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Random matrices, non-colliding processes and queues

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This is survey of some recent results connecting random matrices, non-colliding processes and queues.

1 Introduction

It was recently discovered by Baik, Deift and Johansson [4] that the asymptotic distribution of the length of the longest increasing subsequence in a permutation chosen uniformly at random from S_n, properly centred and normalised, is the same as the asymptotic distribution of the largest eigenvalue of an $n \times n$ GUE random matrix, properly centred and normalised, as $n \to \infty$. This distribution had earlier been identified by Tracy and Widom [54] in the random matrix context, and it is now known as the Tracy-Widom law.

The length of the longest increasing subsequence in a random permutation can be thought of as a 'last-passage time' for a certain directed percolation problem on the plane; this directed percolation problem is closely related to the one-dimensional totally asymmetric exclusion process (with low density of particles travelling at high speed) or equivalently, an infinite series of $M/M/1$ queues in tandem (with low density of customers and high service rates). On the other hand, the eigenvalues of a GUE random matrix of dimension n have the same law as the positions, after a unit length of time, of n independent standard Brownian motions started from the origin and conditioned (in the sense of Doob) never to collide. These remarks are for the purpose of convincing the reader that there might be some interesting connections between random matrices, non-colliding processes and queues. Indeed there are, and that is the topic of this paper.

Let us concentrate on the following, more exact, connection between directed percolation and random matrices which was more recently observed by Baryshnikov [5] and Gravner, Tracy and Widom [26]. Let $B = (B_1, \ldots, B_n)$ be a standard n-dimensional Brownian motion, and write $B(s,t) = B(t) - B(s)$ for $s < t$.

Theorem 1 [Baryshnikov; Gravner-Tracy-Widom] *The random variable*

$$M_n = \sup_{0 \le s_1 \le \cdots \le s_{n-1} \le 1} \{B_n(0, s_1) + \cdots + B_1(s_{n-1}, 1)\}, \tag{1}$$

has the same law as the largest eigenvalue of an n-dimensional GUE random matrix.

The proofs of this fact given in [5] and [26] are based on the Robinson-Shensted-Knuth (RSK) correspondence, and do not make use of the facts that

(a) M_n has a queueing interpretation [24], and

(b) the largest eigenvalue of a GUE random matrix has an interpretation in terms of non-colliding Brownian motions [21, 25].

In [47], a proof is given of a more general result which is based entirely on these interpretations. This more general result identifies a path-transformation $\Gamma_n(B)$ of B, which has the same law as that of n independent Brownian motions conditioned (in the sense of Doob) never to collide. This process, which we denote by $\hat{B}$, is the eigenvalue process associated with Hermitian Brownian motion and $\hat{B}(1)$ has the same distribution as the eigenvalues of an n-dimensional GUE random matrix [21, 25]. The transformed process $\Gamma_n(B)$ has the property that its largest component at time 1 is given by M_n, and so Theorem 1 follows.

At the heart of the proof, which will be outlined in this paper, is a celebrated theorem of classical queueing theory, which states that, in equilibrium, the output of a stable M/M/1 queue is Poisson. This result is usually attributed to Burke, who gave the first proof in 1956. In what follows, we shall also refer to it as an 'output theorem'. It follows from the reversibility of the M/M/1 queue. There is an easy extension of this theorem, which can be proved by similar reversibility arguments; when phrased in the setting of 'max-plus algebra', this extension immediately yields the two-dimensional result. The result in higher dimensions is then obtained by considering a series of queues in tandem and applying an induction argument. The statement of Theorem 1 seems considerably less mysterious (to me at least!) in the setting of queues and non-colliding processes.

In the case $n = 2$, the fact that $\Gamma_n(B)$ and $\hat{B}$ have the same law is equivalent to Pitman's representation for the three-dimensional Bessel process. In the case $n = 3$, it yields a representation for planar Brownian motion conditioned to stay forever in a wedge of angle $\pi/3$; a partial converse of this result was discovered earlier by Biane [6].

As we remarked above, the process $\hat{B}$ has the same law as the eigenvalue process associated with Hermitian Brownian motion. Bougerol and Jeulin [8] have (independently) obtained a similar representation for this process, by completely different methods, which is also consistent with Theorem 1. Their results are presented in the more general setting of Brownian motion on symmetric spaces. The relationship between these two representations will be discussed elsewhere.

There are certain symmetries in the max-plus algebra which play a crucial role, and these symmetries also exist in other algebras, including the conventional algebra on the reals. As a consequence, there are analogues of these output theorems in many different settings, and it seems that there are some general phenomena at work. In all cases, symmetry and reversibility play a key role. We will present some of these examples and their implications.

We will also describe briefly how output theorems and the study of tandem systems can be used to obtain first order asymptotic results for various directed percolation

and directed polymer models.

The outline of the paper is as follows. In the next section we present some background material on random matrices and non-colliding Brownian motions. In section 3 we give a precise statement of the representation theorem obtained in [47], and briefly discuss the special cases $n = 2$ and $n = 3$. In section 4 we describe the output theorem for the M/M/1 queue, how it is proved using reversibility, and how it can be extended using similar reversibility arguments. In section 5 we show how the extended version of Burke's theorem can be used to obtain a representation for non-colliding Poisson processes, and how the representation for non-colliding Brownian motions follows. In section 6 we discuss output theorems generally and give some examples. In section 7 we describe how output theorems can be used to obtain first order asymptotic results for various directed percolation and directed polymer models.

2 Random matrices and non-colliding Brownian motions

Recall that a $n \times n$ GUE random matrix $A \in \mathbb{C}^{n\times n}$ is constructed as follows: it is Hermitian, that is, $A = A^*(= (\bar{A})^t)$; the entries $\{A_{ij},\ i \le j\}$ are independent; on the diagonal A_{ii} are standard real normal random variables; below the diagonal, $\{A_{ij},\ i < j\}$ are standard complex normal random variables, that is, the real and imaginary parts of A_{ij} are independent centered real normal random variables, each with variance 1/2; above the diagonal we set $A_{ji} = \bar{A}_{ij}$. Here, $\bar{z} = x - iy$ denotes the complex conjugate of $z = x+iy$. The joint density (with respect to Lebesgue measure on $\mathbb{R}^n$) of the eigenvalues of A is given by Weyl's formula (see, for example, [39]):

$$f_{GUE}(\lambda_1, \ldots, \lambda_n) = Z^{-1} \prod_{i\neq j}(\lambda_i - \lambda_j) \prod_i \exp(-\lambda_i^2/2), \tag{2}$$

where $Z = (2\pi)^{n/2} \prod_{j=1}^{n-1} j!$.

This distribution has an interpretation in terms of non-colliding Brownian motions. The (Vandermonde) function h defined by

$$h(x) = \prod_{i<j}(x_j - x_i) \tag{3}$$

is harmonic on $\mathbb{R}^n$, and moreover, is a strictly positive harmonic function for Brownian motion killed when it exits the Weyl chamber

$$W = \{x \in \mathbb{R}^n :\ x_1 < x_2 < \cdots < x_n\}. \tag{4}$$

For $x \in \mathbb{R}^n$, let $\mathbb{P}_x$ denote the law of B started at x; for $x \in W$, let $\hat{\mathbb{P}}_x$ denote the law of the h-transform of B started at x, where h is given by (3). The laws $\hat{\mathbb{P}}_x$ and $\mathbb{P}_x$ are related as follows. If T denotes the first exit time of B from W, and $\mathcal{F}_t$ the natural filtration of B, then for $A \in \mathcal{F}_t$,

$$\hat{\mathbb{P}}_x(A) = \mathbb{E}_x\left(\frac{h(B_t)}{h(x)} 1_{A\cap\{T>t\}}\right). \tag{5}$$

It is well-known, and easy to check using the Karlin-MacGregor formula (see, for example, [35]) that

$$\lim_{W \ni x \to 0} \hat{\mathbb{P}}_x(X_t \in dy) = C_t h(y)^2 \mathbb{P}_0(X_t \in dy), \tag{6}$$

where

$$C_t = \left[t^{n(n-1)/2} \prod_{j=1}^{n-1} j! \right]^{-1} \tag{7}$$

is a normalisation constant. Note that the RHS of (6) is equal to $f_{GUE}(y)$ when $t = 1$.

There is a more sophisticated connection at the process level. Note that we can define the law $\hat{\mathbb{P}}_{0+}$, for $A \in \mathcal{T}_t = \sigma(B_u,\ u \geq t)$, $t > 0$, by

$$\hat{\mathbb{P}}_{0+}(A) = \mathbb{P}_0 \left[C_t h(B_t)^2 \hat{\mathbb{P}}_{B_t}(\theta_t A) \right], \tag{8}$$

where θ is the shift operator (so that $\theta_t A \in \mathcal{T}_0$). The fact that this is well-defined follows from (6). Now, *Hermitian Brownian motion* is constructed in the same way as a GUE random matrix, but with Brownian motions instead of normal random variables. It is a fact that the law of process of eigenvalues of Hermitian Brownian motion is given by $\hat{\mathbb{P}}_{0+}$ (see, for example, [21, 25, 39]).

For related work on non-colliding diffusions and random matrices, see [6, 14, 30, 35], and references therein.

3 A generalisation of Theorem 1

Let $D_0(\mathbb{R}_+)$ denote the space of cadlag paths $f : \mathbb{R}_+ \to \mathbb{R}$ with $f(0) = 0$. For $f, g \in D_0(\mathbb{R}_+)$, define $f \otimes g \in D_0(\mathbb{R}_+)$ and $f \odot g \in D_0(\mathbb{R}_+)$ by

$$(f \otimes g)(t) = \inf_{0 \leq s \leq t} [f(s) + g(t) - g(s)], \tag{9}$$

and

$$(f \odot g)(t) = \sup_{0 \leq s \leq t} [f(s) + g(t) - g(s)]. \tag{10}$$

Unless otherwise deleniated by parentheses, the default order of operations is from left to right; for example, when we write $f \otimes g \otimes h$, we mean $(f \otimes g) \otimes h$. Define a sequence of mappings $\Gamma_k : D_0(\mathbb{R}_+)^k \to D_0(\mathbb{R}_+)^k$ by

$$\Gamma_2(f, g) = (f \otimes g, g \odot f), \tag{11}$$

and, for $k > 2$,

$$\begin{aligned} \Gamma_k(f_1, \ldots, f_k) = (&f_1 \otimes f_2 \otimes \cdots \otimes f_k, \\ &\Gamma_{k-1}(f_2 \odot f_1, f_3 \odot (f_1 \otimes f_2), \ldots, f_k \odot (f_1 \otimes \cdots \otimes f_{k-1}))). \end{aligned} \tag{12}$$

The operations $\otimes$ (and $\odot$) arise naturally in a queueing context; they can also be regarded as operator products in the min-plus (resp. max-plus) algebra. We refer the reader to [2] for more about max-plus algebra and its applications.

Let $\hat{B}$ be a realisation of $\hat{\mathbb{P}}_{0+}$, as defined in the previous section. Then the largest eigenvalue of a $n \times n$ GUE random matrix has the same law as $\hat{B}_n(1)$, and Theorem 1 states that M_n and $\hat{B}_n(1)$ have the same law.

Now observe that, if $\Gamma_n(B)_1$ denotes the first component in the n-dimensional process $\Gamma_n(B)$, then

$$\begin{aligned} \Gamma_n(B)_1(t) &= (B_1 \otimes \cdots \otimes B_n)(t) && (13) \\ &= \inf_{0 \le s_1 \le \cdots \le s_{n-1} \le t} \left\{ B^{(1)}_{(0,s_1)} + \cdots + B^{(n)}_{(s_{n-1},t)} \right\}; && (14) \end{aligned}$$

thus, by symmetry, Theorem 1 states that $\Gamma_n(B)_1(1)$ has the same law as the smallest eigenvalue of an n-dimensional GUE random matrix or, equivalently, the random variable $\hat{B}_1(1)$. We remark that, although it is not immediately obvious, it can also be shown that

$$\Gamma_n(B)_n = B_n \odot \cdots \odot B_1, \tag{15}$$

from which it also follows that $M_n = \Gamma_n(B)_n(1)$ has the same law as $\hat{B}_n(1)$. In fact [47]:

Theorem 2 *The processes $\Gamma_n(B)$ and $\hat{B}$ have the same law.*

The proof of this identity in law, given in [47], is based on an extension of the output theorem for a stable $M/M/1$ queue in equilibrium, and certain related symmetries in the max-plus algebra.

In the next section we discuss the output theorem for M/M/1 queues. In Section 5 we show how one can deduce the analogue of Theorem 2 for non-colliding Poisson processes. This is first done in the case of unequal rates, where the non-collision conditioning is non-singular. The case of equal rates follows, and it is then purely a technical exercise to apply Donsker's theorem and recover the Brownian version. We remark that the Poisson case with equal rates is interesting in its own right, as it is closely related to the Charlier orthogonal polynomial ensemble.

As we remarked in the introduction, the case $n = 2$ is equivalent to Pitman's representation for the three-dimensional Bessel process, which states that, if X is a standard one-dimensional Brownian motion and $M(t) = \max_{0<s<t} X(t)$, then $2M - X$ is a three-dimensional Bessel process. The three-dimensional Bessel process, being the h-transform of Brownian motion on $\mathbb{R}_+$ with $h(x) = x$, can also be interpreted as Brownian motion conditioned to stay positive. To see the connection, first note that in this case, $R = (\hat{B}_2 - \hat{B}_1)/\sqrt{2}$ is a three-dimensional Bessel process and $(\hat{B}_2 + \hat{B}_1)/\sqrt{2}$ is a standard Brownian motion, independent of R. On the other hand,

$$\Gamma_2(f,g)_2 + \Gamma_2(f,g)_1 = g + f,$$

and

$$\Gamma_2(f,g)_2 - \Gamma_2(f,g)_1 = g \odot f - f \otimes g = 2m - x,$$

where $x = g - f$ and $m(t) = \max_{0<s<t} x(s)$.

We now consider the statement of Theorem 2 in the case $n = 3$. In this case Brownian motion in the Weyl chamber

$$W_3 = \{x \in \mathbb{R}^3 : \ x_1 < x_2 < x_3\}$$

can be mapped onto Brownian motion in a cone of angle $\pi/3$ on the plane. Let

$$C = \{z \in \mathbb{C} : \ |\arg z| < \pi/6\},$$

and define $\phi : W_3 \to C$ by

$$\phi(x_1, x_2, x_3) = \left(\frac{x_3 - x_1}{\sqrt{2}}, \frac{2x_2 - x_1 - x_3}{2\sqrt{3}} \right).$$

If B is a standard Brownian motion in $\mathbb{R}^3$, then $\phi(B)$ is a standard Brownian motion in $\mathbb{C}$, and moroever, $\phi(W_3) = C$ (see, for example, [6, 45]). It therefore follows from Theorem 2 that $\phi(\Gamma_3(B))$ is a Brownian motion conditioned (in the sense of Doob) to stay in C forever. As remarked by Biane [6], there is only one way to condition on this event.

In fact, Biane [6] proved that, if X is a standard Brownian motion in $\mathbb{C}$ conditioned to stay in C forever, and $J_t = \inf_{u>t} X(u)$, then $\Im(3J - X)$ is a standard one-dimensional Brownian motion. Thus, if we set $X = \phi(\Gamma_3(B))$, then $\Im(3J - X)$ is a standard one-dimensional Brownian motion. The expression $\Im(3J - X)$ can be simplified considerably, but I do not see how to write it as a linear function of B; thus, it is not clear at this point how to recover Biane's result directly from Theorem 2.

4 An extension of Burke's theorem

The stationary M/M/1 queue can be constructed as follows. Let A and S be independent Poisson processes on $\mathbb{R}$ with respective intensities $0 < \lambda < \mu$. For intervals I, open, half-open or closed, we will denote by $A(I)$ the measure of I with respect to dA; for $I = (0, t]$ we will simply write $A(t)$, with the convention that $A(0) = 0$. Similarly for S and any other point process we introduce. For $t \in \mathbb{R}$, set

$$Q(t) = \sup_{s \le t} [A(s,t] - S(s,t]]^+, \tag{16}$$

and for $s < t$,

$$D(s,t] = A(s,t] + Q(s) - Q(t). \tag{17}$$

In the language of queueing theory, A is the *arrivals* process, S is the *service* process, Q is the *queue-length* process, and D is the *departure* process. With this construction it is also natural (and indeed very important for what follows) to define the *unused service* process by

$$U = S - D. \tag{18}$$

We will use the following notation for reversed processes. For a point process X, the reversed process $\bar{X}$ is defined by $\bar{X}(s,t) = X(-t,-s)$. The reversed queue-length process $\bar{Q}$ is defined to be the right-continuous modification of $\{Q(-t), \ t \in \mathbb{R}\}$.

Burke's theorem states that D is a homogeneous Poisson process with intensity λ. On a historical note, this fact was anticipated by O'Brien [43] and Morse [40], and proved in 1956 by Burke [11]. Independently, it was also proved by Cohen [15]. In 1957, Reich [50] gave the following very elegant proof which uses reversibility. The process Q is reversible (in fact, all stationary birth and death processes are reversible). It follows that the joint law of A and D is the same as the joint law of $\bar{D}$ and $\bar{A}$. In particular, $\bar{D}$, and hence D, is a Poisson process with intensity λ.

Burke also proved that, for each t, $\{D(s,t],\ s \leq t\}$ is independent of $Q(t)$. This property is now called *quasi-reversibility*. Note that it also follows from Reich's reversibility argument. Discussions on Burke's theorem and related material can be found in the books of Brémaud [9, 10], Kelly [34] and Robert [51].

Set

$$T = A + U. \tag{19}$$

Theorem 3 *The point processes D and T are independent Poisson processes with respective intensities λ and μ.*

Proof. First note that, given Q, U is a homogeneous Poisson process with intensity μ on the set $I = \{s \in \mathbb{R} :\ Q(s) = 0\}$, and if we let V be another Poisson process with intensity μ on the complement of I, which is conditionally independent of U given Q, then (unconditionally) $N = U + V$ is a homogeneous Poisson process with intensity μ on $\mathbb{R}$ which is independent of Q. Now, (A, S) can be written as a simple function of (Q, N), $(A, S) = \varphi(Q, N)$ say. By construction, we have $(\bar{D}, \bar{T}) = \varphi(\bar{Q}, \bar{N})$. Now we use the reversibility of Q and N to deduce that $(\bar{D}, \bar{T})$, and hence (D, T), has the same law as (A, S), as required. □

Note that

$$Q(t) = \sup_{u>t}[D(t,u) - T(t,u)]. \tag{20}$$

We also have, on $\{Q(0) = 0\}$,

$$\{(D(t), T(t)),\ t \geq 0\} = \{\Gamma_2(A, S)(t),\ t \geq 0\}. \tag{21}$$

Theorem 3 has the following multi-dimensional extension, which relates to a sequence of M/M/1 queues in tandem. Let $A, S_1, \ldots, S_n$ be independent Poisson processes with respective intensities $\lambda, \mu_1, \ldots, \mu_n$, and assume that $\lambda < \min_{i \leq n} \mu_i$. Set $D_0 = A$ and, for $k \geq 1$, $t \in \mathbb{R}$, set

$$Q_k(t) = \sup_{s \leq t}[D_{k-1}(s,t] - S_k(s,t]]^+, \tag{22}$$

and for $s < t$,

$$D_k(s,t] = D_{k-1}(s,t] + Q_k(s) - Q_k(t), \tag{23}$$

$$T_k(s,t] = S_k(s,t] - Q_k(s) + Q_k(t). \tag{24}$$

Theorem 4 *The point processes $D_n, T_1, \ldots, T_n$ are independent Poisson processes with respective intensities $\lambda, \mu_1, \ldots, \mu_n$.*

Proof. By Theorem 3, D_1, T_1 and S_2 are independent Poisson processes with respective intensities λ, μ_1 and μ_2. Applying Theorem 3 again we see that D_2 and T_2 are independent Poisson processes with respective intensities λ and μ_2, and since D_2 and T_2 are determined by D_1 and S_2 they are independent of T_1. Thus D_2, T_1, T_2 and S_3 are independent Poisson processes with respective intensities λ, μ_1, μ_2 and μ_3. And so on. The condition $\lambda < \min_{i \le n} \mu_i$ ensures that this procedure is well-defined. □

By repeated iteration of (22) and (23), we obtain (almost surely)

$$Q_1(0) + \cdots + Q_n(0) = \sup_{s \ge 0}[\bar{A}(s) - (\bar{S}_1 \otimes \cdots \otimes \bar{S}_n)(s)]. \tag{25}$$

Remark. This formula, and variants of it, has been known for some time now. The first formula of this kind was observed by [41] and later extended in [53, 23]. Formulas of this kind can be used to compute first order asymptotics for directed percolation (and polymer) variables, following a program introduced by Seppäläinen [52] (see [44] for a survey). See also [27, 46] for applications in a Brownian context. We give a brief outline of these ideas in Section 7.

Iterating (23) we obtain, for each $k \le n$,

$$D_k(t) + Q_1(t) + \cdots + Q_k(t) = A(t) + Q_1(0) + \cdots + Q_k(0). \tag{26}$$

We also have, by (20),

$$Q_k(t) = \sup_{u > t}[D_k(t,u) - T_k(t,u)]. \tag{27}$$

Applying this repeatedly we obtain

$$Q_1(0) + \cdots + Q_n(0) = \sup_{t > 0}[D_n(t) - (T_1 \otimes \cdots \otimes T_n)(t)]. \tag{28}$$

Note that, on $\{Q_1(0) + \cdots + Q_n(0) = 0\}$,

$$D_n(t) = (A \otimes S_1 \otimes \cdots \otimes S_n)(t), \tag{29}$$

and

$$T_k(t) = (S_k \odot (A \otimes S_1 \otimes \cdots \otimes S_{k-1}))(t), \tag{30}$$

for $t \ge 0$, $k \le n$.

5 Representations for non-colliding processes

We will start by showing how the two-dimensional analogue of Theorem 2 for Poisson processes is an immediate consequence of Theorem 3, the extension of Burke's theorem given in the previous section, and the symmetry formula (20). As before, let A and S be Poisson processes with respective intensities λ and μ.

Theorem 5 *The conditional law of* $\{(A(t), S(t)),\ t \ge 0\}$ *given that* $A(t) \le S(t)$ *for all* $t \ge 0$ *is the same as the unconditional law of* $\{\Gamma_2(A,S)(t),\ t \ge 0\}$.

Proof. By Theorem 3 and the formula (20), the conditional law of $\{(A(t), S(t)),\ t \geq 0\}$ given that $A(t) \leq S(t)$ for all $t \geq 0$ is the same as the conditional law of $\{(D(t), T(t)),\ t \geq 0\}$ given that $Q(0) = 0$. But when $Q(0) = 0$, $(D(t), T(t)) = \Gamma_2(A, S)(t)$ for $t \geq 0$. Moreover, by (16) and the independence of increments of A and S, $\{\Gamma_2(A, S)(t),\ t \geq 0\}$ is independent of $Q(0)$. Therefore, the conditional law of $\{(A(t), S(t)),\ t \geq 0\}$ given that $A(t) \leq S(t)$ for all $t \geq 0$ is the same as the unconditional law of $\{\Gamma_2(A, S)(t),\ t \geq 0\}$, as required. □

Note that we can immediately deduce a discrete version of Pitman's theorem, as follows. If we set $X = S - A$, and $M(t) = \max_{0 \leq s \leq t} X(s)$, then X is a simple random walk with positive drift, and the law of

$$2M - X = \Gamma_2(A, S)_2 - \Gamma_2(A, S)_1$$

is the same as that of X conditioned to stay non-negative.

Now let $N^{(\mu)} = (N_1^{(\mu_1)}, \ldots, N_n^{(\mu_n)})$ be the counting functions of n independent Poisson processes on $\mathbb{R}_+$ with respective intensities $\mu_1 < \mu_2 < \cdots < \mu_n$. That is, $N_k^{(\mu_k)}(t)$ is the measure induced by the kth Poisson process on the interval $(0, t]$, with the convention that $N_k^{(\mu_k)}(0) = 0$. The following result was obtained in [47].

Theorem 6 *The conditional law of $N^{(\mu)}$, given that*

$$N_1^{(\mu_1)}(t) \leq \cdots \leq N_n^{(\mu_n)}(t), \quad \textit{for all } t \geq 0,$$

is the same as the unconditional law of $\Gamma_n(N^{(\mu)})$.

Proof. We prove this by induction on n; the proof for $n = 2$ is given above. Assume that Theorem 3 is true as stated for a particular value of n, and moreover holds for any choice of $\mu_1 < \ldots < \mu_n$. In the above setting we have, by Theorem 4, that $D_n, T_1, \ldots, T_n$ are independent Poisson processes with respective intensities $\lambda, \mu_1, \ldots, \mu_n$. Assume that $\lambda < \mu_1 < \ldots < \mu_n$. By the induction hypothesis, the conditional law of

$$\{(D_n(t), T_1(t), \ldots, T_n(t)),\ t \geq 0\}, \tag{31}$$

given that $T_1(t) \leq \cdots \leq T_n(t)$ for all $t \geq 0$, is the same as the (unconditional) law of

$$\{(D_n(t), \Gamma_n(T_1, \ldots, T_n)(t)),\ t \geq 0\}; \tag{32}$$

therefore, the conditional law of

$$\{(D_n(t), T_1(t), \ldots, T_n(t)),\ t \geq 0\}, \tag{33}$$

given that $D_n(t) \leq T_1(t) \leq \cdots \leq T_n(t)$ for all $t \geq 0$, is the same as the conditional law of

$$\{(D_n(t), \Gamma_n(T_1, \ldots, T_n)(t)),\ t \geq 0\}, \tag{34}$$

given that $D_n(t) \leq (T_1 \otimes \cdots \otimes T_n)(t)$ for all $t \geq 0$. But, by (28), this is precisely the condition that $Q_1(0) + \cdots + Q_n(0) = 0$ or, equivalently, $Q_1(0) = \cdots = Q_n(0) = 0$, and in this case we have, by (29) and (30),

$$(D_n(t), \Gamma_n(T_1, \ldots, T_n)(t)) = \Gamma_{n+1}(A, S_1, \ldots, S_n)(t) \tag{35}$$

for $t \geq 0$; since this latter expression, by independence of increments, is independent of $Q_1(0) + \cdots + Q_n(0)$, we are done. □

Now let $N = (N_1, \ldots, N_n)$ be a collection of independent unit-rate Poisson processes, with $N(0) = (0, \ldots, 0)$. The function h given by (3) is a strictly positive harmonic function for the restriction of the transition kernel of N to the discrete Weyl chamber $E = W \cap \mathbb{Z}^n$. This is well-known to harmonic analysists; a proof is given in [36]. (In fact, h is harmonic for any random walk, or any Lévy process, with exchangeable increments, provided the required moments exist.) Let $x^* + \hat{N}$ be a realisation of the corresponding Doob h-transform, started at $x^* = (0, 1, \ldots, n-1) \in E$ (so that $\hat{N}(0) = (0, \ldots, 0)$). Apart from providing a convenient framework in which to apply Donsker's theorem and deduce Theorem 2 from Theorem 6, the process $\hat{N}$ is interesting in its own right. In [36] it is shown that the random vector $\hat{N}(1)$ is distributed according to the *Charlier* ensemble, a discrete orthogonal polynomial ensemble. That is,

$$P(\hat{N}(t) = y) = C_t h(x^* + y)^2 P(N(t) = y), \tag{36}$$

where C_t is the same (!) normalisation constant as in the Brownian case, given by (7). Thus, the next result, which follows from Theorem 6, yields a representation for the Charlier ensemble. For more on discrete orthogonal polynomial ensembles, see [32].

Theorem 7 *The processes $\hat{N}$ and $\Gamma_n(N)$ have the same law.*

Some elementary potential theory is needed to deduce this from Theorem 6, since the function h is not the only positive harmonic function for the restriction of the transition kernel of N to the discrete Weyl chamber $E = W \cap \mathbb{Z}^n$. We refer the reader to [47] for details of the proof, and to [36] for the necessary asymptotic analysis of the Green's function. We remark that, for any $y \in \mathbb{R}^n_+$, the function defined by $g_y(x) = \mathrm{Schur}_x(y)$ is a positive harmonic function for N on E, and note that g_y is a multiple of h if all the components of y are equal. This follows from the analysis of the Green's function presented in [36] (alternatively it can be deduced from more general representation theoretic arguments, as in [7]).

Theorem 2 (the Brownian version) follows from Theorem 7 by careful application of Donsker's theorem (see [47] for details).

The analogues of Theorems 6 and 7 for discrete-time walks are presented in [36]; in this case, replace 'Poisson' by 'binomial', 'Charlier' by 'Krawtchouk', and modify the definition of Γ_n. The proofs are similar, but there are additional complications due to the fact that the walkers can jump simultaneously; the asymptotic analysis of the Green's function is also more difficult in this case.

6 Output theorems generally

Theorem 3 seems to be a special case of a more general phenomenon. In this section we will present several examples of 'output theorems'; in all cases, reversibility and symmetry play a key role.

The analogue of Theorem 3 holds for Brownian motions with drift. Suppose now that A and S are independent Brownian motions, indexed by $\mathbb{R}$, with respective drifts $\lambda < \mu$. That is, for each s, $\{A(s+t) - A(s),\ t \geq 0\}$ is a Brownian motion with drift λ. Set $A(s,t) = A(t) - A(s)$, $S(s,t) = S(t) - S(s)$,

$$Q(t) = \sup_{s \leq t}[A(s,t) - S(s,t)],$$

$$D(t) = A(t) + Q(0) - Q(t),$$

$$T(t) = S(t) - Q(0) + Q(t),$$

for $t \in \mathbb{R}$. Then the pair (D,T) has the same law as (A,S). Several proofs of this fact are given in [46]; an equivalent result was earlier presented in [29] (see below).

Note that Theorem 3 can also be stated as follows, by considering the process $X = S - A$ and its transform $\tilde{X} = T - D$.

Theorem 8 *Let X be a simple random walk (in continuous time) with positive drift, indexed by $\mathbb{R}$, and set*

$$Q(t) = \sup_{-\infty < s < t} X(s) - X(t),$$

$$\tilde{X}(t) = X(t) + 2[Q(t) - Q(0)].$$

Then $\tilde{X}$ has the same law as X.

Note that, in this setting, the 'symmetry formula' (20) becomes

$$Q(t) = \sup_{u > t} \tilde{X}(t) - \tilde{X}(u). \tag{37}$$

Again, the Brownian version of this result holds: this was the Brownian version of Burke's theorem presented in [29].

The analogue of Theorem 4 can also be shown to hold for Brownian motions with drifts, by exactly the same argument given in the proof of Theorem 4.

We will now describe some output theorems, which are completely analogous to those described above, but in the usual algebra (as opposed to the max-plus algebra). These were given in [46], and motivated by the recent extensions of Pitman's $2M - X$ theorem obtained by Matsumoto and Yor [37] in the context of exponential functionals of Brownian motion. In words, the analogues of Theorems 3 and 8 hold for Brownian motions if we replace 'sup' by '$\log \int \exp$' in the definition of Q. More precisely, suppose that A and S are Brownian motions with respective drifts $\lambda < \mu$ and we set, for $t \in \mathbb{R}$,

$$Q(t) = \log \int_{-\infty}^{t} \exp[A(s,t) - S(s,t)]ds$$

$$D(t) = A(t) + Q(0) - Q(t),$$

$$T(t) = S(t) - Q(0) + Q(t).$$

Then the pair (D, T) has the same law as (A, S). As discussed in [46], symmetry and reversibility play a big role here. In particular, the process Q is reversible, and the analogue of the symmetry formula (37) holds:

$$Q(t) = \log \int_t^\infty \exp[D(t,u) - T(t,u)]du. \tag{38}$$

Let us return to the max-plus version for a moment: intuitively, the statement of Theorem 3 can almost be seen as a direct consequence of the reversibility of the process Q combined with the symmetry formula (20); the only thing which makes this difficult is the non-invertibility of the transformation. This problem does not exist in the usual algebra, and in fact the statement that (D, T) has the same law as (A, S) (when everything is defined using $\log \int \exp$) is actually equivalent to the statement that Q is reversible. For a more precise statement of this equivalence, which holds quite generally, see [46]. The version of Pitman's $2M - X$ theorem obtained in this context by Matsumoto and Yor [37] states that, if X is a Brownian motion with drift and

$$2M(t) = \log \int_0^t e^{2X(s)} ds,$$

then $2M - X$ is a certain diffusion process. Note that by following the arguments given at the start of the last section to deduce Pitman's theorem for random walks from the output theorem 3, we can (formally) recover the Matsumoto-Yor representation and, moreover, see that, in some suitable sense, the law of the diffusion $2M - X$ can be interpreted as the conditional law of X, given that

$$\int_0^\infty e^{2X(s)} ds = 1.$$

Similarly, the analogue of Theorem 4 can be shown to hold in the $\log \int \exp$-world for Brownian motions with drifts, by exactly the same extension argument given in the previous section. Note also that the arguments given in the proof of Theorem 6 can also be (formally) carried over to obtain a $\log \int \exp$-analogue of Theorem 2. This has the following form: define a mapping Π_n analogously with Γ_n but replacing sup by $\log \int \exp$ (and inf by $-\log \int \exp(-)$), in the definitions of $\otimes$ and $\odot$. Then, by these formal arguments, the law of $\Pi_n(B)$ can be interpreted as the conditional law of B, given that

$$\int_0^\infty \exp[B_i(s) - B_{i+1}(s)]ds = 1,$$

for each $i = 1, \ldots, n-1$.

In [28], the following *non-Markovian* version of Pitman's theorem is obtained using similar ideas: reversibility arguments do not require the Markov property.

Theorem 9 *Let $(\xi_k,\ k \geq 0)$ be a Markov chain on $\{-1, +1\}$ with $\xi_0 = 1$ and transition probabilities $P(\xi_{k+1} = 1 |\ \xi_k = 1) = a$ and $P(\xi_{k+1} = -1 |\ \xi_k = -1) = b < a$. Set $X_0 = 0$, $X_n = \xi_1 + \cdots + \xi_n$ and $M_n = \max_{0 \leq k \leq n} X_k$. The process $2M - X$ has the same law as that of X conditioned to stay non-negative.*

The continuous analogue of the above theorem, with ξ replaced by a zig-zag process, is also given in [28].

We end this section with one final example, quite different from the above, to give an indication of how generally these output theorems, and the corresponding representation theorems, hold. It is related to the first-order autoregressive process. Let $(b_n,\ n \in \mathbb{Z})$ be a sequence of independent standard normal random variables, and fix $0 < a < 1$. Consider the stationary process $(X_n,\ n \in \mathbb{Z})$ defined by $X_n = \sum_{j=0}^{\infty} a^{-j} b_{n-j}$. Then $X_{n+1} = aX_n + b_n$ for all n. Now observe that, if we set

$$\hat{b}_n = X_n - aX_{n+1},$$

then $X_n = aX_{n+1} + \hat{b}_n$ and $X_n = \sum_{j=0}^{\infty} a^{-j} \hat{b}_{n+j}$. Now, by the symmetry of this construction, and the fact that the process X is reversible, we see that:

Theorem 10 *The sequences b and $\hat{b}$ have the same law; that is, $(\hat{b}_n,\ n \in \mathbb{Z})$ is a sequence of independent standard normal random variables.*

This is the output theorem. For the analogue of Pitman's theorem in this case, set $Y_0^{(x)} = x$ and, for $n \geq 0$, $Y_{n+1}^{(x)} = aY_n^{(x)} + b_n$,

$$d_n^{(x)} = Y_n^{(x)} - aY_{n+1}^{(x)}.$$

Consider the conditional law of $(b_n,\ n \geq 0)$ given that $\sum_{n=0}^{\infty} a^n b_n = x$. (It is easy to see that this is well-defined.) From the output theorem, and the symmetry formula $X_0 = \sum_{n=0}^{\infty} a^{-n} \hat{b}_n$, this is the same as the law of $(\hat{b}_n,\ n \geq 0)$ given that $X_0 = x$. But, when $X_0 = x$, $\hat{b}_n = d_n^{(x)}$ and, moreover, $(d_n^{(x)},\ n \geq 0)$ is independent of X_0; it follows that

Theorem 11 *The law of the sequence $(d_n^{(x)},\ n \geq 0)$ is the same as the conditional law of $(b_n,\ n \geq 0)$ given that $\sum_{n=0}^{\infty} a^n b_n = x$.*

7 First-order asymptotics for directed percolation and directed polymers

The formula

$$Q_1(0) + \cdots + Q_n(0) = \sup_{s \geq 0} [\bar{A}(s) - (\bar{S}_1 \otimes \cdots \otimes \bar{S}_n)(s)], \tag{39}$$

introduced in Section 4, is very useful for computing first-order asymptotic results for directed percolation variables.

In Section 4, $A, S_1, \ldots, S_n$ are independent Poisson processes with respective rates $\lambda, \mu_1, \ldots, \mu_n$, with $\lambda < \min_i \mu_i$. In this case, the random variables $Q_i(0)$ are independent and geometrically distributed with respective parameters λ/μ_i. Assume $\lambda < 1$.

If we let $\mu_i = 1$, for each i, and divide (39) by n, we can (in principle) let $n \to \infty$ to obtain:

$$\frac{\lambda}{1-\lambda} = \sup_{x>0}[\lambda x - \gamma(x)], \tag{40}$$

where, almost surely,

$$\gamma(x) = \lim_{n\to\infty} \frac{1}{n}(S_1 \otimes \cdots \otimes S_n)(xn). \tag{41}$$

(Here we have implicitly used the reversibility of A, and of S_i, for each i.) The existence of γ follows from Kingman's subadditive ergodic theorem. But (40) is essentially a Legendre transform. In particular, it can be inverted to obtain

$$\gamma(x) = (\sqrt{x} - 1)^2 1_{x>1}.$$

This ingenuous idea is due to Seppäläinen [52], who used it to compute asymptotics for a certain last-passage Bernoulli directed percolation problem. See [44] for a survey on the application of this technique to a variety of directed percolation problems. In each context, there is an associated 'growth model', and these limit theorems can also be interpreted as determining the limiting shape of the associated growth model. See also [31] for related work in this area.

For the Brownian model, we have, almost surely,

$$\lim_{n\to\infty} \frac{1}{n}(B_1 \otimes \cdots \otimes B_n)(xn) = 2\sqrt{x}, \tag{42}$$

where $B_1, B_2, \ldots$ are independent standard Brownian motions. This is described in [46] and proved with sharp uniform concentration estimates in [27]. The corresponding corner growth model is also discussed in that paper. In this case, the formula (39) holds with $A, S_1, \ldots, S_n$ replaced by independent standard Brownian motions with respective drifts $\lambda, 1, \ldots, 1$ and the $Q_i(0)$ are i.i.d. exponentially distributed with parameter λ. Note that the limiting result (42) can also be deduced from Theorem 1 by applying standard results from random matrix theory.

The formula (39) is also valid when 'sup' is replaced by '$\log \int \exp$'. In this context it is used in [46] to compute (formally) the free energy density associated with a certain directed polymer in a random environment. More precisely, for $\beta > 0$, if

$$Z_n(\beta) = \int_{-n<s_1<\cdots<s_{n-1}<0} ds_1 \ldots ds_{n-1} \exp\left\{\beta(B_1(-n, s_1) + \cdots + B_n(s_{n-1}, 0))\right\},$$

where $B_i(s,t) = B_i(t) - B_i(s)$, then, almost surely,

$$f(\beta) = \lim_{n\to\infty} \frac{1}{n} \log Z_n(\beta) = -g(-\beta^2) - 2\log\beta,$$

where

$$g(x) = \sup_{y>0}[xy + \Psi(y)],$$

for $x < 0$, and $\Psi = \Gamma'/\Gamma$ is the digamma function.

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