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Some aspects of the Laplace operator in negative curvature


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ALLEMAGNE
Let $M$ be a compact Riemannian manifold of negative sectional curvature with universal covering $\tilde{M}$. This covering is diffeomorphic to $\mathbb{R}^n$ and admits a natural compactification by adding a sphere $\partial \tilde{M}$ at infinity. The Wiener measure on paths induces for every $x \in \tilde{M}$ a probability measure $\omega^x$ on $\partial \tilde{M}$ which can naturally be considered as a Borel-probability measure on the fibre $T^1_x \tilde{M}$ at $x$ of the unit tangent bundle $T^1 \tilde{M}$ of $\tilde{M}$ ([P], [A], [AS]). These measures are equivariant under the action of the fundamental group $\Gamma$ of $M$ on $T^1 \tilde{M}$ and hence we can define a Borel-probability measure $\omega^*$ on the unit tangent bundle $T^1 M$ of $M$ by $\omega^*(A) = \int_{x \in M} \omega^x(A \cap T^1_x M) dx$ (here $dx$ is the normalized Lebesgue measure).

Recall that $T^1 M$ admits foliations $W^i$ which are invariant under the action of the geodesic flow $\Phi^t$ ($i = s, ss$), the stable foliation $W^s$ and the strong stable foliation $W^{ss}$. Each leaf of $W^s$ is locally diffeomorphic to $M$ and hence the Riemannian metric on $M$ lifts to a Riemannian $g^*$ on the leaves of $W^s$. The Laplace operator on the leaves with respect to this metric then induces a globally defined second order differential operator $\Delta^s$ on $T^1 M$ with continuous coefficients.

**LEDRAPIER. —** The measure $\omega^*$ is the unique harmonic measure for $\Delta^s$, i.e. the unique measure such that $\int \Delta^s \varphi \, d\omega^* = 0$ for all smooth functions $\varphi$ on $T^1 M$ (up to a constant, see [L3]).

Similarly we obtain an operator $\Delta^{ss}$ for $W^{ss}$ and an invariant measure $\omega^{ss}$. The projection of $\omega^{ss}$ is in the Lebesgue measure class and its conditionals on the fibres of $T^1 M \rightarrow M$ are the (non-normalized) Patterson-Sullivan measures ([L3], [Kn], [Y]).

Let $\lambda$ be the Lebesgue-Liouville measure on $T^1 M$. Call $M$ asymptotically harmonic if $\omega^{ss} = \lambda$.

Equivalent are (see [L1], [L2], [L3]) :

i) $M$ is asymptotically harmonic.

ii) $\omega^* = \lambda$.

iii) $\omega^* = \omega^{ss}$.

iv) For every Busemann function $\theta$ on $\tilde{M}$, the function $e^{-h\theta}$ is minimal harmonic.

v) Let $G(x, y)$ be the Green's function on $\tilde{M}$. Then there are positive constants $C > 0, h > 0$ such that

$$\lim_{d(x, y) \rightarrow \infty} G(x, y) e^{h d(x, y)} = C .$$

Here the constant $h$ equals the topological entropy of the geodesic flow $\Phi^t$ on $T^1 M$. 

vi) The top of the $L^2$-spectrum of $A$ on $\tilde{M}$ equals $-\frac{h^2}{4}$. (This and related results can be found implicitly in [L1], [L3] and [H4]).

For $\dim M \leq 4$, locally symmetric spaces are the only asymptotically harmonic ones ([H4], [L4]). In fact in certain cases more can be said:

vii) If $\dim M = 2$ and if any two of the measures $\omega^*, \omega^{**}$ and $\lambda$ are equivalent (i.e. if they have the same measure zero sets) then $M$ is asymptotically harmonic (and hence has constant curvature) ([K1], [K2], [L2], [H3]).

viii) If $\omega^{**}$ and $\lambda$ are equivalent then $M$ is asymptotically harmonic ([H4]).

**Conjecture.** — If any two of the measures $\omega^*, \omega^{**}, \lambda$ are equivalent then $M$ is locally symmetric.

References


