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Atsushi Katsuda

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Closed orbits of Anosov flows in homology classes

Atsushi KATSUDA

Okayama University
Department of Mathematics
3-1 Tsushima-naka
OKAYAMA 700
JAPON

This note consists of a summary of the joint works with Sunada and an explanation of the present situation of related problems.

1. Results.

Let (X, φ_t) be a smooth, transitive and weakly mixing Anosov flow on a compact manifold X. For each $\alpha \in H_1(X, \mathbb{Z}) = H$ and $x > 0$, let

$$\pi(x, \alpha) \sim \#\{\mathfrak{p} : \text{ prime closed orbits } [\mathfrak{p}] = \alpha, \; l(\mathfrak{p}) < x\},$$

where $[\mathfrak{p}]$ denotes the homology class and $l(\mathfrak{p})$ the least period of $\mathfrak{p}$. Employing an idea in analytic number theory, the Dirichlet density theorem for arithmetic progressions, namely Dirichlet L-functions, we give an asymptotic estimate of $\pi(x, \alpha)$ as x goes to infinity. In contrast with the case of number theory, our "Galois group" H, is possibly of infinite order, so that some extra phenomenon will appear.

We denote by h the topological entropy of the flow and by m a (unique) invariant probability measure on X of maximal entropy. Let Z be the vector field generating the flow.

We define the winding cycle Φ, which is a linear functional on the space of closed one-forms on X, by

$$\Phi(\omega) = \int_X \langle \omega, Z \rangle \; dm.$$

Since Φ (exact forms) $= 0$, the linear functional Φ yields a homology class in $H_1(X, \mathbb{R}) = \mathrm{Hom}\,\big(H^1(X, \mathbb{R}), \mathbb{R}\big)$. The winding cycle Φ can be regarded as the average of the "homological" direction by the following

$$\Phi(\omega) = \lim_{t \longrightarrow \infty} \frac{1}{t} \int_0^t \langle \omega, Z \rangle \; (\varphi_\tau x) d\tau \quad a.e. x.$$

which is derived from the ergodicity of the flow.

THEOREM 1. — *If Φ vanishes on $H^1(X, \mathbb{R})$, then*

$$\pi(x, \alpha) \sim C \frac{e^{hx}}{x^{b/2+1}} \quad \text{as } x \nearrow \infty, \text{ where } b = \operatorname{rank} H$$

THEOREM 2. — *If $\Phi\big(H^1(X, \mathbb{R})\big) \not\equiv 0$, then*

$$\frac{\pi(x, \alpha)}{e^{hx}} = o(x^{-N}) \quad \text{as } x \nearrow \infty \quad \text{for any } N > 0$$

Typical example of Theorem 1 is the geodesic flow (UM, φ_t) of negatively curved manifold M. Theorem 1 is a generalization of Parry-Pollicott, Adachi-Sunada for the case when H is finite and Phillips-Sarnak and ours, for the case of compact Riemann manifold with constant negative curvature.

To prove these results, we examine the singularities of the integral of logarithmic derivative of L-functions over the character group of $H_1(X, \mathbb{Z})$. For details, see [1]. In the case of constant negative curvature, this is closely related to the pertubation of the Laplacian. Although this case is special but the idea is essentially same and technically simple, so see also [2]. Note that the special case in Theorem 1 is also obtained by Pollicott and Lalley independently.

2. Problems or remarks.

(1) For all examples satisfying conditions in Theorem 2 (as far as I know) $\pi(x, \alpha)$ is finite. Can you improve Theorem 2?

(2) In Theorem 1, how to explain geometrically the exponent $b/2$ (not b) in the denominator of right hand side? (*cf.* [2] p.146)

(3) Evidently, it is rather difficult to consider the more precise asymptotic estimate of $\pi(x, \alpha)$. This is related to analytic continuation and estimate of zero free region of zeta or L-functions (see chap. 10 in [4]). There is another formulation as usual, which is called "effective version", see [7] in the case of number theory.

(4) One can consider the other flows. This is a medley of my knowledge. It should be noted that there are cases where the Prime Orbits Theorem (P.O.T. see [4]) is established but not the Dirichlet Density Theorem (D.D.T. = Theorem 1).

a) The geodesic flow on hyperbolic manifold (the sectional curvature $\equiv -1$). In the case where vol $< +\infty$, P.O.T. and D.D.T. hold (C. Epstein). In the case where vol $= +\infty$, P.O.T. for dim $M = 2$ is due to Guillopé and Lalley. P.O.T. for dim $M \geqslant 3$ and D.D.T. are not known.

b) The geodesic flow on compact non positively curved manifold. In the case of rank 1, Knieper obtained weak form. Moreover, Hamenstädt said (in my understanding), if the geodesic flow is mixing, then P.O.T. can be proved by the same line of the argument of Margulis for P.O.T. in the case of negatively curved manifold. D.D.T. is not known. If the rank is greater than 1, we do not know except [8], which is very weak.

c) Billiard system. The simplest case dim $= 2$, without trapping, P.O.T. is obtained by T. Morita [9]. For general survey see [10].

d) Teichmüller geodesic flow. This flow is "measurably Anosov". Only weak form is known. See [11].

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(For further references, see references in [1], [3], [4], [10],...)