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AN APPLICATION OF RAMANUJAN GRAPHS
TO C*-ALGEBRA TENSOR PRODUCTS, II

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Let $A, B$ be C*-algebras; denote by $A \otimes B$ the algebraic tensor product of $A$ and $B$. According to the general theory of tensor products of C*-algebras, C*-norms on $A \otimes B$ lie between a minimal C*-norm, denoted by $\| \cdot \|_{\text{min}}$, and a maximal C*-norm, denoted by $\| \cdot \|_{\text{max}}$ (see [Tak79] for all this). Denote by $H$ an infinite-dimensional separable Hilbert space, and by $B(H)$ the C*-algebra of linear, bounded operators on $H$. It was an old problem to decide whether or not all C*-norms coincide on $B(H) \otimes B(H)$; in a remarkable paper [JP95], M. Junge et G. Pisier give a negative answer to that question. More precisely, to describe quantitatively the discrepancy between the maximal and minimal C*-norms on $B(H) \otimes B(H)$, they introduce for any $n \in \mathbb{N}$ the number

$$\lambda(n) = \sup \{ \frac{\|u\|_{\text{max}}}{\|u\|_{\text{min}}} : u \text{ tensor of rank } n \text{ in } B(H) \otimes B(H) \}$$

and they prove:

**Theorem 1.** There exists a constant $c > 0$ such that, for any $n \in \mathbb{N}$:

$$c \sqrt{n} \leq \lambda(n) \leq \sqrt{n}.$$ 

In [JP95], Junge and Pisier ask for the precise asymptotic behaviour of $\lambda(n)$ for $n \to \infty$. Actually they provide an excellent way of getting lower bounds on $\lambda(n)$ by introducing another constant $C_n$ as follows:

**Definition 1.** Let $F_n$ denote the free group on $n$ generators $a_1, a_2, \ldots, a_n$. Let $C_n$ denote the infimum of all numbers $C > 0$ for which there exists a sequence $(\pi_k)_{k \geq 1}$ of unitary, finite-dimensional representations of $F_n$ such that:

$$\| \sum_{i=1}^{n} (\pi_k \otimes \pi_m)(a_i) \| \leq C \text{ for all } k \neq m.$$
(where $\pi_m$ denotes the contragredient representation of $\pi_m$).

It is then proved in [JP95] that the constants $\lambda(n)$ and $C_n$ are related through the following inequality:

$$\frac{n}{C_n} \leq \lambda(n) \quad (*)$$

In [Pis], Pisier proved:

**PROPOSITION 1.** — For any $n \geq 2$, one has

$$C_n \geq 2\sqrt{n-1}.$$  

On the other hand, using Ramanujan graphs (see [Lub94]), I proved in [Val]:

**PROPOSITION 2.** — Let $q$ be a prime power. Then

$$C_{q+1} \leq 2\sqrt{q}.$$  

From these two results, it is natural to conjecture that $C_n = 2\sqrt{n-1}$ for any $n \geq 2$. I shall confirm this conjecture by proving that it holds asymptotically. I am grateful to U. Haagerup and G. Skandalis for suggesting to me the possibility of such a proof.

**PROPOSITION 3.**

$$\lim_{n \to \infty} \frac{C_n}{2\sqrt{n}} = 1$$

**Proof:** It follows from Proposition 3 in [Val] that the sequence $(\frac{C_n}{2\sqrt{n}})_{n \geq 1}$ is bounded, so let $K$ be an upper bound for that sequence. It is also known that the function $n \to C_n$ is sub-additive (see the lemma in [Val]). Denote by $p_k$ the $k$-th prime. For fixed $n$, let $k$ be such that $p_k + 1 \leq n < p_{k+1}$.

Then:

$$C_n \leq C_{p_k+1} + C_{n-p_k-1}$$

Using Proposition 2, we get:

$$C_n \leq 2\sqrt{p_k} + 2K \sqrt{n-p_k-1} \leq 2\sqrt{p_k} + 2K \sqrt{p_{k+1} - p_k}$$

Hence:

$$\frac{C_n}{2\sqrt{n}} \leq \frac{C_n}{2\sqrt{p_k}} \leq 1 + K \sqrt{\frac{p_{k+1}}{p_k}} - 1.$$  

By the prime number theorem of Hadamard and de la Vallée-Poussin, the ratio $\frac{p_{k+1}}{p_k}$ tends to 1 for $k \to \infty$. Thus:

$$\limsup_{n \to \infty} \frac{C_n}{2\sqrt{n}} \leq 1.$$
On the other hand, it follows from Proposition 1 that we also have
\[
\liminf_{n \to \infty} \frac{C_n}{2\sqrt{n}} \geq 1,
\]
so that the result is proved.

From Proposition 3 and the inequality (*), it immediately follows that:

**Corollary 1.**

\[
\liminf_{n \to \infty} \frac{\lambda(n)}{\sqrt{n}} \geq \frac{1}{2}.
\]

**References**


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