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Some aspects of the Laplace operator in negative curvature

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Let M be a compact Riemannian manifold of negative sectional curvature with universal covering $\widetilde{M}$. This covering is diffeomorphic to $\mathbf{R}^n$ and admits a natural compactification by adding a *sphere* $\partial\widetilde{M}$ *at infinity*. The Wiener measure on paths induces for every $x \in \widetilde{M}$ a probability measure ω^x on $\partial\widetilde{M}$ which can naturally be considered as a Borel-probability measure on the fibre $T^1_x\widetilde{M}$ at x of the unit tangent bundle $T^1\widetilde{M}$ of $\widetilde{M}$ ([P], [A], [AS]). These measures are equivariant under the action of the fundamental group Γ of M on $T^1\widetilde{M}$ and hence we can define a Borel-probability measure ω^s on the unit tangent bundle T^1M of M by $\omega^s(A) = \int_{x\in M}\omega^x(A\cap T^1_xM)dx$ (here dx is the normalized Lebesgue measure).

Recall that T^1M admits foliations W^i which are invariant under the action of the geodesic flow Φ^t $(i = s, ss)$, the *stable foliation* W^s and the *strong stable foliation* W^{ss}. Each leaf of W^s is locally diffeomorphic to M and hence the Riemannian metric on M lifts to a Riemannian g^s on the leaves of W^s. The Laplace operator on the leaves with respect to this metric then induces a globally defined second order differential operator Δ^s on T^1M with continuous coefficients.

LEDRAPPIER. — *The measure ω^s is the unique harmonic measure for Δ^s, i.e. the unique measure such that $\int \Delta^s\varphi \, d\omega^s = 0$ for all smooth functions φ on T^1M (up to a constant, see [L3]).*

Similarly we obtain an operator Δ^{ss} for W^{ss} and an invariant measure ω^{ss}. The projection of ω^{ss} is in the Lebesgue measure class and its conditionals on the fibres of $T^1M \longrightarrow M$ are the (non-normalized) Patterson-Sullivan measures ([L3], [Kn], [Y]).

Let λ be the Lebesgue-Liouville measure on T^1M. Call M *asymptotically harmonic* if $\omega^{ss} = \lambda$.

Equivalent are (see [L1],[L2], [L3]) :

i) M is asymptotically harmonic.

ii) $\omega^s = \lambda$.

iii) $\omega^s = \omega^{ss}$.

iv) For every Busemann function θ on $\widetilde{M}$, the function $e^{-h\theta}$ is minimal harmonic.

v) Let $G(x,y)$ be the Green's function on $\widetilde{M}$. Then there are positive constants $C > 0, h > 0$ such that

$$\lim_{d(x,y)\longrightarrow\infty} G(x,y)e^{hd(x,y)} = C\ .$$

Here the constant h equals the topological entropy of the geodesic flow Φ^t on T^1M.

vi) The top of the L^2-spectrum of Δ on $\widetilde{M}$ equals $-\frac{h^2}{4}$. (This and related results can be found implicitely in [L1], [L3] and [H4]).

For $\dim M \leqslant 4$, locally symmetric spaces are the only asymptotically harmonic ones ([H4], [L4]). In fact in certain cases more can be said :

vii) If $\dim M = 2$ and if any two of the measures ω^s, ω^{ss} and λ are equivalent (i.e. if they have the same measure zero sets) then M is asymptotically harmonic (and hence has constant curvature) ([K1], [K2], [L2], [H3]).

viii) If ω^{ss} and λ are equivalent then M is asymptotically harmonic ([H4]).

CONJECTURE. — *If any two of the measures $\omega^s, \omega^{ss}, \lambda$ are equivalent then M is locally symmetric.*

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