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Milnor-Wood inequality for crystallographic groups


0. Introduction

Let $H^2$ be the hyperbolic plane and $\text{Isom}^+ H^2$ the isometry group of $H^2$. A 2-dimensional crystallographic group $\Gamma$ is a cocompact discrete subgroup of $\text{Isom}^+ H^2$. As an abstract group, $\Gamma$ is isomorphic to a unique group of the form

$$\Gamma(g; p_1, \ldots, p_n) = \langle a_1, b_1, \ldots, a_n, b_n, c_1, \ldots, c_n | c_i^{p_i} = 1 \ (i = 1, \ldots, n), \ c_1 \cdots c_n[a_1, b_1] \cdots [a_n, b_n] = 1 \rangle$$

with $g \geq 0, p_i \geq 2$ and $\chi(\Gamma(g; p_1, \ldots, p_n)) < 0$. Here $\chi(\Gamma(g; p_1, \ldots, p_n)) = 2 - 2g - \sum_{i=1}^{n} (p_i - 1)/p_i$ is the rational Euler characteristic of the group $\Gamma(g; p_1, \ldots, p_n)$.

Let $G^r$ be the group of all orientation preserving diffeomorphisms of class $C^r (r = 0, 1, \ldots, \infty)$. For any homomorphism $\phi : \Gamma \rightarrow G^r$, $\Gamma$ acts on the trivial $S^1$ bundle $H^2 \times S^1$ through $\phi$. So we can construct a foliated Seifert bundle $E_\phi = H^2 \times S^1 / \Gamma \rightarrow H^2 / \Sigma_g = S^5$ ($g =$ genus of $\Gamma$). We define the Euler number $eu(\phi)$ of $\phi$ by

$$eu(\phi) = \text{the Euler number of Seifert bundle } E_\phi \rightarrow \Sigma_g$$

$$= eu(E_\phi \rightarrow \Sigma_g).$$

If $\Gamma$ is a surface group, then we have the Milnor-Wood inequality

$$|eu(\phi)| \leq |\chi(\Sigma)| = |\chi(\Gamma)|.$$ 

Moreover, if $\phi_i : \Gamma \rightarrow G^0 (i = 1, 2)$ both have the maximal Euler number $eu(\phi_1) = eu(\phi_2) = \pm \chi(\Gamma)$, then $\phi_1$ is semi-conjugate to $\phi_2$.

In this paper, we shall consider a generalization of the Milnor-Wood inequality for homomorphisms from crystallographic groups to $G^0$, and we also prove that there exists a semi-conjugacy phenomenon in the case that the homomorphism has the maximal Euler number.

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1. Homological definition of Euler number

In this section, we give a homological definition of Euler number $e_u(\phi)$ first. Let $\Gamma = \Gamma(g; p_1, \cdots, p_n)$ be a crystallographic group. $\Gamma$ contains a finite index subgroup $\Gamma'_{g'}$ which is isomorphic to the fundamental group of a closed surface $\Sigma_{g'}$. So we have that the inclusion $i : \Gamma'_{g'} \to \Gamma$ induces an isomorphism

$$i_* : H_2(\Gamma'_{g'}; Q) \to H_2(\Gamma; Q).$$

Since the given presentation of $\Gamma'_{g'}$ determines an orientation of the closed surface $\Sigma_{g'}$, then there exists the fundamental class $[\Gamma'_{g'}] \in H_2(\Gamma'_{g'}; Z) \cong H_2(\Sigma_{g'}).$ We use the notation $[\Gamma'_{g'}]Q$ which is the image of $[\Gamma'_{g'}]$ by Bockstein homomorphism

$$H_2(\Gamma'_{g'}; Z) \to H_2(\Gamma'_{g'}; Q).$$

Now we define the fundamental class $[\Gamma]$ of $\Gamma$ by

$$[\Gamma] = i_*[\Gamma'_{g'}]Q/\text{index}(\Gamma; \Gamma'_{g'}).$$

We can check easily that this definition does not depend on the choice of the finite index subgroup $\Sigma_{g'}$.

2. Cohomological definition of the Euler number

Given a surface group $\Gamma_g$ and a homomorphism $\phi : \Gamma_g \to G^0$, Euler number $e_u(\phi)$ is equal to the pairing

$$e_u(\phi) = \langle \phi^*e, [\Gamma_g] \rangle.$$

Here, $e \in H^2(G^0; Z)$ denotes the universal Euler class. The symbol $e_Q$ denotes the rational universal Euler class which is the image of $e$ by Bockstein homomorphism $H^2(G^0; Z) \to H^2(G^0; Q)$.

**Proposition 2.1.** — For any homomorphism $\phi : \Gamma \to G^0$, we have the formula

$$e_u(\phi) = \langle \phi^*e_Q, [\Gamma] \rangle.$$

In order to prove this proposition, we need the following lemma.

**Lemma 2.2.** — Let $\pi_i : M_i \to \Sigma_i (i = 1, 2)$ be Seifert fibrations. Assume that there exist maps $\hat{h} : M_1 \to M_2$ and $h : \Sigma_1 \to \Sigma_2$ such that $\pi_2 \circ \hat{h} = h \circ \pi_1$, degree($h$) = $k$ and degree($\hat{h}|\text{regular fiber}$) = $l$. Then we have $e(M_1 \to \Sigma_1) = (k/l)e_u(M_2 \to \Sigma_2)$.

**Proof of Proposition 2.1** We take a finite index subgroup $\Gamma'_{g'}$ of $\Gamma$ which is isomorphic to $\pi_1(\Sigma_{g'})$. We put that $k = \text{index}(\Gamma; \Gamma'_{g'})$. So there exist continuous maps
\( h : E_{\phi_{0i}} \to E_\phi \) and \( h : \Sigma_{\phi'} \to \Sigma_{\phi} \) such that \( \pi_\phi \circ h = h \circ \pi_{\phi_{0i}} \), degree(h) = k and degree(\( h|_{\text{regular fiber}} \)) = 1. By using the lemma above, we have

\[
\begin{align*}
eu(\phi) &= \frac{e(\phi \circ i)}{k} \\
&= \frac{e(\phi \circ i)*e_i}{k} \\
&= \frac{e(\phi \circ i)*e_{Q'_i}}{k} \\
&= \frac{e(\phi \circ i)*e_{Q'_i}}{k} \\
&= \frac{(\phi \circ i)*e_Q}{k}.
\end{align*}
\]

The same technique as in the proof of Proposition 2.1 gives us a generalization of the Milnor-Wood inequality for homomorphisms from crystallographic groups to \( G^0 \).

**Theorem 2.3.** — Let \( \Gamma \) be a crystallographic group. For any homomorphism \( \phi : \Gamma \to G^0 \), we have the following inequality

\[ |\text{eu}(\phi)| \leq |\chi(\Gamma)|. \]

**Proof.** We use the same notations as in the proof of Proposition 2.1. Then we have

\[ |\text{eu}(\phi)| = \frac{|\text{eu}(\phi \circ i)|}{k} \leq \frac{|\chi(\Gamma_{\phi'})|}{k} = |\chi(\Gamma)|. \]

The last equality follows from the definition of the rational Euler characteristic \( \chi(\Gamma) \) (see [8]).

\( \Box \)

3. Semi-conjugacy in maximal Euler numbers

Let \( \Gamma \) be a crystallographic group and \( T \to H^2 \) a unit tangent bundle of the hyperbolic plane \( H^2 \). \( \Gamma \) acts on \( T, H^2 \), since \( \Gamma \) acts on \( H^2 \) isometrically. So we can construct a Seifert bundle \( E(\Gamma) = T \to H^2 / \Gamma = \Sigma_{\phi} \) whose total holonomy homomorphism is the identity map \( \phi_T : \Gamma \to \chi(\Gamma) \subset PSL(2, R) \). We know that \( \text{eu}(\phi_T) = \chi(\Gamma) \). The following theorem is a generalization of a theorem of S. Matsumoto to crystallographic groups. In [6], he proved this theorem for surface groups.

**Theorem 3.1.** — Let \( \Gamma \) be as above. For given homomorphism \( \phi : \Gamma \to G^0 \), there exist a continuous degree one map \( h : S^1 \to S^1 \) such that

\[ \phi_T(\gamma) \circ h = h \circ \phi(\gamma) \]

for any \( \gamma \in \Gamma \).

By [5], it suffices to show that

\[ \rho(\phi_T(\gamma)) = \rho(\phi(\gamma)) \]
for any $\gamma \in A$ which is a generating system of $\Gamma$. Here $ho(f) \in S^1$ is rotation number of $f \in G^0$. In order to show this, we need the following formula which is called Milnor's algorithm.

**Lemma 3.2.** — For any homomorphism $\phi : \Gamma(g; p_1, \cdots, p_n) \to G^0$ we can calculate the Euler number $eu(\phi)$ as follows. We choose any lifts $\widetilde{\phi(a_1)}, \widetilde{\phi(b_1)}, \widetilde{\phi(c_1)} \in G^0$. Then, the number

$$
\rho(\widetilde{\phi(a_1)}, \widetilde{\phi(b_1)}) \circ \cdots \circ \widetilde{\phi(a_g)} \circ \widetilde{\phi(b_g)} \circ \widetilde{\phi(c_1)} \circ \cdots \circ \widetilde{\phi(c_n)} + \sum_{i=1}^{n} \rho(\widetilde{\phi(c_i)})
$$

does not depend on the choice of lifts. This number is equal to $eu(\phi)$.

Where, $\rho(f)$ is the translation number of $f$. We can prove the following lemma by Lemma 3.2 with[1],[4] and [7].

**Lemma 3.3.** — For any homomorphism $\phi : \Gamma(g; p_1, \cdots, p_n) \to G^0$, we have that

$$
\rho(\phi(\gamma)) = \begin{cases} 
0 & \text{if } \gamma = a_1, \cdots, a_g b_1, \cdots, b_g \\
[1/p_1] & \text{if } \gamma = c_i (i = 1, \cdots, n)
\end{cases}
$$

if $eu(\phi) = \chi(\Gamma)$.

**References**


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