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On Blaschke manifolds and harmonic manifolds


<http://www.numdam.org/item?id=TSG_1987-1988__6__33_0>
ON BLASCHKE MANIFOLDS AND HARMONIC MANIFOLDS

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0. — A compact riemannian manifold $M$ is called a Blaschke manifold if the diameter of $M$ and the injectivity radius of $M$ coincide. It is known that if $M$ is a Blaschke manifold, then $M$ is diffeomorphic to $S^n$ or $\mathbb{RP}^n$, or $\pi_1(M) = \{0\}$ and $H^*(M, \mathbb{Z}) \cong \mathbb{Z}$, $\mathbb{CP}^n$, $\mathbb{HP}^n$, $\mathbb{CaP}^2$.

The main problem about Blaschke manifolds is to know if the following conjecture, the Blaschke conjecture, is true or not: if $M$ is a Blaschke manifold, then it would be a compact rank one symmetric space.

There are classes of riemannian manifolds related to Blaschke manifolds. A riemannian manifold $M$ is called a globally harmonic manifold if the determinant of $d(\exp_p)_x : T_p M \to T_{\exp_p x} M$ $(p \in M$, $x \in T_p M)$ depends only on the norm $|x|$. A compact riemannian manifold is called a $C_1$-manifold if all of its geodesics are closed and have the same length 1. The relation is as follows:

compact, simply connected, globally harmonic $\implies$ Blaschke $\implies C_1$.

The following results are known:

1. (Green, Berger et al.) — If $(S^n, g)$ is a Blaschke manifold, then it is isometric to the standard one.

2. (Green, Berger et al.) — If $(\mathbb{RP}^n, g)$ is a $C_1$-manifold, then it is isometric to the standard one.

3. (Kiyohara). — Let $P$ be one of the projective spaces $\mathbb{CP}^1$, $\mathbb{HP}^n$ $(n \geq 2)$, $\mathbb{CaP}^2$, and let $(P, g)$ be a $C_\pi$-manifold. If the metric $g$ is sufficiently close to the standard $C_\pi$-metric $g_0$, then $(P, g)$ is isometric to the standard one $(P, g_0)$.

4. (Zoll, Weinstein). — There are non-standard $C_1$-manifolds $(S^n, g)$ for any dimension $n \geq 2$. 
1. — From now on we assume $M$ is a Blaschke manifold, $\pi_1(M) = \{0\}$, $H^*(M,\mathbb{Z}) \cong H^*(CP^n,\mathbb{Z})$ (dim $M = 2n$, $n \geq 2$), and the diameter of $M$ is $\pi/2$. The followings are known about $M$:

1) For any $p \in M$ and any $q \in \text{Cut}(p)$ (the cut locus of $p$), the distance $d(p,q) = \pi/2$.

2) Every cut locus is a submanifold of codimension 2.

3) Let $\rho$ be the bundle projection $TM \to M$, and let $\{\zeta_i\}$ be the geodesic flow on $SM$. Then $\rho \circ \zeta_{\pi/2} : S_p(M) \to \text{Cut}(p)$ is a fibre bundle whose fibres are great circles on $S_pM$.

4) For $p, q \in M$ with $d(p, q) = \pi/2$, we denote by $\Sigma(p, q)$ the union of geodesic orbits through $p$ and $q$. Then $\Sigma(p, q)$ is a 2-dimensional submanifold diffeomorphic to $S^2$.

Now we define a mapping $I : SM \to SM$ as follows: since $H_2(M,\mathbb{Z}) \cong \mathbb{Z}$, we fix a positive generator. Then on each $\Sigma(p, q)$ the orientation is determined. Hence we have an orientation on each fibre $S^1$ of the fibre bundle $\rho \circ \zeta_{\pi/2} : S_pM \to \text{Cut}(p)$, because the fibre $S^1$ over $q \in \text{Cut}(p)$ is nothing but the unit sphere of $T_p\Sigma(p, q)$. So $I : SM \to SM$ is defined by the conditions:

1) If $v \in S_pM$, then $Iv \in S_pM$ and $\rho(\zeta_{\pi/2}v) = \rho(\zeta_{\pi/2}Iv)$.

2) $\langle v, Iv \rangle = 0$.

3) $\{v, Iv\}$ is positive in this order.

We extend the mapping $I$ to $TM\setminus\{0\}$ homogeneously, and let $I_v : T_{p(v)}M \to T_{p(v)}M$ be the differential of $I|T_{p(v)}M\setminus\{0\}$ at $v$. From the definition the mapping $I$ satisfies $I \circ I = (-1)$ identity. So it looks like an almost complex structure, and we have the following

**Proposition A.** — Assume $I_v^2 + 1 = 0$ for all $v \in SM$. Then $I : T_pM\setminus\{0\} \to T_pM\setminus\{0\}$ can be extended to a linear mapping on $T_pM$ for every $p \in M$, i.e. $I$ is an almost complex structure and it is integrable. Therefore $(M, I)$ is a hermitian manifold. Moreover each cut locus is a complex submanifold and is holomorphically isomorphic to $CP^{n-1}$.

**Proposition B.** — Assume dim $M = 4$. If $I_v^2 + 1 = 0$ for all $v \in SM$ and if every cut locus is minimal, then $M$ is isometric to $(CP^2, g_0)$.

**Lemma C.** — If $M$ is moreover globally harmonic, then $(I_v^2 + 1)^{n-1} = 0$ for every $v \in SM$ and every cut locus is minimal (dim $M = 2n$).

**Corollary D.** — If dim $M = 4$ and $M$ is globally harmonic, then $M$ is isometric to $(CP^2, g_0)$.
Remarque. — This corollary is already known by a different method. See [1].

2. — For the proof of propositions we need some lemmas.

**Lemma 1.** There is a Jacobi field $Y(t)$ along the geodesic $\gamma_{\nu}(t) = \rho(\zeta_{t}v)$ such that
\[
\begin{bmatrix} Y(0) \\ Y'(0) \end{bmatrix} = \begin{bmatrix} 0 \\ Iv \end{bmatrix}, \quad \begin{bmatrix} Y(\pi/2) \\ Y'(\pi/2) \end{bmatrix} = \begin{bmatrix} 0 \\ -Iv \end{bmatrix}, \quad \bar{v} = \zeta_{\pi/2}v.
\]
Moreover if a Jacobi field $X(t)$ along $\gamma_{\nu}(t)$ satisfies $X(0) = X(\pi/2) = 0$ then $X(t)$ is a constant multiple of $Y(t)$.

For $X, Y \in T_p M, Y \neq 0$, we put $\nabla_X I \cdot Y = \nabla_{\theta/\theta t}(IY_t)_{t=0}$, where we take a curve $c(t)$ in $M$ such that $c'(0) = X$, and $Y_t$ is the parallel displacement of $Y$ along $c(t)$. $\nabla_X I \cdot Y$ is linear in $X$, but not necessarily in $Y$.

**Lemma 2.** Let $Y(t)$ be a periodic Jacobi field along the geodesic $\gamma_{\nu}(t)$, $\nu \in SM$. Then we have a periodic Jacobi field $Z(t)$ along the geodesic $\gamma_{e^{s}I \nu}(t)$ ($e^{s}v = v \cos s + Iv \sin s$) such that
\[
\begin{bmatrix} Z(0) \\ Z'(0) \end{bmatrix} = \begin{bmatrix} Y(0) \\ (\cos s + \sin s I_v)v Y'(0) + \sin s (\nabla I \cdot v)Y(0) \end{bmatrix},
\]
\[
\begin{bmatrix} Z(\pi/2) \\ Z'(\pi/2) \end{bmatrix} = \begin{bmatrix} Y(\pi/2) \\ (\cos s + \sin s I_v)v Y'(\pi/2) - \sin s (\nabla I \cdot \bar{v})Y(\pi/2) \end{bmatrix},
\]
where $(\nabla I \cdot v)Y(0) = \nabla_Y(0)I \cdot v$, etc.

**Lemma 3.**
1) There are Jacobi fields $Y_1(t), Y_2(t)$ along $\gamma_{\nu}(t)$ such that
\[
\begin{bmatrix} Y_1(0) \\ Y_1'(0) \end{bmatrix} = \begin{bmatrix} I_v \\ -\nabla_v I \cdot v \end{bmatrix}, \quad \begin{bmatrix} Y_1(\pi/2) \\ Y_1'(\pi/2) \end{bmatrix} = \begin{bmatrix} -Iv \\ \nabla_{\bar{v}} I \bar{v} \end{bmatrix},
\]
\[
\begin{bmatrix} Y_2(0) \\ Y_2'(0) \end{bmatrix} = \begin{bmatrix} 2\nabla_v I \cdot v \\ R(I_v, v) - \nabla_v^2 I \cdot v \end{bmatrix}, \quad \begin{bmatrix} Y_2(\pi/2) \\ Y_2'(\pi/2) \end{bmatrix} = \begin{bmatrix} -2\nabla_{\bar{v}} I \cdot \bar{v} \\ -R(I_{\bar{v}}, \bar{v}) + \nabla_{\bar{v}}^2 I \cdot \bar{v} \end{bmatrix}.
\]
2) $\nabla_{e^{s}I \nu} I \cdot d^t I v = \nabla_{\nu} I \cdot v$.

**Lemma 4.** Let $Y(t)$ be a periodic Jacobi field along $\gamma_{\nu}(t)$. Then there is a periodic Jacobi field $Z(t)$ along $\gamma_{\nu}(t)$ such that
\[
\begin{bmatrix} Z(0) \\ Z'(0) \end{bmatrix} = \begin{bmatrix} I_v Y(0) \\ I_v Y'(0) + (\nabla I \cdot v - \nabla v \cdot I)Y(0) + \{(Y(0), \nabla_v I \cdot v) + (Y'(0), I v)\}v \end{bmatrix},
\]
\[
\begin{bmatrix} Z(\pi/2) \\ Z'(\pi/2) \end{bmatrix} = \begin{bmatrix} I_{\bar{v}} Y(\pi/2) \\ -I_{\bar{v}} Y'(\pi/2) - (\nabla I \bar{v} - \nabla \bar{v} I)Y(\pi/2) - \{(Y(\pi/2), \nabla_v I \bar{v}) + (Y'(\pi/2), I \bar{v})\} \bar{v} \end{bmatrix}.
\]

3. Proof of Proposition A. — Fix $p \in M$ and consider the $S^1$–principal bundle $\rho \circ \zeta_{\pi/2} : S_p M \to \text{Cut}(p)$, where the $S^1$–action is given by $e^{s}I$, $0 \leq s \leq 2\pi$. 
We define a 1-form $\omega$ on $S_p M$ by
$$\omega(X) = \langle X, Iv \rangle, \quad X \in T_v(S_p M) = \{ Y \in T_p M \mid \langle v, Y \rangle = 0 \}.$$
As is easily seen, $\omega$ is a connection form, i.e. invariant under the $S^1$-action. We have
$$d\omega(X, Y) = \langle (I_\ast v - t I_\ast v) X, Y \rangle.$$
So there is a unique closed 2-form $\Omega$ on $\text{Cut}(p)$ such that $(\rho \circ \zeta_{\pi/2})^* \Omega = d\omega$. We can see that $[(1/2\pi) \Omega]$ is a generator of $H^2(\text{Cut}(p), \mathbb{Z}) \cong \mathbb{Z}$. Therefore
$$(1/2\pi)^{n-1} \int_{\text{Cut}(p)} \Omega^{n-1} = 1$$
derived under a proper orientation of $\text{Cut}(p)$, and thus
$$\int_{S_p M} \omega \wedge (d\omega)^{n-1} = (2\pi)^n.$$
Now put $J_\ast = I_\ast v - t I_\ast v, \quad S_\ast = I_\ast v + t I_\ast v$. Then $2I_\ast v = J_\ast + S_\ast$ and
$$I_\ast v + 1 = 0 \iff J_\ast^2 + S_\ast^2 + 4 + J_\ast S_\ast + S_\ast J_\ast = 0. \quad (#)$$
Let $e_1, \ldots, e_{2n-2}$ be an orthonormal basis of the orthogonal complement to $R_v + R_\ast v$ in $T_p M$ such that $J_\ast e_{2i-1} = \lambda_i e_{2i}, \quad J_\ast e_{2i} = -\lambda_i e_{2i-1}, \quad \lambda_i \geq 0, \quad i = 1, \ldots, n - 1$. By (#) we have
$$-\lambda_i^2 + |S_\ast e_{2i}|^2 + 4 = 0.$$
Hence $\lambda_i \geq 2$, and $\lambda_i = 2$ for every $i$ if and only if $S_\ast = 0$. Then
$$(\omega \wedge (d\omega)^{n-1})(I_\ast v, e_1, \ldots, e_{2n-2}) = (n - 1)! \prod_{i=1}^{n-1} \lambda_i \geq 2^{n-1}(n-1)!,$$
and the equality holds if and only if $S_\ast = I_\ast v + t I_\ast v = 0$. Therefore we have
$$(2\pi)^n = \int_{S_p M} \omega \wedge (d\omega)^{n-1} \geq 2^{n-1}(n-1)! \text{vol}(S_p M).$$
But $\text{vol}(S_p M)$ is just $2\pi^n/(n-1)!$. So the equality holds in the above inequality. Hence we have $S_\ast = I_\ast v + t I_\ast v = 0$ for any $v \in SM$. Since $I_\ast^2 v + 1 = 0$, it follows that $t_\ast v I_\ast v = 1$. This implies that the mapping $I : S_p M \to S_p M$ is an isometry, and therefore the restriction of a linear orthogonal transformation of $T_p M$. Hence $I$ is extended as a tensor field of type (1,1) with $I^2 = -1$, i.e. an almost complex structure on $M$, and $(M, I)$ is an almost hermitian manifold.

By using the square of the endomorphisms on the space of Jacobi fields in Lemma 4, one gets
$$\langle \{ I(\nabla I \cdot v - t \nabla I \cdot v) - (\nabla I \cdot v - t \nabla I \cdot v) I \} X, Y \rangle = 0, \quad X, Y \perp v, I_\ast v.$$
Moreover Lemma 3 (2) gives
$$\langle \{ I(\nabla I \cdot v + t \nabla I \cdot v) - (\nabla I \cdot v + t \nabla I \cdot v) I \} X, Y \rangle = 0, \quad X, Y \perp v, I_\ast v.$$
These formula gives
\[ \nabla_{IX} I = I \nabla_X I \]
for any vector \( X \). By this it is easy to see that the Nijenhuis’ tensor vanishes, and \( (M, I) \) turns out to be a hermitian manifold.

It is now clear that each cut locus is a complex submanifold of \( M \) and the \( S^1 \)-fibration \( \rho \circ \zeta_{\pi/2} : S_p M \to \text{Cut}(p) \) is nothing but the standard Hopf fibration: \( S^{2n-1} \to \mathbb{C}P^{n-1} \). Hence the last statement of the proposition follows.

**Proof of Proposition B.** For \( v \in SM \) we define the symmetric endomorphism \( \Phi_v \) of \( T_{\rho(v)} M \) by \( \Phi_v v = \Phi_v I v = 0 \) and
\[
\langle \Phi_v X, Y \rangle = -\langle h(X, Y), v \rangle, \quad X, Y \in T_{\rho(v)} M, \quad X, Y \perp v, I v,
\]
where \( h \) is the second fundamental form of \( \text{Cut}(\rho \zeta_{\pi/2} v) \) in \( M \) at \( \rho(v) \). If we take a curve \( c(t) \) in \( \text{Cut}(\rho \zeta_{\pi/2} v) \) such that \( c'(0) = X \), and a normal vector field \( v_t \) to \( \text{Cut}(\rho \zeta_{\pi/2} v) \) along \( c(t) \), we have
\[
\langle \Phi_v X, Y \rangle = \langle \nabla_{\partial / \partial t} v_t |_{t=0}, Y \rangle.
\]
So the following lemma is clear.

**Lemma 5.** \( \Phi_I v X = I \Phi_v X + (\nabla_X I)v, \) \( X \in T_{\rho(v)} M, \) \( X \perp v, I v \).

Since every cut locus is minimal, it follows that \( \text{tr} \Phi_v = 0 \) for any \( v \in Sm \), \( \text{tr} \) being the trace. Hence in view of Lemma 5 one gets
\[ \text{tr} (\nabla I)v = 0 . \]
This together with the formula \( \nabla_{IX} I = I \nabla_X I \), shown in the proof of Proposition A, implies that \( \nabla I = 0 \), i.e. \( (M, I) \) is kählerian.

By applying Lemma 1 to the Jacobi field \( Y_z \) in Lemma 3,
\[ R(Iv, v)v = c(v) Iv, \quad v \in SM, \]
where \( c \) is a function on \( SM \) satisfying \( c(\zeta_{\pi/2} v) = c(v) \). As is easily seen, \( c(v) \) is pointwise constant, i.e. if \( v_1 \) and \( v_2 \) are based at the same point on \( M \), then \( c(v_1) = c(v_2) \). Using the fact that for any two points \( p \) and \( q \) on \( M \), there is a point \( m \) such that \( d(p, m) = d(q, m) = \pi/2 \), we see the constancy of \( c(v) \).

Since \( (M, I) \) is kählerian and has constant holomorphic sectional curvature, it must be holomorphically isometric to \( (\mathbb{C}P^2, g_0) \).

Lemma C is an immediate consequence of Lemma 4.

**Reference**


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