Milnor-Wood inequality for crystallographic groups


MILNOR-WOOD INEQUALITY FOR CRYSTALLOGRAPHIC GROUPS

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0. Introduction

Let $H^2$ be the hyperbolic plane and $\text{Isom}^+ H^2$ the isometry group of $H^2$. A 2-dimensional crystallographic group $\Gamma$ is a cocompact discrete subgroup of $\text{Isom}^+ H^2$. As an abstract group, $\Gamma$ is isomorphic to a unique group of the form

$$\Gamma(g; p_1, \ldots, p_n) = \langle a_1, b_1, \ldots, a_n, b_n, c_1, \ldots, c_n | c_i^{p_i} = 1 \ (i = 1, \ldots, n), c_1 \cdots c_n[a_1, b_1] \cdots [a_n, b_n] = 1 \rangle$$

with $g \geq 0, p_i \geq 2$ and $\chi(\Gamma(g; p_1, \ldots, p_n)) < 0$. Here $\chi(\Gamma(g; p_1, \ldots, p_n)) = 2 - 2g - \sum_{i=1}^{n} (p_i - 1)/p_i$ is the rational Euler characteristic of the group $\Gamma(g; p_1, \ldots, p_n)$.

Let $G^r$ be the group of all orientation preserving diffeomorphisms of class $C^r (r = 0, 1, \ldots, \infty)$. For any homomorphism $\phi : \Gamma \rightarrow G^r$, $\Gamma$ acts on the trivial $S^1$ bundle $H^2 \times S^1$ through $\phi$. So we can construct a foliated Seifert bundle $E_\phi = H^2 \times S^1/\Gamma \rightarrow H^2/\Gamma = \Sigma_g (g = \text{genus of } \Gamma)$. We define the Euler number $eu(\phi)$ of $\phi$ by

$$eu(\phi) = \begin{cases} 
\text{the Euler number of Seifert bundle } E_\phi \rightarrow \Sigma_g \\
= eu(E_\phi \rightarrow \Sigma_g).
\end{cases}$$

If $\Gamma$ is a surface group, then we have the Milnor-Wood inequality

$$|eu(\phi)| \leq |\chi(\Sigma)| = |\chi(\Gamma)|.$$  

Moreover, if $\phi_i : \Gamma \rightarrow G^0 (i = 1, 2)$ both have the maximal Euler number $eu(\phi_1) = eu(\phi_2) = \pm \chi(\Gamma)$, then $\phi_1$ is semi-conjugate to $\phi_2$.

In this paper, we shall consider a generalization of the Milnor-Wood inequality for homomorphisms from crystallographic groups to $G^0$, and we also prove that there exists a semi-conjugacy phenomenon in the case that the homomorphism has the maximal Euler number.

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1. Homological definition of Euler number

In this section, we give a homological definition of Euler number $eu(\phi)$ first. Let $\Gamma = \Gamma(g; p_1, \cdots, p_n)$ be a crystallographic group. $\Gamma$ contains a finite index subgroup $\Gamma_{g'}$ which is isomorphic to the fundamental group of a closed surface $\Sigma_{g'}$. So we have that the inclusion $i : \Gamma_{g'} \to \Gamma$ induces an isomorphism

$$i_* : H_2(\Gamma_{g'}; Q) \to H_2(\Gamma; Q).$$

Since given presentation of $\Gamma_{g'}$ determines an orientation of the closed surface $\Sigma_{g'}$, then there exists the fundamental class $[\Gamma_{g'}] \in H_2(\Gamma_{g'}; Z) \cong H_2(\Sigma_{g'})$. We use the notation $[\Gamma_{g'}]_Q$ which is the image of $[\Gamma_{g'}]$ by Bockstein homomorphism

$$H_2(\Gamma_{g'}; Z) \to H_2(\Gamma_{g'}; Q).$$

Now we define the fundamental class $[\Gamma]$ of $\Gamma$ by

$$[\Gamma] = i_*[\Gamma_{g'}]_Q / \text{index}(\Gamma; \Gamma_{g'}).$$

We can check easily that this definition does not depend on the choice of the finite index subgroup $\Sigma_{g'}$.

2. Cohomological definition of the Euler number

Given a surface group $\Gamma_g$ and a homomorphism $\phi : \Gamma_g \to G^0$, Euler number $eu(\phi)$ is equal to the pairing

$$eu(\phi) = \langle \phi^*e, [\Gamma_g] \rangle.$$

Here, $e \in H^2(G^0; Z)$ denotes the universal Euler class. The symbol $e_Q$ denotes the rational universal Euler class which is the image of $e$ by Bockstein homomorphism $H^2(G^0; Z) \to H^2(G^0; Q)$.

**Proposition 2.1.** — For any homomorphism $\phi : \Gamma \to G^0$, we have the formula

$$eu(\phi) = \langle \phi^*e_Q, [\Gamma] \rangle.$$

In order to prove this proposition, we need the following lemma.

**Lemma 2.2.** — Let $\pi_i : M_i \to \Sigma_i (i = 1, 2)$ be Seifert fibrations. Assume that there exist maps $\tilde{h} : M_1 \to M_2$ and $h : \Sigma_1 \to \Sigma_2$ such that $\pi_2 \circ \tilde{h} = h \circ \pi_1$, degree($h$) = $k$ and degree($\tilde{h}|_{\text{regular fiber}}$) = $l$. Then we have $e(M_1 \to \Sigma_1) = (k/l)eu(M_2 \to \Sigma_2)$.

**Proof of Proposition 2.1** We take a finite index subgroup $\Gamma_{g'}$ of $\Gamma$ which is isomorphic to $\pi_1(\Sigma_{g'})$. We put that $k = \text{index}(\Gamma; \Gamma_{g'})$. So there exist continuous maps...
Theorem 2.3. — Let $\Gamma$ be a crystallographic group. For any homomorphism $\phi: \Gamma \to G^0$, we have the following inequality

$$|eu(\phi)| \leq |\chi(\Gamma)|.$$  

Proof. We use the same notations as in the proof of Proposition 2.1. Then we have

$$|eu(\phi)| = |eu(\phi \circ i)/k| \leq |\chi(\Gamma)|/k = |\chi(\Gamma)|.$$

The last equality follows from the definition of the rational Euler characteristic $\chi(\Gamma)$ (see [8]). \qed

3. Semi-conjugacy in maximal Euler numbers

Let $\Gamma$ be a crystallographic group and $T_1 H^2$ a unit tangent bundle of the hyperbolic plane $H^2$. $\Gamma$ acts on $T_1 H^2$, since $\Gamma$ acts on $H^2$ isometrically. So we can construct a Seifert bundle $E(\Gamma) = T_1 H^2/\Gamma \to H^2/\Gamma = \Sigma_\rho$ whose total holonomy homomorphism is the identity map $\phi_T: \Gamma \to \Gamma \subset PSL(2, R)$. We know that $eu(\phi_T) = \chi(\Gamma)$. The following theorem is a generalization of a theorem of S. Matsumoto to crystallographic groups. In [6], he proved this theorem for surface groups.

Theorem 3.1. — Let $\Gamma$ be as above. For given homomorphism $\phi: \Gamma \to G^0$, there exist a continuous degree one map $h: S^1 \to S^1$ such that

$$\phi_T(\gamma) \circ h = h \circ \phi(\gamma)$$

for any $\gamma \in \Gamma$.

By [5], it suffices to show that

$$\rho(\phi_T(\gamma)) = \rho(\phi(\gamma))$$
for any $\gamma \in A$ which is a generating system of $\Gamma$. Here $\rho(f) \in S^1$ is rotation number of $f \in G^0$. In order to show this, we need the following formula which is called Milnor's algorism.

**Lemma 3.2.** — For any homomorphism $\phi : \Gamma(g; p_1, \ldots, p_n) \to G^0$ we can calculate the Euler number $\text{eu}(\phi)$ as follows. We choose any lifts $\phi(a_1), \phi(b_1), \phi(c_1) \in G^0$. Then, the number

$$\rho([\phi(a_1), \phi(b_1)] \circ \cdots \circ [\phi(a_g), \phi(b_g)] \circ \phi(c_1) \circ \cdots \circ \phi(c_n)) + \sum_{i=1}^{n} \rho(\phi(c_i))$$

does not depend on the choice of lifts. This number is equal to $\text{eu}(\phi)$.

Where, $\rho(\tilde{f})$ is the translation number of $\tilde{f}$. We can prove the following lemma by Lemma 3.2 with [1], [4] and [7].

**Lemma 3.3.** — For any homomorphism $\phi : \Gamma(g; p_1, \ldots, p_n) \to G^0$, we have that

$$\rho(\phi(\gamma)) = \begin{cases} 0 & \text{if } \gamma = a_1, \ldots, a_g b_1, \ldots, b_g \\ [1/p_i] & \text{if } \gamma = c_i(i = 1, \ldots, n) \end{cases}$$

if $\text{eu}(\phi) = \chi(\Gamma)$.

**References**


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