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Lines of curvature and umbilic points on surfaces


The principal configuration on an oriented surface immersed in $\mathbb{R}^3$ is defined by the triple formed by the set of its umbilic points and the foliations by minimal and maximal lines of principal curvature on the complement of umbilic points.

The study of principal configurations on surfaces has well established historical roots in Differential Geometry and Differential Equations. Its sources can be traced back to the initial contributions of Monge, Dupin and Darboux, among others.

Monge [M], determined the principal configuration on the ellipsoid with three different axes, which exhibits 4 umbilic points, connected pairwisely by umbilic separatrices, and closed principal cycles packed into cylinders. This is the first non-trivial global example of principal configuration. It was extended for general surfaces which belong to triply orthogonal families of surfaces by Dupin [St], [Sp]. The beautiful picture which illustrates this configuration can be found in Hilbert Cohn-Vossen [H], [CV] and Fischer [Fi].

The principal configuration near a generic umbilic point of analytic surfaces was determined by Darboux [D].

This paper contains a discussion of recent work focusing on the unifying effect that crucial ideas originating from Differential Equations and Dynamical Systems, such as Structural Stability, Bifurcation and Genericity, have in the study of so classical geometric objects such as the principal configurations on surfaces.

In fact, most of the results and problems discussed here focus on the study of how the principal configuration changes when the surface is slightly deformed.

The first purpose here is to identify the class of surfaces which are structurally stable (or robust) in the sense that their principal configurations remain qualitatively unchanged after such deformations.

The starting point for this research can be linked to Monge's Ellipsoid (which characterizes the structurally stable principal configurations on compact quadrics) and
to Darbouxian Umbilics (which characterize, in terms of the 3-jet of the surface, the
domination stable principal configurations at umbilic points).

In [G-S;1,8] has been given exhibited a $C^3$-open class, $\mathcal{S}$, of structurally stable
principal configurations on compact smooth surfaces immersed into $\mathbb{R}^3$. This class is
defined in terms of a) the umbilic points, assumed Darbouxian, b) the principal cycles,
assumed hyperbolic, c) the absence of umbilic separatrix connections and d) the absence
of non-trivially recurrent principal lines.

It was proved in [G-S;2,8] that $\mathcal{S}$ is $C^2$-dense in the space of immersions. To raise
the class from 2 to 3 in this density result, seems to be a difficult “Closing Lemma”
problem, precisely to achieve condition d) by means of $C^3$-small deformation.

These results also provide the first examples of surfaces with isolated principal
cycles. In fact, in all the examples of classical surface theory these cycles appear packed
in cylinders or tori.

Extensions of the structural stability results to surfaces with singularities have
been carried out in [G-S;4] and [Ga-S;1].

See [Ga] for a partial extension to principal configurations on smooth hypersur-
faces of $\mathbb{R}^4$.

A secondary purpose here is to describe the simplest patterns of qualitative
changes (bifurcations) of principal configurations on families of surfaces depending
on a real parameter. This leads to the definition of three hypersurfaces $\mathcal{S}_i(i), i = a, b, c$
in the space of surfaces immersed into $\mathbb{R}^3$, on which condition $i$ is violated in the mildest
possible way while the other conditions are respected. Deformations of surfaces along
these hypersurfaces preserve the qualitative properties of principal configurations.

Deformations on surfaces, transverse to $\mathcal{S}_i(i)$, produce a qualitative changes on
their principal configuration that can be easily described in terms of the splitting,
disappearance or exchange of types of umbilic points, principal cycles and umbilic
connections. See [G-S;5,6,7,9] and [Ga-S;2].

The systematic study of the simplest non-trivial recurrent principal lines, which
amounts to the violation of condition d), and their bifurcations have not been developed
yet. See [G-S;2,8] for the simplest examples of this phenomenon on surfaces of genus
0 and 1.

Other results concerning the local and global principal configurations on surfaces
with constant mean curvature and Weingarten surfaces have been obtained respectively
in [G-S;3], [G-S et al] and [S]. The principal cycles of these surfaces, however, never
appear isolated. This fact is due to the existence of an invariant transversal measure for
their principal foliations.

Acknowledgement. — This paper is based on a lecture given in 1991 on the
Séminaire de Théorie Spectrale et Géométrie, Institut Fourier, Université de Grenoble.
It was written during a visit in 1992 to the Laboratoire de Topologie, Université de
Bourgogne.
References


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