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OCCUPATION TIMES OF LÉVY PROCESSES AS QUADRATIC VARIATIONS

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Abstract: Bouleau and Yor [BY] have shown that the occupation time of a continuous martingale can be obtained as a quadratic variation. We extend this result to a large class of Lévy processes.

1 - Introduction

We first recall a result established by Bouleau and Yor [BY]. Let Y be a semimartingale such that $\sum_{0\leq s\leq t}|\Delta Y_s| < \infty$. Y admits then the following decomposition

$$Y = Y_0 + M + A \tag{1}$$

where M is a continuous local martingale and A a process with bounded variation.
Let $(\pi_n)_{n\in\mathbf{N}}$ be a sequence of partitions of an interval $[a,b]$ of $I\!R$ such that $|\pi_n|$ converges to 0 as n tends to ∞.
They show that $(\sum_{x_i\in\pi_n}(\int_0^t 1_{(x_i,x_{i+1}]}(Y_{s-})dY_s)^2; n\in\mathbf{N})$ converges in probability uniformly in t on any compact of $I\!R^+$, to

$$\int_0^t 1_{(a,b]}(Y_{s-})d[Y]_s + 2\int_0^t dA_s \int_{(s,t]} dA_u 1_{(a,b]}(Y_{s-})1_{(Y_{u-}=Y_{s-})}. \tag{2}$$

In the particular case when Y is a continuous square-integrable martingale, they obtain the following convergence .

$$\sum_{x_i\in\pi_n}(\int_0^t 1_{(x_i,x_{i+1}]}(Y_s)dY_s)^2 \xrightarrow[n\to\infty]{L^2} \int_0^t 1_{(a,b]}(Y_{s-})d[Y]_s \tag{3}$$

uniformly in t on compacts.
In the special case when Y is a Brownian motion, (3) has also been proved by Perkins [Per].
We establish here a similar result for a large class of Lévy processes.

Theorem 1: *Let $(Y_t, t\geq 0)$ be a Lévy process starting from 0. Assume that Y is not a pure step process and that $\sum_{0\leq s\leq t}|\Delta Y_s| < \infty$.*
Let $(\pi_n)_{n\in\mathbf{N}}$ be a sequence of partitions of an interval $[a,b]$ of $I\!R$ such that $|\pi_n|$ converges to 0 as n tends to ∞. Then

$$\lim_{n\to\infty}\sum_{x_i\in\pi_n}(\int_0^t 1_{(x_i,x_{i+1}]}(Y_{s-})dY_s)^2 = \int_0^t 1_{(a,b]}(Y_{s-})d[Y]_s$$

in probability, uniformly in t on any bounded interval of $\mathbb{R}^+$.

The assumption on the jumps of the Lévy process Y is equivalent to

$$\int (1 \wedge |x|)\nu(dx) < \infty$$

where ν is the Lévy measure of Y.

Remark : If Y is a pure step process, we have immediately thanks to (2)

$$\lim_{n\to} \sum_{x_i \in \pi_n} (\int_0^t 1_{(x_i,x_{i+1}]}(Y_{s-})dY_s)^2$$
$$= \int_0^t 1_{(a,b]}(Y_{s-})d[Y]_s + \sum_{\substack{0\le u,v\le t \\ u\neq v}} 1_{\{a\le Y_{u-}=Y_{v-}\le b\}}(\Delta Y_u)(\Delta Y_v)$$

in probability. There are simple examples of compound Poisson processes such that the second term on the right hand side is different from 0.

Applications : Let $(Y_t\ , t \geq 0)$ be a stable process of index $\alpha \in (0,2)$. We have :

$$[Y]_t = \sum_{0\le s\le t} (\Delta Y_s)^2$$

We note that for any $\lambda > 0$

$$([Y]_{\lambda t}\ , t \geq 0) \overset{(\text{loi})}{=} (\lambda^{2/\alpha}[Y]_t\ , t \geq 0)$$

Consequently $[Y]$ is a stable subordinator of index $\alpha/2$. If α is in $(0,1)$, Y satisfies the assumptions of Theorem 1. We see then that $[Y]$ can be obtained as the limit in probability of $(\sum_{x_i\in\pi_n}(\int_0^t 1_{(x_i,x_{i+1}]}(Y_{s-})dY_s)^2, t \geq 0)$ as $|\pi_n|$ tends to 0, where $(\pi_n, n \in \mathbf{N})$ is a sequence of subdivisions of $\mathbb{R}$.

2 - Proof of Theorem 1

From now on Y is a Lévy process satisfying the assumptions of Theorem 1. Consider the process W defined by : $W_t = Y_t - \sum_{0\le s\le t} \Delta Y_s$. This process is a continuous Lévy process. Let σ be the constant such that : $\mathbb{E}(W_t) = \sigma t$. The process $(W_t - \sigma t, t \geq 0)$ is hence a continuous martingale. Consequently, the process A associated to Y, in the decomposition (1), is equal to $(\sum_{0\le s\le t}(\Delta Y_s) + \sigma t, t \geq 0)$.
Similarly, for every $\alpha > 0$, there exists a constant b_α such that :

$$Y_t = M_t^\alpha + b_\alpha t + V_t^\alpha$$

where : $V_t^\alpha = \sum_{0<s\le t} \Delta Y_s 1_{|\Delta Y_s|\geq\alpha}$,
and M^α is a martingale with bounded jumps such that : $< M^\alpha >_t = c_\alpha t$ (for more details about this general decomposition , see for example [Pro] p.32)
We have to prove that : $\int_0^t dA_s \int_{(s,t]} dA_u 1_{(a,b]}(Y_{s-}) 1_{(Y_{u-}=Y_{s-})}$ is equal to 0.

We start by showing that

$$\int_0^t dV_s^\alpha \int_{(s,t]} dV_u^\alpha 1_{(a,b]}(Y_{s-}) 1_{(Y_{u-}=Y_{s-})} \tag{4}$$

is equal to 0. We write for a fixed $t > 0$:

$$\begin{aligned} &\int_0^t 1_{(v>0)} 1_{\{0\}}(Y_{v-}) dY_v \\ &= \int_0^t 1_{(v>0)} 1_{\{0\}}(Y_{v-}) dM_v^\alpha + \int_0^t 1_{(v>0)} 1_{\{0\}}(Y_{v-}) dV_v^\alpha + b_\alpha \int_0^t 1_{\{0\}}(Y_{v-}) dv \end{aligned} \tag{5}$$

Since M^α is a square integrable martingale, we have :

$$\begin{aligned} \mathbb{E}[(\int_0^t 1_{(v>0)} 1_{\{0\}}(Y_{v-}) dM_v^\alpha)^2] &= \mathbb{E}[\int_0^t 1_{(v>0)} 1_{\{0\}}(Y_{v-}) d[M^\alpha]_v] \\ &= c_\alpha \mathbb{E}[\int_0^t 1_{\{0\}}(Y_{v-}) dv] \end{aligned}$$

Since Y is not a pure step process, we have thanks to Theorem 1 of [BR] : $\mathbb{P}(Y_{v-} = 0) = 0$.
Hence we obtain : $\int_0^t 1_{\{0\}}(Y_{v-}) dv = 0$ and $\int_0^t 1_{(v>0)} 1_{\{0\}}(Y_{v-}) dM_v^\alpha = 0$ a.s.
We note then that V^α converges to 0 as α tends to ∞. Consequently, thanks to (5) :

$$\int_0^t 1_{(v>0)} 1_{\{0\}}(Y_{v-}) dY_v = 0$$

Making use once more of (5), we obtain :

$$\int_0^t 1_{(v>0)} 1_{\{0\}}(Y_{v-}) dV_v^\alpha = 0 \quad \text{a.s.} \tag{6}$$

Thanks to the right continuity of V^α, the process $(\int_0^t 1_{(v>0)} 1_{\{0\}}(Y_{v-}) dV_v^\alpha, t \geq 0)$ is a.s. identically equal to 0. This result remains true if the function $1_{\{0\}}$ is replaced by $1_{\{X\}}$ with X any random variable , independent of Y.
We define now the sequence of stopping times $(T_n)_{n\geq 1}$ by :

$$T_1 = \inf\{s \geq 0 : |\Delta Y_s| \geq \alpha\}$$

$$T_{n+1} = \inf\{s \geq T_n : |\Delta Y_s| \geq \alpha\}$$

Let n be a fixed integer. Conditionally on $\{T_n < +\infty\}$, the process $\tilde{Y} = (Y_{T_n+t} - Y_{T_n}, t \geq 0)$ is independent of $\mathcal{F}_{T_n}$, and has the law of Y. Similarly , we can write the following decomposition of $\tilde{Y}$

$$\tilde{Y}_t = \tilde{M}_t^\alpha + b_\alpha t + \tilde{V}_t^\alpha$$

Note that : $\tilde{V}_t^\alpha = V_{T_n+t}^\alpha - V_{T_n}^\alpha$.
Since the variable ΔY_{T_n} is independent of $\tilde{Y}$, we have , thanks to (6)

$$\int_0^\cdot 1_{(v>0)} 1_{\{-\Delta Y_{T_n}\}}(\tilde{Y}_{v-}) d\tilde{V}_v^\alpha = 0$$

which means that a.s.

$$\int_{T_n}^{\cdot} 1_{(v>T_n)} 1_{\{Y_{(T_n)-}\}}(Y_{v-}) dV_v^\alpha = 0$$

We come back now to the expression (4) :

$$\int_0^t 1_{(a,b]}(Y_{u-}) dV_u^\alpha \int_0^t 1_{(v>u)} 1_{(Y_{u-}=Y_{v-})} dV_v^\alpha$$
$$= \sum_{T_n \le t} 1_{(a,b]}(Y_{T_n-}) \Delta Y_{T_n} \int_{T_n}^t 1_{(v>T_n)} 1_{(Y_{T_n-}=Y_{v-})} dV_v^\alpha$$

Consequently, we have obtained

$$\int_0^t 1_{(a,b]}(Y_s) dV_s^\alpha \int_{(s,t]} 1_{(Y_{s-}=Y_{v-})} dV_v^\alpha = 0 \tag{7}$$

Similarly, we have for every $n > 0$

$$\int_{T_n}^{\cdot} 1_{\{Y_{(T_n)-}\}}(Y_{v-}) dv = 0 \quad \text{a.s.}$$

which leads to

$$\sum_{T_n \le t} 1_{(a,b]}(Y_{T_n-}) \Delta Y_{T_n} \int_{T_n}^t 1_{(Y_{T_n-}=Y_{v-})} dv = 0.$$

We have actually obtained

$$\int_0^t 1_{(a,b]}(Y_{s-}) dV_s^\alpha \int_{(s,t]} 1_{(Y_{s-}=Y_{v-})} dv = 0 \tag{8}$$

The previous argument made at the stopping time T_n, can similarly be written for a fixed time s such that $0 < s < t$. We hence obtain

$$\int_s^t 1_{(v>s)} 1_{(Y_{v-}=Y_{s-})} dV_v^\alpha = 0 \quad \text{and} \quad \int_s^t 1_{(Y_{v-}=Y_{s-})} dv = 0 \quad \text{a.s.}$$

which lead to

$$\int_0^t 1_{(a,b]}(Y_s) ds \int_s^t 1_{(v>s)} 1_{(Y_{v-}=Y_{s-})} dV_v^\alpha = 0 \quad \text{and} \quad \int_0^t 1_{(a,b]}(Y_s) ds \int_s^t 1_{(Y_{v-}=Y_{s-})} dv \tag{9}$$

We set then

$$A_t^\alpha = V_t^\alpha + \sigma t$$

Thanks to (7),(8) and (9), we can write

$$\int_0^t 1_{(a,b]}(Y_s) dA_s^\alpha \int_s^t 1_{(v>s)} 1_{(Y_{v-}=Y_{s-})} dA_v^\alpha = 0$$

Letting α tend to 0, we finally obtain, by dominated convergence

$$\int_0^t 1_{(a,b]}(Y_s) dA_s \int_s^t 1_{(v>s)} 1_{(Y_{v-}=Y_{s-})} dA_v = 0 \quad \square$$

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