

COLIN D. WALTER

**The Ambiguous Class Group and the Genus Group of  
Certain Non-Normal Extensions**

*Séminaire de théorie des nombres de Grenoble*, tome 7 (1978-1979), exp. n° 7, p. 1-26

[http://www.numdam.org/item?id=STNG\\_1978-1979\\_\\_7\\_\\_A7\\_0](http://www.numdam.org/item?id=STNG_1978-1979__7__A7_0)

© Institut Fourier – Université de Grenoble, 1978-1979, tous droits réservés.

L'accès aux archives du séminaire de théorie des nombres de Grenoble implique l'accord avec les conditions générales d'utilisation (<http://www.numdam.org/conditions>). Toute utilisation commerciale ou impression systématique est constitutive d'une infraction pénale. Toute copie ou impression de ce fichier doit contenir la présente mention de copyright.

NUMDAM

Article numérisé dans le cadre du programme  
Numérisation de documents anciens mathématiques  
<http://www.numdam.org/>

Grenoble

THE AMBIGUOUS CLASS GROUP AND THE GENUS GROUP  
OF CERTAIN NON-NORMAL EXTENSIONS

par

Colin D. WALTER

In an article generalising work of Roquette and Zassenhaus, Connell and Sussman [2] have demonstrated the importance of certain prime ideals in a number field  $k_0$  for estimating the  $\ell$ -rank of the class group of an extension  $k$ . These ideals have a power prime to  $\ell$  which is principal and they have prime factors in  $k$  with ramification index divisible by  $\ell$ . The products of the prime divisors of these ideals in the normal closure  $K$  of  $k/k_0$  are invariant under  $\text{Gal}(K/k_0)$ . Thus certain roots in  $k$  of the ideals in  $k_0$  are fixed by the Galois group. This leads to the concept of ambiguous ideals in an extension  $k/k_0$  which is not necessarily normal.

Of particular interest is the case when  $K/k_0$  is metacyclic. Then  $k/k_0$  is almost a cyclic extension and many of the theorems of cyclic fields have analogues which apply. Since the genus number and the ambiguous class number are equal for a cyclic extension it is worth comparing them in  $k/k_0$ . In fact, there they are usually different and this can be seen from the class group description of the genus field. A character theoretic description can also be given for the genus group

and this is useful for computing the genus number.

Estimates for the genus number and ambiguous class number have been combined for dihedral extensions by several authors, including Barrucand and Cohn [1] for pure cubic fields. This is done here for pure fields of any odd prime degree over the rational field  $\mathbb{Q}$ . Indeed, applications to pure fields are the motivating force in this work, and much of the inspiration comes from the class rank estimates of Fröhlich [3] which generalise those of Holzer [8].

1. Ambiguous Classes for Frobenius Extensions

Let  $G$  be a Frobenius group with normal kernel  $N$  and a complement  $F$ . Then  $G$  is a semi-direct product of  $N$  and  $F$  for which the distinct conjugates of  $F$  intersect pairwise in the identity. Consequently, if  $n$  and  $f$  are the orders of  $N$  and  $F$  respectively then the conjugacy classes of  $N-1$  under  $F$  all have order  $f$ . Hence  $f$  divides  $n-1$  and is coprime to  $n$ .

Suppose  $K/k_0$  is a normal extension of number fields whose Galois group is  $G$ . Let  $L = K^N$  and  $k = K^F$  be the fixed subfields of the subgroups  $N$  and  $F$ . There are many similarities between  $k/k_0$  and its lifting by  $L$  to the normal extension  $K/L$ , but the structure of the latter is generally easier to describe. In this study of the extension  $k/k_0$  the analogy between it and the classical case of  $K/L$  can be drawn by assuming  $f = 1$  so that  $k/k_0$  becomes normal.

Denote the (classical) class group of a field  $\Omega$  by  $H_\Omega$ , its class number by  $h_\Omega$ , the  $n$ -subgroup of  $H_\Omega$  by  $C_\Omega$ , and the maximal subgroup with order prime to  $n$  by  $C_\Omega'$ . Thus  $H_\Omega = C_\Omega \times C_\Omega'$ . A class of  $k$  will be called ambiguous (over  $k_0$ ) if its image in  $H_k$  is fixed by  $N$  (which generates all the conjugates of  $k/k_0$ ), or, equivalently, by  $G$ . The subgroups of such classes are written  $H_k^G$ ,  $C_k^G$ , and  $C_k'^G$ . Likewise an ideal of  $k$  is called ambiguous if its extension to  $K$  is fixed under  $N$  or, equivalently, under  $G$ . A class of  $H_k$  is called strongly ambiguous if it contains an ambiguous ideal. These terms are just the standard ones when  $k/k_0$  is normal,

and they can easily be generalised still further.

1.1 Theorem The group of ambiguous classes for  $k/k_0$  is the direct product  $H_k^G = C_k^G \times C_k'^G$ . Here  $C_k'^G$  is the isomorphic image of  $C_{k_0}'$  in  $C_k'$  under the natural embedding given by extension of ideals; and under extension of ideals  $C_k^G$  is isomorphic to  $C_K^G$ , the group of ambiguous classes in  $K/k_0$  with  $n$ -order. Thus

$$H_k^G \cong C_K^G \times C_{k_0}'.$$

Proof In Theorem 5.1 of [11] it was shown that the natural maps induced by extension of ideals provide an exact sequence

$$1 \rightarrow C_{k_0}' \rightarrow C_k' \rightarrow C_K'^F / C_K'^G \rightarrow 1.$$

Hence any class of  $C_k'$  which has its image in  $C_K'$  fixed by  $G$  comes from a class in  $C_{k_0}'$ , and vice versa.

Since  $n$  is prime to  $[K:k]$  there is a natural embedding  $C_k \hookrightarrow C_K$  which restricts to  $C_k^G \hookrightarrow C_K^G$ . This is an isomorphism because the inverse map is obtained by applying the idempotent  $e_F = f^{-1} \sum_{g \in F} g$  and restriction of ideals, i.e. a suitable power of the norm.

Thus the basic observation that provides information about the ambiguous class group of  $k/k_0$  is this:

1.2 Lemma  $C_k^G$  is isomorphic to the direct summand of the  
ambiguous n-class group  $C_K^N$  of  $K/L$  given by the projection  
 $e_F$ , viz.  $C_K^G$ .

1.3 Lemma If  $\mathcal{O}$  is an ambiguous ideal of  $k/k_0$  then the extension  
of  $N_{k/k_0}\mathcal{O}$  is equal to  $\mathcal{O}^n$ .

Proof The extension of  $N_{k/k_0}\mathcal{O}$  to  $K$  is just the product of the  
conjugates of the extension of  $\mathcal{O}$  under  $N$ . However, the extension of  
 $\mathcal{O}$  is fixed under the action of  $N$  and so the product of  
conjugates is just the  $n$ th power. The same equality holds on restriction  
to  $k$ .

Let  $I_\Omega$  be the multiplicative group of non-zero fractional ideals  
of a field  $\Omega$ , extended to  $K$  wherever necessary;  $P_\Omega$  the subgroup  
of principal ideals;  $I_\Omega^\Gamma$  the subgroup of ideals which are fixed by a  
subgroup  $\Gamma$  of  $G$  when extended to  $K$ ; and  $I_\Omega^{\Gamma*}$  the subgroup of  
ideals which lie in a class of  $K$  fixed by  $\Gamma$ . With this notation  
the isomorphic groups  $C_k^G$  and  $C_K^G$  are the  $n$ -subgroups of  
 $I_k^{G*}/P_k$  and  $I_K^{G*}/P_K$  respectively. The most accessible parts  
of these groups are the subgroups  $I_k^G P_k/P_k$  and  $I_K^G P_K/P_K$  of strongly  
ambiguous classes, and in many cases they give the whole group (vid.  
Corollary 1.9).

Let  $\mathfrak{p}$  be a prime ideal of  $k_0$  with prime divisors  $\mathfrak{a}_j$  in  $k$   
and below the prime  $\mathfrak{P}$  of  $K$ . Suppose  $e, e', e_j$ , and  $e'_j$  are the

VII.6

ramification indices for these primes in  $K/L$ ,  $L/k_0$ ,  $k/k_0$ , and  $K/k$  respectively. The equality  $e_j e_{j'} = ee'$  gives  $\mathfrak{p}^n = N_{k/k_0} \mathfrak{p} = \prod_j (N_{k/k_0} \mathfrak{q}_j)^{ee'/e_j}$ . Hence any common factor between the  $e'/e_j$  divides both  $n$  and  $f$  and so equals 1. Thus  $\mathfrak{a} = \prod \mathfrak{q}_j^{e'/e_j}$  has no roots in  $k$ . Any divisor of  $\mathfrak{p}$  in  $k$  which is fixed by  $G$  must decompose in  $K$  as a power of  $\mathfrak{A} = \prod_{\mathfrak{g} \in H \setminus G} \mathfrak{P}^{\mathfrak{g}}$  where  $H$  is the decomposition group of  $\mathfrak{P}$  over  $k_0$ . Therefore such a divisor is a power of  $\mathfrak{a} = \mathfrak{A}^{e'}$  and the generators above  $\mathfrak{p}$  of  $I_K^G$  and  $I_k^G$  are  $\mathfrak{A}$  and  $\mathfrak{a}$  respectively. Since the extensions of  $\mathfrak{p}$  are equal to  $\mathfrak{a}^e$  for  $k$  and  $\mathfrak{A}^{ee'}$  for  $K$  the powers of  $\mathfrak{A}$  and  $\mathfrak{a}$  cannot generate ideal classes with  $n$ -order in  $H_K$  or  $H_k$  other than those of the powers of the extensions of  $\mathfrak{p}$  unless  $e > 1$ , i.e. the prime ideal  $\mathfrak{p}$  ramifies in  $K/L$ . Hence  $I_K^G$  and  $I_k^G$  are generated (the former up to an index prime to  $n$ ) by  $I_L$  and  $I_{k_0}$  respectively, together with the ideals  $\mathfrak{A}$  and  $\mathfrak{a}$  respectively which divide the prime ideals  $\mathfrak{p} \in I_{k_0}$  which are ramified in  $K/L$ .

Put  $e_{\mathfrak{p}}$  for the ramification index in  $K/L$  of a prime ideal  $\mathfrak{p} \in I_{k_0}$ . Then,

1.4 Lemma  $[I_k^G : I_{k_0}^G] = \prod_{\mathfrak{p}} e_{\mathfrak{p}}$ .

1.5 Remark There are potentially more classes in  $k$  to be found from the decomposition of ramified primes: each divisor  $\mathfrak{q}_j$  of  $\mathfrak{p}$  in  $k$  yields some class, but the ideal  $\mathfrak{a}$  may only generate certain products of these classes.

From here on suppose  $N$  is cyclic, with generator  $\sigma$ . Then  $F$  is also cyclic, with generator  $\phi$  say, because it is a subgroup of the cyclic automorphism group of each subgroup of  $N$  with prime order. Thus  $G$  is metacyclic and, because  $f > 1$ ,  $n$  is odd. Write  $\tilde{S}$  for the sum in the integral group ring  $\mathbb{Z}[G]$  of the elements in a subset  $S$  of  $G$ . Define  $\tilde{f} \in \mathbb{Z}[G]$  by  $(1-\sigma)\tilde{f} = \tilde{F}(1-\sigma)$  and  $e_{\tilde{f}} = f^{-1}\tilde{f}$ . Then  $\tilde{f}$  is determined uniquely up to a multiple of  $\tilde{N}$ , so that  $e_{\tilde{f}}$  is really an idempotent of  $\mathbb{Z}[G]/\mathbb{Z}[G]\tilde{N}$  which is conjugate to  $e_F$ . We have

$$e_F = f^{-1}\tilde{F} \quad \text{and} \quad (1-\sigma)e_{\tilde{f}} = e_F(1-\sigma).$$

Finally, let  $E_\Omega$  denote the unit group of a field  $\Omega$ ,  $r(\Omega)$  the  $\mathbb{Q}$ -dimension of  $\mathbb{Q} \otimes_{\mathbb{Z}} E_\Omega$  and  $W$  the torsion subgroup of  $E_K$ . From [11] §3.1, it is known that  $W \subset L$  and  $W^F \subset k_0$ .

1.6 Theorem The number of strongly ambiguous classes for  $k/k_0$  is

$$\frac{h_{k_0} \prod_{\mathfrak{p}} e_{\mathfrak{p}}}{|H^1(N, E_K)^{e_{\tilde{f}}}|}$$

where the product is over (finite) prime ideals  $\mathfrak{p}$  of  $k_0$ .

Proof  $I_k^G/P_k/P_k \cong I_k^G/(I_k^G \cap P_k) \cong (I_k^G/P_{k_0})/(P_k^G/P_{k_0})$ . The numerator has order  $[I_k^G : I_{k_0}^G][I_{k_0}^G : P_{k_0}^G] = h_{k_0} \prod_{\mathfrak{p}} e_{\mathfrak{p}}$  by 1.4. Since by 1.3 its exponent divides  $n$ , the denominator is  $P_k^G/P_{k_0} \cong (P_k^N/P_L)^{e_F} \cong (\{\alpha \in K | \alpha^{1-\sigma} \in E_K\}/L^{\times}E_K)^{e_F} \cong ((K^{1-\sigma} \cap E_K)/E_K^{1-\sigma})^{e_{\tilde{f}}} = H^1(N, E_K)^{e_{\tilde{f}}}$ .

1.7 Corollary The number of strongly ambiguous classes in  $k/k_0$  is a multiple of

$$i) \quad \frac{(h_{k_0} \prod_p e_p) [K^{1-\sigma} \wedge E_k : E_K^{1-\sigma} \wedge k]}{n [E_L : N_{K/L} E_K]}$$

$$ii) \quad \frac{h_{k_0} \prod_p e_p}{n^{r(L)+1} [W : W^G W^n]}$$

$$iii) \quad \frac{h_{k_0} \prod_p e_p}{[k^{n-\tilde{N}} \wedge E_k : E_k^{n-\tilde{N}}] [W : W^G W^n]}$$

The number of strongly ambiguous classes in  $k/k_0$  is a divisor of

$$\frac{h_{k_0} \prod_p e_p}{[k^{1-\sigma} \wedge W : E_K^{1-\sigma} \wedge W]}$$

Proof Define  $\beta_i \in \mathbb{Z}[G]/\mathbb{Z}[G]\tilde{N}$  by  $\beta_i = (1-\sigma)^{-i} \tilde{F} (1-\sigma)^i$ . Then from [11] §1.7, there is a direct sum decomposition  $\mathbb{Z}[G]/\mathbb{Z}[G]\tilde{N} = \bigoplus_{0 \leq i < f} \mathbb{Z}[G]\beta_i$  which yields

$$H^1(N, E_K) = \bigoplus_{0 \leq i < f} H^1(N, E_K)^{\beta_i}.$$

Here  $\beta_0$  and  $\beta_1$  can be replaced by  $e_F$  and  $e_{\tilde{F}}$  respectively so that  $|H^1(N, E_K)^{e_{\tilde{F}}}|$  divides  $|H^1(N, E_K)| |H^1(N, E_K)^F|^{-1}$ . The second factor is just  $[K^{1-\sigma} \wedge E_k : E_K^{1-\sigma} \wedge k]$  whilst the first can be translated

using the value  $Q(E_K) = n^{-1}$  for the Herbrand quotient given, for example, in [13]. Thus  $|H^1(N, E_K)| = n|H^0(N, E_K)| = n[E_L : N_{K/L} E_K]$ . This gives (i) from Theorem 1.6.

For (ii) the part of the denominator of 1.6 due to torsion in  $E_K$  must be extracted. It is  $[k^{1-\sigma} \cap W : E_k^{1-\sigma} \cap W]$ . The non-torsion part divides  $H^1(N, E_K/W) = n[E_L/W : N_{K/L} E_K/W]$  which itself divides  $n^{r(L)+1}$ . For  $\zeta \in k^{1-\sigma} \cap W$  choose  $\alpha \in k$  such that  $\zeta = \alpha^{1-\sigma}$ . Then  $\zeta^n = \zeta^{\tilde{N}} = \alpha^{(1-\sigma)\tilde{N}} = 1$  because  $W \subset K^N$ . Clearly  $k_0(\zeta, \alpha)/k_0$  is normal. But  $G$  has no normal subgroups other than those containing or contained by  $N$ . Thus  $\alpha \notin k_0$  implies  $L = k_0(\zeta)$ . Also  $\zeta \in k_0$  implies  $\alpha \in k_0$  and hence  $\zeta = 1$ . So  $(k^{1-\sigma} \cap W)/(E_k^{1-\sigma} \cap W)$  is trivial unless possibly when  $L \subset k_0(\sqrt[n]{1})$ , and then its order divides  $[W : W^G W^n]$ . In particular, if  $k = k_0(\sqrt[n]{\alpha})$  and a prime not dividing  $n$  is ramified in  $k/k_0$  then  $\alpha$  cannot be a unit and  $[k^{1-\sigma} \cap W : E_k^{1-\sigma} \cap W] = n$ .

For the other parts consider the denominator of 1.6 again. It comes from  $P_k^G/P_{k_0} \cong \{\alpha \in k \mid \alpha^{1-\sigma} \in E_K\}/k_0 \times E_k \cong (k^{1-\sigma} \cap E_K)/E_k^{1-\sigma}$ . This has the factor group  $(k^{1-\sigma} \cap E_K)/E_k^{1-\sigma} (k^{1-\sigma} \cap W) \cong (k^{1-\sigma} \cap E_K)^{(n-\tilde{N})/(1-\sigma)}/E_k^{n-\tilde{N}} \subset (k^{n-\tilde{N}} \cap E_K)/E_k^{n-\tilde{N}}$  where the isomorphism is given by the class of  $\alpha^{1-\sigma} \in k^{1-\sigma} \cap E_K$  mapping to the class of  $\alpha^{n-\tilde{N}}$ . This is well-defined: firstly because  $\alpha^{1-\sigma}$  determines  $\alpha$  up to an element  $\beta \in L^x \cap k^x = k_0^x$  and  $(\alpha\beta)^{n-\tilde{N}} = \alpha^{n-\tilde{N}}$  for such  $\beta$ ; and secondly because if  $\alpha^{1-\sigma} = \zeta \in W$  then  $\alpha^{n-\tilde{N}} = \zeta^{(n-\tilde{N})/(1-\sigma)} = \zeta^{n(n-1)/2} = 1$  by the oddness of  $n$ . The map is certainly surjective. For the injectivity suppose  $\alpha^{1-\sigma} \in k^{1-\sigma} \cap E_K$  maps to  $E_k^{n-\tilde{N}}$ . Then  $(\alpha\epsilon)^{n-\tilde{N}} = 1$  for some  $\epsilon \in E_k$ . Without loss of generality  $\alpha^{n-\tilde{N}} = 1$  so that  $(\alpha^{1-\sigma})^n = (\alpha^n)^{1-\sigma} = \alpha^{\tilde{N}(1-\sigma)} = 1$

whence  $\alpha^{1-\sigma} \in k^{1-\sigma} \cap W$  represents the trivial class. The subgroup initially quotiented out was  $(k^{1-\sigma} \cap W)/(E_k^{1-\sigma} \cap W)$  which has order dividing  $[W:W^{G_W^n}]$ , as was shown above. This completes the proof of (iii) and gives the last part.

Remarks When  $n=\ell$  is prime and  $h_{k_0}$  is prime to  $\ell$  these estimates give lower bounds for the order of an elementary abelian  $\ell$ -group within the class group of  $k$  and hence also a lower bound for the minimal number of generators of its  $\ell$ -Sylow subgroup. Part (iii) and its approximation  $h_{k_0} \prod_{p \neq \ell} e_p / n^{r(k)-r(k_0)+1}$  therefore generalise Frohlich's Theorem 1 in [3] and its proof. This approximation yields the result of Connell and Sussman's Theorem 1 in [2] for  $k/k_0$  when the degree is prime; but the analogue for general  $n$  may be weaker (vid. 1.5). However,  $r(L) + 1 \leq r(k) - r(k_0)$  with equality possible only when  $f = n-1$ . Therefore the estimate in (ii) is at least as good as that from (iii) and the rank interpretation for (ii) generalises Gerth's Proposition 3.4 in [4].

A good knowledge of the unit group of  $K$  allows one to obtain still better estimates for the divisibility of  $h_k$ :

1.8 Theorem The quotient of ambiguous ideal classes modulo strongly ambiguous classes is isomorphic to

$$\left( (N_{K/L} K^\times \cap E_L) / N_{K/L} E_K \right)^{e_K}.$$

Proof  $(I_k^{G^*}/P_k)/(I_k^G/P_k) \cong (I_k^{N^*})^{e_F}/I_k^{N_P K}$   
 $\cong (I_k^{N^*})^{e_F(1-\sigma)}/(I_k^{N_P K})^{1-\sigma} = (I_k^{N^*})^{(1-\sigma)e_{\mathfrak{F}}}/P_k^{1-\sigma}$   
 $= \{(\alpha) | N_{K/L}\alpha \in E_L\}^{e_{\mathfrak{F}}}/P_k^{1-\sigma} \cong \{\alpha \in K | N_{K/L}\alpha \in E_L\}^{e_{\mathfrak{F}}}/E_K K^{1-\sigma}$   
 $\cong (N_{K/L}K \wedge E_L)^{e_{\mathfrak{F}}}/N_{K/L}E_K.$  The first isomorphism is by Lemma 1.2. The subsequent maps are precisely those used by Hasse in [7] Ia §13: multiplication by  $1-\sigma$ , mapping to a generator of a principal ideal, and applying the norm for  $K/L$ . The isomorphisms are proved by him and are straight-forward when Hilbert's Theorem 90 is borne in mind and it is observed that  $N_{K/L}$  and  $e_{\mathfrak{F}}$  commute.

1.9 Corollary Suppose  $L/k_0$  has  $u$  unramified infinite primes. Then the quotient of ambiguous classes modulo strongly ambiguous classes has order dividing  $n^{uf/2}[W:W^N]$ . In particular, when  $u=0$  then the quotient is isomorphic to

$$((N_{K/L}K \wedge W)/(N_{K/L}E_K \wedge W))^{e_{\mathfrak{F}}}.$$

Proof Let  $C_i$  be the decomposition group of one infinite prime divisor in  $K$  above the infinite prime  $i$  of  $k_0$ . By hypothesis,  $C_i$  has order 2 for all but  $u$  valuations  $i$ , and without loss of generality  $C_i \subset F$  as  $n$  is odd. When  $C_i$  has order 2 it is generated by  $\gamma = \phi^{f/2}$  which inverts elements of  $N$ . Write  $C_i \mathbb{Z}[G]N$  for the subgroup of  $\mathbb{Z}[G]$  fixed on the left by  $C_i$  and on the right by  $N$ .  $E_L/W$  is torsion free and (vid. e.g. [10] §4) is isomorphic to a right submodule of finite index in

$$M = (\bigoplus_i C_i \mathbb{Z}[G]N)/\mathbb{Z}(\bigoplus_i \tilde{G}).$$

M is generated by the  $\tilde{C}_i g \tilde{N} = g \tilde{C}_i \tilde{N}$  where  $g \in F$  and so the effect of  $e_{\mathfrak{F}}$  is determined by the values of  $\{\tilde{C}_i \tilde{N}\}$ .

Suppose  $\phi \sigma \phi^{-1} = \sigma^r$  so that  $r$  has order  $f$  modulo  $n$  and then set

$$\mathfrak{F} = \sum_{i=0}^{f-1} \left( \sum_{j=0}^{r^i-1} \sigma^j \right) \phi^i - \tilde{N} \sum_{i=0}^{f/2-1} \left( \frac{r^i + r^{i+f/2}}{2n} \right) (\phi^{i+\phi^{i+f/2}}).$$

It is immediately verifiable that  $(1-\sigma)\mathfrak{F} = \tilde{F}(1-\sigma)$  and that  $\tilde{N}\mathfrak{F} = \tilde{N}(\gamma-1) \sum_{i=0}^{f/2-1} \frac{1}{2}(r^{i+f/2} - r^i)\phi^i$ . Hence  $\tilde{C}_i \tilde{N}\mathfrak{F} = 0$  when  $C_i$  has order 2 and  $\gamma \tilde{C}_i \tilde{N}\mathfrak{F} = -\tilde{C}_i \tilde{N}\mathfrak{F}$  for all  $i$ . Thus  $M\mathfrak{F} \otimes_{\mathbb{Z}} \mathbb{Q}$  has dimension at most  $\frac{1}{2}uf$  over  $\mathbb{Q}$  for this choice of  $\mathfrak{F}$ . The same is therefore true of  $(E_L/W)\mathfrak{F} \otimes_{\mathbb{Z}} \mathbb{Q}$  and shows that  $((N_{K/L} K \wedge E_L)W/N_{K/L} E_K \cdot W)^{e_{\mathfrak{F}}}$  has order dividing  $n^{uf/2}$ .

It remains to consider the subgroup  $((N_{K/L} K \wedge W)/(N_{K/L} E_K \wedge W))^{e_{\mathfrak{F}}}$  of the group in 1.8 due to torsion in  $E_K$ .  $W^n$  is contained in the denominator because  $\zeta^n = N_{K/L} \zeta$  for  $\zeta \in W \subset L$ . If  $\zeta \in W^G$  then, modulo elements which fix  $\zeta$  and multiples of  $n$ , we have  $\mathfrak{F} \equiv \sum_{i=0}^{f-1} \sum_{j=0}^{r^i-1} \sigma^j \phi^i \equiv \sum_{i=0}^{f-1} r^i = (r^f-1)/(r-1) \equiv 0$ . So  $(W^G)^{\mathfrak{F}} \subset W^n$  and there is a natural surjection from  $W^G(W \wedge N_{K/L} K)/W^n W^G$  to the group under consideration, given by  $\zeta/W^n W^G \rightarrow (\zeta/(N_{K/L} E_K \wedge W))^{e_{\mathfrak{F}}}$ . Hence the order of the group divides  $[W:W^n W^G]$ . The exact sequence

$$\begin{aligned} 1 \rightarrow (N_{K/L} K \wedge W)/(N_{K/L} E_K \wedge W) &\rightarrow (N_{K/L} K \wedge E_L)/N_{K/L} E_K \\ &\rightarrow (N_{K/L} K \wedge E_L)W/N_{K/L} E_K \cdot W \rightarrow 1 \end{aligned}$$

remains exact when fixed by the idempotent  $e_{\mathfrak{f}}$ . So the above bounds on the outer two groups of

$$1 \rightarrow ((N_{K/L} K \cap W) / (N_{K/L} E_K \cap W))^{e_{\mathfrak{f}}} \rightarrow ((N_{K/L} K \cap E_L) / N_{K/L} E_K)^{e_{\mathfrak{f}}} \\ \rightarrow ((N_{K/L} K \cap E_L) W / N_{K/L} E_K \cdot W)^{e_{\mathfrak{f}}} \rightarrow 1$$

place the required bound on the central group and yield the required isomorphism between the first two groups when  $u = 0$ .

1.10 Corollary Suppose  $L/k_0$  has no unramified infinite primes and  $\zeta$  generates  $W \cap N_{K/L} K$  over  $W \cap N_{K/L} E_K$ . Choose  $\alpha \in K$  such that  $\zeta = N_{K/L} \alpha$  and an ideal  $\mathfrak{a}$  in  $K$  for which  $(\alpha) = \mathfrak{a}^{1-\sigma}$ . Then the class of  $N_{K/k} \alpha$  generates the ambiguous classes of  $k/k_0$  over the strongly ambiguous classes.

Proof Under the maps of 1.8 and 1.9 the image of  $N_{K/k} \alpha$  is  $\zeta^{\mathfrak{f}}$ , which generates the group of 1.9.

1.11 Lemma Suppose  $k/k_0$  is a pure field extension of a totally real field. Then the quotient of ambiguous by strongly ambiguous classes is isomorphic to

$$(N_{K/L} K \cap W) / (N_{K/L} E_K \cap W).$$

Proof Here  $L$  is obtained from  $k_0$  by adjoining an  $n$ th root of unity  $\zeta$ , and so  $L/k_0$  has no unramified infinite primes. Now  $\zeta$  generates  $W/W^n$  and assuming  $\phi\sigma\phi^{-1} = \sigma^r$  gives  $\zeta^\phi = \zeta^{r^{-1}}$ . So, modulo elements which fix  $\zeta/W^n$ ,  $\mathfrak{F} \equiv \sum_{i=0}^{f-1} \sum_{j=0}^{r^i-1} \sigma^j \phi^i \equiv f$ . Hence  $(W/W^n)^{e_{\mathfrak{F}}} = W/W^n$  and  $e_{\mathfrak{F}}$  acts as an automorphism of the group in 1.9. In fact  $e_{\mathfrak{F}}$  fixes the group.

## 2. The Principal Genus of $k/k_0$ .

Let  $\Omega^*$  denote the Hilbert class field of a field  $\Omega$ , i.e. its maximal abelian unramified extension, and let  $\Omega^{ab}$  be its abelian closure. The (relative) genus field of  $\Omega$  over a subfield  $\Omega_0$  is defined to be  $\Omega^* \cap \Omega_0^{ab}$ ; and the associated genus group is the factor group of the class group of  $\Omega$  corresponding to this extension of  $\Omega$ . The genus group can also be written as a quotient of the group of ideals in  $\Omega$ , and then the subgroup factored out is called the principal genus.

As before, suppose  $K/k_0$  is a metacyclic Frobenius extension. Then  $K/L$  is cyclic of odd degree  $n$  and its (relative) principal genus is known to be  $P_K I_K^{1-\sigma}$  where  $\sigma$  generates  $\text{Gal}(K/L)$  (vid. [13]). Hasse's analogue ([7] Ia §13) of Hilbert's Theorem 90 shows that this is precisely the group  $P_K \text{Ker } N_{K/L}$  where  $\text{Ker } N_{K/L}$  is the kernel of the norm map  $I_K \rightarrow I_L$ . Thus  $\alpha \in I_K$  is in the principal genus if, and only if,  $N_{K/L} \alpha = N_{K/L}(\alpha)$  for some  $\alpha \in K$ . This interpretation also holds for the principal genus of  $k/k_0$  by Theorem 2.2 (iii). However, the genus number and the ambiguous class number, which coincide for  $K/L$  need not be equal for  $k/k_0$ .

The analogue to Hilbert's Theorem 90 for  $k/k_0$  is:

2.1 Lemma    i) If  $\alpha \in k$  and  $N_{k/k_0} \alpha = 1$  then  $\alpha = N_{K/k}(\beta^{1-\sigma})$   
for some  $\beta \in K^\times$ ;

ii) If  $\alpha \in I_k$  and  $N_{k/k_0} \alpha = (1)$  then  $\alpha = N_{K/k}(\mathcal{A}^{1-\sigma})$   
for some  $\mathcal{A} \in I_K$ .

Proof Let  $S$  be a set of representatives for the conjugacy classes of  $N-1$  under  $F$ . If  $N_{k/k_0} \alpha = 1$  then  $\alpha = \beta^{1-\sigma}$  for some  $\beta \in K^\times$  by Hilbert's Theorem 90. Here  $\beta^{1-\sigma}$  is fixed by  $F$  and so  $\alpha = \beta^{1-\sigma} = (\beta^{1-\sigma})^{\tilde{N} - \sum_{h \in F} \sum_{g \in S} h g h^{-1}} = (\beta^{1-\sigma})^{-\tilde{S}\tilde{F}} = (\beta^{-\tilde{S}})^{(1-\sigma)\tilde{F}} = N_{K/k}((\beta^{-\tilde{S}})^{1-\sigma})$ , as required. The second part is analogous using Hasse's lemma (op.cit.).

2.2 Theorem i) The ambiguous class number of  $k/k_0$  is

$$|C_{k_0}'| |C_K^F| / |C_K^{F(1-\sigma)}|.$$

ii) The genus group of  $k/k_0$  is isomorphic to

$$C_{k_0}' \times C_K^F / C_K^{(1-\sigma)F}.$$

iii) The (relative) principal genus of  $k/k_0$  is  $P_{kI_K}^{(1-\sigma)\tilde{F}}$ ,

i.e. the group of ideals  $\mathcal{O} \in I_k$  such that  $N_{k/k_0} \mathcal{O} = N_{k/k_0}(\alpha)$  for some  $\alpha \in k$ .

A comparison of (i) and (ii) shows that for  $k/k_0$  the ambiguous class number will differ from the genus number if  $C_K^{F(1-\sigma)}$  and  $C_K^{(1-\sigma)F}$  have different orders. This is usually the case for pure fields (vid. Section 3).

Proof The first part is just Theorem 1.1 and the exactness of

$$1 \rightarrow C_K^G \rightarrow C_K^F \rightarrow C_K^{F(1-\sigma)} \rightarrow 1.$$

The maximal abelian extension of  $k_0$  unramified over  $k$  and with degree prime to  $n$  is unramified over  $k_0$  and so corresponds to the

class group  $C_{k_0}$ . The maximal abelian  $n$ -extension of  $k_0$  unramified over  $k$  is the maximal abelian  $n$ -extension of  $k_0$  unramified over  $K$ . It is therefore the maximal abelian  $n$ -extension of  $L$  in  $K^*$  which is fixed under  $F$  (i.e. under the action of  $\text{Gal}(L/k_0)$  suitably extended).

The corresponding genus group for this field is  $C_K/C_K^{1-e_F} C_K^{1-\sigma}$  because the group for the class field of  $k$  is  $C_K/C_K^{1-e_F} \cong C_K^F$  and the genus group for  $K/L$  is  $C_K/C_K^{1-\sigma}$ .

Part (ii) now follows from the exactness of

$$1 \rightarrow (C_K^{1-\sigma})^F \rightarrow C_K^F \rightarrow C_K/C_K^{1-e_F} C_K^{1-\sigma} \rightarrow 1.$$

The genus group itself is therefore  $H_k/C_k^{1-e_N} C_K^{(1-\sigma)\tilde{F}}$  where  $e_N = n^{-1}\tilde{N}$ . Hence the principal genus is the group of ideals with class belonging to  $C_k^{1-e_N} C_K^{(1-\sigma)\tilde{F}}$ . From 2.1(ii) this group is included in  $P_k I_K^{(1-\sigma)\tilde{F}}$ . Conversely, if  $\alpha \in I_K$  and  $\alpha^{(1-\sigma)\tilde{F}}$  is in a class of  $C_k$  then  $\alpha^{(1-\sigma)\tilde{F}(n-\tilde{N})} = \alpha^{(1-\sigma)\tilde{F}n}$  is in a class of  $C_k^{1-e_N}$ . So  $\alpha^{(1-\sigma)\tilde{F}}$  is in a class of  $C_k^{1-e_N}$ , and the principal genus is indeed  $P_k I_K^{(1-\sigma)\tilde{F}}$ . The equivalence of the other formulation in (iii) is clear using 2.1(ii).

2.3 Corollary    The genus group of  $k/k_0$  is isomorphic to

$$N_{k/k_0} I_k / N_{k/k_0} P_k.$$

Proof    Apply  $\tilde{N}$  to  $I_k/P_k I_K^{(1-\sigma)\tilde{F}}$ , which is the genus group, and use the alternative definition of the principal genus in 2.2(iii) to show this is a monomorphism.

Now if  $a \in k_0^\times$  and  $a = N_{K/L} \alpha$  then  $a = N_{k/k_0} (a/N_{K/k} \alpha^{(n-1)/f})$ .

Hence:

2.4 Lemma  $a \in k_0^\times$  is a norm in  $k/k_0$  if, and only if, it is a norm in  $K/L$ .

For each prime ideal  $\mathfrak{p}_i$  ( $1 \leq i \leq t$ ) of  $k_0$  which is ramified in  $K/L$  let  $\mathfrak{B}_i$  be a prime of  $L$  above  $\mathfrak{p}_i$  and for  $a \in k_0^\times$  let  $\chi_i(a) = \left( \frac{a, K/L}{\mathfrak{B}_i} \right)$  be the norm residue symbol. This yields a map  $\chi : k_0^\times \rightarrow N^t$  defined by  $\chi(a) = (\chi_1(a), \chi_2(a), \dots, \chi_t(a))$ .

2.5 Lemma  $a \in k_0^\times$  is a norm in  $k/k_0$  if, and only if,  $a \in \ker \chi$ .

Proof  $a$  is a norm in  $k/k_0 \Leftrightarrow a$  is a norm in  $K/L$  (by 2.4)  
 $\Leftrightarrow a$  is a local norm for every completion of  $K/L$  (since  $K/L$  is cyclic)  $\Leftrightarrow a$  is a local norm for each prime ideal of  $L$  ramified in  $K$  (since the oddness of  $n$  ensures that no infinite valuation is ramified)  $\Leftrightarrow \left( \frac{a, K/L}{\mathfrak{B}} \right) = 1$  for each conjugate  $\mathfrak{B}$  of each prime ideal  $\mathfrak{B}_i \Leftrightarrow \left( \frac{a, K/L}{\mathfrak{B}_i} \right) = 1$  for  $1 \leq i \leq t$  (since  $\left( \frac{a, K/L}{\mathfrak{B}_i^\tau} \right) = \tau^{-1} \left( \frac{a, K/L}{\mathfrak{B}_i} \right) \tau$  for  $\tau \in \text{Gal}(L/k_0)$ )  $\Leftrightarrow \chi(a) = 1$ .

Suppose  $N_{k/k_0}^I$  is the group of ideals in  $k$  which have principal norms in  $k_0$ . If  $\alpha \in N_{k/k_0}^I$  and  $N_{k/k_0} \alpha = (a)$  for  $a \in k_0$  then a homomorphism  $\mathcal{N} : N_{k/k_0}^I \rightarrow \chi(k_0) / \chi(E_{k_0})$  can be defined by  $\mathcal{N}(\alpha) = \chi(a) \text{ mod } \chi(E_{k_0})$ .

2.6 Theorem (cf. [5] & [6])  $\ker \chi$  is the principal genus of  $k/k_0$ .

Proof Assume  $\alpha \in N_{k/k_0} I_k$  satisfies  $N_{k/k_0} \alpha = (a)$ . Then by Theorem 2.2(iii)  $\alpha$  is in the principal genus if, and only if,  $a\varepsilon$  is a norm in  $k/k_0$  for some unit  $\varepsilon$  of  $k_0$ , i.e. if, and only if,  $a\varepsilon \in \ker \chi$ .

When the class number of  $k_0$  is prime to  $n$  the map  $\mathcal{N}$  can be extended to the whole of  $I_k$ . Choose  $h \in \mathbb{Z}$  such that  $hh_{k_0} \equiv 1 \pmod{n}$ . For  $\alpha \in I_k$  with  $N_{k/k_0} \alpha^{hh_{k_0}} = (b)$  we must have  $\mathcal{N}(\alpha)^n = 1$  and therefore  $\mathcal{N}(\alpha) = \mathcal{N}(\alpha^{hh_{k_0}}) = \chi(b^h) \text{mod } \chi(E_{k_0})$ . This is consistent with  $\mathcal{N}$  on  $N_{k/k_0} I_k$  as defined above. Clearly for this extended map  $\ker \mathcal{N}$  is the group of ideals whose  $h_{k_0}$ th power is in the principal genus. Hence:

2.7 Theorem When  $h_{k_0}$  is prime to  $n$  the  $n$ -subgroup of the genus group of  $k/k_0$  is isomorphic to  $\mathcal{N}(I_k)$ .

2.8 Corollary When  $h_{k_0}$  is prime to  $n$  the genus number of  $k/k_0$  divides

$$\frac{h_{k_0} n^t}{[E_{k_0} : E_{k_0} \cap N_{k/k_0} k]}$$

Proof  $\mathcal{N}(I_k)$  is a subgroup of  $\chi(k_0^x) / \chi(E_{k_0})$  and this is a subgroup of  $N^t / \chi(E_{k_0})$ , which has order  $n^t / [E_{k_0} : N_{k/k_0} k \cap E_{k_0}]$ . By the theorem this bounds the  $n$ -component of the genus number, and the factor prime to  $n$  is given precisely by Theorem 2.2(ii).

Remark Putting  $f = 1$  and using the product formula for norm residue symbols to replace  $t$  by  $t-1$  in 2.8 provides the familiar formula for the genus number of  $K/L$ .

### 3. Pure Fields of Prime Degree over $\mathbb{Q}$

Let  $\ell$  be an odd rational prime,  $\zeta$  a primitive  $\ell$ th root of unity, and  $m$  a positive  $\ell$ th power free rational integer. For this section let  $k_0 = \mathbb{Q}$ ,  $k = \mathbb{Q}(\sqrt[\ell]{m})$ ,  $L = \mathbb{Q}(\zeta)$ , and  $K = \mathbb{Q}(\sqrt[\ell]{m}, \zeta)$ . These fields satisfy the hypotheses of the earlier sections. So the strongly ambiguous classes are generated by the primes of  $k$  which are totally ramified over  $\mathbb{Q}$ . From Wegner [12] these are the prime ideals dividing  $(m)$  and, if  $m^{\ell-1} \not\equiv 1 \pmod{\ell^2}$ , also the prime ideal above  $(\ell)$ . Hence:

**3.1 Theorem** Let  $\mathfrak{a}$  be an ambiguous ideal of  $k = \mathbb{Q}(\sqrt[\ell]{m})$ . Then  $\mathfrak{a}^\ell = (a)$  for  $a \in \mathbb{Q}$  defined by  $N_{k/\mathbb{Q}} \mathfrak{a} = (a)$ . Here  $a$  is a product of  $\ell$ th powers, primes dividing  $m$ , and, if  $m^{\ell-1} \not\equiv 1 \pmod{\ell^2}$ , also the prime  $\ell$ . In the case that  $\mathfrak{a}$  is principal,  $a$  is a norm.

**3.2 Theorem** For a rational prime  $p$  and  $a \in \mathbb{Q}^\times$  let  $v_p(a) \in \mathbb{Z}$  denote the multiplicity of  $p$  as a factor of  $a$ . Then  $a$  is a norm in  $k/\mathbb{Q}$  if, and only if,

$$\left( \frac{v_p(a) - v_p(m)}{p-1} \right) \pmod{\ell} \equiv 1 \pmod{p}$$

for all primes  $p$  dividing  $m$  with  $p \equiv 1 \pmod{\ell}$ .

**Proof** By Lemma 2.5  $a$  is a norm in  $k/\mathbb{Q}$  if, and only if,

$$\chi_i(a) = \left( \frac{a, K/L}{\beta_i} \right) = 1 \quad \text{for } 1 \leq i \leq t. \quad \text{Since there is only one prime}$$

ideal in  $L$  above  $(\ell)$  the product formula for norm residue symbols permits this prime to be ignored if it occurs. The remaining ramified primes are the  $p \neq \ell$  which divide  $m$ . Using the properties of Hasse's norm residue and power residue symbols (vid. [7] II §11) for the prime

$$\mathfrak{P} \text{ in } L \text{ above } (p) \neq (\ell) \text{ one obtains } \left( \frac{a, K/L}{\mathfrak{P}} \right) = \left( \frac{a, m}{\mathfrak{P}} \right) = \left( \frac{p, a}{\mathfrak{P}} \right)^{-v_p(m)} \left( \frac{m, p}{\mathfrak{P}} \right)^{v_p(a)}$$

$$= \left( \frac{a}{p} \right)^{v_p(m)} \left( \frac{m}{p} \right)^{-v_p(a)} \left( \frac{p, a}{p} \right)^{-v_p(m)} \left( \frac{m, p}{p} \right)^{v_p(a)}$$

Let  $n(p) = (p^{f(p)} - 1)/\ell$  where  $f(p)$  is the order of  $p$  modulo  $\ell$ . Then  $\ell n(p) = N_{L/\mathbb{Q}} \mathfrak{P}^{-1}$ . So  $\left( \frac{x}{\mathfrak{P}} \right) = 1 \iff x^{n(p)} \equiv 1 \pmod{\mathfrak{P}} \iff x^{n(p)} \equiv 1 \pmod{p}$  for  $x \in \mathbb{Q}$ . Thus  $\left( \frac{a, K/L}{\mathfrak{P}} \right) = 1 \iff (m^{v_p(a)} a^{-v_p(m)})^{n(p)} \equiv 1 \pmod{p}$ . This congruence is automatically satisfied when  $n(p) \equiv 0 \pmod{p-1}$ , and therefore when  $\ell$  does not divide  $p-1$ . Otherwise  $p \equiv 1 \pmod{\ell}$ , which gives  $n(p) = (p-1)/\ell$ . The theorem now follows.

**3.3 Corollary** If  $\mathcal{O}$  is an ambiguous ideal of  $k$  with  $\mathcal{O}^\ell = (a)$  and  $a$  does not satisfy all the congruences of Theorem 3.2 then  $\mathcal{O}$  is not principal.

Proof Combine Theorems 3.1 and 3.2.

Let  $\{p_i \mid 1 \leq i \leq t\}$  be the set of ramified primes as described above, and let  $\{p_i \mid 1 \leq i \leq s\}$  be the subset of  $p \equiv 1 \pmod{\ell}$ . Define  $\chi_i'(a) = (m^{v_p(a)} a^{-v_p(m)})^{(p-1)/\ell} \pmod{p}$  for  $p = p_i$  and  $1 \leq i \leq s$ . Then  $\chi'(a) = (\chi_1'(a), \chi_2'(a), \dots, \chi_s'(a))$  provides a homomorphism in effect from  $\mathbb{Q}^\times$  to  $\mathbb{F}_\ell^s$  where  $\mathbb{F}_\ell$  is the finite field of  $\ell$  elements. By 3.2 the kernel of  $\chi'$  is the subgroup of  $a \in \mathbb{Q}^\times$  which are norms in  $k/\mathbb{Q}$ . Composing this with the map  $v: I_k \rightarrow \mathbb{Q}^\times$  given by

$\alpha \mapsto |a|$  for  $N_{k/\mathbb{Q}} \alpha = (a)$  yields a homomorphism  $\mathcal{N}' : I_k \rightarrow \mathbb{F}_\ell^s$ . As in §2 the kernel of  $\mathcal{N}'$  is the group of ideals whose norms are norms of principal ideals. Thus, as in 2.6 and 2.7,

3.4 Theorem  $\ker \mathcal{N}'$  is the principal genus of  $k/\mathbb{Q}$  and  $|\mathcal{N}'(I_k)|$  is the genus number.

3.5 Theorem i) The genus number of  $k/\mathbb{Q}$  is  $\ell^s$ , i.e.  $\mathcal{N}'$  is surjective;

ii) the order of  $\mathcal{N}'(I_k^N)$  is that of the quotient of strongly ambiguous classes by the subgroup of classes representing ideals of the principal genus;

iii) every ambiguous class is strongly ambiguous, if and only if,  $\zeta \in N_{K/L} E_K$  or  $\zeta \notin N_{K/L} K$ .

Remark ([9] Lemma 4)  $\zeta \in N_{K/L} K$  if, and only if,  $p_i^{\ell-1} \equiv 1 \pmod{\ell^2}$  for  $1 \leq i \leq t$  with  $p_i \neq \ell$ . Thus for most  $m$  every ambiguous class is strongly ambiguous.

Proof Fröhlich has already proved (i) in [3]. Alternatively, (c.f. [1], Theorem 4.2), let  $q$  be a rational prime. Fixing the value of  $\chi_i(q)$  only forces  $q$  to belong to certain arithmetic progressions modulo  $p_i$ . Hence  $\chi' : \mathbb{Q}^\times \rightarrow \mathbb{F}_\ell^s$  is surjective even when restricted to primes  $q \equiv 1 \pmod{\ell}$ . But such primes have prime factors  $\sigma_1$  and  $\sigma_{\ell-1}$  of degree 1 and  $\ell-1$  respectively in  $k$ . So  $v(\sigma_1) = q$  and

$\mathcal{N}' = \chi'_{\circ} \nu$  is surjective. Note that the ideals  $\mathcal{O}_{\mathfrak{l}}$  generate the  $\ell^s$  cosets of the principal genus in  $I_k$ , and give rise to an elementary abelian factor group of the class group of  $k$ .

The second part comes from Theorem 3.4 and the last part from Lemma 1.11.

**3.6 Theorem** (c.f. Fröhlich [3] Theorem 3). Let  $\ell^{s'}$  be the order of  $\mathcal{N}'(I_k^N)$ , and let  $\ell^{t'}$  be the number of strongly ambiguous classes. Then  $t' \geq \max(s', t - (\ell + 1)/2)$  and the  $\ell$ -class number of  $k = \mathbb{Q}(\sqrt[\ell]{m})$  is divisible by

$$\ell^{s+t'-s'}.$$

**Proof** By Theorem 3.5(i) the genus group provides  $\ell^s$  cosets of the principal genus and by (ii) of the same theorem the ambiguous ideals provide  $\ell^{t'-s'}$  classes in the principal genus. The lower bound on  $t'$  is just Corollary 1.7(ii) with Theorem 3.5(ii).

**Remark**  $s, t$ , and  $s'$  can be calculated very easily from  $m$  and the definition of  $\mathcal{N}'$  and so the given lower bound for  $t'$  immediately yields a divisor of the  $\ell$ -class number.

References

- 1 P. Barrucand & H. Cohn. "A Rational Genus, Class Number Divisibility, and Unit Theory for Pure Cubic Fields", J. Number Theory, 2 (1970), 7-21, 3 (1971), 226-239.
- 2 I. Connell & D. Sussman. "The p-dimension of class groups of number fields", J. London Math. Soc., (2) 2 (1970), 525-529.
- 3 A. Fröhlich. "On the  $\ell$ -class group of the field  $\mathbb{P}(\ell\sqrt{m})$ ", J. London Math. Soc., 37 (1962), 189-192.
- 4 F. Gerth. "On  $\ell$ -class groups of certain number fields", Mathematika, 23 (1976), 116-123.
- 5 R. Gold. "Genera in Normal Extensions", Pacific J. Math., 63 (1976), 397-400.
- 6 F. Halter-Koch. "Ein Satz über Geschlechter relativ-zyklischer Zahlkörper von Primzahlgrad und seine Anwendung auf biquadratisch-bizyklische Körper", J. Number Theory, 4 (1972), 144-156.
- 7 H. Hasse. Bericht über neuere Untersuchungen und Probleme aus der Theorie der algebraischen Zahlkörper, (Physicia-Verlag, Würzburg/Wien, 1970).
- 8 L. Holzer. "Zur Klassenzahl in reinen Zahlkörpern von ungeraden Primzahlgrade", Acta Math., 83 (1950), 327-348.
- 9 C. Parry & C. Walter. "The class number of pure fields of prime degree", Mathematika, 23 (1976), 220-226, 24 (1977), 122.
- 10 C. Walter. "Brauer's class number relation", Acta Arith., 35 (1979), 33-40.

- 11 C. Walter. "A class number relation in Frobenius extensions of number fields", Mathematika, 24 (1977), 216-225.
- 12 U. Wegner. "Zur Theorie der auflösbaren Gleichungen von Primzahlgrad", J.f. reine u. angew. Math., 168 (1932), 176-190.
- 13 H. Yokoi. "On the class number of a relatively cyclic number field", Nagoya Math. J., 29 (1967), 31-44.

Department of Mathematics,  
University College,  
Belfield,  
Dublin 4, Ireland.

12A35, 12A50: Algebraic Number Theory:  
global fields: metabelian extensions;  
class number.