_i are homotopic (rel B) if and only if ψ extends to a homomorphism

$$\Phi : ((C,B)^I,D) \to (E,d_E).$$

ii) If E is augmented by ε_E then ψ preserves augmentations if and only if ψ_o and ψ_1 do. In this case $\psi_o \simeq \psi_1$ (rel B) if and only if Φ can be chosen to preserve augmentations.

Now recall that E is assumed 0-minimal. Suppose $\psi_0, \psi_1 : (C,d_C) \to (E,d_E)$ are homotopic (rel B) and that Φ and Ψ are both homotopies (rel B) between them. Thus

$$\Phi,\Psi : ((C,B)^I,D) \to (E,d_E)$$

restrict to the same ψ in $(C,B)^{\partial I}$.

Since the sequence (5.26) is a KS extension we can apply definition 5.8 to define the notion of a homotopy (rel $(C,B)^{\partial I}$) between Φ and Ψ. If such exists we say Φ and Ψ are homotopic (rel $(C,B)^{\partial I}$).

Similarly if ψ_0 and ψ_1 preserve augmentations (for some given ε_E) and Φ and Ψ are based homotopies then we can use definition (5.9) to define a based homotopy (rel $(C,B)^{\partial I}$) between Φ and Ψ ; if such exists Φ and Ψ are based homotopic (rel $(C,B)^{\partial I}$).

If E was actually minimal then so is the KS extension (5.26) In this case we can iterate the procedure to obtain homotopies of homotopies of homotopies.... .

5.34.- *Example*.- Suppose E is 0-minimal. Then the homotopy class of the homotopy Φ of lemma 5.11 (resp. the based homotopy class of the homotopy Φ of cor. 5.12) is uniquely determined by the condition $\Phi(\bar{Q}_A) \subset I$.

Indeed if Ψ is a second such homotopy then Φ and Ψ agree in $(C,B)^{\partial I}$ and $\mathrm{Im}(\Phi-\Psi) \subset I$. Thus lemma 5.11 implies that $\Phi \sim \Psi$ (rel $(C,B)^{\partial I}$).

5.35.- *Example*.- The hypothesis of 0-minimality is essential. Indeed consider the contractible complex $\Lambda(x,dx)$ generated by x with $\deg x = 0$. Then

$$\Lambda(x,dx)^I = \Lambda(x,dx,\overline{dx}, D\overline{dx})$$

and

$$i(x) = 0, \quad i(dx) = \overline{dx}, \quad i(\overline{dx}) = i(D\overline{dx}) = 0.$$

Hence

$$\theta(x) = \overline{dx}, \quad \theta(dx) = D\overline{dx}, \quad \theta(\overline{dx}) = \theta(D\overline{dx}) = 0.$$

Thus

$$\lambda_1 x = x + \overline{dx} \quad \text{and} \quad \lambda_1 dx = dx + D\overline{dx}.$$

It follow that $\mathrm{Im}\,\lambda_1$ and $\mathrm{Im}\,\lambda_o$ generate all of $\Lambda(x,dx)^I$, and so lemma 5.28 fails in this case.

Moreover the augmentation $\varepsilon : \Lambda(x,dx) \to k$ and the inclusion $i : k \to \Lambda(x,dx)$ satisfy $i \circ \varepsilon \sim \iota$. But $i \circ \varepsilon$ does not coincide with ι in $\Lambda(x,dx)^o$. So lemma 5.24 fails as well.

Chapter 6

Models

In this chapter we consider a homomorphism of c.g.d.a.'s

$$\gamma : (B, d_B) \to (E, d_E).$$

We assume B is augmented by ε_B and that $H^0(B) = k = H^0(E)$. The main result (existence and uniqueness of models) is stated in the next three theorems.

6.1.- *Theorem (existence)*.- There is a minimal KS extension

$$\mathcal{E} : B \xrightarrow{i} C \xrightarrow{\rho} A$$

and a homomorphism $\psi : (C, d_C) \to (E, d_E)$ such that :

i) $\psi \circ i = \gamma$.

ii) ψ^* is an isomorphism.

Moreover, if E is augmented by ε_E and γ preserves augmentations, then ψ can be chosen to preserve augmentations.

Now assume there are two minimal KS extensions

$$\mathcal{E} : B \xrightarrow{i} C \xrightarrow{\rho} A \qquad \text{and} \qquad \check{\mathcal{E}} : B \xrightarrow{\check{i}} \check{C} \xrightarrow{\check{\rho}} \check{A},$$

and suppose,

$$\psi : (C, d_C) \to (E, d_E) \quad \text{and} \quad \check{\psi} : (\check{C}, d_{\check{C}}) \to (E, d_E)$$

both satisfy i) and ii) of theorem 6.1.

6.2.- *Theorem (uniqueness)*. With the hypotheses above there is a commutative diagram of homomorphisms of c.g.d.a.'s

$$\begin{array}{ccccc} B & \xrightarrow{\check{i}} & \check{C} & \xrightarrow{\check{\rho}} & \check{A} \\ \| & & \cong\ \downarrow \bar{\psi} & & \cong\ \downarrow \bar{\alpha} \\ B & \xrightarrow[i]{} & C & \xrightarrow[\rho]{} & A \end{array}$$

such that $\bar{\psi}$ and $\bar{\alpha}$ are isomorphisms, and $\psi \circ \bar{\psi} \sim \check{\psi}$ (rel B).

Moreover if γ, ψ and $\check{\psi}$ are augmentation preserving with respect to an augmentation ε_E of E, then $\bar{\psi}$ can be chosen to be augmentation preserving and so that $\psi\, \bar{\psi}$ and $\check{\psi}$ are based homotopic (rel B). In this case $\bar{\alpha}$ is also augmentation preserving.

6.3.- *Theorem (uniqueness of isomorphism)*. With the hypotheses and notation of theorem 6.2 assume that

$$\begin{array}{ccccc} B & \xrightarrow{\check{i}} & \check{C} & \xrightarrow{\check{\rho}} & \check{A} \\ \| & & \downarrow \chi & & \downarrow \alpha \\ B & \xrightarrow[i]{} & C & \xrightarrow[\rho]{} & A \end{array}$$

is a commutative diagram of homomorphisms of c.g.d.a.'s such that $\psi \circ \chi \sim \check{\psi}$ (rel B).

Then χ and α are isomorphisms and

$$\chi \sim \bar{\psi} \text{ (rel B)} \quad \text{and} \quad \alpha \sim \bar{\alpha}.$$

Moreover, if E is augmented by ε_E, if all homomorphisms preserve augmentations, and if $\psi \circ \chi \simeq \check{\psi}$ (rel B) then χ and $\bar{\psi}$ are based homotopic (rel B) and α and $\bar{\alpha}$ are based homotopic.

6.4.- *Proof of theorem 6.1.*

We shall construct C in the form $C = B \otimes \Lambda X$ where each X^n is decomposed in the form

$$X^n = \sum_{p \geq 0} X_p^n \qquad , \qquad n \geq 0,$$

and d_C extends d_B and satisfies

(6.5) $$d_C(1 \otimes X_q^m) \subset B \otimes \Lambda(X^{<m} \otimes X_{<q}^m), \qquad m \geq 0, \quad q \geq 0.$$

We shall simultaneously construct $\psi : B \otimes \Lambda X \to E$ so that

(6.6) $$\psi\, d_C = d_E\, \psi.$$

If γ preserves augmentations (for a given ε_E) we shall also arrange that

(6.7) $$\psi(1 \otimes X_q^o) \subset \ker \varepsilon_E, \qquad q \geq 0.$$

6.8.- *The spaces* X_p^o. Set

$$X_o^o = \ker(H^1(B) \xrightarrow{\gamma^*} H^1(E)).$$

Define d_C in $1 \otimes X_o^o$ so that $d_C(1 \otimes x)$ is a cocycle in B representing x :

$$[d_C(1 \otimes x)] = x\ , \qquad x \in X_o^o.$$

Extend ψ to X_o^o so that (6.6) holds (and (6.7) if ε_E is given).

Suppose X_q^o has been defined for $q < p$ and ψ and d_C have been extended to $B \otimes \Lambda X_{<p}^o$ so that (6.5) and (6.6) hold (and (6.7) if ε_E is given). Let

$$X_p^o = \ker(H^1(B \otimes \Lambda X_{<p}^o) \xrightarrow{\psi^*} H^1(E))$$

and further extend ψ and d_C just as we did when $p = 0$.

6.9.- *Lemma*.-

i) $H^o(B \otimes \Lambda X^o) = k$.

ii) $\psi^* : H^1(B \otimes \Lambda X^o) \to H^1(E)$ is injective.

Proof.-

i) We need only show that $H^o(B \otimes \Lambda X^o_{\le p}) = k$ for all p. Assume this holds for $q < p$; then $H^o(B \otimes \Lambda X^o_{<p}) = k$. Let $\check{B} = B \otimes \Lambda X^o_{<p}$ and write

$$B \otimes \Lambda X^o_{\le p} = \check{B} \otimes \Lambda X^o_p .$$

If $\Phi \in (\check{B} \otimes \Lambda X^o_p)^o$ is a cocycle write $\Phi = \sum_{j=0}^m \Phi_j$, $\Phi_j \in \check{B}^o \otimes \Lambda^j X^o_p$, $\Phi_m \neq 0$. Since $d_C\Phi = 0$ it is clear that

$$(d_{\check{B}} \otimes \iota)\Phi_m = 0 \tag{6.10}$$

It follows by our induction hypothesis that $\Phi_m \in 1 \otimes \Lambda^m X^o_p$.

Now by construction, d_C injects X^o_p onto a space of cocycles in $\check{B}^1$ which does not meet $d_{\check{B}}(\check{B}^o)$. On the other hand since $d_C\Phi = 0$, (6.10) shows that

$$d_C\Phi_m + (d_{\check{B}} \otimes \iota)\Phi_{m-1} = 0.$$

Since $\Phi_m \in 1 \otimes \Lambda X^o_p$ this implies

$$d_C\Phi_m = 0.$$

Consider the contractible complex $\Lambda X^o_p \otimes \Lambda DX^o_p$ generated by X^o_p. Since $d_C : X^o_p \longrightarrow \check{B}$ is injective, it follows that $D\Phi_m = 0$. Because $\Lambda X^o_p \otimes \Lambda DX^o_p$ is acyclic, the only D-cocycles in $\Lambda X^o_p \otimes 1$ are scalars. Hence $m = 0$ and $\Phi = \Phi_m \in k$.

ii) Assume $\Phi \in (B \otimes \Lambda X^o)^1$, $d_C\Phi = 0$, and $\psi\,\Phi$ is a coboundary. Then $\Phi \in (B \otimes \Lambda X^o_{\le p})^1$, some p, and so (by definition of X^o_{p+1}) for some

$x \in X^o_{p+1}$,

$$\Phi - d_C(1 \otimes x) \in d_C(B \otimes \Lambda X^o_{\leq p}).$$

In particular Φ is a coboundary in $B \otimes \Lambda X^o$.

Q.E.D.

6.10.- The spaces X^n_p, $n > 0$.

Suppose that for some $n > 0$ X^m_q is defined for $m < n$ and $q \geq 0$, and that d_C and ψ are extended to $B \otimes \Lambda X^{<n}$ so that (6.5) and (6.6) hold.

Assume as well that

$$\psi^* : H(B \otimes \Lambda X^{<n}) \to H(E) \text{ is } (n-1)\text{-regular.} \tag{6.11}$$

Define spaces W^n_o and Y^n_o by

$$W^n_o = \mathrm{Coker}(H^n(B \otimes \Lambda X^{<n}) \xrightarrow{\psi^*} H^n(E))$$

and

$$Y^n_o = \mathrm{Ker}(H^{n+1}(B \otimes \Lambda X^{<n}) \xrightarrow{\psi^*} H^{n+1}(E)).$$

Let $X^n_o = W^n_o \oplus Y^n_o$ and extend d_C to X^n_o by setting

$$\begin{cases} d_C(1 \otimes w) = 0, \quad w \in W^n_o \\ \text{and} \\ d_C(1 \otimes y) \text{ is a cocycle in } (B \otimes \Lambda X^{<n}) \text{ representing } y, \\ \qquad y \in Y^n_0. \end{cases} \tag{6.12}$$

Extend ψ in the obvious way so that ψ^* is surjective in degree n and (6.6) holds.

Next, if X^n_q is defined for $q < p$ and ψ and d_C are extended to $B \otimes \Lambda X^{<n} \otimes \Lambda X^n_{<p}$ so that (6.5) and (6.6) hold, set (for $p \geq 1$)

$$X^n_p = \mathrm{Ker}(H^{n+1}(B \otimes \Lambda X^{<n} \otimes \Lambda X^n_{<p}) \xrightarrow{\psi^*} H^{n+1}(E)).$$

Extend d_C to X^n_p so that

(6.13) $\quad d_C(1 \otimes x) \in B \otimes \Lambda X^{<n} \otimes \Lambda X^n_{<p}$ is a cocycle representing x, $\quad x \in X^n_p$.

Extend ψ to X^n_p so that (6.6) holds.

6.14.- Lemma.

$\psi^* : H(B \otimes \Lambda X^{\leq n}) \to H(E)$ is n-regular.

Proof.-

Since $(B \otimes \Lambda X^{\leq n})^m = (B \otimes \Lambda X^{<n})^m$, $m < n$, it follows from (6.11) that ψ^* is an isomorphism in degrees less than n. We show next that it is injective in degree n.

Suppose Φ is a cocycle in $(B \otimes \Lambda X^{\leq n})^n$ and $\psi\Phi$ is a coboundary. Note that $\Phi \in B \otimes \Lambda X^{<n} \otimes \Lambda X^n_{\leq p}$, some p. Let $\check{B} = B \otimes \Lambda X^{<n} \otimes \Lambda X^n_{<p}$; it is d_C-stable by construction. Write $B \otimes \Lambda X^{<n} \otimes \Lambda X^n_{\leq p} = \check{B} \otimes \Lambda X^n_p$, and write

$$\Phi = \Psi + \Omega , \qquad \Psi \in \check{B}^o \otimes X^n_p, \qquad \Omega \in \check{B}^n.$$

Since $d_C\Phi = 0$ we conclude that $(d_{\check{B}} \otimes \iota)\Psi = 0$. But since $H^o(E) = k$, (6.11) shows that $H^o(\check{B}) = k$. Hence

$$\Psi \in 1 \otimes X^n_p.$$

It follows that $d_C \Psi \in d_{\check{B}}(\check{B}^n)$, which, in view of our construction of X^n_p, implies $\Psi = 0$.

Hence $\Phi \in \check{B}$. Continuing in this way we eventually obtain $\Phi \in B \otimes \Lambda X^{<n}$. Now (6.11) shows that Φ is a coboundary. Thus ψ^* is injective in degree n.

Finally ψ^* is surjective in degree n by the definition of $W^n_o \subset X^n_o$. It is injective in degree n+1 by the same argument as used in lemma 6.9 ii).

Q.E.D.

6.15.- *Proof of theorem 6.1. cont'd.*

Above we have defined a c.g.d.a. (C,d_C) with $C = B \otimes \Lambda X$. We augment it by ε_C, where ε_C extends ε_B and vanishes on X.

If $I = \ker \varepsilon_B \otimes \Lambda X$ we write $A = C/I$ and let $\rho : C \to A$ be the projection. d_C and ε_C factor over ρ to make A into an augmented c.g.d.a. It follows from (6.5) that

$$\mathcal{E} : B \xrightarrow{i} C \xrightarrow{\rho} A.$$

is a minimal KS extension.

We also defined a homomorphism $\psi : (C,d_C) \to (E,d_E)$; lemma 6.14 shows that ψ^* is an isomorphism. Moreover $\psi \circ i = \gamma$ by definition.

Finally, if γ was augmentation preserving with respect to ε_E, then (6.7) applies and shows that ψ is also augmentation preserving.

Q.E.D.

6.16.- *Remarks.*

1) Cor 3.10 shows that

$$H(A) \cong \Lambda X^o \otimes H(\Lambda X^+)$$

for suitable choice of X^+.

2) Cor 3.9 shows that γ^* is n-regular if and only if A is n-connected.

6.17.- *Proof of theorem 6.2.*

Consider the diagram

$$\begin{array}{ccc} C & \xrightarrow{\psi} & E \\ {\scriptstyle i}\uparrow & & \uparrow{\scriptstyle \check{\psi}} \\ B & \xrightarrow[\check{i}]{} & \check{C} \end{array} .$$

Since ψ^* is an isomorphism, theorem 5.19 applies and yields a homomorphism

$$\bar{\psi} : (\check{C}, d_{\check{C}}) \to (C, d_C)$$

such that $\bar{\psi} \circ \check{i} = i$ and $\psi \circ \bar{\psi} \sim \check{\psi}$ (rel B).

In particular $\bar{\psi}(\check{i}(\ker \varepsilon_B) \cdot \check{C}) \subset i(\ker \varepsilon_B) \cdot C$.

This can be rewritten as $\bar{\psi}(\ker \check{\rho}) \subset \ker \rho$. Thus $\bar{\psi}$ factors over $\check{\rho}$ to yield a commutative diagram of homomorphisms of c.g.d.a.'s

$$\begin{array}{ccccc} B & \xrightarrow{\check{i}} & \check{C} & \xrightarrow{\check{\rho}} & \check{A} \\ \| & & \downarrow{\scriptstyle \bar{\psi}} & & \downarrow{\scriptstyle \bar{\alpha}} \\ B & \xrightarrow[i]{} & C & \xrightarrow[\rho]{} & A. \end{array}$$

Moreover, since $\psi \circ \bar{\psi} \sim \check{\psi}$ (rel B), $\psi^* \circ \bar{\psi}^* = \check{\psi}^*$. Because ψ^* and $\check{\psi}^*$ are isomorphisms, so is $\bar{\psi}^*$.

Next set $\tilde{\varepsilon}_1 = \varepsilon_C \circ \bar{\psi}$ and $\tilde{\varepsilon}_2 = \varepsilon_A \circ \bar{\alpha}$; these are (possibly new) augmentations in $\check{C}$ and $\check{A}$. It is trivial to check that $B \xrightarrow{\check{i}} \check{C} \xrightarrow{\check{\rho}} \check{A}$, together with ε_B, $\tilde{\varepsilon}_1$, $\tilde{\varepsilon}_2$, is still a minimal KS extension. Denote it by $\tilde{\mathcal{E}}$.

(In fact if (X,f) is a structure for $\check{\mathcal{E}}$ set $\tilde{X} = \{x - \tilde{\varepsilon}_2(x)1 \mid x \in X\}$; then $(\tilde{X}, f)$ is a structure for $\tilde{\mathcal{E}}$. If (1.4) holds for a basis $\{x_\alpha\}_{\alpha \in I}$ for X, then it holds for the basis $\{x_\alpha - \tilde{\varepsilon}_2(x_\alpha)1\}_{\alpha \in I}$ for $\tilde{X}$.)

Observe that by definition

$$(\iota, \bar{\psi}, \bar{\alpha}) : \tilde{\mathcal{E}} \to \mathcal{E}$$

is a morphism of minimal KS extensions. Since $\bar{\psi}^*$ is an isomorphism, theorem 4.5 shows that $\bar{\alpha}$ is an isomorphism. Since ι and $\bar{\psi}^*$ are isomorphisms, theorem 4.6 shows that $\bar{\psi}$ is an isomorphism.

Finally, suppose $\varepsilon_E \circ \gamma = \varepsilon_B$ for some given augmentation ε_E of $\mathcal{E}$, and ψ and $\check{\psi}$ preserve augmentations. Then theorem 5.20 applies (just as theorem 5.19 applied above) to yield a homomorphism

$$\bar{\psi} : (\check{C}, d_{\check{C}}, \varepsilon_{\check{C}}) \to (C, d_C, \varepsilon_C)$$

such that $\bar{\psi} \circ i = i$ and $\psi \circ \bar{\psi}$ is based homotopic (rel B) to $\check{\psi}$.

As above $\bar{\psi}$ induces $\bar{\alpha} : \check{A} \to A$ and this time without making changes

$$(\iota, \bar{\psi}, \bar{\alpha}) : \check{E} \longrightarrow E$$

is a morphism. It follows exactly as above that $(\iota, \bar{\psi}, \bar{\alpha})$ is an isomorphism.

Q.E.D.

6.18.- *Proof of theorem 6.3*. Change $\check{E}$ to $\tilde{E}$ by changing the augmentations of $\check{C}$ and $\check{A}$ so that $(\iota, \chi, \alpha) : \tilde{E} \to E$ is a morphism (as in the proof of theorem 6.2 above). Since $\psi \circ \chi \sim \check{\psi}$ (rel B), χ^* is an isomorphism. Thus theorems 4.5 and 4.6 imply that χ and α are isomorphisms.

Moreover since $\psi \circ \chi \sim \check{\psi} \sim \psi \circ \bar{\psi}$ (rel B) we obtain $\psi \circ \chi \sim \psi \circ \bar{\psi}$ (rel B). Since ψ^* is an isomorphism we may use prop 5.15 ii) to conclude $\chi \sim \bar{\psi}$(rel B). If Φ is a homotopy (rel B) from $\bar{\psi}$ to χ then Φ factors over the projection

$$\rho \otimes \iota : (\check{C}, B)^I \longrightarrow \check{A}^I$$

to yield a homotopy from α to $\bar{\alpha}$.

Finally, if E is augmented by ε_E, all homomorphisms preserve augmentations and $\psi \circ \chi \underset{\sim}{\sim} \check{\psi}$ (rel B) then

$$\psi \circ \chi \underset{\sim}{\sim} \check{\psi} \underset{\sim}{\sim} \psi \circ \bar{\psi} \text{ (rel B)}.$$

It follows that $\psi \circ \chi \underset{\sim}{\sim} \psi \circ \bar{\psi}$ (rel B). Since ψ^* is an isomorphism prop. 5.16 ii) yields $\chi \underset{\sim}{\sim} \bar{\psi}$ (rel B).

Again, as above, this implies $\alpha \underset{\sim}{\sim} \bar{\alpha}$.

Q.E.D.

Theorems 6.1 and 6.2 motivate the following definitions.

6.19.- *Definition.*

Let $\gamma : (B,d_B) \to (E,d_E)$ be a homomorphism of c.g.d.a.'s such that $H^0(B) = k = H^0(E)$. Assume B is augmented by ε_B.

Then a KS extension, $\mathcal{E} : B \xrightarrow{i} C \xrightarrow{\rho} A$, together with a homomorphism $\psi : (C,d_C) \to (E,d_E)$ is called a model for γ if $\psi \circ i = \gamma$ and ψ^* is an isomorphism.

If $\mathcal{E}$ is minimal we call $(\mathcal{E},\psi)$ a minimal model for γ.

6.20.- *Definition.*

Let $\gamma : (B,d_B,\varepsilon_B) \to (E,d_E,\varepsilon_E)$ be a homomorphism of augmented c.g.d.a.'s such that $H^0(B) = k$ and $H^0(E) = k$.

Then a KS extension $\mathcal{E} : B \xrightarrow{i} C \xrightarrow{\rho} A$ together with a homomorphism

$$\psi : (C,d_C,\varepsilon_C) \to (E,d_E,\varepsilon_E)$$

is called a model for γ if $\psi \circ i = \gamma$ and ψ^* is an isomorphism. If $\mathcal{E}$ is a minimal then $(\mathcal{E},\psi)$ is called a minimal model for γ.

In the case $B = k$ these definitions specialize as follows :

6.21.- *Definition.*

Let (E,d_E) be a c.g.d.a. with $H^0(E) = k$. A model for (E,d_E) is a KS complex (C,d_C) together with a homomorphism $\psi : (C,d_C) \to (E,d_E)$ such that ψ^* is an isomorphism. If (C,d_C) is minimal then we call this a minimal model for (E,d_E).

6.22.- *Remarks.*-

1. Theorem 6.1 and theorem 6.2 show that minimal models exist, and are uniquely determined up to isomorphism.

2. If (E,d_E) is a c.g.d.a with $H^0(E) = k$ then the minimal model is automatically connected. Thus if E is augmented ψ automatically

preserves augmentations.

3. By "abus de langage" we may refer to $\mathcal{E}$ rather than $(\mathcal{E},\psi)$ as a model for γ and we may refer to (C,d_C) as a model for (E,d_E).

4. Let $(\mathcal{E},\psi)$ be a model for $\gamma : (B,d_B,\varepsilon_B) \to (E,d_E,\varepsilon_E)$. Then it is a model for $\gamma : (B,d_B) \to (E,d_E)$. Thus in particular as we vary the augmentation of E we do not change the isomorphism class of $\mathcal{E}$ or the (unbased) homotopy class (rel B) of ψ.

6.23.- Definition.

Let (E,d_E) and (F,d_F) be c.g.d.a.'s such that $H^o(E) = k$ and $H^o(F) = k$. Then (E,d_E) and (F,d_F) are called C-equivalent if there is a sequence (E_i,d_i) of c.g.d.a.'s $(i=0,1,\dots,n)$ such that

i) $(E_o,d_o) = (E,d_E)$ and $(E_n,d_n) = (F,d_F)$.

ii) For each i $(i=0,\dots,n-1)$ there is either a homomorphism $\psi_i : (E_i,d_i) \to (E_{i+1},d_{i+1})$ such that ψ_i^* is an isomorphism, or there is a homomorphism $\psi_i : (E_{i+1},d_{i+1}) \to (E_i,d_i)$ such that ψ^* is an isomorphism.

6.24.- Theorem.

(E,d_E) and (F,d_F) are C-equivalent if and only if they have isomorphic minimal models.

Proof.-

Clearly an isomorphism of models defines a C-equivalence. To prove the converse it is enough to consider the case that there is a homomorphism $\psi : (E,d_E) \to (F,d_F)$ such that ψ^* is an isomorphism.

But then if $(C,d_C) \xrightarrow{\psi_E} (E,d_E)$ is a minimal model, by definition $(C,d_C) \xrightarrow{\psi\circ\psi_E} (F,d_F)$ is a minimal model. Hence by theorem 6.2 it is isomorphic with any other.

Q.E.D.

6.25.- Morphisms. Suppose

$$\mathcal{E}_1 : B_1 \xrightarrow{i_1} C_1 \xrightarrow{\rho_1} A_1 \quad \text{and} \quad \mathcal{E}_2 : B_2 \xrightarrow{i_2} C_2 \xrightarrow{\rho_2} A_2$$

are KS extensions. Assume $\varphi : (C_1, d_{C_1}) \to (C_2, d_{C_2})$ restricts to $\psi : B_1 \to B_2$, and that ψ preserves augmentations.

Then, as we observed in the proof of theorem 6.2, $\varphi(\ker \rho_1) \subset \ker \rho_2$ and so φ factors over ρ_1 to give a commutative diagram of homomorphisms of c.g.d.a.'s

$$\begin{array}{ccccc} B_1 & \longrightarrow & C_1 & \longrightarrow & A_1 \\ \downarrow{\scriptstyle\psi} & & \downarrow{\scriptstyle\varphi} & & \downarrow{\scriptstyle\alpha} \\ B_2 & \longrightarrow & C_2 & \longrightarrow & A_2 \end{array}$$

It is a morphism of extensions if and only if φ preserves augmentations.

6.26.- Definition.

(ψ, φ, α) will be called a free morphism from $\mathcal{E}_1$ to $\mathcal{E}_2$.

6.27.- Remarks.

1. A free morphism need not be a morphism !

2. If (ψ, φ, α) is a free morphism then there are unique augmentations (namely $\varepsilon_{C_2} \circ \psi$ and $\varepsilon_{A_2} \circ \alpha$) in C_1 and A_1 such that φ and α are augmentation preserving.

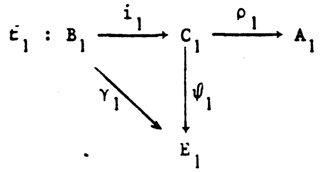
As we observed in the proof of theorem 6.2, with these new augmentations $B_1 \to C_1 \to A_1$ becomes a new KS extension $\tilde{\mathcal{E}}_1$ (minimal if $\mathcal{E}_1$ was) and $(\psi, \varphi, \alpha) : \tilde{\mathcal{E}}_1 \to \mathcal{E}_2$ is a morphism.

Now consider a commutative diagram

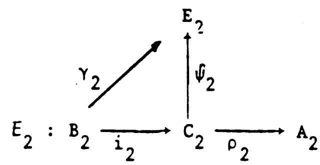
$$\begin{array}{ccc} B_1 & \xrightarrow{\gamma_1} & E_1 \\ \downarrow{\scriptstyle\psi} & & \downarrow{\scriptstyle\eta} \\ B_2 & \xrightarrow{\gamma_2} & E_2 \end{array} \tag{6.28}$$

of homomorphisms of c.g.d.a.'s. Assume B_i is augmented by ε_{B_i} $(i=1,2)$ and that ψ preserves augmentations. Assume $H^o(B_i) = k = H^o(E_i)$, $i=1,2$.

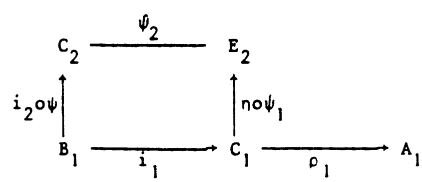
Let



and



be models for γ_1 and γ_2. Combining them with (6.28) we arrive at the commutative diagram



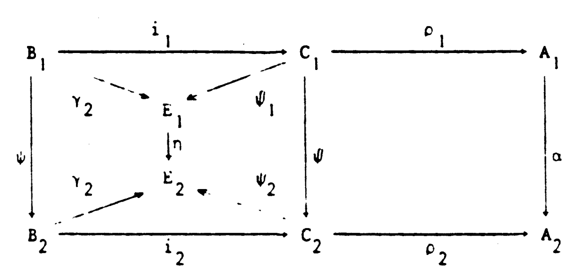
Since φ_2^* is an isomorphism we can apply theorem 5.19 and obtain a homomorphism $\varphi : (C_1,d_{C_1}) \to (C_2,d_{C_2})$ such that

$$\varphi \circ i_1 = i_2 \circ \psi \quad \text{and} \quad \varphi_2\varphi \sim \eta \circ \varphi_1 \text{ (rel } B_1).$$

Moreover the homotopy class of ψ (rel B_1) is uniquely determined.

In particular ψ determines a free morphism (ψ,φ,α) from $\mathcal{E}_1$ to $\mathcal{E}_2$, and we have the diagram

(6.29)



,

in which the triangles and left and right squares commute and the central square homotopy commutes (rel B_1).

6.30.- Remarks.-

1. Assume

$$\begin{array}{ccc} B_2 & \xrightarrow{\gamma_2} & E_2 \\ {\scriptstyle \bar{\psi}}\downarrow & & \downarrow{\scriptstyle \bar{\eta}} \\ B_3 & \xrightarrow[\gamma_3]{} & E_3 \end{array}$$

is a second commutative square of homomorphisms of c.g.d.a.'s. Suppose B_3 is augmented, $\bar{\psi}$ preserves augmentations, and $H^o(B_3) = H^o(E_3) = k$.

Let $\mathcal{E}_3 : B_3 \to C_3 \to A_3$ and $\varphi_3 : C_3 \to E_3$ be a model for γ_3. Repeating the above construction yields a free morphism $(\bar{\psi}, \bar{\varphi}, \bar{\alpha})$ from $\mathcal{E}_2$ to $\mathcal{E}_3$.

Clearly $\bar{\varphi}\varphi i_1 = i_3\bar{\psi}\psi$. Moreover, prop. 5.15 i) shows that $\varphi_3\bar{\varphi}\varphi \sim \bar{\eta}\eta\varphi_1$ (rel B_1). Hence $\bar{\varphi}\varphi$ is a lift for $\bar{\eta}\eta$. (Note that by prop. 5.15 i) the homotopy class of $\bar{\varphi}\varphi$ (rel B_1) depends only on the classes of φ and $\bar{\varphi}$.)

2. Assume in (6.28) that E_1 and E_2 are augmented, and that γ_1 , γ_2 and η respect the augmentations. Then (by definition 6.20) so do φ_1 and φ_2. In this case we can apply theorem 5.20 to obtain a morphism $(\psi, \varphi, \alpha) : \mathcal{E}_1 \to \mathcal{E}_2$ such that $\varphi_2\varphi \overset{\sim}{\sim} \eta\varphi_1$ (rel B_1). The based homotopy classes of φ and of α are uniquely determined by η.

Chapter 7

Isomorphism theorems

In this chapter we consider a morphism,

$$\begin{array}{ccccc} \check{B} & \xrightarrow{\check{i}} & \check{C} & \xrightarrow{\check{\rho}} & \check{A} \\ \downarrow{\scriptstyle\psi} & & \downarrow{\scriptstyle\varphi} & & \downarrow{\scriptstyle\alpha} \\ B & \xrightarrow[i]{} & C & \xrightarrow[\rho]{} & A \end{array} ,$$

of KS extensions $\check{E}$ and E. We always assume that

$$H^o(\check{B}) = H^o(\check{C}) = k \qquad \text{and} \qquad H^o(B) = H^o(C) = k.$$

In view of remark 6.27.2 many of the results extend to free morphisms. The main theorems are as follows :

7.1.- *Theorem*.- Any two of the following three conditions implies the third :

i) ψ^* is an isomorphism.

ii) φ^* is an isomorphism.

iii) Either α^* or $Q(\alpha)^*$ is an isomorphism.

If these conditions hold then both α^* and $Q(\alpha)^*$ are isomorphisms.

7.2.- *Theorem*.- Assume $\check{E}$ and E are minimal. Then any two of the following three conditions implies the third :

i) ψ^* is an isomorphism.

ii) ψ^* is an isomorphism.

iii) $Q(\alpha)$ is an isomorphism.

If they hold, then α and each $Q_p^n(\alpha)$ is an isomorphism.

7.3.- <u>*Theorem*</u>.- Assume $\check{E}$ and E are minimal. Then any two of the following three conditions implies the third :

i) ψ is an isomorphism.

ii) φ is an isomorphism.

iii) $Q(\alpha)$ is an isomorphism.

7.4.- <u>*Remark*</u>.-

These theorems contain theorems 4.5 and 4.6, which we used in chap. 6. Now we use the results of chap 6.

7.5.- <u>*Proof of theorem 7.2*</u>.-

First observe that the final assertion of the theorem, as well as i) and ii) => iii) are proved in theorem 4.5.

i) and iii) => ii) : Consider the minimal model,

$$\begin{array}{ccccc} & & C & & \\ & \overset{\psi}{\nearrow} & \uparrow{\scriptstyle \bar{\psi}} & & \\ \mathcal{E}' : \check{C} & \xrightarrow[j]{} & E & \xrightarrow[p]{} & F \end{array},$$

of the homomorphism $\psi : (\check{C}, d_{\check{C}}, \varepsilon_{\check{C}}) \to (C, d_C, \varepsilon_C)$. Let $I \subset E$ be the ideal generated by $j \circ \check{i}$ (ker $\varepsilon_{\check{B}}$) ; set $G = E/I$ and let $\pi : E \to G$ be the projection. Use π to make G into an augmented c.g.d.a.

It is easy to see that

$$\bar{\mathcal{E}} : \check{B} \xrightarrow{j\check{i}} E \xrightarrow{\pi} G$$

is a KS extension. Moreover, because $H^0(\check{B}) = k$ and $H^0(E) = k$ (since $\bar{\psi}^*$ is an isomorphism) we can apply theorem 2.2. This yields a minimal KS extension $\tilde{\mathcal{E}} : \check{B} \to \tilde{E} \to \tilde{G}$ and a morphism $(\iota, \tilde{\psi}, \tilde{\alpha}) : \tilde{\mathcal{E}} \to \bar{\mathcal{E}}$ such that $\tilde{\psi}^*$ is an isomorphism and (cf. cor. 2.4)

$$Q(\tilde{\alpha})^* : Q(\tilde{G}) \xrightarrow{\cong} H(Q(G), Q(d_G)).$$

On the other hand $\bar{\psi}$ induces a morphism $(\psi, \bar{\psi}, \bar{\alpha})$ from $\bar{E}$ to E. Thus $(\psi, \bar{\psi}\tilde{\psi}, \bar{\alpha}\tilde{\alpha})$ is a morphism between minimal KS extensions and ψ^* and $\bar{\psi}^*\tilde{\psi}^*$ are isomorphisms. Hence theorem 4.5 implies that $Q(\bar{\alpha}\tilde{\alpha})$ is an isomorphism.

But $Q(\bar{\alpha}\tilde{\alpha}) = Q(\bar{\alpha}\tilde{\alpha})^* = Q(\bar{\alpha})^* \, Q(\tilde{\alpha})^*$. Since $Q(\tilde{\alpha})^*$ is an isomorphism, so is $Q(\bar{\alpha})^*$.

Finally observe that j factors over $\check{\rho}$ and p factors over π to yield a commutative diagram,

(7.6)
$$\begin{array}{ccccc} \check{C} & \xrightarrow{j} & E & \xrightarrow{p} & F \\ {\scriptstyle\check{\rho}}\downarrow & & \downarrow{\scriptstyle\pi} & & \| \\ \check{A} & \xrightarrow[\bar{j}]{} & G & \xrightarrow[\bar{p}]{} & F \end{array}$$

which in fact is a morphism of minimal KS extensions.

Since $\check{E}$ and E' are minimal cor 2.4 gives

$$Q(d_{\check{A}}) = 0 \qquad \text{and} \qquad Q(d_F) = 0.$$

Thus the lower row of (7.6) gives rise to the short exact sequence

(7.7) $$0 \to (Q(\check{A}), 0) \to (Q(G), Q(d_G)) \to (Q(F), 0) \to 0.$$

Moreover, it is immediate from the definitions that the diagram

$$\begin{array}{ccc} Q(\check{A}) & \xrightarrow{\qquad} & Q(G) \\ {\scriptstyle Q(\alpha)} \searrow & & \swarrow {\scriptstyle Q(\bar{\alpha})} \\ & Q(A) & \end{array}$$

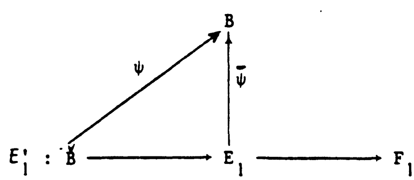
commutes.

But $Q(\alpha)$ is an isomorphism by hypothesis, while $Q(\bar{\alpha})^*$ has just been shown to be an isomorphism. This implies that inclusion induces an isomorphism $Q(\check{A}) \xrightarrow{\cong} H(Q(G),Q(d_G))$. In view of the exact sequence (7·7) we may conclude

$$Q(F) = 0.$$

It follows that $F = k$ and so $\check{C} = E$ and j is the identity map. Hence $\bar{\vartheta} = \vartheta$. But $\bar{\vartheta}^*$ was an isomorphism, and so ϑ^* must be one.

<u>ii) and iii) => i)</u>: Consider the minimal model



of the homomorphism $\psi : (\check{B},d_{\check{B}},\varepsilon_{\check{B}}) \to (B,d_B,\varepsilon_B)$. Let

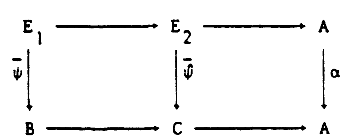
$$E_2 = E_1 \otimes_{\check{B}} \check{C} \; (= E_1 \otimes \check{A}) \qquad ;$$

then the sequence of augmented c.g.d.a.'s

$$\mathcal{E}_1 : E_1 \to E_2 \to \check{A}$$

is a minimal KS extension.

Combine $\bar{\psi}$ and the morphism (ψ,ϑ,α) to get a morphism



of extensions.

We show now that $H^0(E_2) = k$. In fact, write $E_2 = E_1 \otimes \Lambda\check{X}$ and let x_λ be a well ordered homogeneous basis of $\check{X}$ satisfying (1.4) for $\bar{E}_1$. Now $H^0(E_1) = H^0(B) = k$. Assume by induction that $H^0(E_1 \otimes (\Lambda\check{X})_{\leq\lambda}) = k$.

The existence of a non scalar degree zero cocycle in $E_1 \otimes (\Lambda X)_{\leq\lambda}$ would imply (as in previous chapters) that $\deg x_\lambda = 0$ and

$$d(1 \otimes x_\lambda + z) = 0 \qquad , \qquad \text{some } z \in (E_1 \otimes \Lambda X_{<\lambda})^0.$$

We may assume z is in the augmentation ideal, and so, since $H^0(C) = k$, $\bar{\psi}(1 \otimes x_\lambda + z) = 0$. Thus projecting $1 \otimes x_\lambda + z$ to $Q(\check{A})$ and following by $Q(\alpha)$ gives zero. Since $Q(\alpha)$ is an isomorphism we obtain

$$\zeta_A(x_\lambda) \in \zeta_A((\Lambda X)_{<\lambda})$$

which is impossible.

This proves $H^0(E_2) = k$, and so we can apply the second part of the theorem (i) and iii) => ii)) to the morphism above to obtain that $\bar{\psi}^*$ is an isomorphism.

On the other hand, write

$$E_2 = \check{C} \otimes_{\check{B}} E_1 = \check{C} \otimes F_1 .$$

This exhibits

$$\check{C} \xrightarrow{\ j\ } E_2 \xrightarrow{\ \pi\ } F_1$$

as a minimal KS extension, where $\pi = \varepsilon_{\check{C}} \otimes p$, and j is the obvious inclusion. Since $\bar{\psi} j = \check{\psi}$ and $\bar{\psi}^*$ and ψ^* are isomorphisms, we conclude that so is j^*. Hence cor. 3.9 gives $F_1 = k$, $\check{B} = E_1$, $\bar{\psi} = \psi$; in particular ψ^* is an isomorphism.

Q.E.D.

7.8.- *Proof of theorem 7.1*. In view of theorem 2.2 we can find minimal extensions

$$\check{E}_1 : \check{B}_1 \longrightarrow \check{C}_1 \longrightarrow \check{A}_1 \quad \text{and} \quad E_1 : B_1 \longrightarrow C_1 \longrightarrow A_1$$

and morphisms

$$(\psi_1, \psi_1, \alpha_1) : \check{E}_1 \longrightarrow \check{E} \quad \text{and} \quad (\psi_2, \psi_2, \alpha_2) : E \rightarrow E_1$$

such that ψ_i^*, ψ_i^*, α_i^* and $Q(\alpha_i)^*$ are isomorphisms $(i = 1,2)$.

Moreover $(\psi_2\psi\psi_1, \psi_2\psi\psi_1, \alpha_2\alpha\alpha_1) : \check{E}_1 \rightarrow E_1$ is a morphism between minimal KS extensions, to which we can apply theorem 7.2. The final assertion together with i) and ii) => iii) follow immediately. It also follows at once that if $Q(\alpha)^*$ is an isomorphism and either ψ^* or ψ^* is, then both ψ^* and ψ^* are isomorphisms.

Finally, suppose α^* is an isomorphism. Define connected minimal KS complexes $\check{F}$ and F by

$$\check{F} = \check{A}_1 \Big/ (\ker \varepsilon_{\check{A}_1})^o \,.\, \check{A}_1 \quad \text{and} \quad F = A_1 \Big/ (\ker \varepsilon_{A_1})^o \,.\, A_1 \,.$$

Then cor 3.10 shows that

$$H^o(\check{A}_1) = \check{A}_1^o \quad , \quad H^o(A_1) = A_1^o$$

and that the sequences

$$0 \rightarrow H^o(\ker \varepsilon_{\check{A}_1}).H(\check{A}_1) \rightarrow H(\check{A}_1) \rightarrow H(\check{F}) \rightarrow 0$$

and

$$0 \rightarrow H^o(\ker \varepsilon_{A_1}).H(A_1) \rightarrow H(A_1) \rightarrow H(F) \rightarrow 0$$

are short exact.

We may conclude the $\alpha_2\alpha\alpha_1$ is an isomorphism in degree zero. (and so also $Q(\alpha_2\alpha\alpha_1)$ is an isomorphism in degree zero.) We may also conclude that the homomorphism $\bar{\alpha} : \check{F} \to F$ induced from $\alpha_2\alpha\alpha_1$ satisfies : $\bar{\alpha}^*$ is an isomorphism.

Applying theorem 7.2 to the morphism $(\iota,\bar{\alpha},\bar{\alpha})$ from $k \to \check{F} \to \check{F}$ to $k \to F \to F$ we conclude that $Q(\bar{\alpha})$ is an isomorphism. But clearly $Q(\bar{\alpha}) = Q(\alpha_2\alpha\alpha_1)$ in degrees ≥ 1. We have thus shown that $Q(\alpha_2\alpha\alpha_1)$ is an isomorphism. It follows that so is $Q(\alpha)^*$.

Q.E.D.

7.9.- <u>Proof of theorem 7.3</u>.- That i) and ii) imply iii) is shown in theorem 7.2. If i) and iii) hold then theorem 7.2 shows that ψ^* is an isomorphism. Hence theorem 4.6 shows that ψ is an isomorphism.

Finally, assume ii) and iii) hold. Then by theorem 7.2 each $Q_p^n(\alpha)$ is an isomorphism. Thus lemma 4.8 asserts that we can write

$$\psi = \psi \otimes \alpha : \check{B} \otimes \check{A} \to B \otimes A.$$

Since ψ and α are isomorphisms (cf lemma 4.8) it follows that so is ψ .

Q.E.D.

Chapter 8

The ψ-homotopy spaces : $\pi_\psi^*(\gamma)$.

8.1. - Introduction.- In this chapter we consider a homomorphism

(8.2) $$\gamma : (B,d_B,\varepsilon_B) \to (E,d_E,\varepsilon_E)$$

between augmented c.g.d.a.'s such that $H^0(B) = H^0(E) = k$. We shall associate with γ a graded space $\pi_\psi^*(\gamma)$.

8.3. - Lemma.- Assume $\alpha_0,\alpha_1 : (\check{A},d_{\check{A}},\varepsilon_{\check{A}}) \to (A,d_A,\varepsilon_A)$ are homomorphisms of KS complexes, and that α_0 and α_1 are based homotopic. Then

$$Q(\alpha_0)^* = Q(\alpha_1)^*.$$

Proof : Let $\Phi : (\check{A}^I,D,\varepsilon) \to (A,d_A,\varepsilon_A)$ be a based homotopy from α_0 to α_1. Since $\check{A}^I$ is a KS complex we can write

$$Q(\alpha_i)^* = Q(\Phi)^* \circ Q(\lambda_i)^*, \qquad i = 0,1.$$

On the other hand

$$Q(\check{A}^I) = Q_{\check{A}} \oplus \bar{Q}_{\check{A}} \oplus Q(D)(\bar{Q}_{\check{A}})$$

and it follows that $Q(\pi^*) : H(Q(\check{A}^I)) \to H(Q(\check{A}))$ is an isomorphism. Since $Q(\pi) \circ Q(\lambda_i) = \iota$ we obtain

$$Q(\lambda_0)^* = (Q(\pi)^*)^{-1} = Q(\lambda_1)^*,$$

and so $Q(\alpha_0)^* = Q(\alpha_1)^*$.

Q.E.D.

Now let

$$\begin{array}{cccccc} \mathcal{E} : & B & \xrightarrow{i} & C & \xrightarrow{\rho} & A \\ & & \overset{\gamma}{\searrow} & \downarrow \psi & & \\ & & & E & & \end{array}$$

be a model for the homomorphism (8.2) - cf. definition 6.20. Then we set

$$\pi^*_\psi(\gamma;\mathcal{E},\psi) = H^*(Q(A),Q(d_A)) \ . \tag{8.4}$$

Suppose next that

$$\begin{array}{cccccc} \mathcal{E}' : & B & \xrightarrow{i'} & C' & \xrightarrow{\rho'} & A' \\ & & \overset{\gamma}{\searrow} & \downarrow \psi' & & \\ & & & E & & \end{array}$$

is a second model. Theorem 5.20 applies to the commutative square

$$\begin{array}{ccc} C & \xrightarrow{\psi} & E \\ i \uparrow & & \uparrow \psi' \\ B & \xrightarrow[i']{} & C' \end{array}$$

and yields a morphism $(\iota,\bar{\psi},\bar{\alpha}) : \mathcal{E}' \to \mathcal{E}$ such that $\psi\bar{\psi}$ is based homotopic to ψ' (rel B). Moreover the based homotopy classes of $\bar{\psi}$ and $\bar{\alpha}$ are uniquely determined.

In particular there is a canonical homomorphism

$$Q(\bar{\alpha})^* : H(Q(A'),Q(d_{A'})) \longrightarrow H(Q(A),Q(d_A))$$

which is independent of the choice of $\bar{\vartheta}$ and $\bar{\alpha}$. (Apply lemma 8.3.)

Moreover, because $\vartheta\bar{\vartheta} \sim \vartheta'$ (rel B) we have that $\vartheta^* \bar{\vartheta}^* = (\vartheta')^*$, and so $\bar{\vartheta}^*$ is an isomorphism. Thus theorem 7.1 shows that $Q(\bar{\alpha})^*$ is an isomorphism.

We have thus a canonical isomorphism

(8.5) $$\pi_{\psi}^*(\gamma;E',\vartheta') \xrightarrow{\cong} \pi_{\psi}^*(\gamma;E,\vartheta),$$

which depends only on (E',ϑ') and (E,ϑ). If (E'',ϑ'') is a third model for γ then the isomorphism

$$\pi_{\psi}^*(\gamma;E'',\vartheta'') \xrightarrow{\cong} \pi_{\psi}^*(\gamma;E,\vartheta)$$

is obtained by composing the isomorphisms

$\pi_{\psi}^*(\gamma;E'',\vartheta'') \xrightarrow{\cong} \pi_{\psi}^*(\gamma;E',\vartheta')$ and $\pi_{\psi}^*(\gamma;E',\vartheta') \xrightarrow{\cong} \pi_{\psi}^*(\gamma;E,\vartheta)$,

(cf. Remark 6.30.1). Moreover, if $(E',\vartheta') = (E,\vartheta)$ the isomorphism (8.5) is the identity.

Now fix γ, and consider the family of graded spaces $\pi_{\psi}^*(\gamma;E,\vartheta)$ indexed by the models of γ. In view of our remarks above the isomorphisms (8.5) can be used to identify all these spaces as a single graded space.

8.6. - *Definition*.- The graded space obtained by identifying the $\pi_{\psi}^*(\gamma;E,\vartheta)$ will be denoted by $\pi_{\psi}^*(\gamma)$ and will be called the <u>ψ-homotopy space of</u> γ.

8.7. - *Definition*.- If (E,d_E,ε_E) is an augmented c.g.d.a. with $H^o(E) = k$, and if $\gamma : k \to E$ is the inclusion, the graded space $\pi_{\psi}^*(\gamma)$ will be denoted by $\pi_{\psi}^*(E,d_E,\varepsilon_E)$ and called the <u>ψ-homotopy space for</u> E.

8.8. - <u>*Remarks*</u> 1.- Let $(E : B \to C \to A, \Psi)$ be a model for γ. We shall often say we identify $H^*(Q(A))$ with $\pi_\psi^*(\gamma)$; <u>note that this identification depends on</u> Ψ !

2.- We shall frequently further abuse notation and write $\pi_\psi^*(A) = H^*(Q(A))$, even though it may not be the case that $H^0(A) = k$.

3.- If (E, Ψ) is a minimal model then

$$\pi_\psi^*(\gamma) = Q(A).$$

4.- We shall frequently abuse notation and write (in definition 8.7) $\pi_\psi^*(E)$ for $\pi_\psi^*(E, d_E, \varepsilon_E)$.

5.- Observe that the isomorphism class of $\pi_\psi^*(E)$ is independent of the choice of augmentations.

8.9. - <u>*Morphisms*</u>.- Suppose now that

$$(8.10)\qquad \begin{array}{ccc} B_1 & \xrightarrow{\gamma_1} & E_1 \\ \psi \downarrow & & \downarrow \eta \\ B_2 & \xrightarrow[\gamma_2]{} & E_2 \end{array}$$

is a commutative square of homomorphisms augmented c.g.d.a.'s. Assume $H^0(B_i) = H^0(E_i) = k$, $i = 1,2$. Let (E_1, Ψ_1) be a model for γ_1 and (E_2, Ψ_2) be a model for γ_2.

Then (cf. remark 6.30.2 and diagram (6.29)) there is a morphism $(\psi, \hat{\Psi}, \alpha)$: E_1 to E_2 such that $\Psi_2 \hat{\psi} \simeq \eta \Psi_1$ (rel B_1), and this completely determines the based homotopy class of α.

Hence by lemma 8.3 the linear map

$$Q(\alpha)^* : \pi_\psi^*(\gamma_1;E_1,\vartheta_1) \to \pi_\psi^*(\gamma_2;E_2,\vartheta_2)$$

is independ nt of the choice of ϑ and α. Moreover (cf. remark 6.30.1) if (E_1',ϑ_1') and (E_2',ϑ_2') are also models for γ_1 and γ_2 then the diagrams,

$$\begin{array}{ccc} \pi_\psi^*(\gamma_1;E_1',\vartheta_1') & \xrightarrow{\quad Q(\alpha')^* \quad} & \pi_\psi^*(\gamma_2;E_2',\vartheta_2') \\ \cong \downarrow & & \downarrow \cong \\ \pi_\psi^*(\gamma_1;E_1,\vartheta_1) & \xrightarrow[\quad Q(\alpha)^* \quad]{} & \pi_\psi^*(\gamma_2;E_2,\vartheta_2) \end{array} ,$$

commute. (The vertical arrows are the isomorphisms (8.5)·

It follows that the linear maps $Q(\alpha)^*$ define a linear map $\pi_\psi^*(\gamma_1) \to \pi_\psi^*(\gamma_2)$ which depends only on the square (8.10).

8.11.- *Definition*.- The linear map defined above will be denoted by

$$(\psi,\eta)^* : \pi_\psi^*(\gamma_1) \to \pi_\psi^*(\gamma_2).$$

If $B_1 = B_2 = k$ we write it simply as

$$\eta^* : \pi_\psi^*(E_1) \to \pi_\psi^*(E_2).$$

8.12.- *Proposition*.- i) Assume that

$$\begin{array}{ccc} B_2 & \xrightarrow{\quad \gamma_2 \quad} & E_2 \\ \psi' \downarrow & & \downarrow \eta' \\ B_3 & \xrightarrow{\quad \gamma_3 \quad} & E_3 \end{array}$$

is a second commutative square of homomorphisms of augmented c.g.d.a.'s. Assume $H^o(B_3) = H^o(E_3) = k$. Then

$$(\psi'\psi, \eta'\eta)^{\#} = (\psi', \eta')^{\#} \circ (\psi, \eta)^{\#} .$$

ii) If $B_1 = B_2$, $E_1 = E_2$, $\gamma_1 = \gamma_2$, and $\psi = \iota$, $\eta = \iota$ we have

$$(\iota, \iota)^{\#} = \iota .$$

Proof : Apply remark 6.30.1.

Q.E.D.

Chapter 9

Λ-extensions.

9.1. - *Definition*.- A <u>Λ-extension</u> is a KS extension $\mathcal{E} : B \xrightarrow{i} C \xrightarrow{\rho} A$ such that (B, d_B, ε_B) is itself a KS complex. The Λ-extension is called <u>Λ-minimal</u> if B is a minimal KS complex and $\mathcal{E}$ is a minimal KS extension.

9.2. - *Remarks* 1.- A Λ-extension may be minimal as a KS extension, without being Λ-minimal.

2.- Suppose $\mathcal{E} : B \xrightarrow{i} C \xrightarrow{\rho} A$ is a Λ-extension. Let (X,f) be a structure for $\mathcal{E}$ and assume $\{x_\alpha\}_{\alpha \in I}$ is a well ordered homogeneous basis for X such that (1.4) holds. Let $Y \subset \ker \varepsilon_B$ be a graded space such that $B = \Lambda Y$ and assume $\{y_\gamma\}_{\gamma \in J}$ is a well ordered homogeneous basis of Y such that (1.4) holds.

Then the triple (Y,X,f) determines a commutative diagram

$$
(9.3) \qquad
\begin{array}{ccccc}
\Lambda Y & \longrightarrow & \Lambda Y \otimes \Lambda X & \longrightarrow & \Lambda X \\
\| & & \cong \big\downarrow f & & \| \\
B & \xrightarrow[i]{} & C & \xrightarrow[\rho]{} & A
\end{array}
$$

of homomorphisms of graded algebras. This diagram, together with the equations

$$
(9.4) \qquad d_B y_\gamma \in (\Lambda Y)_{<\gamma} \quad \text{and} \quad d_C(1 \otimes x_\alpha) \in \Lambda Y \otimes (\Lambda X)_{<\alpha}
$$

exhibits (C, d_C, ε_C) as a KS complex.

3.- Suppose C is a minimal KS complex. Then by cor. 2.4 $Q(d_C) = 0$ and hence (9.3) implies that $Q(d_A) = 0$ and $Q(d_B) = 0$. Thus (again by cor. 2.4) B and $\mathcal{E}$ are minimal ; i.e., $\mathcal{E}$ is Λ-minimal. The converse, however, may fail : $\mathcal{E}$ may be Λ-minimal while C is not minimal.

Now consider a Λ-extension $E : B \xrightarrow{i} C \xrightarrow{\rho} A$.
As in chap. 5, denote by $\bar{Q}_B,\dots$ the suspensions of $Q_B,\dots,$ and suspend $Q(i)$ and $Q(\rho)$ to linear maps $\overline{Q(i)}$ and $\overline{Q(\rho)}$. These extend uniquely to homomorphisms

$$\Lambda\bar{Q}_B \otimes \Lambda D\bar{Q}_B \xrightarrow{\bar{i}} \Lambda\bar{Q}_C \otimes \Lambda D\bar{Q}_C \xrightarrow{\bar{\rho}} \Lambda\bar{Q}_A \otimes \Lambda D\bar{Q}_A$$

between the contractible complexes generated by $\bar{Q}_B, \bar{Q}_C$ and $\bar{Q}_A$.

Next, tensor this sequence with

$$B \xrightarrow{i} C \xrightarrow{\rho} A$$

to obtain a sequence

(9.5) $$E^I : B^I \xrightarrow{i^I} C^I \xrightarrow{\rho^I} A^I$$

in which $i^I = i \otimes \bar{i}$ and $\rho^I = \rho \otimes \bar{\rho}$.

Choose now (Y,X,f) and well ordered homogeneous bases $\{y_\gamma\}_{\gamma\in J}$ for Y and $\{x_\alpha\}_{\alpha\in I}$ for X, satisfying the conditions of remark 9.2.2. From (9.3) we obtain the commutative diagram

(9.6) $$\begin{array}{ccccc} Y & \longrightarrow & Y \otimes X & \longrightarrow & X \\ \zeta_B \downarrow \cong & & \cong \downarrow \zeta_C \circ f & & \cong \downarrow \zeta_A \\ Q_B & \xrightarrow[Q(i)]{} & Q_C & \xrightarrow[Q(\rho)]{} & Q_A \end{array} .$$

Thus the choice of (Y,X,f) determines isomorphisms

(9.7) $$Q_C \cong Q_B \otimes Q_A \quad \text{and} \quad \bar{Q}_C \cong \bar{Q}_B \otimes \bar{Q}_A$$

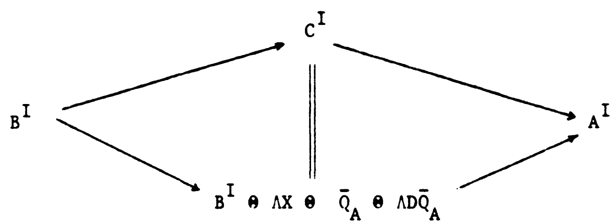
compatible with $Q(i)$ and $Q(\rho)$ (resp. $\overline{Q(i)}$ and $\overline{Q(\rho)}$). Using (9.3) and (9.7) we may write

$$C^I = B \otimes \Lambda X \otimes \Lambda\bar{Q}_B \otimes \Lambda D\bar{Q}_B \otimes \Lambda\bar{Q}_A \otimes \Lambda D\bar{Q}_A$$

(9.8)
$$= B^I \otimes \Lambda X \otimes \Lambda\bar{Q}_A \otimes \Lambda D\bar{Q}_A \ .$$

Moreover, the diagram

(9.9)



commutes. Thus E^I is exhibited as a Λ-extension.

We may also write

(9.10)
$$C^I = \Lambda Y \otimes \Lambda X \otimes \Lambda\bar{Q}_B \otimes \Lambda D\bar{Q}_B \otimes \Lambda\bar{Q}_A \otimes \Lambda D\bar{Q}_A \ .$$

Recall the degree -1 derivation i and the degree 0 derivation θ defined in chap. 5. We apply this definition to the KS complex C to obtain a degree -1 derivation i_C and a degree zero derivation θ_C in C^I given by

$$i_C = 0 \quad \text{in} \quad \bar{Q}_B, D\bar{Q}_B, \bar{Q}_A, D\bar{Q}_A,$$

$$i_C(y) = \bar{y} \ (\in \bar{Q}_B), \quad i_C(x) = \bar{x}(\in \bar{Q}_A), \quad y \in Y,\ x \in X$$

and
$$\theta_C = Di_C + i_C D \ .$$

Then i_C and θ_C restrict to derivations i_B and θ_B in B^I, and project to derivations i_A and θ_A in A^I.

Next denote the standard inclusions and projections by

$$(\lambda_o)_B : B \to B^I, \quad (\lambda_o)_C : C \to C^I, \quad (\lambda_o)_A : A \to A^I$$

and
$$\pi_B : B^I \to B, \quad \pi_C : C^I \to C, \quad \pi_A : A^I \to A.$$

Set $(\lambda_1)_B = e^{\theta_B} \circ (\lambda_o)_B, \ldots$. Then

$$((\lambda_o)_B,(\lambda_o)_C,(\lambda_o)_A) : E \to E^I,$$

$$(\pi_B,\pi_C,\pi_A) : E^I \to E,$$

and
$$((\lambda_1)_B,(\lambda_1)_C,(\lambda_1)_A) : E \to E^I$$

are morphisms of extensions.

9.11. - Remark.- The ideals $\ker \pi_B \subset B^I$ and $\ker \pi_C \subset C^I$ are acyclic, and $i^I(\ker \pi_B) \subset \ker \pi_C$. Moreover,

$$\operatorname{Im}((\lambda_o)_B - (\lambda_1)_B) \subset \ker \pi_B \quad \text{and} \quad \operatorname{Im}((\lambda_o)_C - (\lambda_1)_C) \subset \ker \pi_C.$$

Finally, let $((C,B)^I,D)$ be the c.g.d.a. defined in chap. 5 with inclusions $\lambda_o,\lambda_1 : C \to (C,B)^I$ and projection $\pi : (C,B)^I \to C$. Consider the projection

$$P : C^I \to (C,B)^I$$

defined by

$$P(z) = z, \ z \in C,$$

$$P = 0 \text{ in } \bar{Q}_B \text{ and } D\bar{Q}_B,$$

$$P(\bar{x}) = \bar{x} \text{ and } PD\bar{x} = D\bar{x}, \ \bar{x} \in \bar{Q}_A.$$

Evidently P is a surjective homomorphism of augmented c.g.d.a.'s, and P^* is an isomorphism ; i.e.

$$H(\ker P) = 0.$$

It follows from the definitions that

$$P \circ (\lambda_i)_C = \lambda_i \circ P, \quad i = 0,1.$$

Suppose now that $\eta : (G,d_G) \to (E,d_E)$ is a homomorphism of c.g.d.a.'s. Assume that

$$(9.12) \qquad \begin{array}{ccc} (B,d_B) & \xrightarrow{\psi_j} & (G,d_G) \\ {\scriptstyle i}\downarrow & & \downarrow{\scriptstyle \eta} \\ (C,d_C) & \xrightarrow[\vartheta_j]{} & (E,d_E) \end{array} \quad , \quad j = 0,1,$$

are commutative squares of homomorphisms.

9.13. - *Definition*.- The pair (ψ_o,ϑ_o) is called homotopic to the pair (ψ_1,ϑ_1) if there is a commutative square of homomorphisms

$$\begin{array}{ccc} (B^I,D) & \xrightarrow{\Psi} & (G,d_G) \\ {\scriptstyle i^I}\downarrow & & \downarrow{\scriptstyle \eta} \\ (C^I,D) & \xrightarrow[\Phi]{} & (E,d_E) \end{array}$$

such that $\Psi \circ (\lambda_i)_B = \psi_i$ and $\Phi \circ (\lambda_i)_C = \vartheta_i$, $i = 0,1$. We write $(\psi_o,\vartheta_o) \sim (\psi_1,\vartheta_1)$.

The pair (Ψ,Φ) is called a homotopy from (ψ_o,ϑ_o) to (ψ_1,ϑ_1).

Now suppose that E and G are augmented by ε_E and ε_G, and that ψ_o, ψ_1, ϑ_o, ϑ_1 and η preserve augmentations.

9.14.- *Definition*.- (ψ_o,ϑ_o) is called <u>based homotopic</u> to (ψ_1,ϑ_1) if there is a homotopy (Ψ,Φ) from (ψ_o,ϑ_o) to (ψ_1,ϑ_1) such that Ψ and Φ preserve augmentations. The pair (Ψ,Φ) is then called a <u>based homotopy from</u> (ψ_o,ϑ_o) <u>to</u> (ψ_1,ϑ_1) and we write $(\psi_o,\vartheta_o) \stackrel{\sim}{\sim} (\psi_1,\vartheta_1)$.

9.15.- *Remarks* 1.- We shall see in prop. 9.17. that these definitions do not depend on the choice of structure used to define $(\lambda_1)_B, \dots$.

2.- Assume $(\psi_o,\vartheta_o) \sim (\psi_1,\vartheta_1)$. Then $\psi_o \sim \psi_1$ and $\vartheta_o \sim \vartheta_1$, and hence $\psi_o^* = \psi_1^*$ and $\vartheta_o^* = \vartheta_1^*$. However, the converse is not always true ; it may be the case that $\psi_o \sim \psi_1$ and $\vartheta_o \sim \vartheta_1$ without (ψ_o,ϑ_o) and (ψ_1,ϑ_1) being homotopic.

3.- Suppose $\vartheta_o,\vartheta_1 : (C,d_C) \to (E,d_E)$ satisfy $\vartheta_j \circ i = \eta \circ \psi$, $j = 0,1$, for some $\psi : (B,d_B) \to (G,d_G)$. A homotopy (Ψ,Φ) from (ψ,ϑ_o) to (ψ,ϑ_1) is said to be <u>constant in</u> B if

$$\Psi = \psi \circ \pi_B .$$

It follows easily that a homotopy is constant in B if and only if it has the form $(\psi \circ \pi_B, \Omega \circ P)$ where P is the projection defined above and Ω is a homotopy (rel B) from ϑ_o to ϑ_1. Moreover, if Ω is any homotopy (rel B) from ϑ_o to ϑ_1 then $(\psi \circ \pi_B, \Omega \circ P)$ is a homotopy from (ψ,ϑ_o) to (ψ,ϑ_1) constant in B.

This defines a bijection between homotopies (rel B) and homotopies constant in B. In particular, if $\vartheta_o \sim \vartheta_1$ (rel B) then $(\psi,\vartheta_o) \sim (\psi,\vartheta_1)$, but the reverse implication does not always hold.

4.- Homotopy extension property.- Assume $\psi_o, \psi_1 : (B,d_B) \to (G,d_G)$ and $\vartheta_o : (C,d_C) \to (E,d_E)$ are homomorphisms and that $\vartheta_o \circ i = \eta \circ \psi_o$. Suppose further that Ψ is a homotopy from ψ_o to ψ_1.

Then a bijection between homotopies (Ψ,Φ) starting at (ψ_o,ϑ_o) (and restricting to the given Ψ in B^I) on the one hand, and degree zero linear maps $\bar{Q}_A \to E$ on the other is given by restricting Φ to $\bar{Q}_A$.

In particular (take $\Phi(\bar{Q}_A) = 0$) there is always a homomorphism $\vartheta_1 : (C,d_C) \to (E,d_E)$ such that

$$\vartheta_1 \circ i = \eta \circ \psi_1 \quad \text{and} \quad (\psi_o,\vartheta_o) \sim (\psi_1,\vartheta_1) .$$

5.- Remark 3 applies verbatim to based homotopies, and homomorphisms of augmented c.g.d.a.'s. Remark 4 applies with the single change that the bijection is with degree zero linear maps $\bar{Q}_A \to \ker \varepsilon_E$.

9.16.- Lemma.- Assume given the commutative squares (9.12) with the property that

$$\mathrm{Im}(\psi_1 - \psi_o) \subset I_G \quad \text{and} \quad \mathrm{Im}(\vartheta_1 - \vartheta_o) \subset I_E \quad ,$$

where I_G and I_E are acyclic ideals in G and in E, and $\eta(I_G) \subset I_E$.

Then $(\psi_o,\vartheta_o) \sim (\psi_1,\vartheta_1)$, and the homotopy (Ψ,Φ) can be chosen so that $\Psi(\bar{Q}_B) \subset I_G$ and $\Phi(\bar{Q}_A) \subset I_E$.

Moreover if the homomorphisms of (9.12) preserve augmentations and $I_G \subset \ker \varepsilon_G$ and $I_E \subset \ker \varepsilon_E$, then (Ψ,Φ) is a based homotopy.

Proof : Use the proof of lemma 5.11.

Q.E.D.

9.17. - *Proposition.*-

i) Definitions 9.13 and 9.14 of homotopic and based homotopic are independent of the choice of (Y,X,f).

ii) Homotopy and based homotopy are equivalence relations on commutative squares of the form (9.12) (with $B \xrightarrow{i} C$ and $G \xrightarrow{\eta} E$ fixed !).

Proof : Use lemma 9.16 the same way lemma 5.11 and cor. 5.12 were used to prove prop. 5.14.

Q.E.D.

Next suppose $\check{E} : \check{B} \xrightarrow{\check{i}} \check{C} \xrightarrow{\check{\rho}} \check{A}$ is a second Λ-extension, and that $\check{\eta} : (\check{G}, d_{\check{G}}) \to (\check{E}, d_{\check{E}})$ is a homomorphism of c.g.d.a.'s. Assume given commutative squares of homomorphisms of c.g.d.a.'s

$$\begin{array}{ccc} \check{B} & \xrightarrow{\chi_B} & B \\ {\scriptstyle \check{i}}\downarrow & & \downarrow{\scriptstyle i} \\ \check{C} & \xrightarrow[\chi_C]{} & C \end{array} \quad \text{and} \quad \begin{array}{ccc} G & \xrightarrow{\gamma_G} & \check{G} \\ {\scriptstyle \eta}\downarrow & & \downarrow{\scriptstyle \check{\eta}} \\ E & \xrightarrow[\gamma_E]{} & \check{E} \end{array}$$

9.18. - *Proposition.*- Consider the commutative squares (9.12) in conjunction with the squares above :

i) If $(\psi_0, \vartheta_0) \sim (\psi_1, \vartheta_1)$ then $(\gamma_G \psi_0, \gamma_E \vartheta_0) \sim (\gamma_G \psi_1, \gamma_E \vartheta_1)$ and $(\psi_0 \chi_B, \vartheta_0 \chi_C) \sim (\psi_1 \chi_B, \vartheta_1 \chi_C)$.

ii) If γ_G^* and γ_E^* are isomorphisms then $(\psi_0, \vartheta_0) \sim (\psi_1, \vartheta_1)$ if and only if $(\gamma_G \psi_0, \gamma_E \vartheta_0) \sim (\gamma_G \psi_1, \gamma_E \vartheta_1)$.

iii) If all c.g.d.a.'s are augmented and all homomorphisms preserve augmentations, then i) and ii) hold with "based homotopic" replacing "homotopic".

Proof : Part i) is trivial (cf. the proof of prop. 5.15). To prove ii) we may assume $(\gamma_G \psi_o, \gamma_E \psi_o) \sim (\gamma_G \psi_1, \gamma_E \psi_1)$. Then, as in the proof of prop. 5.15 ii) define a commutative square

$$\begin{array}{ccc} G \otimes \Lambda F \otimes \Lambda DF & \xrightarrow{\;\;\bar\gamma_G\;\;} & \check{G} \\ {\scriptstyle \eta \otimes \eta_1}\downarrow & & \downarrow{\scriptstyle \check\eta} \\ E \otimes \Lambda L \otimes \Lambda DL & \xrightarrow[\;\;\bar\gamma_E\;\;]{} & \check{E} \end{array}$$

where $F = \check{G}$, $L = \check{E}$ and η_1 is the obvious homomorphism extending $\check\eta$.

Denote the left hand side of the above square by

$$\bar G \xrightarrow{\;\bar\eta\;} \bar E$$

and let the obvious inclusions and projections be written

$$G \underset{p_G}{\overset{j_G}{\rightleftarrows}} \bar G \qquad\qquad E \underset{p_E}{\overset{j_E}{\rightleftarrows}} \bar E \;.$$

Now suppose $(\check\Psi, \check\Phi)$ is a homotopy from $(\gamma_G \psi_o, \gamma_E \psi_o)$ to $(\gamma_G \psi_1, \gamma_E \psi_1)$. Choose linear maps (of degree zero)

$$\Psi : \bar Q_B \to \bar G \quad \text{and} \quad \Phi : \bar Q_A \to \bar E$$

so that $\bar\gamma_G \Psi = \check\Psi$ and $\bar\gamma_E \Phi = \check\Phi$. These extend to a unique homotopy (Ψ, Φ) starting at $(j_G \psi_o, j_E \psi_o)$.

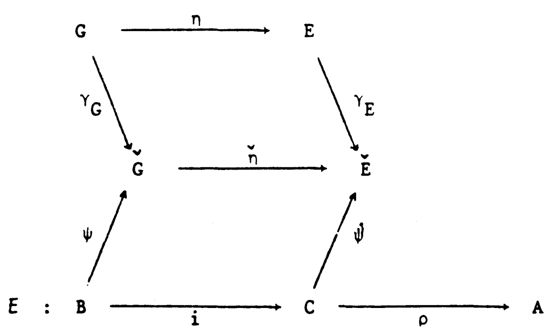
Use lemma 9.16 (as lemma 5.11 is used in prop. 5.15) to conclude the proof.

To prove iii) modify the proof of prop. 5.16 the way the proof of prop. 5.15 was modified above.

Q.E.D.

Suppose that $\mathcal{E} : B \xrightarrow{i} C \xrightarrow{\rho} A$ is a fixed Λ-extension.

9.19. - *Theorem*.- Assume



is a commutative diagram of homomorphisms of c.g.d.a.'s such that

i) γ_G^* and γ_E^* are isomorphisms.

ii) $\mathcal{E}$ is a Λ-extension.

Then there are homomorphisms $\psi_o : (B,d_B) \to (G,d_G)$ and $\dot{\psi}_o : (C,d_C) \to (E,d_E)$ such that

$$\dot{\psi}_o \circ i = \eta \circ \psi_o \quad \text{and} \quad (\gamma_G\psi_o, \gamma_E\dot{\psi}_o) \sim (\psi, \dot{\psi})$$

If $(\psi_1, \dot{\psi}_1)$ also satisfies these conditions, then

$$(\psi_1, \dot{\psi}_1) \sim (\psi_o, \dot{\psi}_o).$$

Proof : *Existence*.- By theorem 5.19, we can find $\psi_o : (B,d_B) \to (G,d_G)$ so that $\gamma_G\psi_o \sim \psi$. By remark 9.15.4 we can find $\bar{\psi} : (C,d_C) \to (\check{E},d_{\check{E}})$ so that

$$\bar{\psi} \circ i = \check{\eta}\,\gamma_G\psi_o \quad \text{and} \quad (\gamma_G\psi_o, \bar{\psi}) \sim (\psi, \dot{\psi}).$$

Now theorem 5.19 applies again and yields $\psi_o : (C,d_C) \to (E,d_E)$ such that

$$\psi_o \circ i = \eta \circ \psi_o \quad \text{and} \quad \gamma_E \psi_o \sim \bar{\psi} \text{ (rel B)}.$$

By remark 9.15.3 this implies $(\gamma_G \psi_o, \gamma_E \psi_o) \sim (\gamma_G \psi_o, \bar{\psi})$ and since homotopy is an equivalence relation we obtain $(\gamma_G \psi_o, \gamma_E \psi_o) \sim (\psi, \psi)$.

Uniqueness.- Apply prop. 9.18 ii).

Q.E.D.

9.20.- *Theorem*.- Assume all the homomorphisms in the diagram of theorem 9.19 are homomorphisms of augmented c.g.d.a.'s. Then (ψ_o, ψ_o) may be chosen to preserve augmentations, as well as to satisfy

$$\hat{\psi}_o \circ i = \eta \circ \psi_o \quad \text{and} \quad (\gamma_G \psi_o, \gamma_E \psi_o) \sim (\psi, \psi).$$

Moreover, these conditions determine (ψ_o, ψ_o) up to based homotopy.

Proof : Modify the proof of theorem 9.19 , using theorem 5.20 and prop. 9.18 iii).

Q.E.D.

Finally, consider free morphisms

$$(\psi_i, \psi_i, \alpha_i) : \check{E} \to E, \quad i = 0,1,$$

where

$$\check{E} : \check{B} \xrightarrow{\check{i}} \check{C} \xrightarrow{\check{\rho}} \check{A} \quad \text{and} \quad E : B \xrightarrow{i} C \xrightarrow{\rho} A$$

are Λ-extensions. If (Ψ, Φ) is a homotopy from (ψ_o, ψ_o) to (ψ_1, ψ_1) then in particular

$$\Psi : \check{B}^I \to B.$$

Now if Ψ is a <u>based</u> homotopy then we have $\Psi(\ker \varepsilon) \subset \ker \varepsilon_B$ and it follows that

$$\Phi(\ker \check{\rho}^I) \subset \ker \rho .$$

Thus Φ factors over $\check{\rho}^I$ to yield a homomorphism

$$\Xi : \check{A}^I \to A,$$

which is a homotopy from α_o to α_1.

9.22. - *<u>Definition</u>*.- If Ψ is a based homotopy we say that $(\psi_o,\varphi_o,\alpha_o)$ and $(\psi_1,\varphi_1,\alpha_1)$ are <u>homotopic</u> by the homotopy (Ψ,Φ,Ξ). We write $(\psi_o,\varphi_o,\alpha_o) \sim (\psi_1,\varphi_1,\alpha_1)$.

If $(\psi_i,\varphi_i,\alpha_i)$ are actual morphisms and Φ is also a based homotopy, then

$$(\Psi,\Phi,\Xi) : \check{E}^I \to E$$

is a morphism. We call it a based homotopy from $(\psi_o,\varphi_o,\alpha_o)$ and $(\psi_1,\varphi_1,\alpha_1)$, and write $(\psi_o,\varphi_o,\alpha_o) \approx (\psi_1,\varphi_1,\alpha_1)$.

Chapter 10

Λ-models.

In this chapter, we consider a homomorphism of c.g.d.a.'s

$$\eta : (G, d_G) \to (E, d_E)$$

such that $H^0(G) = k$ and $H^0(E) = k$. We adapt the theorems of chap. 6.

10.1. - Theorem (existence).- There is a commutative diagram of homomorphisms of c.g.d.a.'s

$$\begin{array}{ccccccc} & & G & \xrightarrow{\ \eta\ } & E & & \\ & & \uparrow{\scriptstyle \psi} & & \uparrow{\scriptstyle \vartheta} & & \\ \mathcal{E} & : & B & \xrightarrow[\ i\ ]{} & C & \xrightarrow[\ \rho\ ]{} & A \end{array}$$

such that

i) $\mathcal{E}$ is a Λ-minimal Λ-extension.

ii) ψ^* and ϑ^* are isomorphisms.

Moreover, if ε_G and ε_E augment G and E and $\varepsilon_E \eta = \varepsilon_G$, then ψ and ϑ can be chosen to preserve augmentations.

Proof : Let $\psi : B \to G$ be the minimal model for (G, d_G) (or for (G, d_G, ε_G)). Let $(\mathcal{E}, \vartheta)$ be the minimal model for $\eta\psi : B \to E$ (or for $\eta\psi : (B, d_B, \varepsilon_B) \to (E, d_E, \varepsilon_E)$).

Q.E.D.

10.2. - Remark.- Note that $H^0(B) = H^0(G) = k$. Thus in the KS extension $k \to B \to B$ the inclusion of k is 0-regular. Thus cor. 3.9. shows that B is connected. In particular B has exactly one augmentation.

Next suppose

$$E : B \xrightarrow{i} C \xrightarrow{\rho} A \quad \text{and} \quad \check{E} : \check{B} \xrightarrow{\check{i}} \check{C} \xrightarrow{\check{\rho}} \check{A}$$

are Λ-minimal Λ-extensions. Assume *

$$\begin{array}{ccc} G & \xrightarrow{\eta} & E \\ \psi \uparrow & & \uparrow \vartheta \\ B & \xrightarrow[i]{} & C \end{array} \qquad \begin{array}{ccc} G & \xrightarrow{\eta} & E \\ \check{\psi} \uparrow & & \uparrow \check{\vartheta} \\ \check{B} & \xrightarrow[\check{i}]{} & \check{C} \end{array}$$

are commutative squares of homomorphisms of c.g.d.a.'s such that ψ^*, ϑ^*, $\check{\psi}^*$ and $\check{\vartheta}^*$ are isomorphisms.

10.3. - *Theorem* (*uniqueness*).- Under the hypotheses above there is a commutative diagram of homomorphisms of c.g.d.a.'s

$$\begin{array}{ccccc} \check{B} & \xrightarrow{\check{i}} & \check{C} & \xrightarrow{\check{\rho}} & \check{A} \\ \bar{\psi} \downarrow \cong & & \bar{\vartheta} \downarrow \cong & & \cong \downarrow \bar{\alpha} \\ B & \xrightarrow[i]{} & C & \xrightarrow[\rho]{} & A \end{array} ,$$

such that $\bar{\psi}$, $\bar{\vartheta}$ and $\bar{\alpha}$ are isomorphisms, and $(\psi\bar{\psi}, \vartheta\bar{\vartheta}) \sim (\check{\psi}, \check{\vartheta})$.

If η, ϑ, $\check{\vartheta}$, ψ, $\check{\psi}$ are augmentation preserving (for given augmentations of G and E) then $(\bar{\psi}, \bar{\vartheta}, \bar{\alpha})$ can be chosen to be augmentation preserving and so that $(\psi\bar{\psi}, \vartheta\bar{\vartheta}) \approx (\check{\psi}, \check{\vartheta})$.

Proof : Apply theorem 9.19. (or in the augmented case theorem 9.20.) to obtain homomorphisms $\bar{\psi}$, $\bar{\vartheta}$ such that $(\psi\bar{\psi}, \vartheta\bar{\vartheta}) \sim (\check{\psi}, \check{\vartheta})$.

By remark 10.2., $\bar{\psi}$ is automatically augmentation preserving.

Hence $\bar{\vartheta}$ factors to give $\bar{\alpha}$.

Now because $(\psi\bar{\psi}, \vartheta\bar{\vartheta}) \sim (\check{\psi}, \check{\vartheta})$ we have that $\bar{\psi}^*$ and $\bar{\vartheta}^*$ are isomorphisms. Apply theorem 7.2 to the morphism

$$(\iota, \bar{\psi}, \bar{\psi}) : (k \longrightarrow \check{B} \xrightarrow{\cong} \check{B}) \to (k \longrightarrow B \xrightarrow{\cong} B)$$

to obtain that $\bar{\psi}$ is an isomorphism.

On the other hand, (cf. remark 6.27.2), we can reaugment $\check{C}$ and $\check{A}$ so that $(\bar{\psi}, \bar{\vartheta}, \bar{\alpha})$ is a morphism. Since $\bar{\psi}$ is an isomorphism and $\bar{\vartheta}^*$ is an isomorphism, theorem 7.2. implies that $\bar{\alpha}$ is an isomorphism. Hence theorem 7.3. shows that so is $\bar{\vartheta}$.

Q.E.D.

10.4. - Theorem (uniqueness of isomorphism). - With the hypotheses and notation of theorem 10.3. assume that $(\psi_o, \vartheta_o, \alpha_o) : \check{E} \to E$ is a free morphism such that

$$(\psi\psi_o, \vartheta\vartheta_o) \sim (\check{\psi}, \check{\vartheta}).$$

Then $(\psi_o, \vartheta_o, \alpha_o)$ is an isomorphism, and $(\psi_o, \vartheta_o) \sim (\bar{\psi}, \bar{\vartheta})$.

Moreover, if E and G are augmented and all homomorphisms preserve augmentations, and if $(\psi\psi_o, \vartheta\vartheta_o) \simeq (\check{\psi}, \check{\vartheta})$ then $(\psi_o, \vartheta_o, \alpha_o) \simeq (\bar{\psi}, \bar{\vartheta}, \bar{\alpha})$.

Proof : Note that ϑ_o^* and ψ_o^* are isomorphisms and argue as above that $(\psi_o, \vartheta_o, \alpha_o)$ is therefore an isomorphism. Prop. 9.18. ii) implies (because $(\psi\psi_o, \vartheta\vartheta_o) \sim (\check{\psi}, \check{\vartheta}) \sim (\psi\bar{\psi}, \vartheta\bar{\vartheta})$) that $(\psi_o, \vartheta_o) \sim (\bar{\psi}, \bar{\vartheta})$.

In the augmented case prop. 9.18 iii) implies that $(\psi_o, \vartheta_o) \simeq (\psi_1, \vartheta_1)$ whence (cf. definition 9.21) $(\psi_o, \vartheta_o, \alpha_o) \simeq (\psi_1, \vartheta_1, \alpha_1)$.

Q.E.D.

10.5. - *Definition*.- Let $\eta : (G,d_G) \to (E,d_E)$ be a homomorphism of c.g.d.a.'s such that $H^o(G) = k = H^o(E)$. Then a Λ-model for η is a Λ-extension $\mathcal{E} : B \xrightarrow{i} C \xrightarrow{\rho} A$ together with homomorphisms

$$\psi : (B,d_B) \to (G,d_G) \quad \text{and} \quad \hat{\psi} : (C,d_C) \to (E,d_E)$$

such that $\hat{\psi} \circ i = \eta \circ \psi$ and ψ^* and $\hat{\psi}^*$ are isomorphisms.

If $\mathcal{E}$ is Λ-minimal then $(\mathcal{E},\psi,\hat{\psi})$ is called a Λ-minimal Λ-model for η.

10.6. - *Definition*.- Let $\eta : (G,d_G,\varepsilon_G) \to (E,d_E,\varepsilon_E)$ be a homomorphism of augmented c.g.d.a.'s such that $H^o(G) = k = H^o(E)$. Then a Λ-model for η is a Λ-extension $\mathcal{E} : B \xrightarrow{i} C \xrightarrow{\rho} A$ together with homomorphisms

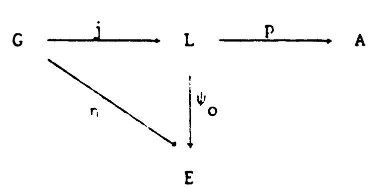
$$\psi : (B,d_B,\varepsilon_B) \to (G,d_G,\varepsilon_G) \quad \text{and} \quad \hat{\psi} : (C,d_C,\varepsilon_C) \to (E,d_E,\varepsilon_E)$$

such that $\hat{\psi} \circ i = \eta \circ \psi$ and ψ^* and $\hat{\psi}^*$ are isomorphisms.

If $\mathcal{E}$ is Λ-minimal then $(\mathcal{E},\psi,\hat{\psi})$ is called a Λ-minimal Λ-model for η.

10.7. - *Remarks* 1.- Theorems 10.1 and 10.3 show that Λ-minimal Λ-models always exist, and are unique up to isomorphism.

2.- Another way of constructing the Λ-minimal Λ-model of η is as follows (if G is augmented). Let



be the minimal model of η. Then let

$$(B, d_B, \varepsilon_B) \xrightarrow{\ \psi\ } (G, d_G, \varepsilon_G)$$

be the minimal model of G. Finally, let

$$\begin{array}{cccccc} E\ : & B & \xrightarrow{\ \ i\ \ } & C & \xrightarrow{\ \ \rho\ \ } & A \\ & {\scriptstyle\psi}\big\downarrow & & \big\downarrow{\scriptstyle\vartheta_1} & & \\ & G & \xrightarrow[\ \ j\ \ ]{} & L & & \end{array}$$

be the minimal model of $j\psi$.

Then $(E, \psi, \vartheta_0\vartheta_1)$ is a (and so the) Λ-minimal Λ-model of η. Note that because ψ preserves augmentations, ϑ_1 factors to yield

$$\begin{array}{ccccc} B & \longrightarrow & C & \longrightarrow & A \\ {\scriptstyle\psi}\big\downarrow & & \big\downarrow{\scriptstyle\vartheta_1} & & {\scriptstyle\cong}\big\downarrow{\scriptstyle\alpha} \\ G & \longrightarrow & L & \longrightarrow & \check{A} \end{array} ,$$

where α is an isomorphism by theorem 7.2.

Thus the fibres of the Λ-minimal Λ-model of η and of the minimal model of η coïncide.

3.- Suppose (E, ψ, ϑ) is a Λ-model for $\eta : G \to E$. Then $\psi : B \to G$ is a model for G and (E, ϑ) is a model for $\eta \circ \psi$. If (E, ψ, ϑ) is Λ-minimal then both these are the minimal models.

On the other hand $\vartheta : C \to E$ is also a model, and even if (E, ψ, ϑ) is Λ-minimal this model for E need not be.

10.8. - Morphisms.- Consider a commutative square

$$(10.9)\qquad \begin{array}{ccc} G_1 & \xrightarrow{\eta_1} & E_1 \\ {\scriptstyle \gamma_G}\downarrow & & \downarrow{\scriptstyle \gamma_E} \\ G_2 & \xrightarrow[\eta_1]{} & E_2 \end{array}$$

of homomorphisms of c.g.d.a.'s. Assume $H^o(G_i) = H^o(E_i) = k$, $i = 1,2$. Combine (10.9) with Λ-models $(E_i, \psi_i, \vartheta_i)$ for η_i $(i = 1,2)$. This gives a diagram

$$(10.10)\qquad \begin{array}{ccccccccc} B_1 & & & \xrightarrow{\quad i_1 \quad} & & & C_1 & \xrightarrow{\rho_1} & A_1 \\ & \searrow^{\psi_1} & & & & \swarrow^{\vartheta_1} & & & \\ & & G_1 & \xrightarrow{\eta_1} & E_1 & & & & \\ {\scriptstyle \psi}\downarrow & & {\scriptstyle \gamma_G}\downarrow & & \downarrow{\scriptstyle \gamma_E} & & \downarrow{\scriptstyle \vartheta} & & \downarrow{\scriptstyle \alpha} \\ & & G_2 & \xrightarrow[\eta_2]{} & E_2 & & & & \\ & \nearrow_{\psi_2} & & & & \nwarrow_{\vartheta_2} & & & \\ B_2 & & & \xrightarrow[\quad i_2 \quad]{} & & & C_2 & \xrightarrow[\rho_2]{} & A_2 \end{array} .$$

Here ψ and ϑ are constructed so that $(\gamma_G \psi_1, \gamma_E \vartheta_1) \sim (\psi_2 \psi, \vartheta_2 \vartheta)$, (cf. theorem 9.19). Note that all the remaining squares commute. We assume the augmentations in B_1 and B_2 are chosen so that ψ preserves augmentations (this is a vacuous assumption if B_1 is minimal). Then ϑ factors to give α. Finally note that the homotopy class of (ψ, ϑ) is uniquely determined.

10.11. - Remarks 1.- Suppose

$$\begin{array}{ccc} G_2 & \xrightarrow{\eta_2} & E_2 \\ \bar{\gamma}_G \downarrow & & \downarrow \bar{\gamma}_E \\ G_3 & \xrightarrow[\eta_3]{} & E_3 \end{array}$$

is a second commutative square with $H^o(G_3) = H^o(E_3) = k$. Let $(E_3, \psi_3, \vartheta_3)$ be a Λ-model for η_3 and repeat the above construction to obtain a free morphism $(\bar{\psi}, \bar{\vartheta}, \bar{\alpha}) : E_2 \to E_3$.

Then $(\bar{\psi}\psi, \bar{\vartheta}\vartheta, \bar{\alpha}\alpha) : E_1 \to E_3$ is the free morphism corresponding to $(\bar{\gamma}_G \circ \gamma_G, \bar{\gamma}_E \circ \gamma_E)$.

2.- Assume in (10.9) that the c.g.d.a.'s are augmented and that the homomorphisms preserve augmentations. Let $(E_i, \psi_i, \vartheta_i)$ be a Λ-model for $\eta_i : (G_i, d_{G_i}, \varepsilon_{G_i}) \to (E_i, d_{E_i}, \varepsilon_{E_i})$, $i = 1,2$.

Then using theorem 9.20 , we can choose $(\psi, \vartheta, \alpha)$ to be a morphism such that $(\gamma_G\, \psi_1, \gamma_E\, \vartheta_1) \simeq (\psi_2\psi, \vartheta_2\vartheta)$. Moreover this uniquely determines the based homotopy class of $(\psi, \vartheta, \alpha)$.

10.12. - The exact ψ-homotopy sequence.- Suppose $E : B \xrightarrow{i} C \xrightarrow{\rho} A$ is a Λ-extension. Then the sequence of differential spaces

$$0 \to (Q(B), Q(d_B)) \xrightarrow{Q(i)} (Q(C), Q(d_C)) \xrightarrow{Q(\rho)} (Q(A), Q(d_A)) \to 0$$

is short exact, as follows from (9.6). Hence it gives rise to an exact triangle

(10.13)
$$\begin{array}{ccc} H(Q(B)) & \xrightarrow{\quad Q(i)^* \quad} & H(Q(C)) \\ & {\scriptstyle \partial^*}\nwarrow \qquad \swarrow{\scriptstyle Q(\rho)^*} & \\ & H(Q(A)) & \end{array}$$

in which the connecting homomorphism, ∂^*, has degree +1.

A morphism of Λ-extensions induces a morphism of short exact sequences and hence a morphism of exact cohomology triangles. According to lemma 8.3 this last only depends on the based homotopy class of the original morphism.

Now suppose that (E, ψ, φ) is a Λ-model for a homomorphism

$$\eta : (G, d_G, \varepsilon_G) \longrightarrow (E, d_E, \varepsilon_E)$$

of augmented c.g.d.a.'s. Then

$$\psi : (B, d_B, \varepsilon_B) \to (G, d_G, \varepsilon_G) \quad \text{and} \quad \varphi : (C, d_C, \varepsilon_C) \to (E, d_E, \varepsilon_E)$$

are models.

Thus, as described in chap. 8, we have canonical identifications

(10.14) $$H(Q(B)) \xrightarrow{\cong} \pi_\psi^*(G) \quad \text{and} \quad H(Q(C)) \xrightarrow{\cong} \pi_\psi^*(E).$$

On the other hand, (E, ι) is clearly a model for i and hence we have a canonical identification

(10.15) $$H(Q(A)) \xrightarrow{\cong} \pi_\psi^*(i).$$

Moreover, because $\varphi \circ i = \eta \circ \psi$ we can apply definition 8.11 to obtain a linear map

$$(\psi,\varphi)^{\#} \; : \pi_\psi^*(i) \longrightarrow \pi_\psi^*(\eta) .$$

Since ψ^* and φ^* are isomorphisms theorem 7.2 shows that so is $(\psi,\varphi)^{\#}$. Thus composing (10.15) with $(\psi,\varphi)^{\#}$ gives an isomorphism

$$H(Q(A)) \xrightarrow{\;\cong\;} \pi_\psi^*(\eta) \; . \tag{10.16}$$

We can thus express the exact triangle (10.13) in the form

$$\begin{array}{ccc} \pi_\psi^*(G) & \xrightarrow{\;\eta^{\#}\;} & \pi_\psi^*(E) \\ & {}_{\partial^*}\nwarrow \quad \swarrow & \\ & \pi_\psi^*(\eta) & \end{array} \tag{10.17}$$

If (E',ψ',φ') is a second Λ-model then theorem 9.20 gives a morphism $(\bar\psi,\bar\varphi,\bar\alpha) \; : \; E' \to E$ such that $(\psi\bar\psi,\varphi\bar\varphi) \simeq (\psi',\varphi')$. This completely determines the based homotopy class of the morphism. Moreover, ψ^* and $\bar\varphi^*$ are isomorphisms, and hence $Q(\bar\psi)^*, Q(\bar\varphi)^*$ and $Q(\bar\alpha)^*$ are canonical isomorphisms.

If we identify the "primed triangle" analogous to (10.13) with (10.17) as described above then the two identifications are identified by $Q(\bar\psi)^*, Q(\bar\varphi)^*$ and $Q(\bar\alpha)^*$.

Finally observe that a commutative square

$$\begin{array}{ccc} G_1 & \xrightarrow{\;\eta_1\;} & E_1 \\ {\scriptstyle \gamma_G}\downarrow & & \downarrow{\scriptstyle \gamma_E} \\ G_2 & \xrightarrow[\;\eta_2\;]{} & E_2 \end{array}$$

of homomorphisms of augmented c.g.d.a.'s defines a commutative diagram

$$\begin{array}{ccccccccccc}
\to & \pi_\psi^p(G_1) & \xrightarrow{\eta_1^\#} & \pi_\psi^p(E_1) & \longrightarrow & \pi_\psi^p(\eta_1) & \xrightarrow{\partial^*} & \pi_\psi^{p+1}(G_1) & \to \\
 & \Big\downarrow \gamma_G^\# & & \Big\downarrow \gamma_E^\# & & \Big\downarrow (\gamma_G,\gamma_E)^\# & & \Big\downarrow \gamma_G^\# & \\
\to & \pi_\psi^p(G_2) & \xrightarrow[\eta_2^\#]{} & \pi_\psi^p(E_2) & \longrightarrow & \pi_\psi^p(\eta_2) & \xrightarrow[\partial^*]{} & \pi_\psi^{p+1}(G_2) & \to & .
\end{array} \qquad (10.18)$$

10.19.- Example.- Suppose the Λ-extension $\mathcal{E}$ is in fact Λ-minimal. Then the short exact sequence reads

$$0 \longrightarrow (Q(B),0) \longrightarrow (Q(C),Q(d_C)) \longrightarrow (Q(A),0) \longrightarrow 0$$

as follows from cor. 2.4.

Thus $Q(d_C)(Q(B)) = 0$ and $\operatorname{Im} Q(d_C) \subset Q(B)$.
Hence $Q(d_C)$ factors to produce a linear map

$$\overline{Q(d_C)} : Q(A) \to Q(B) ,$$

and this by definition is exactly the connecting homomorphism ∂_*.

In other words, $\partial^* = 0$ if and only if $Q(d_C) = 0$.
Thus (cf. cor. 2.4) ∂^* exactly measures the failure of the middle KS complex in the Λ-minimal Λ-extension $\mathcal{E}$ to be itself a minimal KS complex.

Now if $\eta : G \to E$ is a homomorphism of augmented c.g.d.a.'s with $H^o(G) = H^o(E) = k$, then η has a unique Λ-minimal Λ-model. The middle term of this will be the minimal model for E if and only if the connecting homomorphism ∂^* in (10.17) is zero.

Chapter 11

Automorphisms.

In this chapter (A,d_A) denotes a connected KS complex and $\zeta_A : A^+ \to Q(A)$ is the projection (cf. 1.1). Thus if we write $A = \Lambda X$ then $\zeta_A : X \xrightarrow{\cong} Q(A)$.

11.1.- The Hurewicz map. Recall (2.1) that a differential, $Q(d_A)$, is defined in $Q(A)$ by $Q(d_A)\zeta_A = \zeta_A \circ d_A$. Thus ζ_A induces a map

$$\zeta_A^* : H^+(A) \to H(Q(A),Q(d_A)) ;$$

this map is called the Hurewicz map.

If $\check{\psi} : (A,d_A) \to (\check{A},d_{\check{A}})$ is a homomorphism of connected KS complexes, then it induces a linear map $Q(\check{\psi}) : Q(A) \to Q(\check{A})$ and the diagram

$$(11.2) \qquad \begin{array}{ccc} H^+(A) & \xrightarrow{\zeta_A^*} & H(Q(A)) \\ \Big\downarrow{\scriptstyle \check{\psi}^*} & & \Big\downarrow{\scriptstyle Q(\check{\psi})^*} \\ H^+(\check{A}) & \xrightarrow[\zeta_{\check{A}}^*]{} & H(Q(\check{A})) \end{array}$$

commutes.

In particular, suppose $\psi : (A,d_A) \to (E,d_E,\varepsilon_E)$ is a model for an augmented c.g.d.a. (E,d_E,ε_E). If we use ψ to identify

$$H(A) = H(E) \qquad \text{and} \qquad H(Q(A)) = \pi_\psi^*(E)$$

(cf 8.8.1) then ζ_A^* defines a map

(11.3) $$H^{+}(E) \to \pi_{\psi}^{*}(E).$$

Diagram (11.2) shows that (11.3) does not depend on the choice of model ; it is called the Hurewicz map for (E, d_E, ε_E).

Now suppose that (A, d_A) is minimal (so that $Q(d_A) = 0$ - cf. cor 2.4). Then

$$\zeta_A^{*} : H^{+}(A) \to Q(A) .$$

On the other hand, recall from 3.2 (with $B = k$ and $C = A$) the filtrations

$$Q_o^n(A) \subset Q_1^n(A) \subset \dots \subset Q^n(A), \qquad n \geq 1.$$

According to theorem 3.4 ζ_A induces isomorphisms

$$\bar{\zeta}_A : Z_p^n \big/ A_{p-1,n}^n \xrightarrow{\cong} Q_p^n(A) \big/ Q_{p-1}^n(A) \qquad , \quad p \geq 1$$

(11.4) and

$$\bar{\zeta}_A : Z_o^n \big/ A_{-1,n}^n \xrightarrow{\cong} Q_o^n(A).$$

By definition the second restricts to an isomorphism (cf. prop 3.8)

(11.5) $$\bar{\zeta}_A : \operatorname{Im} \beta \xrightarrow{\cong} \operatorname{Im} \zeta_A^{*} .$$

11.6.- *Locally nilpotent families*. Let $\{\psi_\gamma\}_{\gamma\in\Gamma}$ be a family of linear transformations of a vector space W. We say the family is locally nilpotent if there is a well ordered basis $\{w_\alpha\}_{\alpha\in I}$ of W such that for $\gamma \in \Gamma$, $\alpha \in I$ $\psi_\gamma w_\alpha$ is a linear combination of the w_β with $\beta < \alpha$. If V is a subspace of W which is stable under each ψ_γ then the family is locally nilpotent in W if and only if the induced families of transformations of V and W/V are locally nilpotent. If W is a direct sum of subspaces W_λ and each W_λ is stable under the φ_γ then $\{\varphi_\gamma\}$ is locally nilpotent if and only if for each λ $\{\varphi_\gamma | W_\lambda\}$ is locally nilpotent.

Finally, if $\{\varphi_\gamma - \iota\}$ is locally nilpotent we say $\{\varphi_\gamma\}$ is <u>locally unipotent</u>.

11.7.- *<u>Proposition</u>*. Let (A,d_A) be a minimal connected KS complex. Assume $\varphi_\gamma : (A,d_A) \to (A,d_A)$, $\gamma \in \Gamma$, is a family of c.g.d.a. homomorphisms. Then the following are equivalent :

i) $\{\varphi_\gamma\}$ is a locally unipotent family.

ii) $\{\varphi_\gamma^*\}$ is a locally unipotent family.

iii) $\{Q(\varphi_\gamma)\}$ is a locally unipotent family.

iv) The restrictions of the $\{Q(\varphi_\gamma)\}$ to $\mathrm{Im}\ \zeta_A^*$ form a locally unipotent family.

<u>Proof</u>.- The following implications are evident from the remarks above :

$$\begin{array}{ccccc} & & \text{ii)} & & \\ & \nearrow & & \searrow & \\ \text{i)} & & & & \text{iv)} \\ & \searrow & & \nearrow & \\ & & \text{iii)} & & \end{array}$$

It is also easy to verify that iii) $\Rightarrow$ i).

Now assume iv) holds. We apply prop. 3.8 (parts iii) and iv)) together with the isomorphisms (11.4) and (11.5). Together these show that if i) holds for some $A_{p-1,n}$ then iii) holds for $A_{p,n}$. Since iii) $\Rightarrow$ i) the proposition follows.

Q.E.D.

Next, consider a locally unipotent family $\{\varphi_\gamma\}$ of automorphisms of a minimal connected KS complex (A,d_A). Since the $Q(\varphi_\gamma)$ are locally unipotent and respect the filtration

$$Q_o^n(A) \subset Q_1^n(A) \subset \dots\dots$$

we can find a well ordered homogeneous basis $\{z_\alpha\}_{\alpha\in I}$ of $Q(A)$ subject to the following conditions :

(11.8) $\beta < \alpha \Rightarrow \deg z_\beta \leq \deg z_\alpha$.

(11.9) $\beta < \alpha$, $\deg z_\beta = n = \deg z_\alpha$, $z_\alpha \in Q_p^n(A) \Rightarrow z_\beta \in Q_p^n(A)$.

(11.10) For all γ and all α, $(Q(\psi_\gamma)-\iota)z_\alpha$ is a linear combination of the z_β with $\beta < \alpha$.

11.11.- *Proposition*. Let z_α satisfy the above conditions and let $x_\alpha \in A$ be homogeneous vectors such that $\zeta_A x_\alpha = z_\alpha$. Then

i) The x_α are a basis for a graded space $X \subset A^+$, and $A = \Lambda X$.

ii) $d_A x_\alpha \in (\Lambda X)_{<\alpha}$.

iii) For each γ and each α, $(\psi_\gamma-\iota)x_\alpha \in (\Lambda X)_{<\alpha}$.

Proof.- i) and iii) are immediate from the connectivity of A. ii) follows from theorem 3.4 (because of (11.8) and (11.9)).

Q.E.D.

11.12.- *The circle construction*. Assume (A,d_A) is minimal. Fix a locally unipotent automorphism, ψ, of (A,d_A). For each $z \in A$ there is some n_z such that

$$(\psi-\iota)^{n_z} z = 0.$$

Thus we can define

$$\theta = \log \psi = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n} (\psi-\iota)^n \quad ;$$

it is a locally nilpotent derivation of A, homogeneous of degree zero, and commuting with d_A. Note that $\psi = e^\theta$.

We can also define a linear map of degree zero, commuting with d_A : $\psi : A \to A$, by

$$\psi = -\sum_{n=0}^{\infty} \frac{1}{(n+1)!}\,\theta^n.$$

It is locally unipotent, and hence a linear automorphism. It satisfies

$$\theta\psi = \psi\theta = \iota - \phi\ .$$

Now let $\{x_\alpha\}_{\alpha\in I}$ satisfy the conclusions of prop. 11.11 (for ϕ) and let X be their span ; then $A = \Lambda X$. Define a c.g.d.a. $(\Lambda u \otimes A, D_\phi)$, (where deg $u = 1$) by

$$D_\phi u = 0$$

and

$$D_\phi(1 \otimes a) = 1 \otimes d_A a + u \otimes \theta a, \qquad a \in A.$$

Apply prop 11.11 ii) and iii) to conclude that

$$d_A x_\alpha \in (\Lambda X)_{<\alpha} \qquad \text{and} \qquad \theta x_\alpha \in (\Lambda X)_{<\alpha} \qquad ;$$

it follows that $(\Lambda u \otimes A,\ D_\phi)$ is a minimal KS complex. It will be called the circle construction for ϕ.

On the other hand denote by $\Lambda(t,dt)$ the contractible KS complex generated by an element t of degree zero. Define

$$\pi_0,\ \pi_1\ :\ \Lambda(t,dt) \otimes A \to A$$

by $\pi_0(t) = 0$, $\pi_1(t) = 1$ and $\pi_0(a) = \pi_1(a) = a$, $a \in A$.

Let (K_ϕ,d) be the c.g.d.a defined by

$$K_\phi = \ker(\pi_1 - \phi\pi_0).$$

Then a homomorphism of c.g.d.a.'s

$$\sigma_\phi\ :\ (\Lambda u \otimes A, D_\phi) \to (K_\phi, d) \tag{11.13}$$

is defined by $\sigma_\phi(u) = dt$, $\quad \sigma_\phi(a) = \sum_{n=0}^{\infty} \frac{1}{n!}\, t^n \otimes \theta^n(a)$.

11.14.- *Proposition.* σ_ψ^* is an isomorphism. Thus (11.13) is the minimal model of (K_ψ, d).

Proof.- Define a short exact sequence of differential spaces

$$0 \to A \xrightarrow{\alpha} \Lambda u \otimes A \xrightarrow{\beta} A \to 0 \tag{11.15}$$

by $\alpha(a) = -u \otimes \psi^{-1}(a)$ and $\beta(u \otimes a + 1 \otimes a_1) = a_1$. On the other hand we have the short exact sequence

$$0 \to K_\psi \xrightarrow{j} \Lambda(t,dt) \otimes A \xrightarrow{\pi_1 - \psi\pi_0} A \to 0, \tag{11.16}$$

where j is the inclusion.

In the long exact cohomology sequence arising from (11.15) the connecting map $\partial_0 : H^{p+1}(A) \to H^{p+1}(A)$ is given by

$$\partial_0 = \psi^* - \iota$$

as is easy to check. On the other hand, if $a \in A$ then

$$(\pi_1 - \psi\pi_0)\left(-\sum_{n=0}^{\infty} \frac{t^{n+1}}{(n+1)!} \otimes \theta^n \psi^{-1} a\right) = a.$$

It follows that the connecting map ∂_1 for the long exact sequence arising from (11.16) is given by

$$\partial_1 = \sigma_\psi^* \alpha^* .$$

Thus the diagram

$$\begin{array}{ccccccccc}
\to H^p(A) & \xrightarrow{\alpha^*} & H^{p+1}(\Lambda u \otimes A) & \xrightarrow{\beta^*} & H^{p+1}(A) & \xrightarrow{-\partial_0} & H^{p+1}(A) \to \\
\cong \downarrow \iota & & \downarrow \sigma_\psi^* & & \cong \downarrow (\pi_0^*)^{-1} & & \cong \downarrow \iota \\
\to H^p(A) & \xrightarrow[\partial_1]{} & H^{p+1}(K_\psi) & \xrightarrow[j^*]{} & H^{p+1}(\Lambda(t,dt) \otimes A) & \xrightarrow[(\pi_1 - \psi\pi_0)^*]{} & H^{p+1}(A) \to
\end{array}$$

commutes and the proposition follows.

Q.E.D.

Next recall from chap. 5 the c.g.d.a. A^I. (Let $B = k$, $C = A$.) We have the projection $\pi : A^I \to A$ and the inclusions $\lambda_0, \lambda_1 : A \to A^I$. Assume $\Phi : A^I \to A$ is a homotopy from ψ to $\bar{\psi}$. Then $\bar{\psi}^* = \psi^*$, so $\bar{\psi}$ is also locally unipotent. Let $\bar{\theta} = \log \bar{\psi}$.

Regard π_i as projections $\Lambda(t,dt) \otimes A^I \to A^I$, $i = 0,1$, and define

$$P = \pi \circ \pi_1 - \Phi \circ \pi_0 : \Lambda(t,dt) \otimes A^I \to A.$$

Then set $K_\Phi = \ker P$; it is a sub c.g.d.a.

Consider the row exact commutative diagram

$$\begin{array}{ccccccc}
0 & \longrightarrow & K_\psi & \longrightarrow & \Lambda(t,dt) \otimes A & \xrightarrow{\pi_1 - \psi\pi_0} & A \\
 & & \downarrow{\scriptstyle \gamma} & & \downarrow{\scriptstyle 1\otimes\lambda_0} & & \\
0 & \longrightarrow & K_\Phi & \longrightarrow & \Lambda(t,dt) \otimes A^I & \xrightarrow{P} & A \\
 & & \uparrow{\scriptstyle \bar{\gamma}} & & \uparrow{\scriptstyle 1\otimes\lambda_1} & & \\
0 & \longrightarrow & K_{\bar{\psi}} & \longrightarrow & \Lambda(t,dt) \otimes A & \xrightarrow{\pi_1 - \bar{\psi}\pi_0} & A
\end{array} \quad ;$$

from it we deduce that γ^* and $\bar{\gamma}^*$ are isomorphisms.

Now identify u with dt. Then

$$\begin{array}{ccc}
 & & K_\Phi \\
 & \nearrow & \uparrow{\scriptstyle \gamma \circ \sigma_\psi} \\
\Lambda u & \longrightarrow & (\Lambda u \otimes A, D_\psi) \to A
\end{array}$$

and

$$\begin{array}{ccc}
 & & K_\Phi \\
 & \nearrow & \uparrow{\scriptstyle \bar{\gamma} \circ \sigma_{\bar{\psi}}} \\
\Lambda u & \longrightarrow & (\Lambda u \otimes A, D_{\bar{\psi}}) \to A
\end{array}$$

are both minimal models for the inclusion $\Lambda u \to K_\Phi$.

Hence by theorem 6.2 there is a c.g.d.a isomorphism

$$\tau : (\Lambda u \otimes A, D_{\psi}) \xrightarrow{\cong} (\Lambda u \otimes A, D_{\bar\psi})$$

such that $\tau(u) = u$. Clearly τ must have the form

$$\tau(1 \otimes a) = -u \otimes \tilde{s}(a) + 1 \otimes \tau_o(a), \qquad a \in A,$$

where τ_o is an automorphism of (A, d_A).

Straightforward calculations show that

$$\tilde{s}(ab) = \tilde{s}(a)\tau_o(b) + (-1)^{\deg a}\tau_o(a)\tilde{s}(b)$$

and

$$\tilde{s}d_A + d_A\tilde{s} = \tau_o\theta - \bar\theta\tau_o$$

In particular a degree -1 derivation, s, in A is given by

$$s = \tau_o^{-1}\tilde{s}.$$

It satisfies

$$sd_A + d_As = \theta - \tau_o^{-1}\bar\theta\tau_o.$$

We have thus proved

11.17.- *Proposition*. Assume $\bar\psi$ is homotopic to ψ. Then there is a degree -1 derivation, s, of A, and an automorphism τ_o of (A, d_A) such that

$$\log\psi = sd_A + d_As + \tau_o^{-1}\log\bar\psi\,\tau_o.$$

In particular, if $\psi \sim \iota$ then

$$\log\psi = sd_A + d_As.$$

Next suppose Φ is a based homotopy between ψ and $\bar{\psi}$. Since A is connected $A^1 = X^1$ and $A = \Lambda X^1 \otimes \Lambda X^{\geq 2}$. Thus (since A is minimal)

$$\Lambda X^1 \to A \to \Lambda X^{\geq 2}$$

is a minimal KS extension and so we can form the c.g.d.a $(A,\Lambda X^1)^I$. The projection $A^I \xrightarrow{p} (A,\Lambda X^1)^I$ (of chap. 9) satisfies $p \circ \lambda_i = \bar{\lambda}_i$, $i = 0,1$, where $\bar{\lambda}_i : A \to (A,\Lambda X^1)^I$ are the inclusions. Since Φ is based and A is connected, Φ factors over p to yield a homotopy

$$\Psi : (A,\Lambda X^1)^I \to A$$

such that $\Psi \circ \bar{\lambda}_0 = \psi$, $\quad \Psi \circ \bar{\lambda}_1 = \bar{\psi}$.

Now repeat the above construction with Ψ replacing Φ to achieve the commutative diagram

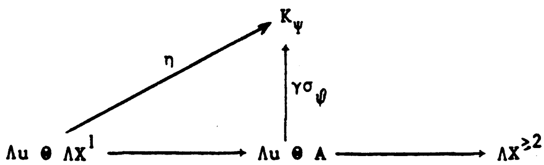
$$\begin{array}{ccccccccc}
0 & \longrightarrow & K_{\psi} & \longrightarrow & \Lambda(t,dt) \otimes A & & \searrow & & \\
 & & \downarrow{\scriptstyle \gamma} & & \downarrow{\scriptstyle \iota \otimes \bar{\lambda}_0} & & & & \\
0 & \longrightarrow & K_{\Psi} & \longrightarrow & \Lambda(t,dt) \otimes (A,\Lambda X^1)^I & \longrightarrow & A & \longrightarrow & 0 \\
 & & \uparrow{\scriptstyle \bar{\gamma}} & & \uparrow{\scriptstyle \iota \otimes \bar{\lambda}_1} & & & & \\
0 & \longrightarrow & K_{\bar{\psi}} & \longrightarrow & \Lambda(t,dt) \otimes A & & \nearrow & &
\end{array}$$

Note that $(A,\Lambda X^1)^I$ is connected. Thus if we augment $\Lambda(t,dt)$ by $t \to 0$ then K_{ψ}, K_{Ψ}, $K_{\bar{\psi}}$ are augmented and γ, $\bar{\gamma}$ preserve the augmentations.

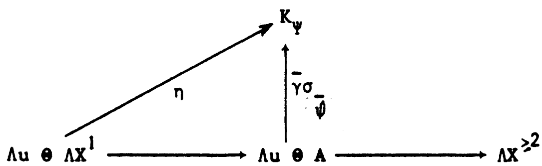
Moreover $\bar{\lambda}_0, \bar{\lambda}_1$ agree in ΛX^1 and hence so do ψ, $\bar{\psi}$ and θ, $\bar{\theta}$. Thus we can write

$$\gamma\sigma_{\psi} = \bar{\gamma}\sigma_{\bar{\psi}} = \eta : \Lambda u \otimes \Lambda X^1 \to K_{\Psi} .$$

In particular



and



are both minimal models for η.

We apply theorem 6.2 to obtain an isomorphism

$$\tau : (\Lambda u \otimes A, D_\psi) \xrightarrow{\cong} (\Lambda u \otimes A, D_{\bar{\psi}})$$

such that $\tau = \iota$ in $\Lambda u \otimes \Lambda X^1$. The resulting derivation s is thus zero in X^1, and we obtain

11.18.- Proposition. Assume $\bar{\psi} \simeq \psi$. Then there is a degree -1 derivation s of A and an automorphism τ_o of (A, d_A) such that

$$\tau_o = \iota \text{ in } \Lambda X^1 \quad \text{and} \quad s = 0 \text{ in } \Lambda X^1$$

and

$$\log \psi = sd_A + d_A s + \tau_o^{-1} \log \bar{\psi}\, \tau_o .$$

In particular, if $\psi \simeq \iota$ then

$$\log \psi = sd_A + d_A s .$$

11.19.- Lomotopies. Again assume (A,d_A) is minimal. For any c.g.d.a. (E,d_E) form the c.g.d.a. $\Lambda(t,dt) \otimes E$ (deg $t = 0$) with the inclusion and projections

$$\ell : E \to \Lambda(t,dt) \otimes E, \quad \pi_0,\pi_1 : \Lambda(t,dt) \otimes E \to E$$

given by $\ell z = 1 \otimes z$, $\pi_0 t = 0$, $\pi_1 t = 1$.

Two c.g.d.a. homomorphisms $\psi_0,\psi_1 : A \to E$ will be called lomotopic if for some c.g.d.a. homomorphism

$$\Phi : A \to \Lambda(t,dt) \otimes E,$$

$\pi_i \circ \Phi = \psi_i$, $i = 0,1$. If ψ_0,ψ_1 preserve augmentations (for a given augmentation ε_E of E) then ψ_0 and ψ_1 are called based lomotopic if Φ can be chosen so that

$$(\iota \otimes \varepsilon_E)\Phi\,(A^+) = 0.$$

11.20.- Proposition. Assume $H^0(E) = k$. Then

i) $\psi_0 \sim \psi_1$ if and only if they are lomotopic

ii) $\psi_0 \overset{\sim}{\sim} \psi_1$ if and only if they are based lomotopic.

Proof.- Since $\mathrm{Im}(\ell\pi_0 - \iota) \subset \Lambda^+(t,dt) \otimes E$, which is an acyclic ideal, we have $\ell\pi_0 \sim \iota$. Similarly $\ell\pi_1 \sim \iota$. Thus $\ell\pi_0 \sim \ell\pi_1$ and by prop 5.15, $\pi_0 \sim \pi_1$. Hence $\pi_0\Phi \sim \pi_1\Phi$. Thus ψ_0,ψ_1 lomotopic $\Rightarrow \psi_0 \sim \psi_1$.

The identical argument shows that ψ_0,ψ_1 based lomotopic $\Rightarrow \psi_0 \overset{\sim}{\sim} \psi_1$, once $\Lambda(t,dt) \otimes E$ is replaced by the augmented c.g.d.a.

$$[\Lambda(t,dt) \otimes \ker \varepsilon_E] \oplus k.$$

Conversely, suppose $\psi_0 \sim \psi_1$ and let $\Psi : A^I \to E$ be a homotopy. Recall that (Chap. 5),

$$A^I = \Lambda X \otimes \Lambda\bar{X} \otimes \Lambda D\bar{X}.$$

Define a degree -1 derivation, h, in $\Lambda(t,dt) \otimes A^I$ by

$$h(x) = t\bar{x} \;;\; h(t) = h(dt) = h(\bar{x}) = h(D\bar{x}) = 0.$$

Denote the differential in $\Lambda(t,dt) \otimes A^I$ also by D.

Then an automorphism of $\Lambda(t,dt) \otimes A^I$ is given by e^{Dh+hD}.

Put

$$\Phi = (\iota \otimes \Psi) \circ e^{Dh+hD} \circ \lambda_o : A \to \Lambda(t,dt) \otimes E.$$

Then $\pi_o\Phi = \psi_o$ and $\pi_1\Phi = \psi_1$. Moreover if ψ was a based homotopy then $(\iota \otimes \varepsilon_E)\,\Phi(A^+) = 0$.

Q.E.D.

11.21.- Theorem. Assume (A,d_A) is a minimal connected KS complex and $\psi : (A,d_A) \to (A,d_A)$ is a homomorphism of c.g.d.a.'s. Then the following are equivalent :

i) $\psi \sim \iota$.

ii) $\psi = e^{sd_A+d_As}$, where s is a degree -1 derivation of A.

Proof.- In prop 11.17 we proved i) => ii). If ii) holds define a degree -1 derivation j in $\Lambda(t,dt) \otimes A$ by

$$j(t) = 0, \quad j(dt) = 0, \quad j(a) = ts(a), \qquad a \in A.$$

Define $\Phi : A \to \Lambda(t,dt) \otimes A$ by

$$\Phi(a) = e^{dj+jd}(1 \otimes a) \qquad , \qquad a \in A,$$

where d is the differential in $\Lambda(t,dt) \otimes A$.

Then $\pi_o\Phi = \iota$ and $\pi_1\Phi = \psi$. Hence prop. 11.20 shows that $\psi \sim \iota$.

Q.E.D.

The same argument gives

11.22.-*Theorem*. Assume (A,d_A) is a minimal connected KS complex and $\psi : (A,d_A) \to (A,d_A)$ is a homomorphism of c.g.d.a.'s. Then the following are equivalent :

i) $\psi \simeq \iota$.

ii) $\psi = e^{sd_A+d_As}$, where s is a degree -1 derivation of A and $s(A^1) = 0$.

Chapter 12

Simplicial sets and local systems.

12.1.- Simplicial sets. For each integer $n \geq 0$ write $[n] = \{0,1,\ldots,n\}$. Let Ord denote the category whose objects are the sets $[0]$, $[1]$, $[2]$, and whose morphisms are given by

$$\mathrm{Ord}([n],[m]) = \{\text{set maps } f : [n] \to [m] \text{ such that } i \leq j \Rightarrow f(i) \leq f($$

Among the elements of $\mathrm{Ord}([n],[n+1])$ we distinguish the face maps

$$\delta_i : \{0,\ldots,n\} \to \{0,\ldots,i-1, \hat{i}, i+1,\ldots,n+1\}$$

and among the elements of $\mathrm{Ord}([n+1],[n])$ we distinguish the degeneracy maps

$$\sigma_i : \{0,\ldots,n+1\} \to \{0,\ldots,i,i,\ldots,n\}.$$

They satisfy the relations

$$\delta_j\delta_i = \delta_i\delta_{j-1} \qquad , i < j$$

$$\sigma_j\sigma_i = \sigma_i\sigma_{j-1} \qquad , i \leq j$$

(12.2) and

$$\sigma_j\delta_i = \begin{cases} \delta_i\sigma_{j-1} & , i < j \\ \iota & , i = j,j+1 \\ \delta_{i-1}\sigma_j & , i > j+1. \end{cases}$$

A simplicial set is a contravariant functor, $\underline{K}$, from Ord to the category of sets. We denote by $\underline{K}_n$ the set $\underline{K}([n])$; it is called the set of n-simplices of K. If $\sigma \in \underline{K}_n$ we write $|\sigma| = \dim \sigma = n$.

Corresponding to the δ_i and σ_i are the <u>face</u> and <u>degeneracy maps</u>

$$\partial_i : \underline{K}_{n+1} \to \underline{K}_n \qquad \text{and} \qquad s_i : \underline{K}_n \to \underline{K}_{n+1} .$$

They satisfy the relations

$$\partial_i \partial_j = \partial_{j-1} \partial_i \quad , \quad i < j,$$

$$s_i s_j = s_{j+1} s_i \quad , \quad i \leq j,$$

(12.3) and

$$\partial_i s_j = \begin{cases} s_{j-1} \partial_i , & i < j \\ 1 & i = j, j+1 \\ s_j \partial_{i-1} , & i > j+1 \end{cases}$$

(A family of sets, S_n, $n \geq 0$, together with maps ∂_i, s_i satisfying (12.3) defines a simplicial set.)

If $\underline{K}$ is a simplicial set the simplices in $\bigcup_{i=0}^{n-1} s_i(\underline{K}_{n-1})$ are called <u>degenerate</u>. The other simplices in $\underline{K}_n$ are called <u>nondegenerate.</u>

A <u>simplicial map</u> $\underline{L} \to \underline{K}$ is a family of set maps $\underline{L}_n \to \underline{K}_n$ commuting with the face and degeneracy maps. If each $\underline{L}_n \to \underline{K}_n$ is the inclusion of a subset, we say $\underline{L}$ is a <u>subsimplicial set</u> of $\underline{K}$.

Suppose $\underline{K}$ is a simplicial set, and $M_n \subset \underline{K}_n$ are subsets such that $\partial_i(M_n) \subset M_{n-1}$. Define subsets $\underline{L}_n \subset \underline{K}_n$ by

(12.4) $$\underline{L}_n = \{ s_{i_1} s_{i_2} \ldots s_{i_p} (\tau) \mid \tau \in M_{n-p},\ 1 \leq p \leq n \} \cup M_n .$$

Using (12.3) it is easy to see that the $\underline{L}_n$ define a subsimplicial set of $\underline{K}$. Note that the non-degenerate simplices of $\underline{L}_n$ are all in M_n ; we have only added degenerate ones.

If we apply this construction with

$$M_m = \begin{cases} \underline{K}_m & , \ m \le n \\ \emptyset & , \ m > n \end{cases}$$

we obtain a subsimplicial set which we denote by $\underline{K}^{(n)}$ and call the n^{th} skeleton of $\underline{K}$. Thus

$$\text{Nondegenerate simplices in } \underline{K}_m^{(n)} = \begin{cases} \text{Nondegenerate simplices in } \underline{K}_m, & m \le n, \\ \emptyset & m > n. \end{cases}$$

(Note that $\underline{K}_m^{(n)} = (\underline{K}^{(n)})_m$.)

A simplicial map $\underline{L} \to \underline{K}$ restricts to simplicial maps $\underline{L}^{(n)} \to \underline{K}^{(n)}$, $n \ge 0$.

12.5.- Lemma. Let $\underline{K}$ be a simplicial set. Assume $\omega, \tau \in \underline{K}_n$ satisfy

$$s_i \tau = s_j \omega = \sigma \quad , \text{ some } j \le i.$$

If $j = i$ then $\tau = \omega$. If $j < i$ then

$$\tau = s_j \partial_i \omega \quad \text{and} \quad \omega = s_{i-1} \partial_i \omega.$$

Proof.- Using (12.3) we find

$$\tau = \partial_{i+1} s_i \tau = \partial_{i+1} s_j \omega = s_j \partial_i \omega,$$

and

$$\omega = \partial_j s_j \omega = \partial_j s_i \tau = s_{i-1} \partial_j \tau.$$

Hence

$$\omega = s_{i-1} \partial_j s_j \partial_i \omega = s_{i-1} \partial_i \omega.$$

Q.E.D.

12.6.- *Example-simplicial complexes*. Let K be a simplicial complex. If K_p is the set of p-simplices then each element of K_p is a subset of K_o containing $p+1$ elements.

An associated simplicial set $\underline{\underline{K}}$ is given as follows :

i) $\underline{\underline{K}}_n$ consists of all the set maps $\sigma : [n] \to K_o$ such that $\text{Im}\,\sigma$ is a simplex of K.

ii) For $f \in \text{Ord}([n],[m])$, $\underline{\underline{K}}(f) : \underline{\underline{K}}_m \to \underline{\underline{K}}_n$ is given by

$$\underline{\underline{K}}(f)(\sigma) = \sigma \circ f.$$

12.7.- *Example - ordered simplicial complexes*. Suppose K is a simplicial complex, and that the vertices of K havec been assigned a (partial) order in such a way that the vertices of any simplex are <u>linearly</u> ordered.

An associated simplicial set, $\underline{K}$, is given as follows :

i) $\underline{K}_n$ consists of all the set maps $\sigma : |n] \to K_o$ such that $\text{Im}\,\sigma$ is a simplex of K and such that

$$\sigma(0) \leq \sigma(1) \leq \ldots \leq \sigma(n).$$

ii) For $f \in \text{Ord}(|n|,|m|)$, $\underline{K}(f) : \underline{K}_m \to \underline{K}_n$ is defined by

$$\underline{K}(f)(\sigma) = \sigma \circ f.$$

Then the set map $\sigma \mapsto \text{Im}\,\sigma$ defines a bijection :

(12.8) Nondegenerate simplices of $\underline{K}_n = K_n$.

In particular, if $K^{(n)}$ is the n-skeleton of K then $\underline{K^{(n)}} = (\underline{K})^{(n)}$.

12.9.- *Example-the standard simplex* Δ^n. Denote the standard basis

$$(1,0,\ldots,0)\ ,\ (0,1,0,\ldots,0)\ ,\ \ldots\ ,\ (0,\ldots,0,1)$$

of R^{n+1} by $v_o,\ldots,v_n$. The standard n-simplex, Δ^n, is the subset of R^{n+1} given by

$$\Delta^n = \{\sum_0^n t_i v_i \mid 0 \leq t_i \leq 1 \; ; \; \sum_0^n t_i = 1\}.$$

The continuous function $b_i : \Delta^n \to \mathbb{R}$ given by

$$b_i(\sum t_j v_j) = t_i$$

is called the i^{th} barycentric coordinate function. If $f : [n] \to [m]$ is any set map we define a continuous map $\Delta(f) : \Delta^n \to \Delta^m$ by

$$\Delta(f)(\sum_0^n t_i v_i) = \sum_0^n t_i v_{f(i)}.$$

Then

(12.10)
$$b_i \circ \Delta(f) = \begin{cases} \sum_{j \in f^{-1}(i)} b_j & , \quad f^{-1}(i) \neq \emptyset, \\ 0 & , \quad \text{otherwise}. \end{cases}$$

A function $\Delta^n \to M$ (M a C^∞ manifold) is called smooth if it extends to a smooth function $U \to M$ in some neighbourhood U of Δ^n in $\mathbb{R}^{n+1}$.

We shall also denote by Δ^n, and call the standard n-simplex the ordered simplicial complex given by

$$(\Delta^n)_o = \{v_o, \ldots, v_n\} \qquad , \quad v_o < \ldots < v_n$$

and

$$(\Delta^n)_p = \{\text{all subsets of } (\Delta^n)_o \text{ with } p+1 \text{ elements}\}.$$

Applying example 12.7 we obtain a simplicial set $\underline{\Delta}^n$; if we identify $[n] = \{v_o, \ldots, v_n\}$ then

(12.11)
$$(\underline{\Delta}^n)_p = \mathrm{Ord}([p],[n]).$$

The $(n-1)$-skeleton of Δ^n is just its boundary $\partial\Delta^r$ (consisting of all simplices of dimension $< n$) ; clearly then $\underline{\partial\Delta}^n$ is the $(n-1)$-skeleton of $\underline{\Delta}^n$:

$$\underline{\partial\Delta}^n = (\underline{\Delta}^n)^{(n-1)} \subset \underline{\Delta}^n .$$

(We regard $\partial\Delta^n$ as an ordered complex !)

Thus (cf. (12.11) or (12.8)) all the non degenerate simplices of $\underline{\partial\Delta}^n$ are of dimension $\leq n-1$. $\underline{\Delta}^n$ has a single additional non-degenerate simplex. It has dimension n and we denote it by $[\Delta^n]$.

Finally, let $\underline{K}$ be any simplicial set. Each $\sigma \in \underline{K}_n$ determines a simplicial map

$$\underline{\sigma} : \underline{\Delta}^n \to \underline{K}$$

by (using (12.11))

$$\underline{\sigma}(f) = \underline{K}(f) \circ \sigma.$$

It satisfies

(12.12) $$\underline{\sigma}([\Delta^n]) = \sigma.$$

12.13.- Example - singular simplices. The singular simplices on a topological space, M, form a simplicial set $\underline{\text{Sing}}(M)$:

$$\underline{\text{Sing}}_n(M) = \{\sigma : \Delta^n \xrightarrow{\text{cont}} M\} \quad ;$$

$$\partial_i(\sigma) = \sigma \circ \delta_i \quad ; \quad s_i(\sigma) = \sigma \circ \sigma_i .$$

A continuous map $\psi : M \to N$ defines a simplicial map

$$S(\psi) : \underline{\text{Sing}}(M) \to \underline{\text{Sing}}(N)$$

by $S(\psi)\sigma = \psi \circ \sigma$.

If M is a C^∞ manifold then smooth singular simplices are a subsimplicial set, $\underline{\text{Sing}}^\infty(M)$.

12.14.- Local systems. Recall that all vector spaces are defined over a given field k of characteristic zero. By the category of n-graded spaces we mean the category whose objects are the n-graded spaces

$$V = \sum_{p_1,\dots,p_n \geq 0} V^{p_1,\dots,p_n}$$

and whose morphisms are the linear maps, homogeneous of multidegree zero.

The category of n-graded differential spaces has for objects pairs (V,d), where V is an n-graded space, d is a linear map of <u>total</u> degree 1, and $d^2 = 0$. The morphisms are the linear maps, homogeneous of multidegree zero, which commute with d.

In the sequel $\mathcal{C}$ will always denote one of the following categories :

I
- i) n-graded spaces.
- ii) The subcategory of n-graded algebras.
- iii) The subcategory of commutative (in the graded sense) n-graded algebras.

II
- iv) n-graded differential spaces.
- v) The subcategory of n-graded differential algebras.
- vi) The subcategory of commutative (in the graded sense) n-graded differential algebras.

In group II the differential is usually denoted by d. We shall almost always restrict ourselves to the case $n = 1$ (graded spaces, etc.) and leave the general case to the reader.

12.15.- Definition. Let $\underline{K}$ be a simplicial set. A <u>local system</u> F on $\underline{K}$ with values in $\mathcal{C}$ is :

i) A family of objects $F_\sigma = \sum_{p\geq 0} F^p_\sigma$ in $\mathcal{C}$, indexed by the simplices σ of $\underline{K}$.

ii) A family of morphisms (called the <u>face</u> and <u>degeneracy operators</u>)

$$\partial_i : F_\sigma \longrightarrow F_{\partial_i\sigma} \quad \text{and} \quad s_i : F_\sigma \longrightarrow F_{s_i\sigma}$$

satisfying the relations (12.3).

12.16.- *Definition.* Let $\psi : \underline{L} \to \underline{K}$ be a simplicial map, and F a local system over $\underline{K}$. The pullback of F to $\underline{L}$ is the local system ψ^*F over $\underline{L}$ given by

$$(\psi^*F)_\sigma = F_{\psi\sigma} \quad ; \quad \partial_i = \partial_i \quad ; \quad s_i = s_i .$$

If $\underline{L} \overset{\psi}{\hookrightarrow} \underline{K}$ is a subset we say ψ^*F is the restriction of F to $\underline{L}$.

Next, let F be a local system over $\underline{K}$ with values in C. Define a graded space $F(\underline{K})$ as follows : an element Φ of $F^p(\underline{K})$ is a function which assigns to each simplex σ of $\underline{K}$ an element $\Phi_\sigma \in F^p_\sigma$ such that for all σ

$$\Phi_{\partial_i\sigma} = \partial_i(\Phi_\sigma) \qquad \text{and} \qquad \Phi_{s_i\sigma} = s_i(\Phi_\sigma).$$

The linear structure is given by

$$(\lambda\Phi+\mu\Psi)_\sigma = \lambda\,\Phi_\sigma + \mu\,\Psi_\sigma\ .$$

If C is a category of algebras (or with differential) we put

$$(\Phi.\Psi)_\sigma = \Phi_\sigma\,.\,\Psi_\sigma \qquad (\text{or} \quad (d\Phi)_\sigma = d(\Phi_\sigma)).$$

If C is a category of n-graded spaces then the finite decompositions

$$F^p_\sigma = \sum_{p_1+\dots+p_n=p} F^{p_1,\dots,p_n}_\sigma$$

define in an obvious way an n-grading in $F(\underline{K})$.

In this way $F(\underline{K})$ becomes an object of C. It is called the space of global sections of F.

If $\psi : \underline{L} \to \underline{K}$ is a simplicial map it determines a morphism $F(\psi) : (\psi^* F)(L) \to F(\underline{K})$ given by

$$(F(\psi)\Phi)_\sigma = \Phi_{\psi\sigma}$$

If ψ is an inclusion of $\underline{L}$ we denote $(\psi^* F)(\underline{L})$ simply by $F(\underline{L})$, and call the morphism $F(\underline{K}) \to F(\underline{L})$ restriction.

Again suppose F is a local system over $\underline{K}$. Assume $M_n \subset \underline{K}_n$ are subsets $(n \geq 0)$ such that $\partial_i : M_n \to M_{n-1}$.

Recall that $\{M_n\}$ generates a subsimplicial set $\underline{L} \subset \underline{K}$ (cf. (12.4)). Moreover, if $s_i\, \sigma \in M_{n+1}$ then $\sigma = \partial_i s_i \sigma \in M_n$.

*12.17.- **Lemma**.* Suppose $\Phi_\sigma \in F_\sigma^p$ $(\sigma \in M_n,\ n \geq 0)$ satisfy $\Phi_{\partial_i\sigma} = \partial_i \Phi_\sigma$ $(\sigma \in M_n,\ n \geq 1)$ and $\Phi_{s_i\sigma} = s_i \Phi_\sigma$ (if σ and $s_i\, \sigma \in M_n$, $n \geq 0$). Then there is a unique element $\Phi \in F^p(\underline{L})$ extending the Φ_σ.

Proof : We show by induction that there are unique elements $\Psi_\sigma \in F_\sigma^p$ $(\sigma \in \underline{L}_n,\ n = 0,1,\dots)$ such that

$$(12.18)_n \qquad \begin{aligned} \Psi_\sigma &= \Phi_\sigma && \text{if } \sigma \in M_n \\ \Psi_{s_i\tau} &= s_i \Psi_\tau && \text{if } \tau \in \underline{L}_{n-1},\ n \geq 1. \\ \Psi_{\partial_i\sigma} &= \partial_i \Psi_\sigma && \text{if } \sigma \in \underline{L}_n,\ n \geq 1. \end{aligned}$$

In fact, for $n = 0$ the first condition defines the Ψ_σ and the others are vacuous. Assume the Ψ_σ are constructed for $\sigma \in \underline{L}_m$, $m \leq n$, such that $(12.18)_m$ holds for $m \leq n$.

If $\sigma \in \underline{L}_{n+1}$ then by (12.4) either $\sigma \in M_{n+1}$ or $\sigma = s_i \tau$, $\tau \in \underline{L}_n$. In the first case put $\Psi_\sigma = \Phi_\sigma$, in the second put $\Psi_\sigma = s_i \Psi_\tau$. To check that Ψ_σ is well defined note first that if $\sigma = s_i \tau \in M_{n+1}$ then $\tau \in M_n$ and so $\Phi_\sigma = s_i \Phi_\tau = s_i \Psi_\tau$.

On the other hand, if $\sigma = s_i\tau = s_j\omega$ with $\tau \neq \omega$, $\tau,\omega \in \underline{L}_n$ then $i \neq j$. Take $j < i$ and apply lemma 12.5 to find

$$s_j\Psi_\omega = s_j\Psi_{s_{i-1}\partial_i\omega} = s_j\, s_{i-1}\, \Psi_{\partial_i\omega}$$

$$= s_i s_j \Psi_{\partial_i\omega} = s_i\Psi_{s_j\partial_i\omega} = s_i\Psi_\tau \; .$$

Finally, the first two parts of $(12.18)_{n+1}$ are true by definition, while the third follows easily from (12.3).

Q.E.D.

Recall (example 12.9) that if $\underline{K}$ is a simplicial set, each simplex $\sigma \in \underline{K}_n$ determines a simplicial map $\underline{\sigma} : \underline{\Delta}^n \to \underline{K}$ with $\underline{\sigma}([\Delta^n]) = \sigma$. In particular, if F is a local system over $\underline{K}$ we can form $\underline{\sigma}^*F$ over $\underline{\Delta}^n$.

12.19.- *Definition*. F is extendable if for each σ the restriction

$$(\underline{\sigma}^*F)(\underline{\Delta}^n) \to (\underline{\sigma}^*F)(\underline{\partial\Delta}^n)$$

is surjective.

12.20.- *Proposition*. If $\varphi : \underline{L} \to \underline{K}$ is a simplicial map and F is an extendable local system over $\underline{K}$, then φ^*F is an extendable local system over $\underline{L}$.

Proof : Obvious.

Q.E.D.

12.21.- *Proposition*. Suppose $\underline{L} \subset \underline{K}$ is a subsimplicial set and F is an extendable local system over $\underline{K}$. Then the restriction morphism

$$F(\underline{K}) \to F(\underline{L})$$

is surjective.

Proof : It is sufficient to show that any $\Phi \in F^p(\underline{L} \cup \underline{K}^{(n-1)})$ can be extended to an element Ψ of $F^p(\underline{L} \cup \underline{K}^{(n)})$. Suppose that $\sigma \in \underline{K}_n$, $\sigma \notin \underline{L}_n$ and σ non degenerate. Then $\underline{\sigma}$ restricts to

$$\partial\underline{\sigma} : \underline{\partial\Delta}^n \to \underline{K}^{(n-1)} \text{ , and}$$

we can use $\partial\underline{\sigma}$ to pull Φ back to an element of $\underline{\sigma}^* F^p(\underline{\partial\Delta}^n)$.

By hypothesis there is an element $\Psi_\sigma \in F^p_\sigma$ which restricts to this pull back of Φ (use the isomorphism 12.12.b) below) ;

$$\partial_i \Psi_\sigma = \Phi_{\partial_i \sigma} \qquad 0 \leq i \leq n.$$

Next, set $\Psi_\sigma = \Phi_\sigma$ if $\sigma \in (\underline{L} \cup \underline{K}^{(n-1)})_m$, $m \geq 0$. Finally, use lemma 12.17 to extend Ψ to all the simplices in $\underline{L} \cup \underline{K}^{(n)}$.

Q.E.D.

12.22.- *Definition*. If F is an extendable local system over $\underline{K}$ and $\underline{L} \subset \underline{K}$ we denote the kernel of $F(\underline{K}) \to F(\underline{L})$ by $F(\underline{K},\underline{L})$, the space of relative global sections. Thus

$$0 \to F(\underline{K},\underline{L}) \to F(\underline{K}) \to F(\underline{L}) \to 0$$

is a short exact sequence.

12.23.- *Example*. Let F be any local system over $\underline{K}$. Put

$$N\underline{K}_n = \{\text{nondegenerate simplices in } \underline{K}_n\}$$

and

$$NF^p_\sigma = \begin{cases} F^p_\sigma & , \quad \sigma \in \underline{K}_o \\ \bigcap_{0 \leq i \leq n} \ker(\partial_i : F^p_\sigma \to F^p_{\partial_i \sigma}), & \sigma \in \underline{K}_n, \quad n \geq 1. \end{cases}$$

Define an inclusion

$$\prod_{\sigma\in \underline{NK}_n} NF^p_\sigma \longrightarrow F^p(\underline{K}^{(n)})$$

as follows. If $\Psi_\sigma \in NF^p_\sigma$, $\sigma \in \underline{NK}_n$, extend by putting $\Psi_\sigma = 0$, if $|\sigma| < n$ and then use lemma 12.17 to extend to a unique element Ψ of $F^p(\underline{K}^{(n)})$. It is easy to verify that the sequence

$$(12.24) \qquad 0 \longrightarrow \prod_{\sigma\in \underline{NK}_n} NF^p_\sigma \to F^p(\underline{K}^{(n)}) \xrightarrow{\text{rest'n}} F^p(\underline{K}^{(n-1)})$$

is exact.

In particular, if F is extendable this inclusion is an isomorphism

$$(12.25) \qquad \prod_{\sigma\in \underline{NK}_n} NF^p_\sigma \xrightarrow{\cong} F^p(\underline{K}^{(n)}, \underline{K}^{(n-1)}) \quad , \quad p = 0,1,\ldots .$$

On the other hand, suppose F is any local system over $\underline{\Delta}^n$. It is an equally easy exercise to deduce that an isomorphism

$$(12.26.a) \qquad F(\underline{\Delta}^n) \xrightarrow{\cong} F_{[\Delta_n]}$$

is defined by $\Phi \mapsto \Phi_{[\Delta^n]}$. In view of (12.24) it restricts to an isomorphism

$$(12.26.b) \qquad \operatorname{Ker}(F(\underline{\Delta}^n) \to F(\underline{\partial\Delta}^n)) \xrightarrow{\cong} NF_{[\Delta^n]} .$$

In the applications we shall consider local systems satisfying certain additional conditions. Aside from extendable we need the following definitions. A local system F over $\underline{K}$ is

i) <u>constant</u> if for some $F_o \in \mathcal{C}$ each $F_\sigma = F_o$ and each ∂_i, s_i is the identity map of F_o.

ii) <u>constant by dimension</u> if for some sequence $F_n \in \mathcal{C}$ $(n \geq 0)$, $F_\sigma = F_n$, $\sigma \in \underline{K}_n$, and ∂_i, s_j depend only on $|\sigma|$.

iii) a local system of coefficients (ℓ.s.c.) if for each σ and each i

$$\partial_i : F_\sigma \to F_{\partial_i \sigma} \qquad \text{and} \qquad s_i : F_\sigma \to F_{s_i \sigma}$$

are isomorphisms.

iv) a local system of differential coefficients if C is a category with differentials and if for each σ , i

$$\partial_i^* : H(F_\sigma) \to H(F_{\partial_i \sigma}) \qquad \text{and} \qquad s_i^* : H(F_\sigma) \to H(F_{s_i \sigma})$$

are isomorphisms (in other words if $\{H(F_\sigma), \partial_i^*, s_i^*\}$ is a local system of coefficients).

Our next goal is the following

12.27.- Theorem. Let F and G be extendable local systems of differential coefficients over $\underline{K}$. Assume we are given morphisms

$$\psi_\sigma : F_\sigma \to G_\sigma \ , \qquad \sigma \in \underline{K} \ ,$$

compatible with the face and degeneracy operators, with each ψ_σ^* an isomorphism.

Then a morphism $\psi : F(\underline{K}) \to G(\underline{K})$ is given by $(\psi\Phi)_\sigma = \psi_\sigma(\Phi_\sigma)$, and

$$\psi^* : H(F(\underline{K})) \to H(G(\underline{K}))$$

is an isomorphism.

12.28.- Lemma. Let E be an extendable local system of differential coefficients over $\underline{\Delta}^n$. For $m \leq n$ let $L^{m,n} \subset \Delta^n$ be the subcomplex generated by all simplices of dimension $\leq m$ which contain the vertex v_o. Then

i) For each $m \leq n$,

$$H(E(\underline{\Delta}^n)) \to H(E(\underline{L}^{m,n}))$$

is an isomorphism.

ii) $H^p(E(\underline{\Delta}^n, \partial\underline{\Delta}^n)) = 0, \quad p < n.$

Proof :

i) If $m = n$ there is nothing to prove and if $m = 0$ formula (12.26.b) reduces the assertion to $H(E_{[\Delta^n]}) \xrightarrow{\cong} H(E_{v_o})$. This is true by hypothesis.

Now assume that $0 < m < n$ and (by induction) that the lemma holds for $L^{p,q} \subset \Delta^q$ whenever $p < m$. Since E is extendable we need only show that (cf. definition 12.22)

$$(12.29) \qquad H(E(\underline{L}^{m,n}, \underline{L}^{m-1,n})) = 0.$$

Let $\sigma_1, \ldots, \sigma_N$ be the m-dimensional simplices of $L^{m,n}$; they are also the non degenerate simplices of $\underline{L}^{m,n}$ (of dimension m). Each $\underline{\sigma}_i : \underline{\Delta}^m \to \underline{L}^{m,n}$ pulls E back to a local system E_i of differential coefficients on $\underline{\Delta}^m$ and E_i is extendable by prop. 12.20.

It is easy to check that the induced morphisms

$$E(\underline{\sigma}_i) : E(\underline{L}^{m,n}) \to E_i(\underline{\Delta}^m)$$

define an isomorphism

$$E(\underline{L}^{m,n}, \underline{L}^{m-1,n}) \xrightarrow{\cong} \prod_{i=1}^{N} E_i(\underline{\Delta}^m, \underline{L}^{m-1,m}).$$

Formula (12.29) follows, and so i) is proved.

ii) Use the embedding $\delta_o : [n-1] \hookrightarrow [n]$ to write $\Delta^{n-1} \subset \Delta^n$ (as the face opposite v_o) and $\underline{\Delta}^{n-1} \subset \underline{\Delta}^n$. Since E is extendable we obtain the short exact sequence

$$0 \to E(\underline{\Delta}^n, \partial\underline{\Delta}^n) \to E(\underline{\Delta}^n, \underline{L}^{n-1,n}) \to E(\underline{\Delta}^{n-1}, \partial\underline{\Delta}^{n-1}) \to 0.$$

But i) shows that $H(E(\underline{\Delta}^n, \underline{L}^{n-1,n})) = 0$, and so there is a linear isomorphism of degree 1

(12.30) $$H(E(\underline{\Delta}^{n-1},\underline{\partial\Delta}^{n-1})) \xrightarrow{\cong} H(E(\underline{\Delta}^{n},\underline{\partial\Delta}^{n})).$$

Q.E.D.

12.31.- *Proof of theorem 12.27.* We show first that for each n,

(12.32) $$\psi^* : H(F(\underline{K}^{(n)})) \to H(G(\underline{K}^{(n)}))$$

is an isomorphism. Assume this is true for $n-1$; to prove it for n we have to show only that

$$\psi^* : H(F(\underline{K}^{(n)},\underline{K}^{(n-1)})) \to H(G(\underline{K}^{(n)},\underline{K}^{(n-1)}))$$

is an isomorphism.

Because of the isomorphism (12.25) it is sufficient to prove that $\psi^*_\sigma : H(NF_\sigma) \to H(NG_\sigma)$ is an isomorphism for $\sigma \in N\underline{K}_n$. Use σ to pull F and G back to $\underline{\Delta}^n$ and obtain (cf. (12.26.b) a commutative row exact diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & NF_\sigma & \longrightarrow & F_\sigma & \longrightarrow & \underline{\sigma}^*F(\underline{\Delta}_n^{(n-1)}) & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle \psi_\sigma} & & \downarrow{\scriptstyle \psi_\sigma} & & \downarrow & & \\ 0 & \longrightarrow & NG_\sigma & \longrightarrow & G_\sigma & \longrightarrow & \underline{\sigma}^*G(\underline{\Delta}_n^{(n-1)}) & \longrightarrow & 0 \end{array}$$

in which the right hand arrow is a cohomology isomorphism by induction and the central arrow is by hypothesis. Hence so is the left hand arrow, and (12.32) follows.

Next observe that by lemma 12.28 if $\sigma \in \underline{K}_n$,

$$H^p(\underline{\sigma}^*F(\underline{\Delta}^n,\underline{\partial\Delta}^n)) = 0 = H^p(\underline{\sigma}^*G(\underline{\Delta}^n,\underline{\partial\Delta}^n)), \qquad p < n.$$

Equations (12.26.a) and (12.26.b) translate this to

(12.33) $$H^p(NF_\sigma) = 0 = H^p(NG_\sigma) , \qquad \sigma \in \underline{K}_n , \qquad p < n.$$

Now the isomorphisms (12.25) yield

$$H^p(F(\underline{K}^{(n)},\underline{K}^{(n-1)}) = 0 = H^p(G(\underline{K}^{(n)},\underline{K}^{(n-1)})), \qquad p < n. \tag{12.34}$$

In particular, let $\Phi \in F^p(\underline{K},\underline{K}^{(n)})$, $p \leq n$, satisfy $d\Phi = 0$. Using equations (12.34) and the extendability of F we can find a sequence $\Psi_i \in F^{p-1}(\underline{K},\underline{K}^{(n+i)})$, $i = 0,1,\ldots$ so that $\Phi - d(\sum_0^m \Psi_i) \in F^p(\underline{K},\underline{K}^{(n+m+1)})$. For each σ set

$$\Psi_\sigma = \sum_0^\infty (\Psi_i)_\sigma \qquad \text{(this is a finite sum !).}$$

Then $\Psi \in F^{p-1}(\underline{K},\underline{K}^{(n)})$, and $\Phi = d\Psi$. Hence

$$H^p(F(\underline{K},\underline{K}^{(n)})) = 0 = H^p(G(\underline{K},\underline{K}^{(n)})), \qquad p \leq n. \tag{12.35}$$

These equations, together with (12.32), complete the proof.

Q.E.D.

Finally, suppose A and B are objects from one of the categories C in our list. If A and B were n-graded then $A \otimes B$ is 2n-graded. If A, B were algebras we set $(a \otimes b).(a' \otimes b') = (-1)^{|b||a'|}aa' \otimes bb'$. If A and B had differentials we set $d(a \otimes b) = da \otimes b + (-1)^{|a|}a \otimes db$. With these conventions we say.

12.36.- *Definition*. The tensor product of local systems E and F over $\underline{K}$ with values in C is the local system $E \otimes F$ given by

$$(E \otimes F)_\sigma = E_\sigma \otimes F_\sigma \quad ; \quad \partial_i = \partial_i \otimes \partial_i \quad ; \quad s_i = s_i \otimes s_i .$$

Our next main result is

12.37.- *Theorem*. The tensor product of extendable local systems is extendable.

12.38.- Proposition. Let E be an extendable local system over $\underline{K}$. Let $\underline{L}, \underline{L}^1, \ldots, \underline{L}^m \subset \underline{K}$ be subsimplicial sets.

i) Suppose $\Phi_i \in E(\underline{L}, \underline{L} \cap \underline{L}^i)$ satisfy $\sum_{i=1}^{m} \Phi_i = 0$. Then each Φ_i extends to $\Psi_i \in E(\underline{K}, \underline{L}^i)$ such that $\sum_{i=1}^{m} \Psi_i = 0$.

ii) If $\Gamma \in E(\underline{K}, \underline{L})$ and $\Gamma = \sum_{i=1}^{m} \Gamma_i$, $\Gamma_i \in E(\underline{K}, \underline{L}^i)$ then also $\Gamma = \sum_{i=1}^{m} \bar{\Gamma}_i$ with $\bar{\Gamma}_i \in E(\underline{K}, \underline{L} \cup \underline{L}^i)$.

Proof : By induction on m. Regard the assertions above as $i)_m$ and $ii)_m$. Note that both are trivial for $m = 1$. We show that

$$ii)_{m-1} \Rightarrow i)_m , \qquad \text{and} \qquad i)_m \Rightarrow ii)_m .$$

$\underline{ii)_{m-1} \Rightarrow i)_m}$. Choose $\Omega_i \in E(\underline{K}, \underline{L}^i)$ extending Φ_i for $i = 1, \ldots, m-1$, and set $\Omega = \sum_{i=1}^{m-1} \Omega_i$. Then

$$\Omega|_{\underline{L} \cap \underline{L}^m} = \left(\sum_{i=1}^{m-1} \Phi_i\right)\Big|_{\underline{L} \cap \underline{L}^m} = -\Phi_m|_{\underline{L} \cap \underline{L}^m} = 0,$$

and

$$\Omega = \sum_{i=1}^{m-1} \Omega_i \quad , \qquad \Omega_i \in E(\underline{K}, \underline{L}^i).$$

Hence by $ii)_{m-1}$ $\qquad \Omega = \sum_{i=1}^{m-1} \bar{\Omega}_i$, where $\bar{\Omega}_i \in E(\underline{K}, \underline{L}^i \cup (\underline{L} \cap \underline{L}^m))$.

In particular, $\bar{\Omega}_i$ is zero in $\underline{L}^m \cap (\underline{L}^i \cup \underline{L})$. Hence by the extendability of E we can find $\Gamma_i \in E(K, \underline{L}^i \cup \underline{L})$ so that

$$\Gamma_i|_{\underline{L}^m} = \bar{\Omega}_i|_{\underline{L}^m} \quad , \qquad i = 1, \ldots, m-1.$$

Put $\Psi_i = \Omega_i - \Gamma_i$ $(i = 1, \ldots, m-1)$ and $\Psi_m = -\sum_{i=1}^{m-1} \Psi_i$. Then

$$\Psi_i\Big|_{\underline{L}^i} = \Omega_i\Big|_{\underline{L}^i} - \Gamma_i\Big|_{\underline{L}^i} = 0 - 0 = 0, \qquad i = 1,\dots,m-1$$

and

$$\Psi_m\Big|_{\underline{L}^m} = \Big(-\sum_1^{m-1} \Omega_i\Big)\Big|_{\underline{L}^m} + \Big(\sum_1^{m-1} \Gamma_i\Big)\Big|_{\underline{L}^m} = -\Omega\Big|_{\underline{L}^m} + \sum_1^{m-1} \bar{\Omega}_i\Big|_{\underline{L}^m}$$

$$= 0.$$

Finally

$$\Psi_i\Big|_{\underline{L}} = \Omega_i\Big|_{\underline{L}} - \Gamma_i\Big|_{\underline{L}} = \Phi_i - 0 = \Phi_i \quad , \quad i = 1,\dots,m-1$$

and so

$$\Psi_m\Big|_{\underline{L}} = -\sum_{i=1}^{m-1} \Psi_i\Big|_{\underline{L}} = -\sum_{i=1}^{m-1} \Phi_i = \Phi_m.$$

<u>$i)_m \Rightarrow ii)_m$.</u> Set $\Phi_i = \Gamma_i\Big|_{\underline{L}}$. Then $\sum_1^m \Phi_i = 0$ and $\Phi_i \in E(\underline{L}, \underline{L} \cap \underline{L}^i)$. Hence by $i)_m$ Φ_i extends to $\Psi_i \in E(\underline{K}, \underline{L}^i)$ such that $\sum_{i=1}^m \Psi_i = 0$. Now set $\bar{\Gamma}_i = \Gamma_i - \Psi_i$.

Q.E.D.

12.39.- <u>Lemma</u>. Let E and F be extendable local systems over $\underline{\Delta}^n$. Suppose $\Omega \in (E \otimes F)(\underline{\Delta}^n)$ satisfies $(\partial_i \otimes \partial_i)\Omega = 0$, $0 \le i \le m$. Then we can write $\Omega = \sum_\alpha \Omega_\alpha$ where for each α :

Either $(\partial_i \otimes \iota)\Omega_\alpha = 0$ or $(\iota \otimes \partial_i)\Omega_\alpha = 0$, $i = 0,\dots,m$.

<u>*Proof*</u> : For each subset $\sigma = \{i_1,\dots,i_p\} \subset [n]$ (including $\sigma = \emptyset$) define subspaces $N_\sigma^E \subset E_{[\Delta^n]}$, $N_\sigma^F \subset F_{[\Delta^n]}$ by

$$N_\sigma^E = \bigcap_{i\in\sigma} \ker \partial_i \qquad \text{and} \qquad N_\sigma^F = \bigcap_{i\in\sigma} \ker \partial_i$$

Choose subspaces $\bar{N}^E_\sigma \subset N^E_\sigma$ and $\bar{N}^F_\sigma \subset N^F_\sigma$ so that for each p

$$\bigoplus_{|\sigma|\geq p} \bar{N}^E_\sigma = \sum_{|\sigma|\geq p} N^E_\sigma \quad \text{and} \quad \bigoplus_{|\sigma|\geq p} \bar{N}^F_\sigma = \sum_{\sigma\geq p} N^F_\sigma .$$

(Here $\oplus$ denotes direct sum, $+$ denotes not necessarily direct sum.)

We show first that in $E_{[\Delta^n]}$ and $F_{[\Delta^n]}$,

(12.40) $$\ker \partial_i = \bigoplus_{\sigma\ni i} \bar{N}^F_\sigma \quad \text{and} \quad \ker \partial_i = \bigoplus_{\sigma\ni i} \bar{N}^F_\sigma .$$

In fact the right hand side is contained by definition in the left. Thus we need to show that $\ker \partial_i \cap \bigoplus_{\sigma\not\ni i} \bar{N}^E_\sigma = 0$.

Recall that $\partial_i(\Delta^n)$ is its i^{th} face ; denote it by L^i and denote $\bigcup_{i\in\sigma} L^i$ by L^σ. These are subcomplexes of Δ^n and so we can form $\underline{L}^\sigma \subset \underline{\Delta}^n$; clearly $N^E_\sigma = E(\underline{\Delta}^n, \underline{L}^\sigma)$.

Now suppose $\Phi = \sum_{\substack{|\sigma|\geq p \\ i\notin\sigma}} \Phi_\sigma$, $\Phi_\sigma \in \bar{N}^E_\sigma$, and that $\partial_i\Phi = 0$. Then

$$\Phi_\sigma\Big|_{\underline{L}^\sigma} = 0 \quad \text{and} \quad \Phi\Big|_{\underline{L}^i} = 0.$$

Hence by prop. 12.38 ii) $\Phi = \sum_{|\sigma|\geq p} \Psi_\sigma$ with $\Psi_\sigma\Big|_{\underline{L}^\sigma \cup \underline{L}^i} = 0.$

In particular $\Psi_\sigma \in N^E_{\sigma\cup\{i\}}$ and since $i \notin \sigma$ we have

$$\Phi \in \sum_{|\tau|\geq p+1} N^E_\tau = \bigoplus_{|\tau|\geq p+1} \bar{N}^E_\tau .$$

Since $E_{[\Delta^n]} = \bigoplus_\sigma \bar{N}^E_\sigma$ this gives $\Phi_\sigma = 0$, $|\sigma| = p$.

Continue in this way to deduce $\Phi = 0$, which completes the proof of (12.40).

Finally, write $\Omega = \sum_{\sigma\tau} \Omega_{\sigma\tau}$, $\Omega_{\sigma\tau} \in \bar{N}^E_\sigma \otimes \bar{N}^E_\tau$.

Using (12.40) we see at once that if $(\partial_i \otimes \partial_i)\Omega = 0$ then $\Omega_{\sigma\tau} = 0$ unless either $i \in \sigma$ or $i \in \tau$. Hence for each σ and τ $(\partial_i \otimes \iota)\Omega_{\sigma\tau} = 0$ or $(\iota \otimes \partial_i)\Omega_{\sigma\tau} = 0$.

Q.E.D.

12.41.- Proof of theorem 12.37. In view of the definition we have only to prove that if E and F are extendable local systems over $\underline{\Delta}^n$ then

$$(E \otimes F)(\underline{\Delta}^n) \to (E \otimes F)(\underline{\partial\Delta}^n)$$

is surjective.

Suppose $\Omega \in (E \otimes F)(\underline{\partial\Delta}^n)$. If $\Omega_{\partial_i\Delta^n} = 0$, $0 \leq i \leq n$, then $\Omega = 0$. So if $\Omega \neq 0$ there is a least r such that $\Omega_{\partial_r\Delta^n} \neq 0$.

In particular, for $i < r$,

$$(\partial_i \otimes \partial_i)\Omega_{\partial_r\Delta^n} = \Omega_{\partial_i\partial_r\Delta^n} = \Omega_{\partial_{r-1}\partial_i\Delta^n}$$

$$= (\partial_{r-1} \otimes \partial_{r-1})\Omega_{\partial_i\Delta^n} = 0.$$

By lemma 12.39 we can write

$$\Omega_{\partial_r\Delta^n} = \sum \Phi_\alpha \quad , \qquad \Phi_\alpha \in E_{\partial_r\Delta^n} \otimes F_{\partial_r\Delta^n}$$

where either $(\partial_i \otimes \iota)\Phi_\alpha = 0$ or $(\iota \otimes \partial_i)\Phi_\alpha = 0$, $i < r$, and we may obviously assume $\Phi_\alpha = \Phi^E_\alpha \otimes \Phi^F_\alpha$. Extend Φ^E_α and Φ^F_α to elements $\Psi^E_\alpha \in E_{\Delta^n}$ and $\Psi^F_\alpha \in F_{\Delta^n}$ so that $\partial_i \Psi^E_\alpha = 0$ whenever $\partial_i \Phi^E_\alpha = 0$ (and similarly for F), $i < r$.

Then $\Psi^E_\alpha \otimes \Psi^F_\alpha$ satisfies $(\partial_i \otimes \partial_i)(\Psi^E_\alpha \otimes \Psi^F_\alpha) = 0$, $i < r$.

Moreover $\Omega - (\sum_\alpha \Psi^E_\alpha \otimes \Psi^F_\alpha)\big|_{\underline{\partial\Delta}^n}$ is zero in $\partial_r\Delta^n$. This completes the proof (by induction on r).

Q.E.D.

12.42.- *Proposition*. Let E, F, G be extendable local systems over $\underline{K}$. Assume for $\sigma \in \underline{K}_n$, $n = 0,1,\ldots$

$$0 \to E_\sigma \xrightarrow{\varphi_\sigma} F_\sigma \xrightarrow{\psi_\sigma} G_\sigma \to 0$$

are short exact sequences, coherent with the face and degeneracy operators.

Then

$$0 \to E(\underline{K}) \xrightarrow{\varphi} F(\underline{K}) \xrightarrow{\psi} G(\underline{K}) \to 0$$

is also short exact.

Proof : Use the identical inductive technique of theorem 12.27.

Q.E.D.

12.43.- *A spectral sequence*. Suppose G and F are local systems over $\underline{K}$ taking values in a category with differentials. Suppose further that

i) G and F are extendable

ii) F is a local system of differential coefficients.

The bigrading on the local system $G \otimes F$ makes $(G \otimes F(\underline{K}),d)$ into a bicomplex. That is we can write

$$d = d_G + d_F$$

where d_E, d_F are the differentials of degrees (1,0) and (0,1) given by

$$[d_G(\Omega)]_\sigma = d_G(\Omega_\sigma) \qquad , \qquad d_G(\Phi_\sigma \otimes \Psi_\sigma) = d_G\,\Phi_\sigma \otimes \Psi_\sigma$$

$$[d_F(\Omega)]_\sigma = d_F(\Omega_\sigma) \qquad , \qquad d_F(\Phi_\sigma \otimes \Psi_\sigma) = (-1)^{|\Phi_\sigma|}\,\Phi_\sigma \otimes d_F\,\Psi_\sigma \,.$$

Now filter (as usual) this bicomplex by the spaces

$$I^p = \sum_{\substack{j\geq p \\ q\geq 0}} (G^j \otimes F^q)(\underline{K}) = \sum_{j\geq p} (G^j \otimes F)(\underline{K}).$$

We obtain in this way a convergent first quadrant spectral sequence

(12.44) $$(E_i, d_i) \Rightarrow H((G \otimes F)(\underline{K}), d).$$

It is standard that

$$(E_o, d_o) = ((G \otimes F)(\underline{K}), d_F)$$

$$(E_1, d_1) = (H\{(G \otimes F)(\underline{K}), d_F\},\ d_G^*)$$

and

$$E_2 = H\{H\{(G \otimes F)(\underline{K}), d_F\ , d_G^*\}.$$

On the other hand, note that we can define local systems $Z(F)$, $B(F)$ and $H(F)$ over $\underline{K}$ by

$$Z(F)_\sigma = \ker d : F_\sigma \to F_\sigma \quad ; \quad B(F)_\sigma = d(F_\sigma)$$

and

$$H(F)_\sigma = H(F_\sigma, d).$$

Moreover because F is a local system of differential coefficients, $H(F)$ is a <u>local system of coefficients</u>.

Notice that $\Omega \in (G \otimes F)(\underline{K})$ satisfies

$$(d_F\Omega)_\sigma = d_F(\Omega_\sigma)$$

and so $d_F\Omega = 0$ if and only if $\Omega_\sigma \in G_\sigma \otimes Z(F)_\sigma$ for all σ. Similarly for any Ω

$$(d_F\Omega)_\sigma \in G_\sigma \otimes B(F)_\sigma.$$

In this way we obtain a commutative diagram

$$(12.45)\qquad \begin{array}{ccccccccc} 0 & \longrightarrow & [G \otimes B(F)](\underline{K}) & \longrightarrow & [G \otimes Z(F)](\underline{K}) & \longrightarrow & [G \otimes H(F)](\underline{K}) & \longrightarrow & 0 \\ & & \uparrow & & \cong\uparrow & & \uparrow & & \\ 0 & \longrightarrow & \operatorname{Im} d_F & \longrightarrow & \operatorname{Ker} d_F & \longrightarrow & H([G \otimes F](\underline{K}), d_F) & \to & 0 \end{array}$$

in which the bottom row is exact and the top row arises from the short exact sequences

$$(12.46)\qquad 0 \to [G \otimes B(F)]_\sigma \to [G \otimes Z(F)]_\sigma \to [G \otimes H(F)]_\sigma \to 0.$$

Our next object is to establish

12.47- Theorem. With the hypotheses above, all the vertical arrows in (12.45) are isomorphisms. Thus the spectral sequence (12.44) satisfies

$$(E_1, d_1) = ((G \otimes H(F))(\underline{K}), d)$$

and

$$E_2 = H[(G \otimes H(F))(\underline{K}), d].$$

12.48.- Lemma.

i) The local system $G \otimes Z(F)$ is extendable.

ii) The inclusion $\operatorname{Im} d_F \longrightarrow G \otimes B(F)(\underline{K})$ is an isomorphism.

iii) The local system $G \otimes B(F)$ is extendable.

Proof : We establish inductively that :

$(12.49)_n$ If $\underline{K} = \underline{\Delta}^n$ then $[G \otimes Z(F)](\underline{\Delta}^n) \to [G \otimes Z(F)](\underline{\partial\Delta}^n)$ is surjective.

$$(12.50)_n\qquad [G \otimes B(F)](\underline{K}^{(n)}, \underline{K}^{(n-1)}) = d_F\{[G \otimes F](\underline{K}^{(n)}, \underline{K}^{(n-1)})\}.$$

$$(12.51)_n\qquad [G \otimes B(F)](\underline{K}^{(n)}) = d_F\{[G \otimes F](\underline{K}^{(n)})\}.$$

In fact, $(12.49)_o$ is vacuous and $(12.50)_o$ coincides with $(12.51)_o$. To prove them let $\Phi \in [G \otimes B(F)](\underline{K}^{(o)})$. For each $\sigma \in \underline{K}_o$ write $\Phi_\sigma = d_F \Psi_\sigma$ and extend the Ψ_σ to a (unique) element Ψ of $(G \otimes F)(\underline{K}^{(o)})$ via lemma 12.17. Then $\Phi = d_F \Psi$.

We now assume the three assertions proved for $m < n$ (some $n \geq 1$) and establish them for n.

$\underline{(12.49)}_n$: let $\Phi \in [G \otimes Z(F)](\underline{\partial\Delta}^n)$. For each σ of $\underline{\partial\Delta}^n$ let $\gamma_\sigma \in G_\sigma \otimes H(F_\sigma)$ be the element represented by Φ_σ. Because $H(F)$ is a local system of coefficients and G is extendable, $G \otimes H(F)$ is extendable. The γ_σ define an element in $[G \otimes H(F)](\underline{\partial\Delta}^n)$, and so we can find an element $\gamma \in [G \otimes H(F)](\underline{\Delta}^n) = G_{[\Delta^n]} \otimes H(F_{[\Delta^n]})$ which restricts to the γ_σ.

Let $\Gamma \in G_{[\Delta^n]} \otimes (ZF)_{[\Delta^n]}$ represent γ, and regard Γ as an element in $[G \otimes Z(F)](\underline{\Delta}^n)$. Then

$$\Phi - \Gamma|_{\underline{\partial\Delta}^n} \in [G \otimes B(F)](\underline{\partial\Delta}^n).$$

Since $\underline{\partial\Delta}^n = (\underline{\partial\Delta}^n)^{(n-1)}$ we can apply $(12.51)_{n-1}$ with $\underline{K} = \underline{\partial\Delta}^n$ to obtain $\Phi - \Gamma|_{\underline{\partial\Delta}^n} = d_F \Psi$, some $\Psi \in (G \otimes F)(\underline{\partial\Delta}^n)$.

Because G and F are extendable so is $G \otimes F$ (theorem 12.37).

Extend Ψ to an element $\Psi \in (G \otimes F)(\underline{\Delta}^n)$; then $\Gamma + d_F \Psi \in [G \otimes Z(F)](\underline{\Delta}^n)$ and extends Φ.

$\underline{(12.50)}_n$: Suppose $\Omega \in [G \otimes B(F)](\underline{K}^{(n)}, \underline{K}^{(n-1)})$. For each nondegenerate $\sigma \in \underline{K}_n$ write $\Omega_\sigma = d_F \Gamma_\sigma$, some $\Gamma_\sigma \in G_\sigma \otimes F_\sigma$.

Recall the simplicial map $\underline{\sigma} : \underline{\Delta}^n \to \underline{K}$ of example 12.9. It pulls G and F back to local systems over $\underline{\Delta}^n$ satisfying the same hypotheses. We may write

$$\Gamma_\sigma \in G_\sigma \otimes F_\sigma = \underline{\sigma}^* G_{[\Delta^n]} \otimes \underline{\sigma}^* F_{[\Delta^n]} = (\underline{\sigma}^* G \otimes \underline{\sigma}^* F)(\underline{\Delta}^n) \quad .$$

Moreover

$$d_F(\partial_i \Gamma_\sigma) = \partial_i \, d_F \, \Gamma_\sigma = \partial_i \, \Omega_\sigma = 0 \quad , \qquad 0 \leq i \leq n,$$

and so

$$\Gamma_\sigma \Big|_{\partial \underline{\Delta}^n} \in \underline{\sigma}^* G \otimes Z(\underline{\sigma}^* F)(\partial \underline{\Delta}^n).$$

By $(12.49)_n$ $\Gamma_\sigma \Big|_{\partial \underline{\Delta}^n}$ extends to an element

$$\Psi_\sigma \in \underline{\sigma}^* G \otimes Z(\underline{\sigma}^* F)(\underline{\Delta}^n) = G_\sigma \otimes Z(F)_\sigma .$$

Let $\Phi_\sigma = \Gamma_\sigma - \Psi_\sigma$; then

$$d_F \, \Phi_\sigma = \Omega_\sigma \qquad \text{and} \qquad \partial_i \, \Phi_\sigma = 0, \qquad 0 \leq i \leq n.$$

The isomorphism (12.25) implies now that the Φ_σ define an element $\Phi \in (G \otimes F)(\underline{K}^{(n)}, \underline{K}^{(n-1)})$, which by definition satisfies $d_F \Phi = \Omega$.

$\underline{(12.51)}_n$: This is now immediate, via the extendability of $G \otimes F$.

We finally complete the proof of the lemma.

Part i) is already established by (12.49).

To prove ii) let $\Phi \in (G \otimes BF)(\underline{K})$. Using (12.50) we may construct a sequence $\Psi_0, \ldots, \Psi_n, \ldots \in (G \otimes F)(\underline{K})$ such that

$$\Psi_n \Big|_{\underline{K}^{(n-1)}} = 0 \qquad \text{and} \qquad \Phi \Big|_{\underline{K}^{(n)}} = d_F \Big(\sum_{i=0}^{n} \Psi_i \Big) \Big|_{\underline{K}^{(n)}} \quad .$$

Then $\Phi = d_F \Psi$, where $\Psi = \sum_{n=0}^{\infty} \Psi_n$.

To prove iii), we may assume $\underline{K} = \underline{\Delta}^n$. If $\Phi \in (G \otimes BF)(\partial\underline{\Delta}^n)$ then by ii) $\Phi = d_F\Psi$. Extend Ψ to $\Gamma \in (G \otimes F)(\underline{\Delta}^n)$; then $d_F\Gamma$ extends Φ.

Q.E.D.

12.52.- Proof of theorem 12.47. Because of lemma 12.48 we know that $G \otimes B(F)$ and $G \otimes Z(F)$ are extendable. Because G is extendable while $H(F)$ is a local system of coefficients, $G \otimes H(F)$ is also extendable. Hence, because the sequences (12.46) are exact, prop. 12.42 implies that the upper row in (12.45) is exact.

The theorem now follows trivially from part ii) of lemma 12.48.

Q.E.D.

Finally, assume $\check{G}$ and $\check{F}$ is a second pair of extendable local systems over $\underline{K}$ taking values in a category with differentials, and such that $\check{F}$ is a local system of differential coefficients. Assume

$$\varphi_\sigma : G_\sigma \to \check{G}_\sigma \quad , \qquad \psi_\sigma : F_\sigma \to \check{F}_\sigma$$

are morphisms,compatible with the face and degeneracy operators.

Then

$$\varphi \otimes \psi : (G \otimes F)(\underline{K}) \to (\check{G} \otimes \check{F})(\underline{K})$$

is a map of bicomplexes and so induces a map of spectral sequences

$$(\varphi \otimes \psi)_i : (E_i, d_i) \to (\check{E}_i, \check{d}_i) \quad , \qquad i \geq 0.$$

The isomorphism of theorem 12.47 identifies

(12.53) $$(\varphi \otimes \psi)_1 = \varphi \otimes \psi^* : [G \otimes H(F)](\underline{K}) \to [\check{G} \otimes H(\check{F})](\underline{K}).$$

We thus arrive at

12.54.- *Theorem*. Assume in addition to the above hypotheses that G and $\check{G}$ are also local systems of differential coefficients, and that each $\check{\psi}_\sigma^*$ and ψ_σ^* is an isomorphism. Then

$$(\check{\psi} \otimes \psi)_i : (E_i, d_i) \to (E_i, d_i)$$

is an isomorphism for $2 \leq i \leq \infty$.

Proof : By (12.53) $(\check{\psi} \otimes \psi)_1$ is identified with the map of global sections

$$\check{\psi} \otimes \psi^* : [G \otimes H(F)](\underline{K}) \to [G \otimes H(F)](\underline{K})$$

determined by the morphisms $\check{\psi}_\sigma \otimes \psi_\sigma^*$. Since G, $\check{G}$ are extendable local systems of differential coefficients, and $H(F)$ and $H(\check{F})$ are local systems of coefficients, $G \otimes H(F)$ and $\check{G} \otimes H(\check{F})$ are extendable local systems of differential coefficients.

Moreover $(\check{\psi}_\sigma \otimes \psi_\sigma^*)^* = \check{\psi}_\sigma^* \otimes \psi_\sigma^*$ is an isomorphism. Thus theorem 12.27 asserts that $(\check{\psi} \otimes \psi^*)^*$ is an isomorphism, whence $(\check{\psi} \otimes \psi)_2$ is.

Q.E.D.

Chapter 13

Differential forms.

13.1 *Differential forms on* Δ^n. Recall that $\Delta^n \subset \mathbb{R}^{n+1}$ (example 12.9.) ; it spans an affine n-plane which for the moment we denote by F^n. A C^∞ p-differential form on Δ^n, Φ, is a family of p-linear skew symmetric maps

$$\Phi_x : \underbrace{T_x(F^n) \times \ldots \times T_x(F^n)}_{p \text{ factors}} \longrightarrow \mathbb{R}, \qquad x \in \Delta^n,$$

which extends to an ordinary C^∞ p-form on the manifold F^n.

These form a real vector space, $A^p_\infty(\Delta^n)$ and we have the obvious restriction map $A^p_\infty(F) \to A^p_\infty(\Delta^n)$, which is surjective by definition. It is a straight forward calculation that in the direct sum

$$A_\infty(\Delta^n) = \sum_{p=0}^{n} A^p(\Delta^n)$$

there is a unique multiplication, $\wedge$, and a unique differential d such that restriction $A_\infty(F^n) \to A_\infty(\Delta^n)$ is a homomorphism of c.g.d.a.'s.

Moreover, the standard proofs of the Poincaré lemma apply to show that $(A_\infty(\Delta^n),d)$ is acyclic :

$$H(A_\infty(\Delta^n),d) = \mathbb{R} .$$

Next, let $f : [n] \to [m]$ be any set map and define a linear map $f : R^{n+1} \to R^{m+1}$ by $v_i \to v_{f(i)}$. It restricts to a linear map $\Delta(f) : \Delta^n \to \Delta^m$ which induces in the standard way a homomorphism of c.g.d.a.'s

$$A_\infty(f) : A_\infty(\Delta^n) \leftarrow A_\infty(\Delta^m) .$$

If f is a face map δ_i or a degeneracy map σ_i (cf. 12.1) we write simply

(13.2) $$A_\infty(\delta_i) = \partial_i \quad \text{and} \quad A_\infty(\sigma_i) = s_i .$$

These satisfy the relations (12.3).

Now each element of $A_\infty^p(\Delta^n)$ may be uniquely written

$$\Phi = \sum_{1\le i_1<\dots<i_p\le n} \Phi_{i_1\dots i_p} db_{i_1} \wedge \dots \wedge db_{i_p} ,$$

where (cf. 12.9) b_j is the j^{th} barycentric coordinate function and each $\Phi_{i_1\dots i_p}$ is C^∞-function in Δ^n. If each $\Phi_{i_1\dots i_p}$ is a polynomial in $b_1,\dots,b_n$ with rational coefficients we say Φ is an element of the rational vector space $A_{\mathbb{Q}}^p(n)$.

The rational vector space

$$A_{\mathbb{Q}}(n) = \sum_o^n A_{\mathbb{Q}}^p(n)$$

is in fact the sub c.g.d.a. of $A_\infty(\Delta^n)$ generated (over $\mathbb{Q}$) by $b_o,\dots,b_n$ (because $b_o = 1 - \sum_{i=1}^n b_i$). We can write

$$A_Q(n) = \Lambda(b_1,\dots,b_n,db_1,\dots,db_n).$$

Hence it is a contractible KS complex and in particular acyclic :

$$H(A_Q(n),d) = \mathbb{Q}.$$

Note as well that if $f : |n| \to |m|$ is a set map the induced map $\Delta(f) : \Delta^n \to \Delta^m$ is linear, and $A_\infty(f)$ restricts to a homomorphism of c.g.d.a.'s

$$A_{\mathbb{Q}}(f) : A(n) \leftarrow A(m)$$

(cf. formula 12.10). In particular, as in (13.2) we write

(13.3) $$A_{\mathbb{Q}}(\delta_i) = \partial_i \quad \text{and} \quad A_{\mathbb{Q}}(\sigma_i) = s_i .$$

Finally, if k is any field extension of $\mathbb{Q}$ we set

(13.4) $$A_k(n) = A_{\mathbb{Q}}(n) \otimes_{\mathbb{Q}} k \ ; \quad \partial_i = \partial_i \otimes \iota, \quad s_i = s_i \otimes \iota .$$

If the field k is fixed throughout we will usually suppress the k and simply write $A(n)$.

13.5 *Differential forms on a simplicial set*.- Let $\underline{K}$ be a simplicial set. Define local systems A_∞ and A (over $\mathbb{R}$, or a given field k) on $\underline{K}$ by

$$(A_\infty)_\sigma = A_\infty(\Delta^n), \quad \partial_i = A_\infty(\delta_i), \quad s_i = A_\infty(\sigma_i), \ \sigma \in \underline{K}_n,$$

and

$$A_\sigma = A(n), \quad \partial_i = A(\delta_i), \quad s_i = A(\sigma_i), \quad \sigma \in \underline{K}_n .$$

13.6 *Definition*.- The c.g.d.a.'s of global sections of A_∞ and A :

$$(A_\infty(\underline{K}),d) \quad \text{and} \quad (A(\underline{K}),d)$$

are called, respectively the c.g.d.a. of C^∞ differential forms and the c.g.d.a. of polynomial differential forms on $\underline{K}$. Because we usually consider $A(\underline{K})$ we often call it simply the c.g.d.a. of differential forms on $\underline{K}$.

13.7 *Proposition*.- If $\varphi : \underline{L} \to \underline{K}$ is simplicial then $\varphi^* A_\infty = A_\infty$ and $\varphi^* A = A$. In particular φ induces homomorphisms

$$A_\infty(\varphi) : A_\infty(\underline{L}) \leftarrow A_\infty(\underline{K}) \quad \text{and} \quad A(\varphi) : A(\underline{L}) \leftarrow A(\underline{K}) .$$

Proof : Clear.

Q.E.D.

13.8 *Proposition*.- A and A_∞ are extendable local systems.

Proof : Given prop. 13.7 we need only prove that

$$A(\underline{\Delta}^n) \to A(\partial\underline{\Delta}^n) \quad \text{and} \quad A_\infty(\underline{\Delta}^n) \to A_\infty(\partial\underline{\Delta}^n)$$

are surjective.

Now an element Φ in $A^p(\partial\underline{\Delta}^n)$ is determined by the elements $\Phi_i = \Phi_{\partial_i\Delta^n} \in A^p(n-1)$, $0 \leq i \leq n$ which must satisfy.

$$\partial_i\Phi_j = \partial_{j-1}\Phi_i\ , \qquad i < j.$$

If $\Phi_i = 0$, $i < r < n$, then $\partial_i\Phi_r = 0$, $i < r$. Write

$$\Phi_r = \sum_{0 \leq j_1 < \ldots < r < \ldots j_p < n} P_{j_1 \ldots j_p}\, db_{j_1} \wedge \ \ldots \wedge db_{j_p}\ ,$$

where $P_{j_1\ldots j_p}$ is a polynomial in $b_o,\ldots,\hat{b}_r,\ldots,b_{n-1}$. This same expression defines an element Ψ_r in $A^p(n)$ which satisfies $\partial_i\Psi_r = 0$, $i < r$ and $\partial_r\Psi_r = \Phi_r$. In this way we reduce to the case $\Phi_i = 0$, $i < n$. Assume this.

Consider the projection $\Delta^n - \{v_n\} \xrightarrow{\pi} \Delta^{n-1}$ given by

$$\pi(\sum_o^n t_i v_i) = \sum_o^{n-1} \frac{t_i}{1-t_n} v_i\ .$$

Then $A_\infty(\pi) : A_\infty(\Delta^{n-1}) \to A_\infty(\Delta^n - \{v_n\})$ satisfies

$$A_\infty(\pi)(b_i) = \frac{b_i}{1-b_n} \quad \text{and} \quad A_\infty(db_i) = \frac{db_i}{1-b_n} + \frac{b_i db_n}{(1-b_n)^2}\ .$$

Because Φ_n is <u>polynomial</u> it follows that for some large N

$$(1-b_n)^N A_\infty(\pi)\Phi_n \in A^p(n) = A^p(\underline{\Delta}^n)$$

It is the desired extension.

In the cas of A_∞ the same proof works, except that $(1-b_n)^N$ has to be replaced by a smooth function, zero near v_n.

Q.E.D.

13.9.- Remark.- Prop. 13.8 and its proof are taken directly from Sullivan [10, p. D3]. Cf. also Grivel [7] who gives a different proof, which he attributes to Karoubi.

Next, let F be any local system on $\underline{K}$. We "regard" F as taking values in a category with differentials, (which may possibily all be zero !). Then we can form the local system $A \otimes F$ (cf. 12.36) whose space of global sections is denoted by $A(\underline{K};F)$.

13.10 Definition.- The graded differential space (or g.d.a. or c.g.d.a.) $(A(\underline{K};F),d)$ is called the space (algebra) of differential forms with values in the local system F.

13.11 Remarks 1.- The canonical inclusions

$$i_\sigma : F_\sigma \to 1 \otimes F_\sigma \subset A(n) \otimes F_\sigma, \quad \sigma \in \underline{K}_n ,$$

are coherent with the face and degenerary operators. In particular they determine a canonical inclusion

$$i_F : F(\underline{K}) \longrightarrow A(\underline{K} ; F) .$$

Moreover, because

$$H(A(n) \otimes F_\sigma) = H(A(n) \otimes H(F_\sigma) = H(F_\sigma),$$

each i_σ^* is an isomorphism.

2.- If F is a local system of differential coefficients then so is $A \otimes F$, since $H((A \otimes F)_\sigma) = H(F_\sigma)$.

3.- If F is extendable so is $A \otimes F$ (by theorem 12.37).

4.- If F is a local system of coefficients, then $A \otimes F$ is extendable. Indeed it is trivial that the pullback of F to $\underline{\Delta}^n$ via any $\sigma \in \underline{K}_n$ is constant, and the tensor product of an extendable with a constant local system is obviously extendable.

13.12 *Theorem*.- Let F be an extendable local system of differential coefficients on $\underline{K}$. Then the canonical inclusion, i_F, induces an isomorphism

$$i_F^* : H(F(\underline{K})) \xrightarrow{\cong} H(A(\underline{K};F)) .$$

Proof : According to the remarks above, $A \otimes F$ is also an extendable local system of differential coefficients. By remark 1 we can apply theorem 12.27.

Q.E.D.

Finally, consider the local system, F, of theorem 13.12. We apply the results of sec. 12.43 to the tensor product $A \otimes F$ to obtain a bicomplex. Filtering by the ideals

$$I^p = \sum_{\substack{j \geq p \\ \text{all } q}} (A^j \otimes F^q)(\underline{K}) = \sum_{j \geq p} A^j(\underline{K};F)$$

we obtain a convergent first quadrant spectral sequence, (E_i, d_i), $i \geq 0$.

According to theorem 13.12 this spectral sequence converges to $H(F(\underline{K}))$. On the other hand, theorem 12.47 gives natural isomorphisms

$$(E_1, d_1) = ([A \otimes H(F)](\underline{K}), d) = (A(\underline{K};H(F)), d)$$

(13.13) and

$$E_2 = H(A(\underline{K};H(F))).$$

As an example, assume F takes values in the category of g.d.a.'s. Define a homomorphism $\lambda : (A(\underline{K}),d) \to (A(\underline{K};F),d)$ by

$$(\lambda\Phi)_\sigma = \Phi_\sigma \otimes 1 \quad , \sigma \in \underline{K} .$$

13.14 *Proposition*.- Assume $H^o(A(\underline{K})) = k = H^o(F_\sigma)$, $\sigma \in \underline{K}$. Then $H^o(A(\underline{K};F)) = H^o(\underline{K})) = k$ and

$$\lambda^* : H^1(A(K)) \longrightarrow H^1(A(\underline{K};F))$$

is injective.

Proof : Filter $A(\underline{K})$ by the ideals $\sum_{j\geq p} A^j(\underline{K})$ to obtain a spectral sequence $(\check{E}_i^{p,q},\check{d}_i)$. Then λ induces a homomorphism $\lambda_i : \check{E}_i \to E_i$ of spectral sequences.

Because of our hypothesis on F, (13.13) yields

$$E_1^{p,o} = (A^p(\underline{K}),d), \quad p \geq 0,$$

and so $\lambda_1 : A(K) \xrightarrow{\cong} E_1^{*,o}$. Hence λ_2 is an isomorphism in degree zero and injective in degree 1. The standard comparison theorem for spectral sequences now implies that λ^* has the same property.

Q.E.D.

Chapter 14

Simplicial cohomology.

14.1. - Definition.- Let $\underline{K}$ be a simplicial set. Define

$$\partial_p^F : \underline{K}_{p+q} \to \underline{K}_p \quad \text{and} \quad \partial_q^B : \underline{K}_{p+q} \to \underline{K}_q$$

by

$$\partial_p^F\sigma = (\partial_{p+1} \circ \dots \circ \partial_{p+q})\sigma \quad \text{and} \quad \partial_q^B\sigma = (\partial_o \underbrace{\circ \dots \circ \partial_o}_{p \text{ factors}})\sigma.$$

The simplices $\partial_p^F\sigma$ and $\partial_q^B\sigma$ are called the front p-face and the back q-face of σ.

Now we recall the g.d.a. $(C(\underline{K};k),\delta)$ of simplicial cochains on $\underline{K}$. $C^p(\underline{K};k)$ is the vector space of all set maps $f : \underline{K}_p \to k$. It is a g.d.a. with multiplication and differential given by

$$(f.g)(\sigma) = f(\partial_p^F\sigma)g(\partial_q^B\sigma) \quad , \quad f \in C^p(\underline{K};k), \; g \in C^q(\underline{K};k), \; \sigma \in \underline{K}_{p+q}$$

and

$$(\delta f)(\sigma) = \sum_{i=0}^{p+1} (-1)^i f(\partial_i\sigma) \quad , \quad f \in C^p(\underline{K};k), \; \sigma \in \underline{K}_{p+1}.$$

The identity is the constant function $\underline{K}_o \to 1$.

The cohomology algebra of $(C(\underline{K};k),\delta)$ is denoted by $H(\underline{K};k)$ and is called the simplicial cohomology of $\underline{K}$.

A simplicial map $\psi : \underline{K} \to \underline{L}$ determines homomorphisms $C(\psi) : C(\underline{K};k) \leftarrow C(\underline{L};k)$ and $C(\psi)^* : H(\underline{K};k) \leftarrow H(\underline{L};k)$ given by $(C(\psi)f)(\sigma) = f(\psi\sigma)$.

More generally, suppose F is a local system of coefficients on $\underline{K}$ (cf. just before theorem 12.27) so that each $\partial_i : F_\sigma \to F_{\partial_i\sigma}$ and $s_j : F_\sigma \to F_{s_j\sigma}$ are isomorphisms. Define a bigraded space

$$C(\underline{K};F) = \sum_{p,q\geq 0} C^p(\underline{K};F^q)$$

as follows : an element of $C^p(\underline{K};F^q)$ is a function which assigns to each $\sigma \in \underline{K}_p$ an element of F^q_σ.

A differential (of bidegree (1,0)) in $C(\underline{K};F)$ is given by

$$(\delta f)(\sigma) = \sum_{i=0}^{p+1} (-1)^i \, \partial_i^{-1} f(\partial_i \sigma) \qquad \sigma \in \underline{K}_{p+1}, \quad f \in C^p(\underline{K};F).$$

If F takes values in a category of algebras we make $C(\underline{K};F)$ into a g.d.a. by putting

$$(f.g)(\sigma) = (-1)^{qr} \left[(\partial_p^F)^{-1} f(\partial_p^F \sigma)\right].\left[(\partial_q^B)^{-1} g(\partial_q^B \sigma)\right],$$

$$f \in C^p(\underline{K};F^r), g \in C^q(\underline{K};F^s).$$

We call $C(\underline{K};F)$ the space (or algebra) of <u>simplicial cochains with coefficients in</u> F. Its cohomology, $H(\underline{K};F)$ is called the <u>simplicial cohomology of</u> $\underline{K}$ <u>with coefficients in</u> F. It is bigraded : $H(\underline{K};F) = \sum_{p,q\geq 0} H^p(\underline{K};F^q)$.

14.2. - <u>The local system</u> C.- We shall interpret $C(\underline{K};k)$ as the global sections of a certain local system over $\underline{K}$ (if 12.14). Recall (12.11) that $\underline{\Delta}^n$ is the simplicial set given by $(\underline{\Delta}^n)_p = \mathrm{Ord}([p],[n])$.

Denote $C(\underline{\Delta}^n;k)$ simply by $C(n)$. Each $\alpha \in \mathrm{Ord}([n],[m])$ defines a simplicial map $\alpha : \underline{\Delta}^n \to \underline{\Delta}^m$ $(\alpha(\sigma) = \alpha \circ \sigma)$ and so determines a g.d.a. homomorphism $C(\alpha) : C(n) \leftarrow C(m)$:

(14.3) $\quad (C(\alpha)f)(\sigma) = f(\alpha \circ \sigma) \qquad f \in C^p(m)$

$$\alpha \in \mathrm{Ord}([n],[m])$$

$$\sigma \in (\underline{\Delta}^n)_p.$$

In particular we write $C(\delta_i) = \partial_i$ and $C(\sigma_j) = s_j$ where δ_i and σ_j are defined in (12.1) ; these homomorphisms satisfy (12.3).

Now proceed exactly as in the case of differential forms to define a local system C over every simplicial set $\underline{K}$ as follows :

$$C_\sigma = C(|\sigma|) \quad , \quad \sigma \in \underline{K}_p, \ p = 0,1,\dots \quad ,$$

and ∂_i, s_j are the homomorphisms just defined. The g.d.a. of global sections is written $(C(\underline{K}),\delta)$. A simplicial map $\varphi : \underline{K} \to \underline{L}$ determines an obsvious homomorphism $C(\varphi) : C(\underline{L}) \to C(\underline{K})$.

More generally let F be any local system of coefficients over $\underline{K}$, such that the differential in each F_σ is zero. Then the local system $C \otimes F$ over $\underline{K}$ is a system of bigraded differential spaces (in which the differentials are simply $\delta \otimes \iota$). We shall describe a canonical isomorphism

$$\text{(14.4)} \qquad \Gamma : (C^p \otimes F^q)(\underline{K}) \xrightarrow{\cong} C^p(\underline{K};F^q),$$

where the right hand side is defined in 14.1. In particular (when $F = k$) we may identify $C(\underline{K})$ with $C(\underline{K};k)$ as g.d.a.'s.

First observe that

$$\begin{aligned} C^p(n) \otimes F^q_\sigma &= \text{Set functions}((\underline{\Delta}^n)_p \to k) \otimes F^q_\sigma \\ &= \text{Set functions}((\underline{\Delta}^n)_p \to F^q_\sigma) \\ &= C^p(\underline{\Delta}^n;F^q_\sigma), \end{aligned}$$

because $(\underline{\Delta}^n)_p$ is a finite set. Thus if $\Phi \in (C^p \otimes F^q)(\underline{K})$ we may interpret each Φ_σ as a set map

$$\Phi_\sigma : (\underline{\Delta}^n)_p \to F^q_\sigma, \ \sigma \in \underline{K}_n.$$

In particular Φ determines set maps

$$\Phi_\sigma : (\underline{\Delta}^p)_p \to F^q_\sigma, \qquad \sigma \in \underline{K}_p .$$

Now recall that $[\Delta^p] \in (\underline{\Delta}^p)_p$ is the distinguished simplex corresponding to the identity map $[p] \to [p]$. Define Γ by

$$(14.5) \qquad (\Gamma\Phi)(\sigma) = \Phi_\sigma([\Delta^p]) \quad , \quad \Phi \in (C^p \otimes F^q)(\underline{K}), \quad \sigma \in \underline{K}_p \; .$$

14.6. - Theorem.- The linear maps

$$\Gamma : (C^p \otimes F^q)(\underline{K}) \to C^p(\underline{K};F)$$

defined by (14.5) are isomorphisms commuting with the differentials. If F takes values in a category of algebras then Γ is an isomorphism of g.d.a.'s.

Proof : Because the face and degeneracy operators of F satisfy (12.3) we can find unique morphisms

$$F(\alpha) : F_\sigma \to F_{\underline{K}(\alpha)\sigma} \quad , \quad \sigma \in \underline{K}_p, \quad \alpha \in \mathrm{Ord}([m],[p]) \; ,$$

such that $F(\delta_i) = \partial_i$, $F(\sigma_j) = s_j$ and $F(\alpha\beta) = F(\beta) \circ F(\alpha)$. Since F is a local system of coefficients, each $F(\alpha)$ is an isomorphism.

Next, fix $\Phi \in (C^p \otimes F^q)(\underline{K})$. Because Φ is compatible with the face and degeneracy operators we have

$$(14.7) \qquad [C(\alpha) \otimes F(\alpha)]\Phi_\sigma = \Phi_{\underline{K}(\alpha)\sigma}, \qquad \sigma \in \underline{K}_n \; , \quad \alpha \in \mathrm{Ord}([m],[n]) .$$

If Φ_σ is interpreted as a function $(\underline{\Delta}^n)_p \to F^q_\sigma$ $(|\sigma| = n)$ then this reads

$$(14.8) \qquad F(\alpha)[\Phi_\sigma(\alpha \circ \tau)] = \Phi_{\underline{K}(\alpha)\sigma}(\tau), \tau \in (\underline{\Delta}^m)_p = \mathrm{Ord}([p],[m]),$$

as follows from (14.3).

Since $F(\alpha)$ is an isomorphism we obtain, finally,

$$(14.9) \qquad \Phi_\sigma(\alpha \circ \tau) = F(\alpha)^{-1}\{\Phi_{\underline{K}(\alpha)\sigma}(\tau)\} \qquad \begin{array}{l} \sigma \in \underline{K}_n \\ \alpha \in \mathrm{Ord}([m],[n]) \\ \tau \in \mathrm{Ord}([p],[m]) \; . \end{array}$$

In particular when $m = p$ and $\tau = [\Delta^p]$ is the identity map of $[p]$, (14.9) reads

(14.10) $$\Phi_\sigma(\alpha) = F(\alpha)^{-1}\{(\Gamma\Phi)(\underline{K}(\alpha)\sigma)\} \qquad \begin{array}{l}\sigma \in \underline{K}_n \\ \alpha \in \mathrm{Ord}([p],[n])\end{array}.$$

This equation shows that if $\Gamma\Phi = 0$ then $\Phi = 0$ and so Γ is injective. Moreover if $f \in C^p(\underline{K};F^q)$ is given, define $\Phi \in (C^p \otimes F^q)(\underline{K})$ by

$$\Phi_\sigma(\alpha) = F(\alpha)^{-1}\{f(\underline{K}(\alpha)\sigma)\} \quad \sigma \in \underline{K}_n, \alpha \in \mathrm{Ord}([p],[n]).$$

Then $\Gamma\Phi = f$ and so Γ is surjective.

It is easy to verify that $\Gamma\circ\delta = \delta\circ\Gamma$ and that Γ preserves products if F takes values in a category of algebras.

Q.E.D.

14.11. - *Proposition*.- The local system C (over $\underline{K}$) has the following properties :

i) If $\varphi : \underline{L} \to \underline{K}$ is a simplicial map then $\varphi^*C = C$. The induced homomorphism $C(\varphi) : C(\underline{L}) \leftarrow C(\underline{K})$ coïncides with that given in 14.1.

ii) $H(C(n),\delta) = k$ for all n. In particular, C is a local system of differential coefficients.

iii) C is extendable.

Proof : i) is obvious, ii) is a classical computation and iii) is an immediate corollary of theorem 14.6 (with $F = k$).

Q.E.D.

14.12. - *Topological spaces*.- Let M be a topological space and consider the simplicial set $\underline{\mathrm{Sing}}\ M$ of singular simplices on M (eg. 12.13) The g.d.a. $(C(\underline{\mathrm{Sing}}\ M),\delta)$ will simply be denoted by $(C(M),\delta)$; by definition it is the g.d.a. of singular cochains on M (with coefficients k). Its cohomology is written $H(M)$ or $H(M;k)$; it is the singular cohomology of M (with coefficients k).

14.13. - Integration and de Rham theorems.- We conclude this chapter with the simplicial version of de Rham's theorem. Recall from 13.1 the definition of $A_\infty(\Delta^n)$. Define a linear map

$$\int_{\Delta^n} : A_\infty^n(\Delta^n) \to \mathbb{R}$$

by

$$\int_{\Delta^n} f\, db_1 \wedge \dots \wedge db_n = \int_{\Delta^n} f(x)dx, \quad f \in C^\infty(\Delta^n),$$

where the right hand side is the ordinary Riemann integral. Stokes'theorem reads

$$(14.14) \qquad \int_{\Delta^n} d\Phi = \sum_{i=0}^{n} (-1)^i \int_{\Delta^{n-1}} \partial_i \Phi\ , \qquad \Phi \in A_\infty^{n-1}(\Delta^n).$$

Now $A_{\mathbb{Q}}^n(n) \subset A_\infty^n(\Delta^n)$ is the subspace defined by : f is a polynomial in the b_i with rational coefficients. It follows that $\int_{\Delta^n}$ restricts to a linear map $A_{\mathbb{Q}}^n(n) \to \mathbb{Q}$. Tensoring with k we obtain a linear map

$$\int_{\Delta^n} : A^n(n) \to k$$

which still satisfies (14.14) . If E is any vector space (over k) we also write

$$\int_{\Delta^n} = \int_{\Delta^n} \otimes \iota : A^n(n) \otimes E \to E.$$

Next, let F be a local system of coefficients over a simplicial set $\underline{K}$. Define linear maps

$$\int : A^p(\underline{K}\ ;\ F^q) \to C^p(\underline{K}\ ;\ F^q)$$

by

(14.15) $$\left(\int \Phi\right)(\sigma) = \int_{\Delta^p} \Phi_\sigma \qquad \sigma \in \underline{K}_p\ .$$

(Note that $\Phi_\sigma \in A^p(p) \otimes E_\sigma$ and so $\int_{\Delta^p} \Phi_\sigma \in F_\sigma$.)

If $\psi : \underline{L} \to \underline{K}$ is a simplicial map and $G = \psi^* F$ then ψ induces maps.

$$A(\psi) : A(\underline{L}\ ;\ G) \leftarrow A(\underline{K}\ ;\ F) \quad \text{and} \quad C(\psi) : C(\underline{L}\ ;\ G) \leftarrow C(\underline{K}\ ;\ F).$$

It is immediate from the definitions that the diagram

(14.16) $$\begin{array}{ccc} A(\underline{L}\ ;\ G) & \xleftarrow{\quad A(\psi)\quad} & A(\underline{K}\ ;\ F) \\ {\scriptstyle\int}\big\downarrow & & \big\downarrow{\scriptstyle\int} \\ C(\underline{L}\ ;\ G) & \xleftarrow[\quad C(\psi)\quad]{} & C(\underline{K}\ ;\ F) \end{array}$$

commutes.

Next we use (14.14) to prove that

(14.17) $$\int \circ\ d = \delta \circ \int\ .$$

Let $\Phi \in A^p(\underline{K}\ ;\ F^q) = (A^p \otimes F^q)(\underline{K})$. Then

$$\left(\int d\Phi\right)(\sigma) = \int_{\Delta^p} d\Phi_\sigma = \sum_{i=0}^{p} (-1)^i \int_{\Delta^{p-1}} \partial_i\, \Phi_\sigma$$

$$= \sum_{i=0}^{p-1} (-1)^i\, \partial_i^{-1} \int_{\Delta^{p-1}} (\partial_i \otimes \partial_i)(\Phi_\sigma).$$

Because $\Phi \in (A^p \otimes F^q)(\underline{K})$, $(\partial_i \otimes \partial_i)\Phi_\sigma = \Phi_{\partial_i\sigma}$. Thus

$$\left(\int d\Phi\right)(\sigma) = \sum_{i=0}^{p+1} (-1)^i \, \partial_i^{-1}\left(\int \Phi\right)(\partial_i\sigma) = \left(\delta\int \Phi\right)(\sigma).$$

<u>Note, however, that even if</u> $F = k$, $\int$ <u>is not an algebra homomorphism</u> !

In view of (14.17) and (14.16) $\int$ induces natural linear maps

$$\int^{*} : H^{p+q}((A \otimes F^q)(\underline{K}), d) \longrightarrow H^p(\underline{K} \,;\, F^q).$$

14.18. *Theorem*.- Suppose F is a local system of coefficients over $\underline{K}$ (with zero differentials). Then

$$\int^{*} : H((A \otimes F)(\underline{K})) \to H(\underline{K} \,;\, F)$$

is an isomorphism of bigraded spaces. If F takes values in a category of algebras then $\int^{*}$ is an isomorphism of algebras.

Proof : Since A and C are extendable local systems (prop 13.8, prop 14.11) and F is a local system of coefficients, $A \otimes F$ and $C \otimes F$ are also extendable. Thus by theorem 12.37, $A \otimes C \otimes F$ is extendable as well.

Because $H(A(n)) = k = H(C(n))$ and F is a local system of coefficients, $A \otimes F$, $C \otimes F$ and $A \otimes C \otimes F$ are all local systems of differential coefficients.

Now consider the inclusions "opposite 1"

$$\varphi_\sigma : (A \otimes F)_\sigma \longrightarrow (A \otimes C \otimes F)_\sigma$$

and

$$\psi_\sigma : (C \otimes F)_\sigma \longrightarrow (A \otimes C \otimes F)_\sigma \;, \qquad \sigma \in \underline{K}_n, \quad n \geq 0.$$

They satisfy the hypotheses of theorem 12.27. Applying this theorem we obtain isomorphisms

$$\varphi^{*} : H[(A \otimes F)(\underline{K})] \xrightarrow{\;\cong\;} H[(A \otimes C \otimes F)(\underline{K})]$$

and

$$\psi^* : H(C(\underline{K} ; F)) \xrightarrow{\cong} H[(A_\bullet \otimes C \otimes F)(\underline{K})],$$

which are multiplicative if F takes values in a category of algebras. (Note that we have used the isomorphism Γ of theorem 14.6 to identify $(C \otimes F)(\underline{K})$ with $C(\underline{K} ; F)$.)

It remains to show that $\int^* = (\psi^*)^{-1} \circ \varphi^*$. Define linear maps

$$I_\sigma : A(n) \otimes C(n) \otimes F_\sigma \to C(n) \otimes F \qquad , \qquad \sigma \in \underline{K}_n,$$

as follows : since $A(n) = A(\underline{\Delta}^n)$ and $C(n) = C(\underline{\Delta}^n)$ the chain map

$$\int : (A(n),d) \to (C(n),\delta)$$

is defined. Let

$$I_\sigma(\Phi \otimes f \otimes z) = \left(\int \Phi\right) . f \otimes z$$

Because $\int d = \delta \int$, I_σ commutes with the differentials. Because $\int$ is natural with respect to simplicial maps (14.16), the I_σ commute with the face and degenerary operators. Hence the I_σ define a map

$$I : (A \otimes C \otimes F)(\underline{K}) \longrightarrow C(\underline{K} ; F).$$

Clearly $I \circ \psi = \iota$ and $I \circ \psi = \int$. Hence $I^* = (\psi^*)^{-1}$ and so $\int^* = (\psi^*)^{-1} \circ \varphi^*$.

Q.E.D.

Chapter 15

Topological spaces and manifolds.

15.1.- Differential forms.- Let M be a topological space and recall (example 12.13) that Sing M denotes the simplicial set of singular simplices on M. Thus we can form the c.g.d.a. (A(Sing M),d).

If $\psi : M \to N$ is a continuous map it defines a simplicial map $S(\psi)$: Sing(M) → Sing(N) and so we obtain a homomorphism

$$A(\psi) : (A(\underline{\text{Sing}}\ M),d) \leftarrow (A(\underline{\text{Sing}}\ N),d).$$

15.2.- Definition.- The c.g.d.a. (A(Sing M),d) will be denoted simply by

$$(A(M),d)$$

and called the c.g.d.a. of differential forms on M. If $i : N \to M$ is the inclusion of a subspace then

$$(A(M,N),d)$$

denotes the ideal of forms which vanish on N ($A(i)\Phi = 0$).

15.3.- Property.- The map $M,\psi \rightsquigarrow A(M),A(\psi)$ is a contravariant functor from spaces to c.g.d.a.'s.

15.4.- Property.- If N is a subspace of M then

$$0 \to A(M,N) \to A(M) \xrightarrow{A(i)} A(N) \to 0$$

is a short exact sequence.

Proof : Since $N \subset M$, $\underline{\text{Sing}}\, N$ is a subset of $\underline{\text{Sing}}\, M$. Apply prop. 12.21 to the extendable local system A.

Q.E.D.

15.5.- *Property*.- A(point) = k. In particular the inclusion of a point in M defines an augmentation $A(M) \to k$.

Proof : Any singular simplex $\sigma \in \underline{\text{Sing}}_n(pt)$ can be written

$$\sigma = s_{i_1} \circ \dots \circ s_{i_n}(\tau)$$

where τ is the unique singular simplex of dimension zero. It follows that $A(pt) \to A(0)$ defined by $\Phi \mapsto \Phi_\tau$ is an isomorphism. But $A(0) = k$.

Q.E.D.

15.6.- *Property*.- Integration defines a natural isomorphism of graded algebras

$$H(A(M),d) \xrightarrow{\cong} H(M;k),$$

where $H(M;k)$ is the singular cohomology.

Proof : By definition $H(M;k) = H(\underline{\text{Sing}}\, M;k)$. Now apply theorem 14.18.

Q.E.D.

15.7.- *Property*.- If $N \subset M$ integration defines a natural isomorphism

$$H(A(M,N),d) \xrightarrow{\cong} H(M,N;k),$$

which identifies the long exact cohomology sequence of the differential form cohomology with that of singular cohomology.

Because $H(A(M),d) = H(M;k)$ it follows that $H^0(A(M),d) = k$ if and only M is path connected. In this case we can apply the results of chap. 6 to the inclusion

$$k \hookrightarrow (A(M),d)$$

to obtain a "unique" minimal model (cf theorems 6.1 , 6.2 , 6.3)

$$\mathcal{V}_M : (\Lambda X_M,d) \longrightarrow (A(M),d) .$$

15.8.- *Definition*.- The minimal model

$$\mathcal{V}_M : (\Lambda X_M,d) \longrightarrow (A(M),d)$$

is called the minimal model for M. Note that ΛX_M is connected.

Next suppose $A(M)$ is augmented (by the inclusion of a fixed base point in M). Recall (definition 8.7 with $A(M) = E$) the ψ-homotopy spaces $\pi_\psi(A(M))$ and note that $\mathcal{V}_M$ gives an explicit identification

$$\pi_\psi^p(A(M)) = Q^p(\Lambda X_M) \ (\cong X_M^p) .$$

15.9.- *Definition*.- The spaces $\pi_\psi^p(A(M))$ will be written simply $\pi_\psi^p(M)$:

$$\pi_\psi^p(M) = Q^p(\Lambda X_M)$$

and called the ψ-homotopy spaces of the path connected based space M.

Next recall that a continuous map $\mathcal{V} : N \to M$ between path connected spaces determines a homomorphism

$$A(\mathcal{V}) : (A(M),d) \to (A(N),d).$$

We regard this as a special case of diagram (6.28) (with $B_1 = B_2 = k$ $E_1 = A(M)$, $E_2 = A(N)$, $\eta = A(\mathcal{V})$).

Then, as described in diagram (6.29) we obtain a homomorphism

$$\bar{\psi} : (\Lambda X_M, d) \to (\Lambda X_N, d)$$

such that the diagram

(15.10)
$$\begin{array}{ccc} A(M) & \xleftarrow{\ \varphi_M\ } & \Lambda X_M \\ {\scriptstyle A(\psi)}\downarrow & & \downarrow{\scriptstyle \bar{\psi}} \\ A(N) & \xleftarrow[\ \varphi_N\ ]{} & \Lambda X_N \end{array}$$

homotopy commutes. Moreover, the homotopy class of $\bar{\psi}$ is uniquely determined.

If ψ preserves base points then the based homotopy class of $\bar{\psi}$ is uniquely determined. In particular, it determines maps between the ψ-homotopy spaces (cf. definition 8.11) which we write

(15.11)
$$\psi^{\#} : \pi_{\psi}^{*}(M) \longrightarrow \pi_{\psi}^{*}(N) \quad ;$$

i.e.,

$$\psi^{\#} = Q(\bar{\psi}).$$

15.12.- Property.- Assume $\psi_o, \psi_1 : N \to M$ are homotopic maps between path connected spaces. Then the induced homomorphisms

$$\bar{\psi}_o, \bar{\psi}_1 : \Lambda X_M \longrightarrow \Lambda X_N$$

are homotopic.

If ψ_o, ψ_1, and the homotopy preserve base points (fixed in M and N) then $\bar{\psi}_o$ and $\bar{\psi}_1$ are based homotopic, and so

$$\psi_o^{\#} = \psi_1^{\#} : \pi_{\psi}^{*}(M) \longrightarrow \pi_{\psi}^{*}(N).$$

Proof : Fix a base point $x \in N$, and consider the c.g.d.a.

$$\tilde{A}(N \times I) = A(N \times I, x \times I) \oplus k.$$

The inclusion $\tilde{A}(N \times I) \hookrightarrow A(N \times I)$ induces a cohomology isomorphism as follows at one from property 15.7 , and $\tilde{A}(N \times I)$ is naturally augmented.

The projection $\pi : N \times I \to N$ induces a homomorphism of augmented c.g.d.a.'s

$$A(\pi) : A(N) \to \tilde{A}(N \times I)$$

which is a cohomology isomorphism. Hence we may take

$$A(\pi) \circ \psi_N : \Lambda X_N \to \tilde{A}(N \times I)$$

as the minimal model ; composing with the inclusion $\tilde{A}(N \times I) \hookrightarrow A(N \times I)$ gives the minimal model for $N \times I$.

On the other hand the inclusions $j_\lambda : N \to N \times \{\lambda\}$ $(\lambda = 0,1)$ define homomorphisms of augmented c.g.d.a.'s

$$A(j_\lambda) : \tilde{A}(N \times I) \to A(N)$$

and $A(j_\lambda) \circ A(\pi) = A(\pi j_\lambda) = \iota$.

Now assume $\Phi : N \times I \to M$ is a homotopy from ψ_0 to ψ_1 which preserves base points : $\Phi(x \times I) = y$. Then Φ determines a homomorphism of augmented c.g.d.a.'s

$$A(\Phi) : \tilde{A}(N \times I) \to A(M)$$

This in turn determines a homomorphism

$$\bar{\Phi} : \Lambda X_N \to \Lambda X_M$$

such that $A(\Phi) \circ \psi_M \simeq A(\pi) \circ \psi_N \circ \bar{\Phi}$.

We thus have

$$\psi_N\bar{\psi}_\lambda \simeq A(\psi_\lambda)\psi_M = A(j_\lambda)A(\Phi)\psi_M$$
$$\simeq A(j_\lambda)A(\pi)\psi_N\bar{\Phi} = \psi_N\bar{\Phi} \; , \qquad \lambda = 0,1.$$

Now prop. 5.16. ii) implies (because ψ_N^* is an isomorphism) that $\bar{\psi}_o \simeq \bar{\Phi} \simeq \bar{\psi}_1$, whence $\bar{\psi}_o \simeq \bar{\psi}_1$.

The "unbased" case is left to the reader.

Q.E.D.

We next consider a number of topological constructions, and find c.g.d.a.'s which "carry" the model of the constructed space.

15.13.- ***Products.*** - Let $\pi_M : M \times N \to M$, $\pi_N : M \times N \to N$ be the projections, and define

$$A(M) \otimes A(N) \quad \to \quad A(M \times N)$$

by $\Phi \otimes \Psi \mapsto A(\pi_M)\Phi . A(\pi_N)\Psi$.

This is a homomorphism of c.g.d.a.'s. Because $\int^*$ is a multiplicative isomorphism, it identifies the induced homomorphism

$$H(A(M)) \otimes H(A(N)) \quad \to \quad H(A(M \times N))$$

with the homomorphism

$$H(M;k) \otimes H(N;k) \quad \to \quad H(M \times N;k)$$

given by $\alpha \otimes \beta \to (\pi_M^*\alpha).(\pi_N^*\beta)$.

If either $H(M;k)$ or $H(N;k)$ has finite type the Kunneth theorem asserts that this is an isomorphism.

Assume this to be the case and let

$$\psi_M : \Lambda X_M \to A(M) \quad \text{and} \quad \psi_N : \Lambda X_N \to A(N)$$

be the minimal models (supposing as well M,N path connected). Then

$$\psi_M \cdot \psi_N : \Lambda X_M \otimes \Lambda X_N \to A(M \times N)$$

$(\psi_M \cdot \psi_N(\Phi \otimes \Psi) = A(\pi_M)\psi_M\Phi \cdot A(\pi_N)\psi_N\Psi)$ induces a cohomology isomorphism. Hence it is the minimal model for $M \times N$.

15.14.- Wedges.- Let $\{M_\alpha, x_\alpha\}_{\alpha\in I}$ be a collection of path connected spaces with base points. Their wedge, $\vee_\alpha M_\alpha$ is the disjoint union with all the x_α's identified.

On the other hand if $\{A_\beta, \varepsilon_\beta\}_{\beta\in I}$ is a collection of augmented c.g.d.a.'s, their wedge is the augmented c.g.d.a.

$$\vee_\beta A_\beta = k \oplus \prod_\beta \ker \varepsilon_\beta .$$

In particular the inclusions $M_\alpha \to \vee_\alpha M_\alpha$ define homomorphisms

$$A(\vee_\alpha M_\alpha) \to \vee_\alpha A(M_\alpha)$$

and

$$H(\vee_\alpha M_\alpha;k) \to \vee_\alpha H(M_\alpha;k).$$

Integration identifies the second of these with the cohomology homomorphism induced by the first : if the second is an isomorphism the first induces a cohomology isomorphism.

Assume this to be the case. Then $A(\vee_\alpha M_\alpha)$ and $\vee_\alpha A(M_\alpha)$ have the same minimal models (cf. theorem 6.24). On the other hand if $\psi_\alpha : \Lambda X_\alpha \to A(M_\alpha)$ is the minimal model then the ψ_α determine

$$V\psi_\alpha : \underset{\alpha}{V}(\Lambda X_\alpha) \to \underset{\alpha}{V} A(M_\alpha)$$

and $(V\psi_\alpha)^*$ is an isomorphism. Hence (again by 6.24) the minimal model of $\underset{\alpha}{V} M_\alpha$ is the minimal model of

$$\underset{\alpha}{V}(\Lambda X_\alpha) = k \oplus \prod_\alpha \Lambda^+ X_\alpha .$$

(Observe that the right hand side is rather horrid.)

15.15.- *Attaching maps*.- Let

$$i : N \to M \quad \text{and} \quad f : N \to P$$

be continuous maps, in which i is the inclusion of a subspace. We use f to attach M to P by identifying x and $f(x)$ $(x \in N)$ in the disjoint union of P and M ; the resulting space is written $P \cup_f M$.

Then these is an obvious map of topological pairs

$$(M,N) \longrightarrow (P \cup_f M, P)$$

and we shall say M is <u>well attached</u> to P if this map induces an isomorphism

(15.16) $$H(M,N;k) \xleftarrow{\cong} H(P \cup_f M, P;k) .$$

Consider the analogous situation for c.g.d.a.'s. Suppose

$$\eta : R \to G \quad \text{and} \quad \gamma : L \to G$$

are homomorphisms of c.g.d.a.'s, and η <u>is surjective</u>. Define the c.g.d.a.

$$L \overset{\bullet}{\otimes}_G R \subset L \otimes R$$

to consist of the pairs $(\Phi \otimes \Psi)$ such that $\gamma\Phi = \eta\Psi$. Note that it fits into the short exact sequence

(15.17) $$0 \to L \otimes_G R \to L \otimes R \xrightarrow{\gamma-\eta} G \to 0$$

(in which $\gamma-\eta$ is not product preserving and sends 1 to zero !).

In particular, if we are in the topological situation described above we can form the exact sequence

$$0 \to A(P) \otimes_{A(N)} A(M) \to A(P) \otimes A(M) \xrightarrow{A(f)-A(i)} A(N) \to 0.$$

Let $j : P \to P \cup_f M$, $\psi : M \to P \cup_f M$ be the obvious maps and define a homomorphism

$$\vartheta : A(P \cup_f M) \longrightarrow A(P) \otimes_{A(N)} A(M)$$

by $\vartheta\Phi = A(j)\Phi \otimes A(\psi)\Phi$.

15.18.- Proposition.- If M is well attached to P then ϑ^* is an isomorphism of cohomology. In particular, the minimal model for $P \cup_f M$ is the minimal model for $A(P) \otimes_{A(N)} A(M)$.

Proof : Observe that $A(M,N) = 0 \otimes A(M,N) \subset A(P) \otimes_{A(N)} A(M)$ and that the diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & A(P \cup_f M, P) & \longrightarrow & A(P \cup_f M) & \longrightarrow & A(P) & \longrightarrow & 0 \\ & & \downarrow & & \downarrow \vartheta & & \downarrow \iota & & \\ 0 & \longrightarrow & A(M,N) & \longrightarrow & A(P) \otimes_{A(N)} A(M) & \longrightarrow & A(P) & \longrightarrow & 0 \end{array}$$

is commutative and row exact. By (15.16) and property 15.7 the left hand arrow is a cohomology isomorphism. Hence ψ^* is an isomorphism.

Q.E.D.

15.19.- *Manifolds*.- Let M be a smooth manifold and consider the standard c.g.d.a. of "de Rham" differential forms, $(A_{DR}(M),d)$. Recall the simplicial set $\underline{\mathrm{Sing}}^{\infty}(M)$ of smooth singular simplices on M. Define a homomorphism (cf. 13.6) of c.g.d.a.'s

$$\gamma_1 : A_{DR}(M) \to A_{\infty}(\underline{\mathrm{Sing}}^{\infty}(M))$$

by $(\gamma_1\Phi)_{\sigma} = A(\sigma)\Phi$.

On the other hand the inclusions $\underline{\mathrm{Sing}}^{\infty}(M) \hookrightarrow \underline{\mathrm{Sing}}(M)$ and $A_{\mathbb{R}}(n) \to A_{\infty}(\Delta^n)$ define homomorphisms of c.g.d.a.'s

$$A_{\infty}(\underline{\mathrm{Sing}}^{\infty}(M)) \longleftarrow A_{\mathbb{R}}(\mathrm{Sing}^{\infty}(M)) \longleftarrow A_{\mathbb{R}}(M) .$$

Moreover the diagram

$$\begin{array}{ccccc}
A_{DR}(M) & \xrightarrow{\ \gamma_1\ } & A_{\infty}(\underline{\mathrm{Sing}}^{\infty}(M)) & \xleftarrow{\ \gamma_2\ } & A_{\mathbb{R}}(M) \\
\int\downarrow & \swarrow \int & & & \downarrow\int \\
C(\underline{\mathrm{Sing}}^{\infty}(M);\mathbb{R}) & \longleftarrow & & & C(\underline{\mathrm{Sing}}(M);\mathbb{R})
\end{array}$$

commutes.

Now the first vertical arrow is a cohomology isomorphism. (This is in fact the De Rham theorem of De Rham !) The other two vertical arrows induce cohomology isomorphisms by theorem 12.27. That the bottom arrow does this is standard topology.

It follows that γ_1^* and γ_2^* are isomorphisms. Hence γ_1 and γ_2 define a c-equivalence between $A_{DR}(M)$ and $A_{\mathbb{R}}(M)$ (cf. 6.23). Hence (theorem 6.24) their minimal models and ψ-homotopy spaces coïncide.

Chapter 16

$\pi_1(M)$-modules and singular cohomology.

16.1.- In this chapter M denotes a fixed path connected topological space with base point x. The fundamental group of M, based at x, is denoted by $\pi_1(M)$.

By a $\pi_1(M)$-module we shall mean a graded vector space (over k) $F = \sum_{q\geq 0} F^q$ together with group homomorphisms $\pi_1(M) \to GL(F^q)$, $q \geq 0$. ($GL(F^q)$ denotes the abstract group of k-linear automorphisms of F^q.) We shall regard $\alpha \in \pi_1(M)$ as an automorphism of F and write simply $\alpha.w \in F$, $w \in F$.

Given a $\pi_1(M)$-module F define submodules

$$F_o \subset F_1 \subset \ldots \subset F_p \subset \ldots \subset F$$

by : $w \in F_p$ if and only if $(\alpha_o - \iota)(\alpha_1 - \iota) \ldots (\alpha_p - \iota)w = 0$ for all $\alpha_o, \ldots, \alpha_p \in \pi_1(M)$. This sequence is called the upper central series for F.

The quotient of a $\pi_1(M)$ module by a submodule is a $\pi_1(M)$-module (in the obvious way). In particular, the inclusions $F_p \subset F_{p+q} \subset F$ define an isomorphism of modules

$$(16.2) \qquad F_{p+q}/F_p \xrightarrow{\cong} (F/F_p)_{q-1}.$$

Next observe that if $G \subset F$ is a submodule then

$$(16.3) \qquad G_p = G \cap F_p , \qquad p \geq 0$$

and so G/G_p is a submodule of F/F_p.

A $\pi_1(M)$-module F is called nilpotent if for each q there is an integer $N(q)$ such that

$$F^q = F^q_{N(q)} .$$

In this case we say $\pi_1(M)$ acts nilpotently in F.

A $\pi_1(M)$-module F has finite type if it has finite type as a graded vector space (each F^q is finite dimensional). It is finitely approximable if it is the union of its submodules of finite type.

16.4.- Lemma.- Suppose F is a finitely approximable $\pi_1(M)$-module with family $\{F^\gamma\}$ of submodules of finite type :

$$F = \bigcup_\gamma F^\gamma.$$

Then

i) Each F_p is finitely approximable and the submodules of finite type are exactly the F^γ_p.

ii) Each F/F_p is finitely approximable and the submodules of finite type are exactly the F^γ/F^γ_p.

iii) Each F_{p+q}/F_p is finitely approximable and the submodules of finite type are exactly the $F^\gamma_{p+q}/F^\gamma_p$.

Proof :

i) follows from (16.3) with G replaced by F^γ.

ii) follows from the observation that F^γ/F^γ_p is a submodule of F/F_p.

iii) is a special case of ii).

Q.E.D.

16.5.- Local systems of coefficients.- Let F be a local system of coefficients (ℓ.s.c.) over Sing M (cf. just after (12.26)). Each $z \in M$ is a singular o-simplex and hence determines a graded vector space F_z. Each path $\lambda : I \to M$ is a singular 1-simplex and hence determines isomorphisms

$$F_{\lambda(0)} \xleftarrow[\cong]{\partial_1} F_\lambda \xrightarrow[\cong]{\partial_o} F_{\lambda(1)}.$$

The composite isomorphism $\partial_o \circ \partial_1^{-1}$ will be denoted by

$$[\lambda] : F_{\lambda(0)} \xrightarrow{\cong} F_{\lambda(1)}$$

and called the path isomorphism determined by λ.

16.6.- *Proposition*.- The path isomorphisms have the following properties :

i) If $\lambda : I \to z$ is the constant path, then $[\lambda]$ is the identity map of F_z.

ii) $[\lambda]$ depends only on the homotopy class (rel. the endpoints) of λ.

iii) Let λ and μ be paths from y to z and from z to w, and let $\mu * \lambda$ be the composite path from y to w. Then $[\mu * \lambda] = [\mu] \circ [\lambda]$.

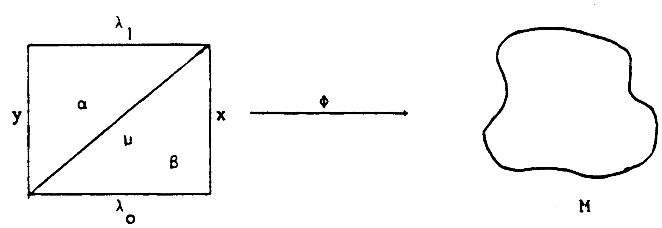
Proof :

i) Regard z as a 0-simplex ; then $\lambda = s_o z$. Hence $\partial_o = \partial_1 = s_o^{-1} : F_\lambda \xrightarrow{\cong} F_z$.

ii) Let $\Phi : I \times I \to M$ be a homotopy such that

$$\Phi(s,0) = \lambda_o(s),\ \Phi(s,1) = \lambda_1(s),\ \Phi(0,t) = y,\ \Phi(1,t) = z.$$

Triangulate $I \times I$ as shown :



and let α,β and μ be the singular simplices (of dimensions 2,2 and 1) obtained by restricting Φ.

The face maps define isomorphisms

$$F_\alpha \xrightarrow{\cong} F_{\lambda_1}, F_\mu \qquad \text{and} \quad F_\beta \xrightarrow{\cong} F_{\lambda_o}, F_\mu .$$

Combine these with i) to achieve the proof.

iii) Construct a singular 2-simplex α such that $\partial_o\alpha = \mu$, $\partial_1\alpha = \mu*\lambda$ and $\partial_2\alpha = \lambda$; then argue as in ii).

Q.E.D.

If λ is a loop based at x, then $[\lambda]$ is an automorphism of F_x. Prop. 16.6 shows that these automorphisms make F_x into a $\pi_1(M)$-module.

16.7.- Definition.- The $\pi_1(M)$-module F_x is called the $\pi_1(M)$-module associated with the ℓ.s.c. F.

Suppose $G \subset F$ is a sub ℓ.s.c. (i.e. $G_\sigma \subset F_\sigma$, $\sigma \in$ Sing M and the face and degeneracy maps of F restrict to those of G). We define the quotient ℓ.s.c., F/G by

$$(F/G)_\sigma = F_\sigma/G_\sigma \quad , \quad \sigma \in \text{Sing } M$$

with the obvious face and degeneracy operators.

In particular note that G_x is a submodule of F_x and the equality

$$(F/G)_x = F_x/G_x$$

is an equality of $\pi_1(M)$-modules.

On the other hand, let $E \subset F_x$ be any sub $\pi_1(M)$-module.

For each $\sigma \in \underline{\text{Sing}}\, M$ choose a path λ from x to $\sigma(0)$. Restriction defines an isomorphism $F_\sigma \xrightarrow{\cong} F_{\sigma(0)}$ and so λ determines a composite isomorphism

$$F_x \xrightarrow[\cong]{[\lambda]} F_{\sigma(0)} \xrightarrow[\cong]{} F_\sigma .$$

Because E is a submodule, the image of E under this isomorphism is a subspace $E_\sigma \subset F_\sigma$ which is independent of the choice of λ. Thus the E_σ define a sub ℓ.s.c. of F. In this way we obtain

16.8.- Proposition.- The correspondence $G \rightsquigarrow G_x$ defines a bijection between sub local coefficient systems of F and sub $\pi_1(M)$-modules of F_x.

16.9.- Example.- The upper central series of F_x :

$$(F_x)_o \subset \ldots \subset (F_x)_p \subset \ldots$$

determines a sequence of local systems of coefficients

$$F_o \subset \ldots \subset F_p \subset \ldots$$

such that $(F_p)_x = (F_x)_p$. It is called the upper central series for F.

Next recall that the ℓ.s.c. F determines the graded space $F(\underline{\text{Sing}}\, M)$ of global sections whose elements Φ are the families $\Phi_\sigma \in F_\sigma$ ($\sigma \in \underline{\text{Sing}}\, M$) compatible with the face and degeneracy operators. For simplicity we write

$$F(\underline{\text{Sing}}\, M) = F(M).$$

Then the correspondence $\Phi \mapsto \Phi_\sigma$ is a linear map

$$e_\sigma : F(M) \to F_\sigma \quad , \quad \sigma \in \underline{\text{Sing}}\, M.$$

If σ and ω are faces of a simplex τ then the face maps define isomorphisms $F_\tau \xrightarrow{\cong} F_\sigma, F_\omega$ and the diagram

(16.10)
$$\begin{array}{ccc} & & F_\sigma \\ & \overset{e_\sigma}{\nearrow} & \uparrow \cong \\ F(M) & \xrightarrow{e_\tau} & F_\tau \\ & \underset{e_\omega}{\searrow} & \downarrow \cong \\ & & F_\omega \end{array}$$

commutes.

Since M is path connected each σ is a face of a simplex τ which contains x as a vertex. Setting $\omega = x$ in (16.10) we see that $e_\sigma \Phi = 0$ if and only if $e_x \Phi = 0$. But in this case $e_\sigma \Phi = 0$ for all σ and so $\Phi = 0$. Hence each e_σ is injective.

Next consider (16.10) when τ is a loop based at x and $\sigma = \omega = x$. It shows that $[\tau] \circ e_x = e_x$ and hence

$$\operatorname{Im} e_x \subset (F_x)_o .$$

On the other hand, if $w \in (F_x)_o$ an argument similar to the proof of prop. 16.8 extends w to an element of $F(M)$. Thus

(16.11) $$e_x : F(M) \xrightarrow{\cong} (F_x)_o .$$

Since $(F_x)_o = (F_o)_x$ a final application of (16.10) shows that

(16.12) $$e_\sigma : F(M) \xrightarrow{\cong} (F_o)_\sigma , \quad \sigma \in \underline{\text{Sing}}\, M.$$

The isomorphisms (16.12) exhibit F_o as a constant system of coefficients.

16.13.- Singular cohomology.- Let F be a local system of coefficients over $\underline{\text{Sing}}\, M$ (with zero differentials !). In 14.1. we defined the bigraded differential space $(C(\underline{\text{Sing}}\, M;F),\delta)$ which we shall denote simply by $(C(M;F),\delta)$. It is called the space of singular cochains on M with coefficients in F.

Because δ is homogeneous of bidegree $(1,0)$ the cohomology space $H(C(M;F),\delta)$ is also bigraded : it is written simply

$$H(M;F) = \sum_{p,q\geq 0} H^p(M;F^q)$$

and called the singular cohomology of M with coefficients in F.

On the other hand if E is a graded vector space we have the bigraded differential space $(C(\underline{\text{Sing}}\, M;E),\delta)$ of singular cochains on M with values in E, which we write $(C(M;E),\delta)$. Its cohomology,

$$H(M;E) = \sum_{p,q\geq 0} H^p(M;E^q),$$

is the ordinary singular cohomology of M with values in E. When $E = k$ we write simply $H(M) = H(M;k)$ - cf. example 14.12. - and call this c.g.a. the singular cohomology algebra of M.

Suppose F is, as above, an ℓ.s.c. Then we have the differential spaces $C(M;F(M))$, $C(M;(F_x)_o)$ and $C(M;F_o)$ obtained by taking $E = F(M),(F_x)_o$ and replacing F by the ℓ.s.c. F_o. If $f \in C^p(M;F^q(M))$ define

$$\bar{e}_x f \in C^p(M;(F^q_x)_o) \quad\text{and}\quad \bar{e}f \in C^p(M;F_o)$$

by

$$(\bar{e}_x f)(\sigma) = e_x(f(\sigma)) = f(\sigma)_x \quad\text{and}\quad (\bar{e}f)(\sigma) = e_\sigma(f(\sigma)) = f(\sigma)_\sigma,\ \sigma \in \underline{\text{Sing}}_p(M).$$

In this way, we obtain isomorphisms of differential spaces

$$(16.14)\qquad C(M;F_o) \underset{\cong}{\xleftarrow{\ \bar{e}\ }} C(M;F(M)) \underset{\cong}{\xrightarrow{\ \bar{e}_x\ }} C(M;(F_x)_o)\ ,$$

as follows from (16.10), (16.11) and (16.12). These isomorphisms induce isomorphisms

$$(16.15) \qquad H(M;F_o) \xleftarrow[\cong]{\bar{e}^*} H(M;F(M)) \xrightarrow[\cong]{\bar{e}_x^*} H(M;(F_x)_o).$$

Finally, recall that an element $f \in C^p(M;F^q)$ associates with each p-simplex σ an element $f(\sigma) \in F^q_\sigma$. Since δ is homogeneous of bidegree (1,0) we have

$$H^o(M;F^q) = C^o(M;F^q) \cap \ker \delta.$$

In particular the linear map $C^o(M;F^q) \to F^q_x$ given by $f \to f(x)$ restricts to a linear map $H^o(M;F^q) \to F^q_x$.

We show that this map is in fact an isomorphism

$$(16.16) \qquad H^o(M;F) \xrightarrow{\cong} (F_x)_o.$$

Indeed, suppose $f \in H^o(M;F^q)$. Let λ be a path from x to y, with face maps $\partial_o : F_\lambda \xrightarrow{\cong} F_y$, $\partial_1 : F_\lambda \xrightarrow{\cong} F_x$. Then

$$0 = (\delta f)(\lambda) = \partial_o^{-1} f(y) - \partial_1^{-1} f(x)$$

and hence

$$f(y) = \partial_o \partial_1^{-1} f(x) = [\lambda] f(x), \qquad y \in M.$$

This shows that $f(x) = 0$ if and only if $f = 0$, and that $f(x) \in (F_x)_o$. Finally, if $v \in (F^q_x)_o$ extend v to an element $\Phi \in F^q(M)$ and observe that $y \to \Phi_y$ is a cocycle in $C^o(M;F^q)$ which restricts to v. Thus (16.16) is established.

16.17.- Direct limits.- Let G be an ℓ.s.c. over Sing M (with zero differentials). Denote by $G^\gamma_x \subset G_x$ the family of sub $\pi_1(M)$-modules of finite type ; each G^γ_x entends to a unique sub ℓ.s.c. $G^\gamma \subset G$.

The modules G_x^γ are directed by inclusion, and if $G_x^\gamma \subset G_x^\lambda$ then $G^\gamma \subset G^\lambda$, so we have an inclusion $C(M;G^\gamma) \subset C(M;G^\lambda)$. Denote the direct limit by $(\varinjlim C(M;G^\gamma),\delta)$. The inclusions $G^\gamma \subset G$ define inclusions $i_\gamma : C(M;G^\gamma) \to C(M;G)$ and so we obtain an inclusion

$$i : (\varinjlim C(M;G^\gamma),\delta) \longrightarrow (C(M;G),\delta)$$

of bigraded differential spaces.

16.18.- Proposition.- Suppose that

i) G_x is a nilpotent $\pi_1(M)$-module.

ii) G_x is finitely approximable : $G_x = \bigcup_\gamma G_x^\gamma$.

iii) Either G_x or $H(M)$ has finite type.

Then i^* is an isomorphism.

16.19.- Lemma.- The proposition is correct when $G_x = (G_x)_o$.

Proof : When $G_x = (G_x)_o$ then the same is true for each G_x^γ. In this case we use the isomorphisms (16.14) to identify i with the inclusion

$$\varinjlim C(M;G_x^\gamma) \longrightarrow C(M;G_x)$$

Now because G_x^γ has finite type there is a canonical isomorphism $C(M) \otimes G_x^\gamma \xrightarrow{\cong} C(M;G_x^\gamma)$. Extend to direct limits and use hypothesis ii) to write

$$\varinjlim C(M;G_x^\gamma) = C(M) \otimes G_x .$$

We are thus reduced to proving that the inclusion $C(M) \otimes G_x \to C(M;G_x)$ induces a cohomology isomorphism. This is a direct consequence of hypothesis iii).

Q.E.D.

16.20.- *Proof of the proposition.*- Consider the short exact sequences of ℓ.s.c.'s

$$0 \to G_p \to G_{p+1} \to G_{p+1}/G_p \to 0$$

Because C is an extendable local system, tensoring with an ℓ.s.c. yields an extendable local system. Hence prop. 12.42 applies to yield the short exact sequences

$$0 \to C(M;G_p) \to C(M;G_{p+1}) \to C(M;G_{p+1}/G_p) \to 0.$$

We make the same construction for each G^γ and pass to direct limits to obtain the commutative row exact diagram of differential spaces

$$\begin{array}{ccccccccc} 0 & \to & \varinjlim_\gamma C(M;G_p^\gamma) & \longrightarrow & \varinjlim_\gamma C(M;G_{p+1}^\gamma) & \longrightarrow & \varinjlim_\gamma C(M;G_{p+1}^\gamma/G_p^\gamma) & \to & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \to & C(M;G_p) & \longrightarrow & C(M;G_{p+1}) & \longrightarrow & C(M;G_{p+1}/G_p) & \to & 0 . \end{array} \tag{16.21}$$

According to lemma 16.4 the $\pi_1(M)$-modules $(G_p)_x$ and $(G_{p+1}/G_p)_x$ are finitely approximable, and their submodules of finite type are exactly the $(G_p^\gamma)_x$ and the $(G_{p+1}^\gamma/G_p^\gamma)_x$. Moreover (16.2) (with $q = 1$) shows that

$$(G_{p+1}/G_p)_x = [(G_{p+1}/G_p)_x]_o .$$

Thus lemma 16.19 implies that the vertical arrow on the right of (16.21) is always a cohomology isomorphism, and the one on the left is when $p = 0$. It follows by induction that for each $p \geq 0$

$$i^* : H(\varinjlim C(M;G_p^\gamma)) \xrightarrow{\cong} H(M;G_p).$$

The proposition follows now from the nilpotence of G_x and the equations

$$H(M;G) = \sum_q H(M;G^q)$$

Q.E.D.

16.22.- *Corollary*.- Under the hypotheses of prop. 16.18 the linear maps i_γ^* define an isomorphism

$$\varinjlim i_\gamma^* : \varinjlim H(M;G^\gamma) \xrightarrow{\cong} H(M;G).$$

16.23.- *The main theorem*.- Suppose now that G and F are local systems of coefficients (with zero differentials) over $\underline{\text{Sing}}$ M, where M is, as always our fixed path connected topological space with base point x. Assume given linear maps

$$\varphi_\sigma : G_\sigma \to F_\sigma , \quad \sigma \in \underline{\text{Sing}}\ M,$$

homogeneous of degree zero and compatible with the face and degeneracy operators.

Define a linear map of bigraded differential spaces,

$$\varphi : (C(M;G),\delta) \to (C(M;F),\delta)$$

by $(\varphi f)(\sigma) = \varphi_\sigma(f(\sigma))$. Denote by $\varphi^* : H(M;G) \to H(M;F)$ the induced map of bigraded spaces.

As above let $G^\gamma \subset G$ be the sub local systems of coefficients such that G_x^γ has finite type. Write

$$\psi_\gamma : \varphi \circ i_\gamma : C(M;G^\gamma) \to C(M;F)$$

and

$$\psi = \varphi \circ i : \varinjlim C(M;G^\gamma) \to C(M;F) ;$$

then $\psi = \varinjlim \psi_\gamma$ and $\psi^* = \varinjlim \psi_\gamma^*$.

16.24.- Theorem.- With the notation and hypotheses above, assume further that

i) F_x and each G_x^γ are nilpotent $\pi_1(M)$-modules.

ii) G_x is finitely approximable : $G_x = \bigcup\limits_\gamma G_x^\gamma$.

iii) Either F_x or $H(M)$ has finite type.

iv) $\psi^* : \varinjlim H^0(M;G^\gamma) \to H^0(M;F)$ is an isomorphism and $\psi^* : \varinjlim H^1(M;G^\gamma) \to H^1(M;F)$ is injective.

Then φ^*, ψ^* and each φ_σ are isomorphisms. In particular, G_x is nilpotent.

Proof : It is enough to prove the theorem separately for each F^q and G^q and so we lose no generality in assuming $F = F^q$, $G = G^q$. We assume this henceforth. In particular, because F is nilpotent,

$$F = F_n \quad , \quad \text{some } n.$$

We show next that φ_x restricts to isomorphisms

$$(\varphi_x)_p : (G_x)_p \xrightarrow{\cong} (F_x)_p \quad , \quad p \geq 0.$$

In fact we obtain from (16.16) the commutative diagram

$$\begin{array}{ccc}
\varinjlim H^0(M;G^\gamma) & \xrightarrow{\cong} & \varinjlim (G_x^\gamma)_0 = (G_x)_0 \\
\psi^* \downarrow & & \downarrow \swarrow (\varphi_x)_0 \\
H^0(M;F) & \xrightarrow[\cong]{} & (F_x)_0
\end{array}$$

Thus hypothesis iv) implies that $(\varphi_x)_0$ is an isomorphism.

Suppose we have proved that $(\varphi_x)_p$ is an isomorphism. It follows at once (because M is path connected) that φ_σ restricts to isomorphisms

$(\psi_p)_\sigma : (G_p)_\sigma \xrightarrow{\cong} (F_p)_\sigma$ and so ψ restricts to an isomorphism $\psi_p : C(M;G_p) \xrightarrow{\cong} C(M;F_p)$.

By hypothesis ii) and lemma 16.4 $(G_p)_x$ is finitely approximable with submodules of finite type exactly the $(G_x^\gamma)_p = (G_p^\gamma)_x$. Because $(\psi_x)_p$ is an isomorphism either $(G_x)_p$ has finite type or $H(M)$ has finite type. Since $(G_x)_p$ is (trivially) nilpotent we may apply prop. 16.18 and conclude that

$$H(\varinjlim_\gamma C(M;G_p^\gamma)) \xrightarrow{\cong} H(M;G_p).$$

Because ψ_p was an isomorphism this implies that the restriction

$$\psi_p : \varinjlim_\gamma C(M;G_p^\gamma) \longrightarrow C(M;F_p)$$

also induces a cohomology isomorphism, ψ_p^*.

Next, exactly as in the proof of prop. 16.18, we have the commutative row exact diagram of differential spaces

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \varinjlim C(M;G_p^\gamma) & \longrightarrow & \varinjlim C(M;G^\gamma) & \longrightarrow & \lim C(M;G^\gamma/G_p^\gamma) & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle \psi_p} & & \downarrow{\scriptstyle \psi} & & \downarrow{\scriptstyle \bar\psi_p} & & \\
0 & \longrightarrow & C(M;F_p) & \longrightarrow & C(M;F) & \longrightarrow & C(M;F/F_p) & \longrightarrow & 0
\end{array}$$

where $\bar\psi_p$ is defined in the obvious way.

Since ψ_p^* is an isomorphism in all degrees we can pass to cohomology and use hypothesis iv) to conclude that

$$\bar\psi_p^* : \varinjlim H^0(M;G^\gamma/G_p^\gamma) \xrightarrow{\cong} H^0(M;F/F_p).$$

As above use the isomorphisms (16.16) to identify this with an isomorphism

$$\varinjlim\, \left[G_x^\gamma/(G_x^\gamma)_p\right]_o \xrightarrow{\cong} \left[F_x/(F_x)_p\right]_o$$

induced by ψ_x.

On the other hand by lemma 16.4 $G_x^\gamma/(G_x^\gamma)_p$ is a submodule of $G_x/(G_x)_p$, and the left hand side is exactly $\left[G_x/(G_x)_p\right]_o$. By (16.2) we may identify this isomorphism with an isomorphism

$$(G_x)_{p+1}/(G_x)_p \xrightarrow{\cong} (F_x)_{p+1}/(F_x)_p$$

(induced by ψ_x). Since $(\psi_x)_p$ is already shown to be an isomorphism it follows that so is $(\psi_x)_{p+1}$.

We have now shown that each $(\psi_x)_p$ is an isomorphism. Since $F_x = (F_x)_n$ we have $(F_x)_n = (F_x)_{n+1}$ and so $(G_x)_n = (G_x)_{n+1}$. This implies that $(G_x^\gamma)_n = (G_x^\gamma)_{n+1}$. Since G_x^γ is assumed to be nilpotent we conclude that $G_x^\gamma = (G_x^\gamma)_n$ for all γ and so

$$G_x = \bigcup_\gamma (G_x^\gamma)_n = (G_x)_n .$$

It follows that ψ_x and each ψ_σ (and hence ψ and ψ^*) are isomorphisms. Moreover G_x is nilpotent and so i^* and ψ^* are also isomorphisms (prop. 16.18).

Q.E.D.

Chapter 17

A Converse to theorem 12.27

17.1.- Introduction. Fix a path connected topological space M with base point x. For any local system, F, over Sing M denote the space of global sections, F(Sing M) simply by F(M). If G is a second local system over Sing M with values in the same category C then a morphism

$$\psi_* : G \longrightarrow F$$

is a family of morphisms $\psi_\sigma : G_\sigma \to F_\sigma$, compatible with the face and degeneracy operators. It determines a morphism

$$\psi : G(M) \to F(M)$$

via $(\psi\Phi)_\sigma = \psi_\sigma \Phi_\sigma$.

Next, recall (chap. 13) the local system A whose global sections are the c.g.d.a A(M) of differential forms on M. Tensor ψ_* with the identity to obtain a morphism of local systems :

$$\bar{\psi}_* = \iota \otimes \psi_* : A \otimes G \to A \otimes F.$$

The induced morphism,

$$\bar{\psi} : A(M ; G) \to A(M ; F),$$

is homogeneous of bidegree zero.

Suppose now that G and F are extendable local systems of differential coefficients. Then local systems of coefficients H(G) and H(F) over Sing M are given by

$$H(G)_\sigma = H(G_\sigma) \qquad \text{and} \qquad H(F)_\sigma = H(F_\sigma).$$

The morphisms ψ_σ define morphisms $\psi_\sigma^* : H(G_\sigma) \to H(F_\sigma)$, which it is convenient to denote by $H(\psi_\sigma)$. Then the $H(\psi_\sigma)$ define a morphism

$$H(\psi_*) : H(G) \to H(F)$$

of local systems. Proceeding as above obtain a morphism of local systems :

$$\overline{H(\psi_*)} = \iota \otimes H(\psi_*) : A \otimes H(G) \to A \otimes H(F)$$

and hence a linear map of bigraded differential spaces :

$$\overline{H(\psi)} : A(M\ ;\ H(G)) \to A(M\ ;\ H(F)).$$

On the other hand, as in 16.23 (replace ψ_σ by $H(\psi_\sigma)$) the morphisms $H(\psi_\sigma)$ define a linear map of bigraded differential spaces,

$$H(\psi) : C(M\ ;\ H(G)) \to C(M\ ;\ H(F)).$$

It is trivial to cherk that the diagram

$$\begin{array}{ccc} A(M\ ;\ H(G)) & \xrightarrow{\overline{H(\psi)}} & A(M\ ;\ H(F)) \\ \int \downarrow & & \downarrow \int \\ C(M\ ;\ H(G)) & \xrightarrow[H(\psi)]{} & C(M\ ;\ H(F)) \end{array}$$

commutes.

Next, filter $A(M\ ;\ G)$ and $A(M\ ;\ F)$ by the subspaces

$${}_G I^p = \sum_{j \geq p} A^j(M\ ;\ G) \quad \text{and} \quad {}_F I^p = \sum_{j \geq p} A^j(M\ ;\ F).$$

This defines spectral sequences

$${}_G E_i^{p,q} \Rightarrow H(A(M\ ;\ G)) \quad \text{and} \quad {}_F E_i^{p,q} \Rightarrow H(A(M\ ;\ F)),$$

as described in 12.43 and the end of chap. 13. Because $\bar{\psi}$ is homogeneous of bidegree zero it defines a homomorphism of spectral sequences

$$\bar{\psi}_i^{p,q} : {}_G E_i^{p,q} \to {}_F E_i^{p,q} \ , \qquad i \geq 0.$$

Since $\bar{\psi}$ is homogeneous of bidegree zero we can identify $\bar{\psi}_o = \bar{\psi}$; ${}_G E_o = A(M\ ;\ G)$; ${}_F E_o = A(M\ ;\ F)$. Thus theorem 12.47 yields the commutative diagram

$$(17.2) \qquad \begin{array}{ccccc} {}_G E_1 & \xrightarrow{\cong} & A(M\ ;\ H(G)) & \xrightarrow{\int} & C(M\ ;\ H(G)) \\ \bar{\psi}_1 \downarrow & & \downarrow \overline{H(\psi)} & & \downarrow H(\psi) \\ {}_F E_1 & \xrightarrow[\cong]{} & A(M\ ;\ H(F)) & \xrightarrow[\int]{} & C(M\ ;\ H(F)) \end{array} .$$

Finally, theorem 14.18 shows that $\int^*$ is an isomorphism. Thus (17.2) yields the commutative diagram

$$(17.3) \qquad \begin{array}{ccc} {}_G E_2^{p,q} & \xrightarrow{\cong} & H^p(M\ ;\ H^q(G)) \\ \bar{\psi}_2^{p,q} \downarrow & & \downarrow (H(\psi)^*)^{p,q} \\ {}_F E_2^{p,q} & \xrightarrow[\cong]{} & H^p(M\ ;\ H^q(F)). \end{array}$$

17.4.- *Direct limits*. Let Γ be a partially ordered set such that for any $\gamma_1, \gamma_2 \in \Gamma$ there is some $\gamma \in \Gamma$ with $\gamma \geq \gamma_1$ and $\gamma \geq \gamma_2$. Assume given :

i) A family of local systems $\{G^\gamma\}_{\gamma \in \Gamma}$ over <u>Sing</u> M with values in the same category C.

ii) A family of morphisms

$$\xi_*^{\gamma_2\gamma_1} : G^{\gamma_1} \longrightarrow G^{\gamma_2} \qquad , \ \gamma_1 \leq \gamma_2 ,$$

such that $\xi_*^{\gamma\gamma} = \iota$ and $\xi_*^{\gamma_3\gamma_2} \circ \xi_*^{\gamma_2\gamma_1} = \xi_*^{\gamma_3\gamma_1}$.

Then $\{G^\gamma ; \xi^{\gamma_2\gamma_1}\}$ is called a <u>directed family</u> of local systems.

Because all our categories C have direct limits we can form the <u>direct limit</u> of such a directed family ; it is the local system $G = \varinjlim_\Gamma G^\gamma$ given by

$$G_\sigma = \varinjlim_\Gamma (G^\gamma)_\sigma \quad ; \quad \partial_i = \varinjlim_\Gamma \partial_i \quad ; \quad s_j = \varinjlim_\Gamma s_j \ .$$

The morphisms $G_\sigma^\gamma \to G_\sigma$ define morphisms

$$\xi_*^\gamma : G^\gamma \to G.$$

The morphisms $\xi^{\gamma_2\gamma_1} : G^{\gamma_1}(M) \to G^{\gamma_2}(M)$ define a directed system of objects in C and the morphisms $\xi^\gamma : G^\gamma(M) \to G(M)$ define a morphism

$$\xi : \varinjlim_\Gamma G^\gamma(M) \to G(M).$$

However, ξ need not be an isomorphism.

Next, assume G and each G^γ are extendable local systems of differential coefficients. As in 17.1 the morphisms $\xi_*^{\gamma_2\gamma_1}$ and ξ_*^γ determine morphisms

$$\bar{\xi}^{\gamma_2\gamma_1} : A(M ; G^{\gamma_1}) \to A(M ; G^{\gamma_2})$$

and

$$\bar{\xi}^\gamma : A(M ; G^\gamma) \to A(M ; G).$$

The first collection of morphisms makes $\{A(M ; G^\gamma)\}$ into a directed family, and the second collection determines a morphism

$$\bar{\xi} : \varinjlim_{\Gamma} A(M\ ;\ G^{\gamma}) \to A(M\ ;\ G).$$

Since the $\bar{\xi}^{\gamma_2\gamma_1}$ are homogeneous of bidegree zero we can bigrade the limit by

$$[\varinjlim_{\Gamma} A(M\ ;\ G^{\gamma})]^{p,q} = \varinjlim_{\Gamma} A^{p}(M\ ;\ (G^{\gamma})^{q}).$$

With this bigradation $\bar{\xi}$ is homogeneous of bidegree zero.

Moreover, if we set $\check{I}^{p} = \sum_{j\geq p}[\varinjlim_{\Gamma} A(M\ ;\ G^{\gamma})]^{j,*}$, we define a filtration which determines a spectral sequence

$$\check{E}_i^{p,q} \Rightarrow H(\varinjlim_{\Gamma} A(M\ ;\ G^{\gamma})).$$

Since $\bar{\xi}$ is homogeneous of bidegree zero it preserves filtrations and induces a map

$$\bar{\xi}_i^{p,q} : \check{E}_i^{p,q} \longrightarrow {}_{G}E_i^{p,q}$$

of spectral sequences.

As in 17.1 the morphisms $\xi_*^{\gamma_2\gamma_1}$ determine morphisms $\overline{H(\xi^{\gamma_2\gamma_1})}$ and $H(\xi^{\gamma_2\gamma_1})$ which make $\{A(M\ ;\ H(G^{\gamma}))\}$ and $\{C(M\ ;\ H(G^{\gamma}))\}$ into directed families. Moreover theorem 12.47 yields the commutative diagram

$$(17.5)\qquad \begin{array}{ccccc} \check{E}_1 & \xrightarrow{\ \cong\ } & \varinjlim_{\Gamma} A(M\ ;\ H(G^{\gamma})) & \xrightarrow{\ \int\ } & \varinjlim_{\Gamma} C(M\ ;\ H(G^{\gamma})) \\ \Big\downarrow{\scriptstyle \bar{\xi}_1} & & \Big\downarrow{\scriptstyle \varinjlim_{\Gamma} \overline{H(\xi^{\gamma})}} & & \Big\downarrow{\scriptstyle \varinjlim_{\Gamma} H(\xi^{\gamma})} \\ {}_{G}E_1 & \xrightarrow[\ \cong\ ]{} & A(M\ ;\ H(G)) & \xrightarrow[\ \int\ ]{} & (C(M\ ;\ H(G)) \end{array}$$

(in which $\overline{H(\xi^\gamma)}$ and $H(\xi^\gamma)$ are the morphisms induced from ξ^γ_*, again as described in 17.1).

Passing to cohomology and using the fact that $\int^*$ is an isomorphism we obtain the commutative diagram

$$(17.6)\qquad \begin{array}{ccc} \check{E}_2^{p,q} & \xrightarrow{\cong} & \varinjlim_{\Gamma} H^p(M\,;\,H^q(G^\gamma)) \\ {\scriptstyle \bar{\xi}_2^{p,q}}\downarrow & & \downarrow {\scriptstyle \varinjlim_{\Gamma}[H(\xi^\gamma)^*]^{p,q}} \\ {}_GE_2^{p,q} & \xrightarrow[\cong]{} & H^p(M\,;\,H^q(G)). \end{array}$$

Finally recall that $G_\sigma = \varinjlim_{\Gamma} G^\gamma_\sigma$. It follows that for $\sigma \in \underline{\text{Sing}}\, M$ the $H(\xi^\gamma_\sigma)$ define an isomorphism

$$(17.7)\qquad \varinjlim_{\Gamma} H(\xi^\gamma_\sigma) : \varinjlim_{\Gamma} H(G^\gamma_\sigma) \xrightarrow{\cong} H(G_\sigma).$$

17.8.- Finite type. Suppose as in 17.4 that $G = \varinjlim G^\gamma$, where G and each G^γ $(\gamma \in \Gamma)$ is an extendable local system of differential coefficients over $\underline{\text{Sing}}\, M$. Then $H(G)$ and $H(G^\gamma)$ are ℓ.s.c.'s.

As in 16.17 let $\{H^\tau \mid \tau \in T\}$ be the family of sub ℓ.s.c.'s of $H(G)$ such that H^τ_* has finite type. Then we have the inclusions $i_\tau : C(M\,;\,H^\tau) \to C(M\,;\,H(G))$ which yield the inclusion

$$i : \varinjlim_{T} C(M\,;\,H^\tau) \to C(M\,;\,H(G)).$$

Now assume that for each $\gamma \in \Gamma$, $H(G^\gamma_*)$ has finite type. Since $H(\xi^\gamma) : H(G^\gamma) \to H(G)$ is a morphism of ℓ.s.c' s, a sub ℓ.s.c. $H^\gamma \subset H(G)$ is defined by

$$(H^\gamma)_\sigma = \operatorname{Im} H(\xi^\gamma_\sigma) \qquad , \qquad \sigma \in \underline{\text{Sing}}\, M.$$

Moreover H_x^γ has finite type ; hence H^γ is one of the H^τ 's considered above.

If $\gamma_1 \le \gamma_2$ then $H(\xi_*^{\gamma_2}) \circ H(\xi_*^{\gamma_2\gamma_1}) = H(\xi_*^{\gamma_1})$ and so $H^{\gamma_1} \subset H^{\gamma_2}$. Moreover (17.7) shows that $H(G_\sigma) = \bigcup_{\gamma\in\Gamma} H_\sigma^\gamma$.

Thus we have the commutative diagram

(17.9)
$$\begin{array}{ccc} \varinjlim_{\Gamma} C(M\,;\,H^\gamma) & \xrightarrow{\ \cong\ } & \varinjlim_{T} C(M\,;\,H^\tau) \\ & {\scriptstyle i}\searrow \quad \swarrow{\scriptstyle i} & \\ & C(M\,;\,H(G)) & \end{array}$$

On the other hand, the morphisms $H(\xi_*^\gamma)$ determine morphisms $\ell_*^\gamma : H(G^\gamma) \to H^\gamma$ by the requirement that the composite

$$H(G^\gamma) \to H^\gamma \to H(G)$$

be $H(\xi_*^\gamma)$. The morphism ℓ_*^γ defines a linear map

$$\ell^\gamma : C(M\,;\,H(G^\gamma)) \to C(M\,;\,H^\gamma)$$

and, passing to the limit we obtain the linear map (of differential spaces)

$$\ell = \varinjlim_{\Gamma} \ell^\gamma : \varinjlim_{\Gamma} C(M\,;\,H(G^\gamma)) \to \varinjlim_{\Gamma} C(M\,;\,H^\gamma).$$

17.10.- Lemma. The linear map ℓ is an isomorphism and makes the diagram

$$\begin{array}{ccc} \varinjlim_{\Gamma} C(M\,;\,H(G^\gamma)) & \xrightarrow[\cong]{\ \ell\ } & \varinjlim_{\Gamma} C(M\,;\,H^\gamma) \\ & {\scriptstyle \varinjlim_{\Gamma} H(\xi^\gamma)}\searrow \quad \swarrow{\scriptstyle i} & \\ & C(M\,;\,H(G)) & \end{array}$$

commute.

Proof. It is obvious that the diagram commutes. By definition

$$\ell_\sigma^\gamma : H(G_\sigma^\gamma) \to H_\sigma^\gamma \qquad , \quad \gamma \in \Gamma \ , \quad \sigma \in \underline{\text{Sing}}\ M,$$

is surjective ; hence so is ℓ^γ and it follows that so is ℓ.

On the other hand, suppose $\ell\Phi = 0$. Then Φ comes from some $\Phi^\gamma \in C(M ; H(G^\gamma))$, and by choosing γ large enough we can arrange that $\ell^\gamma\Phi^\gamma = 0$. We lose no generality in assuming $\Phi^\gamma \in C(M ; H^q(G^\gamma))$.

Now since $H(\xi_*^\gamma) : H(G^\gamma) \to H(G)$ is a morphism of ℓ.s.c.'s an ℓ.s.c. $N^\gamma \subset H(G^\gamma)$ is defined by

$$(N^\gamma)_\sigma = \ker H(\xi_\sigma^\gamma) \qquad , \qquad \sigma \in \underline{\text{Sing}}\ M.$$

Since $\ell^\gamma\Phi^\gamma = 0$ we have $\Phi^\gamma \in C(M ; (N^\gamma)^q)$.

On the other hand, recall from (17.7) that

$$H(G_x) = \varinjlim_\Gamma H(G_x^\gamma).$$

It follows that if $z \in \ker H(\xi_x^\gamma)$ then $H(\xi_x^{\gamma'\gamma})z = 0$ for some $\gamma' \geq \gamma$. Since $(N_x^\gamma)^q$ has finite dimension (because $H^q(G_x^\gamma)$ does) there is some $\gamma_q \geq \gamma$ such that

$$H(\xi_x^{\gamma_q\gamma})(N_x^\gamma)^q = 0.$$

This yields $H(\xi_\sigma^{\gamma_q\gamma})(N_\sigma^\gamma)^q = 0$, $\sigma \in \underline{\text{Sing}}\ M$, whence

$$H(\xi^{\gamma_q\gamma})\Phi^\gamma = 0.$$

It follows that $\Phi = 0$ and ℓ is injective.

Q.E.D.

If we combine (17.6), lemma 17.10, and (17.9), we arrive at the commutative diagram

$$(17.11)\qquad \begin{array}{ccc} \check{E}_2^{p,q} & \xrightarrow{\cong} & \varinjlim H^p(M\ ;\ (H^\tau)^q) \\ \bar{\xi}_2^{p,q}\downarrow & & \downarrow (i^*)^{p,q} \\ {}_G E_2^{p,q} & \xrightarrow[\cong]{} & H^p(M\ ;\ H^q(G)) \end{array}$$

in which the direct limit is over all sub ℓ.s.c.'s $H^\tau \subset H(G)$ such that H_x^τ has finite type, and i^* is the linear map defined in 16.17.

17.12.- *The main theorem*. Recall that M is a path connected topological space with base point x. Assume that

$$\varphi_* : G \to F$$

is a morphism of local systems over Sing M. Assume further that $\{G^\gamma\ ;\ \xi_*^{\gamma_2\gamma_1}\}_{\gamma\in\Gamma}$ is a directed family of local systems over Sing M and that

$$\xi_*^\gamma : G^\gamma \to G$$

are morphisms such that $\xi_*^{\gamma_2} \circ \xi_*^{\gamma_2\gamma_1} = \xi_*^{\gamma_1}$, and

$$\varinjlim_\Gamma \xi_*^\gamma : \varinjlim_\Gamma G^\gamma \xrightarrow{\cong} G.$$

Recall from 17.1 and 17.4 that φ_* and the ξ_*^γ determine morphisms

$$\varphi : G(M) \to F(M) \qquad \text{and} \qquad \xi : \varinjlim_\Gamma G^\gamma(M) \to G(M).$$

The main theorem of this chapter reads.

17.13.- *Theorem*. With the hypotheses and notation above, suppose that

i) F,G and each G^γ are extendable local systems of differential coefficients.

ii) $H(F_x)$ is a nilpotent $\pi_1(M)$-module.

iii) Each $H(G_x^\gamma)$ is a nilpotent $\pi_1(M)$-module of finite type.

iv) Either $H(M)$ or $H(F_x)$ has finite type.

v) There are isomorphisms

$$H^o(F_\sigma) \cong H^o(G_\sigma) \cong H^o(G_\sigma^\gamma) \cong k \qquad \gamma \in \Gamma \quad , \sigma \in \underline{\text{Sing}}\, M,$$

compatible with the face and degeneracy operators.

vi) The composite $\varphi^* \circ \xi^* : \varinjlim_\Gamma H(G^\gamma(M)) \to H(F(M))$ is an isomorphism.

Then ξ^*, φ^* and each $\varphi_\sigma^* : H(G_\sigma) \to H(F_\sigma)$ are isomorphisms.

Proof.- We adopt all the notation defined earlier in this chapter. In particular we have morphisms

$$\bar\xi : \varinjlim_\Gamma A(M\,;G^\gamma) \to A(M\,;G) \quad \text{and} \quad \bar\varphi : A(M\,;G) \to A(M\,;F).$$

Moreover (cf. 13.11 and theorem 13.12) we have a commutative diagram

$$\begin{array}{ccccc} \varinjlim_\Gamma G^\gamma(M) & \xrightarrow{\ \xi\ } & G(M) & \xrightarrow{\ \varphi\ } & F(M) \\ \downarrow & & \downarrow & & \downarrow \\ \varinjlim_\Gamma A(M\,;G^\gamma) & \xrightarrow[\ \bar\xi\ ]{} & A(M\,;G) & \xrightarrow[\ \bar\varphi\ ]{} & A(M\,;F) \end{array}$$

in which the verticalarrows are cohomology isomorphisms. It follows that

$$\bar\varphi^* \circ \bar\xi^* : \varinjlim_\Gamma H(A(M\,;G^\gamma)) \xrightarrow{\ \cong\ } H(A(M\,;F))$$

On the other hand $\bar\varphi$ and $\bar\xi$ define morphisms of convergent spectral sequences. Combining (17.11) and (17.3) we have the commutative diagram

$$(17.14)\qquad \begin{array}{ccccc} \check{E}_2^{p,q} & \xrightarrow{\bar{\xi}_2^{p,q}} & {}_G E_2^{p,q} & \xrightarrow{\bar{\psi}_2^{p,q}} & {}_F E_2^{p,q} \\ \cong \downarrow & & \downarrow \cong & & \downarrow \cong \\ \varinjlim H^p(M\,;\,(H^\tau)^q) & \xrightarrow[(i^*)^{p,q}]{} & H^p(M\,;\,H^q(G)) & \xrightarrow[(H(\psi)^*)^{p,q}]{} & H^p(M\,;\,H^q(F)). \end{array}$$

17.15.- Lemma. Assume that for some q, $\bar{\psi}_2^{o,q} \circ \bar{\xi}_2^{o,q}$ is an isomorphism and $\bar{\psi}_2^{1,q} \circ \bar{\xi}_2^{1,q}$ is injective. Then

i) $\bar{\psi}_2^{p,q}$ and $\bar{\xi}_2^{p,q}$ are isomorphisms for all p.

ii) $\psi_\sigma^* : H^q(G_\sigma) \to H^q(F_\sigma)$ is an isomorphism, $\sigma \in \underline{\text{Sing}}\, M$.

Proof.- First consider the $\pi_1(M)$-module $H^q(G_x)$. Formula (17.7) shows that

$$H^q(G_x) = \varinjlim_{\Gamma} H^q(G_x^\gamma).$$

Since each $H^q(G_x^\gamma)$ is a nilpotent $\pi_1(M)$-module of finite dimension it follows that $H^q(G_x)$ is finitely approximable, and each finite dimensional submodule is nilpotent.

Moreover, by hypothesis $H^q(F_x)$ is a nilpotent $\pi_1(M)$-module, and either $H(M)$ has finite type or $H^q(F_x)$ has finite dimension.

Finally, (17.14) shows that

$$H(\psi)^* \circ i^* : \varinjlim H^o(M\,;\,(H^\tau)^q) \to H^o(M\,;\,H^q(F))$$

is an isomorphism, and

$$H(\psi)^* \circ i^* : \varinjlim H^1(M\,;\,(H^\tau)^q) \to H^1(M\,;\,H^q(F))$$

is injective.

We may thus apply theorem 16.24 to the morphism

$$H(\psi_*) : H^q(G) \to H^q(F)$$

and conclude that $H(\psi)_\sigma$ is an isomorphism (in degree q) for all σ and that

$$H(\psi)^* : H(M ; H^q(G)) \to H(M ; H^q(F))$$

and

$$i^* : \varinjlim H(M ; (H^\tau)^q) \to H(M ; H^q(G))$$

are isomorphisms.

Part i) of the lemma now follows via (17.14) while part ii) follows from the definition $H(\psi_\sigma) = \psi_\sigma^*$.

Q.E.D.

17.16.- Lemma. $\bar{\psi}_2^{p,o} \circ \bar{\xi}_2^{p,o}$ is an isomorphism for each $p \geq 0$.

Proof. Hypothesis v) of the theorem and (17.14) identify $\bar{\psi}_2^{p,o} \circ \bar{\xi}_2^{p,o}$ with the identity map of $H^p(M ; k)$.

Q.E.D.

Theorem 17.13 follows now from the Zeeman–Moore comparison theorem (17.17) below. Indeed, since $\bar{\psi}^* \circ \bar{\xi}^*$ is an isomorphism, lemmas 17.15 and 17.16 allow us to apply the comparison theorem (with $\eta_i^{p,q} = \bar{\psi}_i^{p,q} \circ \bar{\xi}_i^{p,q}$) to conclude that $\bar{\psi}_i^{p,q} \circ \bar{\xi}_i^{p,q}$ is an isomorphism for all $i \geq 2$ and all $p,q \geq 0$.

A second application of lemma 17.15 gives that $\bar{\psi}_2$, $\bar{\xi}_2$ and each ψ_σ^* are isomorphisms. Hence so are $\bar{\psi}^*$, $\bar{\xi}^*$, ψ^* and ξ^*.

Q.E.D.

We complete the chapter by establishing the Zeeman-Moore theorem.

Suppose $\eta_i^{p,q} : \check{E}_i^{p,q} \to E_i^{p,q}$ $(i \geq 2)$ is a morphism of first quadrant spectral sequences in which the i^{th} differentials are homogeneous of degree $(i, 1-i)$. We put

$$\eta_i^{p,q} = \eta_\infty^{p,q} \quad , \quad \check{E}_i^{p,q} = \check{E}_\infty^{p,q} \quad , \quad E_i^{p,q} = E_\infty^{p,q}$$

if $i \geq p+q+2$.

Next assume $\eta^* : \check{H} \xrightarrow{\cong} H$ is an isomorphism of graded spaces. Suppose $\eta^*(F^p(\check{H}^q)) \subset F^p(H^q)$ where

$$H^q = F^o(H^q) \supset F^1(H^q) \supset \ldots \supset F^{q+1}(H^q) = 0 , \qquad q \geq 0$$

and similarly for $\check{H}^q$. We say η_i <u>converges to the isomorphism</u> η^* if there are isomorphisms $\check{E}_\infty^{p,q} \cong F^p\, \check{H}^{p+q}/F^{p+1}\, \check{H}^{p+q}$ and $E_\infty^{p,q} \cong F^p\, H^{p+q}/F^{p+1}\, H^{p+q}$ which identify $\eta_\infty^{p,q}$ with the map induced by η^*.

The comparison theorem we need reads as follows :

17.17.- <u>*Theorem*</u>. Suppose $\eta_i : \check{E}_i \to E_i$, $2 \leq i \leq \infty$ is a map of first quadrant spectral sequences, as described above. Assume that

i) $\eta_2^{p,o}$ is always an isomorphism $(p \geq 0)$.

ii) If for some q, $\eta_2^{o,q}$ is an isomorphism and $\eta_2^{1,q}$ is injective, then $\eta_2^{p,q}$ is an isomorphism for all $p \geq 0$.

iii) η_i converges to an isomorphism.

Then $\eta_i^{p,q}$ is an isomorphism for all $p,q \geq 0$ and $i \geq 2$.

The proof proceeds by a number of lemmas.

17.18.- <u>*Lemma*</u>.

i) $\eta_\infty^{r,o}$ is injective.

ii) If $\eta_\infty^{p,r-p}$ is an isomorphism for $p > m$ then $\eta_\infty^{m,r-m}$ is injective.

iii) If $\eta_\infty^{p,r-p}$ is an isomorphism for $p > 0$ then $\eta_\infty^{o,r}$ is an isomorphism.

<u>*Proof*</u>.- Apply hypothesis iii) of the theorem.

Q.E.D.

17.19.- *Lemma.* $\eta_i^{p,0}$ is surjective for all $i \geq 2$. In particular $\eta_\infty^{p,o}$ is an isomorphism.

Proof.- Apply hypothesis i) of the theorem, and lemma 17.18 i).

Q.E.D.

For the rest of this chapter (only) we establish the notation :

$$B_i^{p,q} = d_i(E_i^{p-i,q+i-1}) \qquad ; \qquad Z_i^{p,q} = E_i^{p,q} \cap \ker d_i$$

and we denote the restrictions of $\eta_i^{p,q}$ by

$$B(\eta)_i^{p,q} : \check{B}_i^{p,q} \longrightarrow B_i^{p,q}$$

and

$$Z(\eta)_i^{p,q} : \check{Z}_i^{p,q} \to Z_i^{p,q} .$$

We also adopt the convention that p and q always denote integers ≥ 0.

17.20.- *Lemma.* Suppose for some $m \geqslant 0$ that $\eta_2^{p,q}$ is an isomorphism, $q \leqslant m$. Then for $i \geq 2$:

$$\eta_i^{p,q} \text{ is an } \begin{cases} \text{isomorphism if } p+q \leq m+1, & q \neq m+1. \\ \text{isomorphism if } p+q = m+2, & p \geq i. \\ \text{injection if } p+q = m+3, & p \geq \max(3,2i-2). \end{cases}$$

Proof.- By induction on i. The lemma is true by hypothesis when $i = 2$. Assume it is proved for some $i \geq 2$. We show that :

$$(17.21) \quad B(\eta)_i^{p,q} \text{ is surjective if} \begin{cases} p+q \leq m+1 \\ \text{or} \\ p+q = m+2, \ p \neq i \\ \text{or} \\ p+q = m+3, \ p \geq 2i. \end{cases}$$

$$(17.22)\qquad \eta_i^{p,q} \text{ is injective if } \begin{cases} p+q \leq m+1, & q \neq m+1 \\ \text{or} & \\ p+q = m+2, & p \geq i \\ \text{or} & \\ p+q = m+3, & p \geq \max(3,2i-2). \end{cases}$$

$$(17.23)\qquad B(\eta)_i^{p,q} \text{ is an isomorphism if } \begin{cases} p+q \leq m+1, & q \neq m+1 \\ \text{or} & \\ p+q = m+2, & p \geq i+1 \\ \text{or} & \\ p+q = m+3, & p \geq \max(3,2i). \end{cases}$$

$$(17.24)\qquad Z(\eta)_i^{p,q} \text{ is an isomorphism if } \begin{cases} p+q \leq m+1, & q \neq m+1 \\ \text{or} & \\ p+q = m+2, & p \geq i. \end{cases}$$

Indeed by induction $\eta_i^{p-i,q+i-1}$ is an isomorphism when (p,q) satisfy one of the conditions of (17.21). Thus (17.21) follows from the surjections $d_i : E_i^{p-i,q+i-1} \to B_i^{p,q}$.

Equation (17.22) simply restates part of the induction hypothesis. Since $B(\eta)_i^{p,q}$ is a restriction of $\eta_i^{p,q}$, (17.23) follows from (17.21) and (17.22).

To prove (17.24) consider the exact sequence

$$0 \to Z_i^{p,q} \longrightarrow E_i^{p,q} \xrightarrow{\ d_i\ } E_i^{p+i,q-i+1}$$

By our induction hypothesis $\eta_i^{p,q}$ is an isomorphism and $\eta_i^{p+i,q-i+1}$ is injective when (p,q) satisfy one of the conditions of (17.24). Equation (17.24) follows.

Finally, consider the exact sequence

$$0 \to B_i^{p,q} \to Z_i^{p,q} \to E_{i+1}^{p,q} \to 0.$$

In view of this sequence equations (17.23) and (17.24) imply that $\eta_{i+1}^{p,q}$ is an isomorphism if $p+q \leq m+1$, $q \neq m+1$ or if $p+q = m+2$, $p \geq i+1$.

Equations (17.22) and (17.23) show that $\eta_{i+1}^{p,q}$ is injective if $p+q = m+3$ and $p \geq \max(3,2i)$.

This closes the induction, and completes the proof.

Q.E.D.

17.25.- *Lemma*. Suppose $\eta_2^{p,q}$ is an isomorphism for $q \leq m$, all p. Then $\eta_\infty^{p,q}$ is an isomorphism for $p+q \leq m+1$.

Proof.- If $q \neq m+1$, this is lemma 17.20. Thus the lemma follows from lemma 17.18 iii).

Q.E.D.

17.26.- *Lemma*. Fix integers $m \geq 0$ and $i \geq 2$. Suppose that $\eta_2^{p,q}$ is an isomorphism if $q \leq m$ (all p), and $\eta_j^{o,m+1}$ is an isomorphism, $j \geq i$.

Then for $\ell \geq 0$ and $j \geq i$:

i) $\eta_{j+\ell}^{2j+\ell,(m+3)-(2j+\ell)}$ is injective.

ii) $B(\eta)_{j+\ell}^{2j+\ell,(m+3)-(2j+\ell)}$ is an isomorphism.

iii) $\eta_{j+\ell}^{j,m+2-j}$ is an isomorphism.

iv) $\eta_{j+\ell+1}^{2j+\ell,(m+3)-(2j+\ell)}$ is injective.

Proof.- We use induction on ℓ. When $\ell = 0$, ii) is equation (17.23) and the other three parts are lemma 17.20. Now assume the lemma proved for $j \geq i$ and $\ell' \leq \ell$, some fixed $\ell \geq 0$. We prove it for $j \geq i$ and $\ell+1$

Part i) : If $\ell = 0$ then lemma 17.20 shows that $\eta_{j+1}^{2j+1,(m+3)-(2j+1)}$ is injective. If $\ell > 0$ then $i)_{j,\ell+1}$ coincides with $iv)_{j+1,\ell-1}$ and so is true by induction.

Part iii) : Consider the short exact sequence

$$0 \to Z_{j+\ell}^{j,m+2-j} \to E_{j+\ell}^{j,m+2-j} \xrightarrow{d_{j+\ell}} B_{j+\ell}^{2j+\ell,(m+3)-(2j+\ell)} \to 0.$$

Because of $iii)_{j,\ell}$ and $ii)_{j,\ell}$ we may conclude that

(17.27) $$Z(\eta)^{j,m+2-j}_{j+\ell} \quad \text{is an isomorphism.}$$

If $\ell > 0$ then $E^{j,m+2-j}_{j+\ell+1} = Z^{j,m+2-j}_{j+\ell}$ and so $\eta^{j,m+2-j}_{j+\ell+1}$ is an isomorphism. If $\ell = 0$ we have the exact sequence

$$0 \to E^{o,m+1}_j \xrightarrow{d_j} Z^{j,m+2-j}_j \longrightarrow E^{j,m+2-j}_{j+1} \to 0.$$

We have assumed that $\eta^{o,m+1}_j$ is an isomorphism (hypothesis of lemma). Thus (17.27) shows that $\eta^{j,m+2-j}_{j+1}$ is an isomorphism, and iii) is proved for $\ell+1$.

<u>part</u> ii) : Observe that $Z^{j,m+2-j}_{j+\ell+1} = E^{j,m+2-j}_{j+\ell+2}$ and consider the exact sequence.

$$0 \to E^{j,m+2-j}_{j+\ell+2} \to E^{j,m+2-j}_{j+\ell+1} \xrightarrow{d_{j+\ell+1}} B^{2j+\ell+1,(m+3)-(2j+\ell+1)}_{j+\ell+1} \to 0.$$

By $iii)_{j,\ell+1}$ and $i)_{j,\ell+1}$ (which are now proved) we have an isomorphism in the middle of this sequence and an injection at the right. Hence we have an isomorphism at the right, which proves $ii)_{j,\ell+1}$.

<u>Part</u> iv) : Let $r = 2j+\ell+1$ and consider the exact sequence

$$0 \to B^{r,m+3-r}_{j+\ell+1} \to Z^{r,m+3-r}_{j+\ell+1} \to E^{r,m+3-r}_{j+\ell+2} \to 0.$$

By $ii)_{j,\ell+1}$ we have an isomorphism at the left and by $i)_{j,\ell+1}$ we have an injection at the centre. Thus we have an injection at the right ; i.e., $\eta^{r,m+3-r}_{j+\ell+2}$ is injective.

The induction is now closed and the proof complete.

Q.E.D.

17.28.- <u>Lemma</u>. Assume $\eta^{p,q}_2$ is an isomorphism for $q \leq m$ and all p. Then $\eta^{o,m+1}_i$ is an isomorphism for all $i \geq 2$.

Proof.- By lemma 17.25 $\eta_\infty^{o,m+1} = \eta_{m+3}^{o,m+1}$ is an isomorphism.

Now assume $\eta_j^{o,m+1}$ is an isomorphism for all $j \geq i+1$, some $i \geq 2$. We show that

(17.29) $$\eta_{i+\ell}^{2i+\ell,(m+3)-(2i+\ell)} \quad \text{is injective,} \quad \ell \geq 0$$

and

(17.30) $$\eta_{i+\ell}^{i,m+2-i} \quad \text{is injective,} \quad \ell \geq 1.$$

In fact (17.29) is lemma 17.20 if $\ell = 0,1$ or 2. If $\ell > 2$ we write

$$\eta_{i+\ell}^{2i+\ell,(m+3)-(2i+\ell)} = \eta_{(i+1)+(\ell-2)+1}^{2(i+1)+\ell-2,(m+3)-(2i+\ell)}$$

and apply lemma 17.26, iv). This proves (17.29).

Next, consider the exact sequences

$$0 \to E_{i+\ell+1}^{i,m+2-i} \to E_{i+\ell}^{i,m+2-i} \xrightarrow{d_{i+\ell}} E_{i+\ell}^{2i+\ell,(m+3)-(2i+\ell)}, \qquad \ell \geq 1.$$

By (17.29) we have an injection on the right. Hence

(17.31) $$\eta_{i+\ell+1}^{i,m+2-i} \text{ injective} \Rightarrow \eta_{i+\ell}^{i,m+2-i} \text{ injective,} \qquad \ell \geq 1.$$

On the other hand for $j \geq i+1$ lemma 17.26 iii) shows that $\eta_\infty^{j,m+2-j}$ is an isomorphism. Hence lemma 17.18 ii) shows that $\eta_\infty^{i,m+2-i}$ is injective. Now (17.30) follows from (17.31).

Finally, consider the exact sequence

$$0 \to E_{i+1}^{o,m+1} \to E_i^{o,m+1} \xrightarrow{d_i} Z_i^{i,m+2-i} \to E_{i+1}^{i,m+2-i} \to 0.$$

Since we have assumed (by induction) that $\eta_{i+1}^{o,m+1}$ is an isomorphism, and since $Z(\eta)_i^{i,m+2-i}$ is an isomorphism by (17.24) and since $\eta_{i+1}^{i,m+2-i}$ is injective by (17.30), we conclude that $\eta_i^{o,m+1}$ is an isomorphism. The lemma follows by induction.

Q.E.D.

17.32.- Lemma. Assume $\eta_2^{p,q}$ is an isomorphism for $q \leq m$ (all p) and $\eta_2^{o,m+1}$ is also an isomorphism. Then for $i \geq 2$:

$$\eta_i^{p,q} \text{ is an } \begin{cases} \text{isomorphism if } p+q \leq m+1. \\ \text{isomorphism if } p+q = m+2, \ p \geq 2. \\ \text{injection if } p+q = m+3, \ p \geq i+1. \end{cases}$$

Proof.- The proof is identical with that of lemma 17.20.

Q.E.D.

17.33.- *Proof of theorem 17.17*. We show by induction on q that $\eta_2^{p,q}$ is an isomorphism (all p). When $q = 0$ this is hypothesis i) of the theorem. Assume we have proved it for $q \leq m$.

By lemma 17.28 $\eta_i^{o,m+1}$ is an isomorphism, $i \geq 2$. In particular $\eta_2^{o,m+1}$ is an isomorphism. Moreover, by lemma 17.26 iii) $\eta_\infty^{i,m+2-i}$ is an isomorphism for $i \geq 2$. Hence by lemma 17.18 ii), $\eta_\infty^{1,m+1}$ is injective.

Consider the exact sequences (for $i \geq 2$)

$$0 \to E_{i+1}^{1,m+1} \to E_i^{1,m+1} \xrightarrow{d_i} E_i^{i+1,(m+3)-(i+1)}$$

Since $\eta_2^{o,m+1}$ is an isomorphism, lemma 17.32 applies and shows we have an injection on the right. Hence

$$\eta_{i+1}^{1,m+1} \text{ injective} \Rightarrow \eta_i^{1,m+1} \text{ injective}$$

Since $\eta_\infty^{1,m+1}$ is injective we conclude that $\eta_2^{1,m+1}$ is injective.

Now hypothesis ii) of the theorem shows that $\eta_2^{p,m+1}$ is an isomorphism for all p. This closes the induction.

Q.E.D.

Chapter 18

The local system of a KS extension.

18.1.- Introduction.- In this chapter M denotes a fixed path connected topological space with base point m. Recall (definition 15.2) the c.g.d.a. $(A(M),d)$ and (property 15.5) the augmentation $A(M) \to k$ determined by m. If F is any local system over $\underline{\text{Sing}}\, M$ we adopt the same convention as for A and denote $F\ (\underline{\text{Sing}}\, M)$ simply by $F(M)$.

Since $A(M) = A(\underline{\text{Sing}}\, M)$ each singular simplex $\sigma : \Delta^n \to M$ determines the evaluation homomorphism (of c.g.d.a.'s)

$$e_\sigma : A(M) \to A(n)$$

given by $e_\sigma \Phi = \Phi_\sigma$ (cf. 13.5) .

Now assume

$$E : A(M) \xrightarrow{i} R \xrightarrow{\rho} T$$

is a KS extension (cf. chap. 1). Use the c.g.d.a. homomorphisms

$$A(M) \xrightarrow{e_\sigma} A(n) \quad \text{and} \quad A(M) \xrightarrow{i} R$$

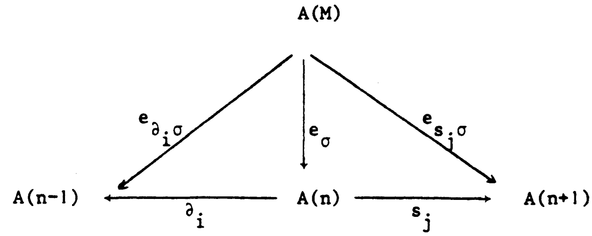
to tensor $A(n)$ with R over $A(M)$: denote the resulting c.g.d.a. by

$$(R_\sigma,d) = (A(n) \otimes_{A(M)} R,d).$$

(Note the dependence on σ comes from the homomorphism e_σ .)

Next observe that by definition the diagrams

(18.2)



commute. It follows that ∂_i and s_j extend to c.g.d.a. homomorphisms

$$\partial_i \otimes \iota : A(n) \otimes_{A(M)} R \to A(n-1) \otimes_{A(M)} R \quad \text{and} \quad s_j \otimes \iota : A(n) \otimes_{A(M)} R \to A(n+1) \otimes_{A(M)} R.$$

We denote these simply by

$$\partial_i : R_\sigma \to R_{\partial_i \sigma} \quad \text{and} \quad s_j : R_\sigma \to R_{s_j \sigma}.$$

Formulae (12.3) are clearly satisfied and so $\{R_\sigma, \partial_i, s_j\}$ is a local system over <u>Sing</u> M. Since it is a local system of c.g.d.a.'s we can pass to cohomology and obtain the local system $\{H(R_\sigma), \partial_i^*, s_j^*\}$.

18.3.- <u>Definition</u>.- The local system $\{R_\sigma, \partial_i, s_j\}$ will be denoted by R_* and called the <u>local system determined by</u> E. The local system $\{H(R_\sigma), \partial_i^*, s_j^*\}$ will be denoted by $H(R_*)$.

The canonical homomorphisms

(18.4) $$\lambda_\sigma : A(n) \to A(n) \otimes_{A(M)} R = R_\sigma, \quad \sigma \in \underline{\text{Sing}}\ M,$$

are compatible with the face and degeneracy operators. Hence the λ_σ define a c.g.d.a. homomorphism

(18.5) $$\lambda : (A(M), d) \to (R_*(M), d),$$

by $(\lambda \Phi)_\sigma = \lambda_\sigma \Phi_\sigma$.

On the other hand the canonical homomorphisms

(18.6) $$\mu_\sigma : R \to A(n) \otimes_{A(M)} R = R_\sigma , \qquad \sigma \in \underline{\text{Sing}}\ M$$

satisfy $\partial_i \circ \mu_\sigma = \mu_{\partial_i \sigma}$ and $s_j \circ \mu_\sigma = \mu_{s_j} \sigma$. Thus a c.g.d.a. homomorphism

(18.7) $$\mu : (R,d) \to (R_*(M),d)$$

is defined by $(\mu\Phi)_\sigma = \mu_\sigma \Phi$.

A short calculation yields $\mu \circ i = \lambda$. Since $e_m : A(M) \to k$ is the augmentation and E is a KS extension, $\ker \rho$ is generated by $i(\ker e_m)$. Thus ρ factors to yields the commutative diagram of c.g.d.a.'s

(18.8) $$\begin{array}{ccccc} & \overset{i}{\nearrow} & R & \xrightarrow{\ \rho\ } & T \\ A(M) & & \downarrow{\scriptstyle \mu} & & \downarrow{\scriptstyle \bar{\mu}} \\ & \underset{\lambda}{\searrow} & R_*(M) & \xrightarrow[(e_R)_m]{} & R_m \end{array}$$

(Here $(e_R)_m : \Phi \longrightarrow \Phi_m$; clearly $(e_R)_m \circ \lambda = e_m$).

18.9.- *Proposition*.- With the hypotheses and notation above, suppose T is connected. Then

i) R_* is an extendable local system of differential coefficients.

ii) $H(R_*)$ is a local system of coefficients.

iii) $\bar{\mu} : T \to R_m$ is an isomorphism.

iv) If T has finite type (as a graded space) then $\mu : R \to R_*(M)$ is an isomorphism.

Proof : Since M is path connected $H^0(M) = k$ and so by theorem 14.18 (with $F = k$) $H^0(A(M)) = k$. We can thus write

(18.10) $$R = A(M) \otimes T$$

so that $\rho = e_n \otimes \iota$ and $d(1 \otimes a) - 1 \otimes d_T a \in A^+(M) \otimes T$, $a \in T$, (cf. 1.14). Moreover we can write $T = \Lambda X$ and find a well ordered homogeneous basis $\{x_\alpha\}$ for X such that (cf. 1.4)

(18.11) $$d(1 \otimes x_\alpha) \in A(M) \otimes (\Lambda X)_{<\alpha} .$$

Use the isomorphism (18.10) to identify

(18.12) $$R_\sigma = A(n) \otimes_{A(M)} A(M) \otimes T = A(n) \otimes T.$$

Thus an augmentation $A(n) \to k$ determines a c.g.d.a. homomorphism $R_\sigma \to T$ and, by (18.11)

$$A(n) \xrightarrow{\lambda_\sigma} R_\sigma \longrightarrow T$$

is a KS extension. For suitable augmentations of $A(n)$ and $A(n-1)$, ∂_i preserves augmentations, and so there is a commutative diagram

$$\begin{array}{ccccc} A(n) & \xrightarrow{\lambda_\sigma} & R & \longrightarrow & T \\ {\scriptstyle \partial_i}\downarrow & & {\scriptstyle \partial_i}\downarrow & & \| \\ A(n-1) & \xrightarrow[\lambda_{\partial_i \sigma}]{} & R_{\partial_i \sigma} & \longrightarrow & T \end{array}$$

This diagram is a morphism of KS extensions. Since T is connected $H^0(R_\sigma) = H^0(R_{\partial_i \sigma}) = k$. Since (trivially) $\partial_i^* : H(A(n)) \xrightarrow{\cong} H(A(n-1))$, theorem 7.1 implies that $\partial_i^* : H(R_\sigma) \xrightarrow{\cong} H(R_{\partial_i \sigma})$. Since $\partial_i^* \circ s_i^* = \iota$, each s_i^* is also an isomorphism. This proves ii).

To complete the proof of i) we need only show that R_* is extendable.

Regard T as a constant local system over <u>Sing</u> M. The isomorphisms (18,12) identity R_* with $A \otimes T$ (in the category of graded spaces). Since T is constant and A is extendable, $A \otimes T$ (and so R_*) is extendable.

Part iii) follows from (18.12) with $\sigma = m$. To prove iv) let $a_1,\ldots,a_r$ be a basis of $\sum_{\ell \le p} T^{\ell}$. If $\Omega \in R_*(M)$ use the isomorphisms (18.12) to write

$$\Omega_\sigma = \sum_{\ell=1}^{r} \Omega_{\sigma,\ell} \otimes a_\ell \quad , \quad \Omega_{\sigma,\ell} \in A(|\sigma|).$$

The isomorphisms (18.12) convert ∂_i and s_j to $\partial_i \otimes \iota$ and $s_j \otimes \iota$ and so the equations $\Omega_{\partial_i\sigma} = \partial_i\Omega_\sigma$ and $\Omega_{s_j\sigma} = s_j\Omega_\sigma$ yield

$$\Omega_{\partial_i\sigma,\ell} = \partial_i\Omega_{\sigma,\ell} \quad \text{and} \quad \Omega_{s_j\sigma,\ell} = s_j\Omega_{\sigma,\ell} \, , \quad \ell = 1,\ldots,p.$$

It follows that elements $\Omega_\ell \in A(M)$, $\ell = 1,\ldots,r$ are defined by $(\Omega_\ell) = \Omega_{\sigma,\ell}$. Clearly $\Omega = \mu(\sum \Omega_\ell \otimes a_\ell)$. Thus μ is surjective. On the other hand if $\mu(\sum \Phi_\ell \otimes a_\ell) = 0$ for elements $\Phi_\ell \in A(M)$ then

$$0 = \mu_\sigma(\sum \Phi_\ell \otimes a_\ell) = \sum (\Phi_\ell)_\sigma \otimes a_\ell .$$

Hence $(\Phi_\ell)_\sigma = 0$ for all ℓ,σ. This implies that $\Phi_\ell = 0$, $\ell = 1,\ldots,r$, and so μ is injective.

Q.E.D.

18.13.- <u>Remark</u>.- Suppose $E : A(M) \xrightarrow{i} R \xrightarrow{\rho} T$ is a KS extension with T connected. Since $H(R_*)$ is a ℓ.s.c., a canonical $\pi_1(M)$-module structure is defined in $H(R_*)_m = H(R_m)$ - cf. definition 16.7.

Now by part iii) of prop. 18.9 $\bar{\mu}^*$ is an isomorphism :

$$\bar{\mu}^* : H(T) \xrightarrow{\cong} H(R_m).$$

Thus $H(T)$ becomes a canonical $\pi_1(M)$-module.

We shall now make this $\pi_1(M)$-module structure explicit. Fix an isomorphism $R = A(M) \otimes T$, a graded subspace $X \subset T^+$, and a well ordered homogeneous basis $\{x_\alpha\}_{\alpha \in I}$ of X such that

$$\begin{aligned} &\text{i)} \quad \rho = e_m \otimes \iota \\ (18.14) \quad &\text{ii)} \quad T = \Lambda X \\ &\text{iii)} \quad d(1 \otimes x_\alpha) \in A(M) \otimes (\Lambda X)_{<\alpha}, \quad \alpha \in I . \end{aligned}$$

We also need the following conventions. We regard S^1 as the unit interval I with endpoints identified, and let $\sigma : I \to S^1$ be the projection. We put $\sigma(0) = \sigma(1) = b$; it is a base point for S^1. Note that σ is a 1-simplex on S^1.

Next write $A(1) = \Lambda(t,dt)$, where t is the barycentric coordinate function b_1 (cf. example 12.9). Then $A(0) = k$ and the face maps $\partial_0, \partial_1 : A(1) \to k$ are given by

$$(18.15) \qquad \partial_0 t = 1 \quad \text{and} \quad \partial_1 t = 0.$$

It is easy to construct a cocycle $u \in A^1(S^1)$ such that $u_\sigma = dt$. Clearly the inclusion

$$(\Lambda u, 0) \longrightarrow (A(S^1), d)$$

induces a cohomology isomorphism.

Now suppose $\tau : I \to M$ is a loop based at m : $\tau(0) = \tau(1) = m$. Then τ factors over σ to yield a continuous map

$$\psi : (S^1, b) \to (M, m).$$

The base point b determines an augmentation $e_b : A(S^1) \to k$ and

$$A(\varphi) : A(M) \to A(S^1)$$

is a homomorphism of augmented c.g.d.a.'s.

In particular we can form the commutative diagram of homomorphisms of augmented c.g.d.a.'s :

$$\begin{array}{ccccc}
A(M) & \xrightarrow{\ i\ } & R & \xrightarrow{\ \rho\ } & \Lambda X \\
{\scriptstyle A(\varphi)}\big\downarrow & & \big\downarrow & & \big\downarrow{\scriptstyle \iota} \\
A(S^1) & \xrightarrow[\ j\ ]{} & A(S^1) \otimes_{A(M)} R & \xrightarrow[\ \eta\ ]{} & \Lambda X
\end{array}$$

in which $j\Phi = \Phi \otimes 1$ and $\eta = e_b \otimes \rho$.

Our fixed isomorphism $R = A(M) \otimes \Lambda X$ defines an isomorphism $A(S^1) \otimes_{A(M)} R \cong A(S^1) \otimes \Lambda X$. Because of (18.14) this identifies the lower row above as a KS extension.

We now apply prop. 1.11 and its corollary to the inclusion $\Lambda u \hookrightarrow A(S^1)$. This yields a <u>second</u> isomorphism

$$f : A(S^1) \otimes \Lambda X \xrightarrow{\ \cong\ } A(S^1) \otimes_{A(M)} R$$

with respect to which

i) $\eta = e_b \otimes \iota$.

ii) $d(1 \otimes x_\alpha) \in A(S^1) \otimes (\Lambda X)_{<\alpha}$, $\alpha \in I$.

iii) $\Lambda u \otimes \Lambda X$ is stable under d.

Identify $A(S^1) \otimes \Lambda X$ with $A(S^1) \otimes_{A(M)} R$ via f. Then iii) implies that

$$d(1 \otimes a) = 1 \otimes d_T a + u \otimes \theta(a), \qquad a \in \Lambda X,$$

where θ is a degree zero derivation of ΛX. The equation $d^2 = 0$ yields

$d_T\theta = \theta d_T$ and so θ induces a derivation θ^* in $H(T)$. Moreover, because of ii) we have

$$\theta x_\alpha \in (\Lambda X)_{<\alpha}, \qquad \alpha \in I.$$

This shows that for any $a \in T$, $\theta^n a = 0$ if n is large enough. Thus $e^\theta = \sum^\infty \theta^n/n!$ is an automorphism of (T,d_T) and

$$(e^\theta)^* = e^{(\theta^*)} .$$

18.16.- Proposition.- The $\pi_1(M)$-module structure of $H(T)$ is given by

$$[\tau] = e^{\theta^*} : H(T) \xrightarrow{\cong} H(T).$$

Proof : Since $\psi \circ \sigma = \tau$ the homomorphisms

$$e_\tau : A(M) \to A(1) \quad \text{and} \quad e_\sigma : A(S^1) \to A(1)$$

are connected by $e_\tau = e_\sigma \circ A(\psi)$. Thus we can write

$$R_\tau = A(1) \otimes_{A(M)} R = A(1) \otimes_{A(S^1)} \left[A(S^1) \otimes_{A(M)} R\right] ,$$

where we use e_σ to make $A(1)$ into an algebra over $A(S^1)$.

Next, use the isomorphism f above, together with this equation to write

$$R_\tau = A(1) \otimes \Lambda X = \Lambda(t,dt) \otimes \Lambda X.$$

Because $u = dt$ the differential in R_τ is then given explicitly by

$$d(1 \otimes a) = 1 \otimes d_T a + dt \otimes \theta a, \quad a \in \Lambda X.$$

It follows that a homomorphism $\psi : (T,d_T) \to (R_\tau,d)$ is given by

$$\psi_a = \sum_{n=0}^{\infty} t^n/n! \otimes \theta^n a.$$

On the other hand, because $\sigma(0) = \sigma(1) = b$ we have

$$\partial_1 \circ e_\sigma = \partial_0 \circ e_\sigma = e_b : A(S^1) \to k.$$

It follows that the diagrams

$$\begin{array}{ccccc}
A(1) \otimes_{A(M)} R & \xrightarrow{\cong} & A(1) \otimes_{A(S^1)} [A(S^1) \otimes_{A(M)} R] & = & A(1) \otimes \Lambda X \\
\Big\downarrow \partial_i \otimes \iota & & \Big\downarrow \partial_i \otimes e_b \otimes \iota & & \Big\downarrow \partial_i \otimes \iota \\
R_m = k \otimes_{A(M)} R & = & k \otimes_{A(M)} R & \xleftarrow{\cong} & \Lambda X
\end{array}$$

commute. Thus the $\pi_1(M)$-module structure of $H(T)$ is given by

$$[\tau] = (\partial_0 \otimes \iota)^* \circ [(\partial_1 \otimes \iota)^*]^{-1} : H(T) \xrightarrow{\cong} H(T).$$

Now use (18.15) to conclude $(\partial_1 \otimes \iota)\psi = \iota$, $(\partial_0 \otimes \iota)\psi = e^\theta$ and hence

$$[\tau] = (\partial_0 \otimes \iota)^* \psi^* = e^{\theta^*} .$$

Q.E.D.

18.17.- Corollary.- If T is connected and finitely generated (as an algebra) then $H(T)$ is a nilpotent $\pi_1(M)$-module.

Proof : We can write (in this case)

$$R = A(M) \otimes \Lambda(x_1, \dots, x_n)$$

with $d(1 \otimes x_i) \in A(M) \otimes \Lambda(x_1, \dots, x_{i-1})$. For each loop τ in M based at m we have

$$[\tau] = (e^{\theta_\tau})^* : H(T) \to H(T)$$

where θ_τ is a derivation of (T,d_T) and

$$\theta_\tau(x_1) = 0, \quad \theta_\tau(x_i) \in \Lambda(x_1,\dots,x_{i-1}), \quad i > 1.$$

These equations imply that if $a \in T$ then for some integer N (depending on a)

$$\theta_{\tau_1} \circ \dots \circ \theta_{\tau_N}(a) = 0$$

for any sequence of N loops $\tau_1,\dots,\tau_N$.

Since T (and hence $H(T)$) has finite type, it follows that $H(T)$ is a nilpotent $\pi_1(M)$-module.

Q.E.D.

18.18.- Direct limits.- Again consider the KS extension

$$E : A(M) \xrightarrow{i} R \xrightarrow{\rho} T$$

of 18.1. Assume T is connected. Fix an isomorphism

(18.19) $$R = A(M) \otimes T ,$$

a graded subspace $X \subset T^+$ and a well ordered homogeneous basis $\{x_\alpha\}_{\alpha\in I}$ of X such that (18.14) holds.

If γ is a subset of I denote by $T^\gamma \subset T$ the subalgebra generated by the x_α, $\alpha \in \gamma$ and denote by $X^\gamma \subset X$ the subspace spanned by the x_α, $\alpha \in \gamma$. Then

$$T^\gamma = \Lambda X^\gamma .$$

Let Γ be the collection of finite subsets $\gamma \subset I$ such that

$R^\gamma = A(M) \otimes T^\gamma$ is stable under d. Then Γ is ordered by inclusion. The properties (18.14) exhibit

$$E^\gamma : A(M) \longrightarrow R^\gamma \longrightarrow T^\gamma$$

as a sub KS extension of E. Moreover $I = \bigcup_{\gamma\in\Gamma} \gamma$ and so the inclusions $R^\gamma \hookrightarrow R$ define an isomorphism,

(18.20) $$\varinjlim_\Gamma R^\gamma \xrightarrow{\cong} R .$$

Similarly the inclusions $R^{\gamma_1} \longrightarrow R^{\gamma_2} \longrightarrow R$ define morphisms of local systems

$$R_*^{\gamma_1} \longrightarrow R_*^{\gamma_2} \xrightarrow{\xi_*^{\gamma_2}} R_*$$

(cf. 17.1). In particular $\{R_*^\gamma\}_{\gamma\in\Gamma}$ is a directed family of local systems over Sing M (cf. 17.4) and the morphisms $\xi_*^\gamma : R_*^\gamma \to R_*$ define a morphism of local systems

$$\xi_* : \varinjlim_\Gamma R_*^\gamma \longrightarrow R_*$$

18.21.- Proposition.- With the notation above :

i) R_* and each R_*^γ are extendable local systems of differential coefficients.

ii) For each γ, $H(R_m^\gamma)$ is a nilpotent $\pi_1(M)$-module of finite type.

iii) The morphism $\varinjlim_\Gamma R_*^\gamma \to R_*$ is an isomorphism of local systems.

iv) The diagram below commutes, with isomorphisms as shown :

$$\begin{array}{ccc}
\varinjlim_{\Gamma} R^{\gamma} & \xrightarrow{\ \cong\ } & R \\
{\scriptstyle \varinjlim_{\Gamma} \mu^{\gamma}}\ \downarrow \cong & & \downarrow \mu \\
\varinjlim_{\Gamma} R_{*}^{\gamma}(M) & \xrightarrow[\ \xi\ ]{} & R_{*}(M) \ .
\end{array}$$

Proof :

i) is prop. 18.9 i). ii) is cor. 18.17 (with remark 18.13). iii) is immediate from the definitions. The commutativity of iv) is obvious and each μ^{γ} is an isomorphism by prop. 18.9 iv).

Q.E.D.

Chapter 19

Serre fibrations

19.1.- Serre fibrations. A Serre fibration is a continuous surjective map of topological spaces,

$$\pi : P \to M,$$

which has the following homotopy lifting property : if

$$\begin{array}{ccc} \Delta^n \times \{0\} & \xrightarrow{\tau_o} & P \\ \big\downarrow & & \big\downarrow \pi \\ \Delta^n \times I & \xrightarrow[\sigma]{} & M \end{array}$$

is a commutative square of continuous maps, then τ_o extends to a continuous map $\tau : \Delta^n \times I \to P$ such that $\pi\tau = \sigma$.

The spaces P and M are the total space and base space of the fibration. If $x \in M$ the fibre, P_x, at x is the subspace $\pi^{-1}(x)$ of P.

19.2.- Fibrations over a simplex. Recall (12.9) that the vertices of the p-simplex Δ^p are denoted by $v_o,\dots,v_p$. Further, the order preserving maps $\alpha \in \mathrm{Ord}([p],[q])$ define continuous maps $\Delta(\alpha) : \Delta^p \to \Delta^q$. In this section it will, in particular, be convenient to denote the face and degeneracy maps respectively by

(19.3) $$\eta_i = \Delta(\delta_i) : \Delta^{p-1} \to \Delta^p \quad \text{and} \quad \zeta_j = \Delta(\sigma_j) : \Delta^p \to \Delta^{p-1}.$$

We recall now the standard triangulation of $\Delta^p \times I$. It is given by the continuous maps

$$\alpha_m : \Delta^{p+1} \longrightarrow \Delta^p \times I \quad , \quad 0 \le m \le p,$$

defined by

$$\alpha_m(z) = (\zeta_m(z), \sum_{i=m+1}^{p+1} b_i(z)), \qquad z \in \Delta^{p+1},$$

where $b_i(z)$ is the i^{th} barycentric coordinate.

Let $\sigma : \Delta^p \to \Delta^n$ be any continuous map. Denote by $\bar{\sigma} : \Delta^p \times I \to \Delta^n$ the continuous map given by

$$(19.4) \qquad \bar{\sigma} \circ \alpha_m(\sum_{i=0}^{p+1} \lambda_i v_i) = \begin{cases} (1-\lambda)\sigma\,(\sum_{i=0}^{m} \frac{\lambda_i}{1-\lambda} v_i) + \lambda v_n, & \lambda \neq 1, \\ v_n, & \lambda = 1, \end{cases}$$

where $\lambda = \sum_{i=m+1}^{p+1} \lambda_i$. Thus

$$(19.5) \qquad \bar{\sigma}(z,0) = \sigma(z) \quad \text{and} \quad \bar{\sigma}(z,1) = v_n, \qquad z \in \Delta^p.$$

The relations (12.2) imply that the diagrams

$$(19.6) \qquad \begin{array}{ccc} \Delta^{p-1} \times I & \xrightarrow{\overline{\partial_i \sigma}} & \\ {\scriptstyle \eta_i \times \iota}\downarrow & & \Delta^n \\ \Delta^p \times I & \xrightarrow{\bar{\sigma}} & \end{array}$$

commute, where (as usual) $\partial_i\sigma = \sigma \circ \eta_i$.

Now consider a Serre fibration

$$P \xrightarrow{\pi} \Delta^n.$$

Using the homotopy lifting property, and (19.6) we obtain

19.7.- <u>Lemma</u>. There are continuous maps

$$\bar{\bar{\tau}} : \Delta^p \times I \longrightarrow P \quad , \qquad \tau \in \underline{Sing_p}(P), \qquad p \geq 0,$$

such that

i) $\bar{\tau}(z,0) = \tau(z)$.

ii) $\pi \circ \bar{\tau} = \overline{\pi o \tau}$, where $\overline{\pi o \tau}$ is defined from $\pi \circ \tau$ by (19.4).

iii) If $\pi\tau(\Delta^p) = v_n$, then $\bar{\tau}(z,t) = \tau(z)$, $t \in I$.

iv) $\bar{\tau} \circ (\eta_i \times \iota) = \overline{\overline{\partial_i \tau}}$, $0 \leq i \leq p$.

Recall next from 19.2 (or 12.9) that an inclusion of simplicial sets

$$\underline{\Delta}^n \subset \mathrm{Sing}(\Delta^n)$$

is defined by $\alpha \longmapsto \Delta(\alpha)$. Thus a subsimplicial set $\underline{P} \subset \underline{\mathrm{Sing}}\, P$ is defined by

(19.8) $$\tau \in \underline{P} \Leftrightarrow \pi \circ \tau \in \Delta^n.$$

We have thus the sequence of inclusions of simplicial sets

$$\underline{\mathrm{Sing}}(P_{v_n}) \xrightarrow{\gamma_2} \underline{P} \xrightarrow{\gamma_1} \underline{\mathrm{Sing}}\,(P).$$

19.9.- *Lemma*. The homomorphisms

$$A(P_{v_n}) \xleftarrow{A(\gamma_2)} A(\underline{P}) \xleftarrow{A(\gamma_1)} A(P)$$

induce isomorphisms of cohomology.

Proof.- In view of theorem 14.18 it is enough to prove that

$$C(P_{v_n}) \xleftarrow{C(\gamma_2)} C(\underline{P}) \xleftarrow{C(\gamma_1)} C(P)$$

induce cohomology isomorphisms. For each $\tau \in \underline{\mathrm{Sing}}_p(P)$ $(p \geq 0)$ let $\bar{\bar{\tau}} : \Delta^p \times I \to P$ be a continuous map such that the conclusions of lemma 19.7 hold. Define $\check{\tau} \in \underline{\mathrm{Sing}}_p(P_{v_n})$ by $\tau(z) = \bar{\bar{\tau}}(z,1)$ - cf. 19.7 ii) and 19.5.

Define linear maps

$$\lambda : C^p(P_{\check{v}_n}) \to C^p(P), \qquad p \geq 0,$$

by $(\lambda f)(\tau) = f(\check{\tau})$. It follows from 19.7 iii) that

(19.10) $$C(\gamma_2) \circ C(\gamma_1) \circ \lambda = \iota.$$

Recall the linear maps $\alpha_m : \Delta^{p+1} \to \Delta^p \times I$ defined above.

Put $h_m(\tau) = \bar{\bar{\tau}} \circ \alpha_m : \Delta^{p+1} \to P$, $\tau \in \underline{\mathrm{Sing}}_p(P)$. Define linear maps

$$h : C^{p+1}(P) \to C^p(P), \qquad p \geq 0,$$

by

$$(hf)(\tau) = \sum_{m=0}^{p} (-1)^m f(h_m(\tau)) \quad , \qquad \tau \in \underline{\mathrm{Sing}}_p(P).$$

Then a classical calculation (via 19.7 iv)) yields

(19.11) $$h\delta + \delta h = \lambda \circ C(\gamma_2) \circ C(\gamma_1) - \iota .$$

This, together with (19.10) shows that $C(\gamma_2)^* \circ C(\gamma_1)^*$ is an isomorphism.

Finally, suppose $\sigma \in \underline{\mathrm{Sing}}_p(\Delta^n)$ is actually in $(\underline{\Delta}^n)_p$.

Then formula (19.4) shows that $\bar{\sigma} \circ \alpha_m \in (\underline{\Delta}^n)_{p+1}$, $m \geq 0$. Hence if $\tau \in \underline{P}$, $h_m(\tau)$ satisfies (by 19.7 ii))

$$\pi \circ h_m(\tau) = \overline{\pi \circ \tau} \circ \alpha_m \in (\underline{\Delta}^n)_{p+1}.$$

This shows that if $\tau \in \underline{P}$ so does each $h_m(\tau)$ and so h restricts to an operator in $C(\underline{P})$. It follows that $C(\gamma_2)^*$ is an isomorphism .

Q.E.D.

19.12.- *Pullbacks*. Consider continuous maps

$$\begin{array}{ccc} & & P \\ & & \downarrow{\scriptstyle \pi} \\ M_1 & \xrightarrow[\varphi]{} & M \end{array}$$

and suppose π is a Serre fibration. Define a subspace $\varphi^* P \subset M_1 \times P$ by

$$\varphi^* P = \{(x,y) \mid \varphi x = \pi y\}.$$

The projections of $M_1 \times P$ onto M_1 and P, when restricted to $\varphi^* P$, give the commutative square

$$\begin{array}{ccc} \varphi^* P & \longrightarrow & P \\ {\scriptstyle \pi_1}\downarrow & & \downarrow{\scriptstyle \pi} \\ M_1 & \xrightarrow[\varphi]{} & M . \end{array}$$

Evidently π_1 is again a Serre fibration ; it is called the pullback of π to M_1 via φ.

Let $\psi : M_2 \to M_1$ be another continuous map. Then $\psi^*(\varphi^* P)$ is the subspace of $M_2 \times M_1 \times P$ of points of the form $(x,\psi x,z)$ with $\varphi\psi x = z$. Thus projection $M_2 \times M_1 \times P \to M_2 \times P$ restricts to a homeomorphism

(19.13) $$\psi^*(\varphi^* P) \xrightarrow{\cong} (\psi\varphi)^*(P).$$

We often identify these spaces via this homeomorphism.

If $\varphi : x \to M$ is the inclusion of a single point then the pullback is just the constant projection $P_x \to x$. More generally, if $\sigma : \Delta^n \to M$ is any singular simplex on M we denote the pullback of π to Δ^n via σ by

(19.14) $$P_\sigma \xrightarrow{\pi_\sigma} \Delta^n.$$

In particular, if ∂_i ans s_j are the face and degeneracy operators in Sing M (cf. 12.13) then the identifications (19.13) yield commutative squares of continuous maps

$$(19.15)\qquad \begin{array}{ccc} P_{\partial_i \sigma} & \xrightarrow{\Delta_i} & P_\sigma \\ {\scriptstyle \pi_{\partial_i\sigma}}\downarrow & & \downarrow{\scriptstyle \pi_\sigma} \\ \Delta^{n-1} & \xrightarrow[\eta_i]{} & \Delta^n \end{array} \quad \text{and} \quad \begin{array}{ccc} P_{s_j \sigma} & \xrightarrow{\Sigma_j} & P_\sigma \\ {\scriptstyle \pi_{s_j\sigma}}\downarrow & & \downarrow{\scriptstyle \pi_\sigma} \\ \Delta^{n+1} & \xrightarrow[\zeta_j]{} & \Delta^n \end{array}$$

(η_i and ζ_j are defined in (19.3).) Moreover Δ_i, Σ_j satisfy the equations (12.2).

19.16. *Local system of forms*. Consider a Serre fibration

$$P \xrightarrow{\pi} M.$$

For each $\sigma \in \underline{Sing}_n(M)$ the pullback

$$P_\sigma \xrightarrow{\pi_\sigma} \Delta^n$$

is a Serre fibration over Δ^n. We can thus apply definition (19.8) to define a simplicial subset $\underline{P_\sigma} \subset \underline{Sing}(P)$.

It follows from the diagrams (19.15) that the simplicial maps $\underline{Sing}(P_{\partial_i\sigma}) \to \underline{Sing}(P_\sigma)$ and $\underline{Sing}(P_{s_j\sigma}) \to \underline{Sing}(P_\sigma)$ defined by Δ_i and Σ_j restrict to simplicial maps

$$\underline{P_{\partial_i\sigma}} \longrightarrow \underline{P_\sigma} \quad \text{and} \quad \underline{P_{s_j\sigma}} \longrightarrow \underline{P_\sigma}.$$

Hence they determine c.g.d.a. homomorphisms

$$\partial_i : A(\underline{P_{\partial_i\sigma}}) \leftarrow A(\underline{P_\sigma}) \quad \text{and} \quad s_j : A(\underline{P_{s_j\sigma}}) \leftarrow A(\underline{P_\sigma}).$$

Because Δ_i and Σ_j satisfy (12.2), ∂_i and s_j satisfy (12.3). Hence a local system of c.g.d.a.'s, F, over $\underline{\mathrm{Sing}}(M)$ is defined by

$$F_\sigma = A(\underline{P_\sigma}), \qquad \partial_i, \; s_j \qquad , \qquad \sigma \in \underline{\mathrm{Sing}}(M).$$

It is called the <u>local system of the fibration</u>.

19.17.- Lemma. The local system, F, of a fibration is an extendable local system of differential coefficients.

Proof.- That F is a local system of differential coefficients is an easy consequence of lemma 19.9.

To show it is extendable, fix $\sigma \in \underline{\mathrm{Sing}}_n(M)$. Denote by $\underline{P_{\partial\sigma}}$ the simplicial subset of these τ such that $\pi_\sigma \circ \tau \in \underline{\partial\Delta}^n$. It is easy to identify the homomorphism

$$\underline{\sigma}^* F(\underline{\Delta}^n) \to \underline{\sigma}^* F(\underline{\partial\Delta}^n)$$

with the restriction $A(\underline{P_\sigma}) \to A(\underline{P_{\partial\sigma}})$, -cf. definition 12.19. But the latter is surjective (prop. 13.8).

Q.E.D.

Since P_σ is the pullback of P we have commutative squares

(19.18)
$$\begin{array}{ccc} P_\sigma & \xrightarrow{\;\xi_\sigma\;} & P \\ {\scriptstyle \pi_\sigma}\downarrow & & \downarrow{\scriptstyle \pi} \\ \Delta^n & \xrightarrow[\;\sigma\;]{} & M \end{array} \qquad , \qquad \sigma \in \underline{\mathrm{Sing}}_n(M).$$

Equations (19.13) yield the relations

(19.19)
$$\xi_\sigma \circ \Delta_i = \xi_{\partial_i\sigma} \quad \text{and} \quad \xi_\sigma \circ \Sigma_j = \xi_{s_j\sigma} .$$

Next, denote by

$$a_\sigma : A(P) \to A(P_{\underline{\sigma}})$$

the composite homomorphism $A(P) \xrightarrow{A(\xi_\sigma)} A(P_\sigma) \to A(P_{\underline{\sigma}})$. From (19.19) we have

$$\text{(19.20)} \qquad \partial_i \circ a_\sigma = a_{\partial_i\sigma} \quad \text{and} \quad s_j \circ a_\sigma = a_{s_j\sigma} \quad , \quad \sigma \in \underline{\text{Sing}}(M).$$

These relations show that a c.g.d.a. homomorphism

$$a : A(P) \to F(M)$$

is defined by $(a\Phi)_\sigma = a_\sigma\Phi$, $\sigma \in \underline{\text{Sing}}(M)$. ($F(M)$ is the c.g.d.a. of global sections of the local system F.)

19.21.- Lemma. The homomorphism, a, is an isomorphism of c.g.d.a.'s

$$a : A(P) \xrightarrow{\cong} F(M).$$

Proof.- If $\sigma \in \underline{\text{Sing}}_n(M)$ and $\alpha \in \text{Ord}([m],[n])$ then $\Delta(\alpha) : \Delta^m \to \Delta^n$ is a continuous map, and we obtain the commutative diagram

$$\begin{array}{ccccc} \Delta(\alpha)^*(P_\sigma) & \xrightarrow{\lambda_\alpha} & P_\sigma & \xrightarrow{\xi_\sigma} & P \\ \downarrow & & \downarrow{\scriptstyle \pi_\sigma} & & \downarrow{\scriptstyle \pi} \\ \Delta^m & \xrightarrow[\Delta(\alpha)]{} & \Delta^n & \xrightarrow[\sigma]{} & M \end{array}$$

If we use (19.13) to identify $\Delta(\alpha)^*(P_\sigma)$ with $P_{\sigma\circ\Delta(\alpha)}$ then

$$\xi_\sigma \circ \lambda_\alpha = \xi_{\sigma\circ\Delta(\alpha)}.$$

(This generalizes (19.19).)

Moreover the simplicial map $\underline{\text{Sing}}(P_{\sigma\circ\Delta(\alpha)}) \to \underline{\text{Sing}}(P_\sigma)$ induced by λ_α restricts to a simplicial map

$$\underline{\lambda_\alpha} : \underline{P_{\sigma o \Delta(\alpha)}} \to \underline{P_\sigma}$$

and hence determines a homomorphism

$$A(\underline{\lambda_\alpha}) : A(\underline{P_{\sigma o \Delta(\alpha)}}) \to A(\underline{P_\sigma}).$$

(Note that ∂_i and s_j are special cases !) Notice that $\underline{\lambda_\alpha}$ is given by

$$\underline{\lambda_\alpha}(\tau) = \lambda_\alpha \circ \tau \qquad \tau \in \underline{P_{\sigma o \Delta(\alpha)}}.$$

Furthermore if $\Omega \in F(M)$ it is coherent with respect to all the $A(\underline{\lambda_\alpha})$; i.e.

(19.22) $$A(\underline{\lambda_\alpha})\Omega_\sigma = \Omega_{\sigma o \Delta(\alpha)} \qquad \sigma \in \underline{\mathrm{Sing}_n}(M), \quad \alpha \in \mathrm{Ord}([m],[n]).$$

Now consider a singular simplex $\omega : \Delta^n \to P$.

It determines the singular simplices

$$\sigma_\omega : \Delta^n \to M \quad \text{and} \quad \tau_\omega : \Delta^n \to P_{\sigma_\omega}$$

given by

$$\sigma_\omega(z) = \pi\omega(z) \quad \text{and} \quad \tau_\omega(z) = (z,\omega(z)) , \qquad z \in \Delta^n.$$

Observe that τ_ω satisfies (and is determined by) the relations

(19.23) $$\pi_{\sigma_\omega} \circ \tau_\omega = \iota \quad \text{and} \quad \xi_{\sigma_\omega} \circ \tau_\omega = \omega .$$

Hence $\tau_\omega \in \underline{P_{\sigma_\omega}}$. Moreover, for $\omega \in \underline{\mathrm{Sing}_n}(P)$ and $\alpha \in \mathrm{Ord}([m],[n])$,

$$\sigma_\omega \circ \Delta(\alpha) = \sigma_{\omega o \Delta(\alpha)}$$

(19.24) and

$$\tau_\omega \circ \Delta(\alpha) = \lambda_\alpha \circ \tau_{\omega o \Delta(\alpha)} = \underline{\lambda_\alpha}(\tau_{\omega o \Delta(\alpha)}) ,$$

as is easily verified. These equations specialize to

$$\partial_i \sigma_\omega = \sigma_{\partial_i \omega} \quad , \quad s_j \sigma_\omega = \sigma_{s_j \omega}$$

(19.25) and

$$\partial_i \tau_\omega = \Delta_i \circ \tau_{\partial_i \omega} \quad , \quad s_j \tau_\omega = \Sigma_j \circ \tau_{s_j \omega} .$$

From (19.23) we deduce (for $\Phi \in A(P)$, $\omega \in \underline{\text{Sing}}(P)$) that

$$(a\Phi)_{\sigma_\omega}(\tau_\omega) = \left[A(\xi_{\sigma_\omega})\Phi\right](\tau_\omega) = \Phi(\xi_{\sigma_\omega} \circ \tau_\omega) = \Phi_\omega \quad ;$$

i.e.,

$$(a\Phi)_{\sigma_\omega}(\tau_\omega) = \Phi_\omega \quad , \quad \omega \in \underline{\text{Sing}}(P).$$

Hence a is injective.

On the other hand, let $\Omega \in F(M)$. For each $\omega \in \underline{\text{Sing}}_n(P)$ define $\Phi_\omega \in A(n)$ by

$$\Phi_\omega = \Omega_{\sigma_\omega}(\tau_\omega)$$

It follows from (19.25) that the Φ_ω are compatible with the face and degeneracy operators, and so define an element $\Phi \in A(P)$.

Now if $\tau \in \underline{P_\sigma}$, some $\sigma \in \underline{\text{Sing}}_n(M)$ then $\pi_\sigma \circ \tau = \Delta(\alpha)$, some $\alpha \in \text{Ord}([m],[n])$ and we can define $\tau' \in \underline{\text{Sing}}_n(P_{\sigma\circ\Delta(\alpha)})$ by the equations

$$\pi_{\sigma\circ\Delta(\alpha)}\tau' = \iota \quad \text{and} \quad \lambda_\alpha \circ \tau' = \tau.$$

Let $\omega = \xi_\sigma \circ \tau = \xi_{\sigma\circ\Delta(\alpha)} \circ \tau'$. Then (cf. (19.23)) $\tau' = \tau_\omega$ and

$$\sigma_\omega = \sigma \circ \Delta(\alpha) \quad , \quad \tau = \lambda_\alpha \circ \tau_\omega .$$

Equations (19.22) yield

$$\Omega_\sigma(\tau) = \Omega_{\sigma_\omega}(\tau_\omega) = \Phi_\omega .$$

But by definition

$$(a\Phi)_\sigma(\tau) = \left[A(\xi_\sigma)\Phi\right](\tau) = \Phi(\xi_\sigma o\tau) = \Phi_\omega.$$

Hence $a\Phi = \Omega$ and a is surjective.

Q.E.D.

Chapter 20

The fundamental theorem.

20.1.- Introduction.- Suppose

$$P \xrightarrow{\pi} M$$

is a Serre fibration in which M is a path connected space with base point m. We adopt the notation of chap. 19. The diagrams (19.15) show that

$$\{H(P_\sigma;k) \; ; \; \Delta_i^* \; ; \; \Sigma_j^*\}_{\sigma\in \underline{\text{Sing}}M}$$

is a local system over $\underline{\text{Sing}}\, M$. Integration (cf. theorem 14.18) identifies this with the local system

$$\{H(A(P_\sigma)) \; ; \; A(\Delta_i)^* \; ; \; A(\Sigma_j)^*\}.$$

Thus by lemma 19.9 it is identified with the local system

$$\{H(A(\underline{P_\sigma})) \; ; \; \partial_i \; ; \; s_j\}$$

and is in particular a local system of coefficients. (cf. also lemma 19.17 1)

Since M is path connected $H(P_m;k)$ becomes a $\pi_1(M)$-module (cf. 16.5) and the identifications

$$H(P_m;k) = H(A(P_m)) = H(A(\underline{P_m}))$$

are isomorphisms of $\pi_1(M)$-modules.

Next, let $j : P_m \to P$ be the inclusion. Then $\pi \circ j$ is the constant map $P_m \to m$ and so

$$A(j) \circ A(\pi) = e_m : A(M) \to A(P_m) \, ,$$

where e_m is the augmentation of $A(M)$ at m. It is given explicitly by

$$e_m \Phi = \Phi_m \quad , \ \Phi \in A(M).$$

Finally, assume P is also path connected. Then $H^o(A(M)) = H^o(A(P)) = k$ and so by theorem 6.1 and theorem 6.2 there is a unique minimal model :

$$\begin{array}{ccccc} & & A(P) & & \\ & \overset{A(\pi)}{\nearrow} & \uparrow \beta & & \\ E : A(M) & \xrightarrow[i]{} & R & \xrightarrow[\rho]{} & T \end{array}$$

for $A(\pi)$; the row denotes a minimal KS extension. Because $\ker \rho$ is the ideal generated by $i\,(\ker e_m)$ we can complete this diagram to a commutative diagram

$$\begin{array}{ccccc} A(M) & \xrightarrow{A(\pi)} & A(P) & \xrightarrow{A(j)} & A(P_m) \\ \| & & \uparrow \beta & & \uparrow \alpha \\ A(M) & \xrightarrow[i]{} & R & \xrightarrow[\rho]{} & T \end{array} \tag{20.2}$$

in which α is a homomorphism of c.g.d.a.'s.

The main theorem of these notes reads

20.3.- <u>*Theorem*</u>.- With the notation and hypotheses above suppose that

i) P_m is path connected.

ii) $H(P_m;k)$ is a nilpotent $\pi_1(M)$-module.

iii) Either $H(P_m;k)$ or $H(M;k)$ has finite type.

Then $\alpha^* : H(T) \to H(A(P_m))$ is an isomorphism, and so (T,α) is the minimal model for P_m.

Proof : Let $\sigma \in \underline{\mathrm{Sing}}_n(M)$. Then $\pi_\sigma : P_\sigma \to \Delta^n$ defines a simplicial map $\underline{\mathrm{Sing}}(P_\sigma) \to \underline{\mathrm{Sing}}(\Delta^n)$ which by definition (cf. 19.8) restricts to a simplicial map

$$\underline{\pi_\sigma} : \underline{P_\sigma} \to \underline{\Delta}^n .$$

Since (12.26 a)) $A(\underline{\Delta}^n) = A(n)$, the resulting homomorphism of differential forms is a homomorphism

$$A(\underline{P_\sigma}) \xleftarrow{A(\underline{\pi_\sigma})} A(n) .$$

Now denote by

$$e_\sigma : A(M) \to A(n)$$

the homomorphism $\Phi \mapsto \Phi_\sigma$. Then a computation shows that the diagram

$$\text{(20.4)} \qquad \begin{array}{ccc} A(\underline{P_\sigma}) & \xleftarrow{a_\sigma} & A(P) \\ {\scriptstyle A(\underline{\pi_\sigma})}\uparrow & & \uparrow{\scriptstyle A(\pi)} \\ A(n) & \xleftarrow[e_\sigma]{} & A(M) \end{array}$$

commutes.

Next, recall the local system $R_* = \{R_\sigma ; \partial_i ; s_j\}$ determined by E, as defined in 18.3. Since $R_\sigma = A(n) \otimes_{A(M)} R$, the diagrams (20.4) show that homomorphisms of c.g.d.a.'s

$$\psi_\sigma : R_\sigma \longrightarrow A(\underline{P_\sigma}), \qquad \sigma \in \underline{\mathrm{Sing}}(M),$$

are defined by

$$\psi_\sigma = A(\underline{\pi_\sigma}) \otimes (a_\sigma \circ \beta).$$

As in 19.16 denote the local system $\{A(\underline{P_\sigma})\}$ by F. Relations (19.20) imply that the ψ_σ are compatible with the face and degeneracy operators. Hence they define a morphism

$$\varphi_* : R_* \to F$$

of local systems. We wish to apply theorem 17.13 to this morphism.

First observe that since P_m is path connected the homomorphism $\pi^* : H^1(M) \to H^1(P)$ is injective. This is therefore also true for $A(\pi)^*$, and it follows by cor. 3.9 that T is connected.

Since T is connected we can apply the results of 18.18. In particular (cf. prop. 18.21) R_* is expressed as the direct limit of a directed family R_*^γ of local systems over $\underline{\mathrm{Sing}}_* M$. We now verify the hypotheses of theorem 17.13 (with R_* playing the role of G) :

i) F is an extendable local system of differential coefficients by lemma 19.17. R_* and R_*^γ are extendable local systems of differential coefficients (prop. 18.21 i)).

ii) Note that $F_m = A(\underline{P_m})$ and so $H(F_m) = H(A(\underline{P_m}))$ is a nilpotent $\pi_1(M)$-module by hypothesis.

iii) Each $H(R_m^\gamma)$ is a nilpotent $\pi_1(M)$-module of finite type (prop. 18.21 ii)).

iv) Either $H(M;k)$ or $H(F_m)$ ($\cong H(P_m;k)$) has finite type by hypothesis.

v) Since $\{H(F_\sigma)\}$ is a local system of coefficients, and P_m is path connected, clearly $H^0(F_\sigma) = H^0(P_m) = k$ and these isomorphisms are compatible with the face and degeneracy operators. The same is true for R_* and R_*^γ because T is connected.

vi) By prop. 18.21 iv) the composite $H(\varinjlim R^\gamma(M)) \to H(F(M))$ can be identified with the composite

$$H(R) \xrightarrow{\mu^*} H(R_*(M)) \xrightarrow{\varphi^*} H(F(M)).$$

By definition, for $z \in R$

$$\begin{aligned} \psi\mu(z)_\sigma &= \psi_\sigma \mu_\sigma(z) \\ &= \left[A(\underline{\pi_\sigma}) \otimes (a_\sigma \circ \beta)\right](1 \otimes z) \\ &= a_\sigma(\beta z) = \left[a\beta(z)\right]_\sigma . \end{aligned}$$

Hence $\psi \circ \mu = a \circ \beta$.

Now a is an isomorphism (lemma 19.21) and β^* is an isomorphism by hypothesis. Hence $\psi^*\mu^*$ is an isomorphism.

We have now verified all the hypotheses of theorem 17.13. We may thus conclude that

$$\psi_m^* : H(R_m) \xrightarrow{\cong} H(A(P_{\underline{m}})).$$

But

$$R_m = T \quad (\text{prop } 18.9), \quad A(P_{\underline{m}}) = A(P_m) \quad (\text{definition})$$

and, clearly, $\psi_m = \alpha$. This completes the proof.

Q.E.D.

20.5.- *Theorem*. Let M be a path connected space with base point m and suppose $P \xrightarrow{\pi} M$ is a Serre fibration satisfying the hypotheses of theorem 20.3. Assume that

$$\begin{array}{ccccc} A(M) & \xrightarrow{A(\pi)} & A(P) & \xrightarrow{A(j)} & A(P_m) \\ \Vert & & \uparrow \beta_1 & & \uparrow \alpha_1 \\ A(M) & \xrightarrow[i_1]{} & R_1 & \xrightarrow[\rho_1]{} & T_1 \end{array}$$

is a commutative diagram of c.g.d.a.'s, in which the bottom row is a KS extension.

Suppose also that $\alpha_1^* : H(T_1) \xrightarrow{\cong} H(A(P_m))$. Then $\beta_1^* : H(R_1) \xrightarrow{\cong} H(A(P))$.

Proof.- Combine (20.2) with the diagram above to achieve the commutative diagram

$$\begin{array}{ccccc} & \nearrow^{i} & R & \xrightarrow{\rho} & T \\ & & \downarrow{\scriptstyle\beta} & & \downarrow{\scriptstyle\alpha} \\ A(M) & \xrightarrow[A(\pi)]{} & A(P) & \xrightarrow[A(j)]{} & A(P_m) \\ & & \uparrow{\scriptstyle\beta_1} & & \uparrow{\scriptstyle\alpha_1} \\ & \searrow_{i_1} & R_1 & \xrightarrow[\rho_1]{} & T_1 \end{array} .$$

By construction β^* is an isomorphism.

Hence we can apply theorem 5.19 to obtain a c.g.d.a. homomorphism

$$\bar{\beta} : R_1 \to R$$

such that $\bar{\beta} \circ i_1 = i$ and $\beta \circ \bar{\beta} \sim \beta$ (rel $A(M)$).

The first condition shows that $\bar{\beta}\, i_1 (\ker e_m) = i(\ker e_m)$ and so $\bar{\beta} (\ker \rho_1) \subset \ker \rho$. Thus $\bar{\beta}$ factors to yield the commutative diagram

$$\begin{array}{ccccc} A(M) & \xrightarrow{i} & R & \xrightarrow{\rho} & T \\ \Vert & & \uparrow{\scriptstyle\bar{\beta}} & & \uparrow{\scriptstyle\bar{\alpha}} \\ A(M) & \xrightarrow[i_1]{} & R_1 & \xrightarrow[\rho_1]{} & T_1 \end{array} .$$

By changing the augmentation in T_1 (if necessary) we can arrange that $\bar{\alpha}$ preserves augmentations. Hence so does $\bar{\beta}$: $(\iota, \bar{\beta}, \bar{\alpha})$ is a morphism of KS extensions.

Now (cf. definition 5.8) the homotopy from $\beta \circ \bar{\beta}$ to β_1 is a homomorphism $\Phi : (R_1, A(M))^I \to A(P)$ such that

$$\Phi \circ \lambda_o = \beta \circ \bar{\beta}, \qquad \Phi \circ \lambda_1 = \beta_1, \qquad \Phi \circ i_1 = A(\pi).$$

In 5.1 is defined a projection $(R_1, A(M))^I \to T_1^I$. The equations above show that $A(j) \circ \Phi$ factors over this projection to yield a homotopy

$$T_1^I \longrightarrow A(P_m)$$

from $\alpha \circ \bar{\alpha}$ to α_1.

It follows that $\alpha^* \circ \bar{\alpha}^* = \alpha_1^*$. Since α_1^* is an isomorphism by hypothesis, and α^* is (by theorem 20.3) so is $\bar{\alpha}^*$. Now theorem 7.1 implies that $\bar{\beta}^*$ is an isomorphism.

Since $\beta_1 \sim \beta \circ \bar{\beta}$ and β^* and $\bar{\beta}^*$ are isomorphisms, $\beta_1^* = \beta^* \circ \bar{\beta}^*$ is also an isomorphism.

Q.E.D.

20.6.- Pullbacks.- Consider a commutative square of continuous maps

$$\begin{array}{ccc} \psi^* P & \longrightarrow & P \\ \pi_1 \Big\downarrow & & \Big\downarrow \pi \\ M_1 & \xrightarrow{\ \psi\ } & M \end{array} \tag{20.7}$$

in which π is a Serre fibration and π_1 is the pullback via ψ.

Choose basepoints $m_1 \in M_1$ and $m \in M$ so that $\psi\, m_1 = m$; then

$$(\psi^* P)_{m_1} = P_m.$$

Now suppose that

i) M and M_1 are path connected.

ii) $\pi_1(M)$ acts nilpotently on $H(P_m;k)$

iii) Either $H(P_m;k)$ or both $H(M;k)$ and $H(M_1;k)$ have finite type.

Let the model of $A(\pi)$ be denoted by

$$(20.8)\qquad \begin{array}{ccccc} A(M) & \longrightarrow & A(P) & \longrightarrow & A(P_m) \\ \uparrow = & & \uparrow \beta & & \uparrow \alpha \\ \mathcal{E} : A(M) & \longrightarrow & R & \longrightarrow & T \end{array} \quad ;$$

then β^* is an isomorphism by construction and α^* is an isomorphism by theorem 20.3.

On the other hand $A(\psi) : A(M) \to A(M_1)$ is a c.g.d.a. homomorphism which preserves augmentations (because $\psi m_1 = m$).

Thus we can use $A(\psi)$ to define the c.g.d.a.

$$R_1 = A(M_1) \otimes_{A(M)} R.$$

Diagrams (20.7) and (20.8) produce in the obvious way the diagram

$$\begin{array}{ccccc} A(M_1) & \longrightarrow & A(\psi^*P) & \longrightarrow & A((\psi^*P)_{m_1}) \\ \uparrow = & & \uparrow \beta_1 & & \uparrow \alpha_1 \\ A(M_1) & \longrightarrow & R_1 & \longrightarrow & T \end{array}$$

in which the bottom row is a KS extension. Moreover the homeomorphism $P_m \overset{\sim}{=} (\psi^*P)_{m_1}$ gives an isomorphism $A(P_m) \overset{\sim}{=} A((\psi^*P)_{m_1})$ which identifies α and α_1. Hence α_1^* is an isomorphism.

New theorem 20.5 applies and shows that β_1^* is an isomorphism :

$$(20.9)\qquad \beta_1^* : H(A(M_1) \otimes_{A(M)} R) \xrightarrow{\cong} H(A(\psi^*P)).$$

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