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Some remarks on $L^\infty$, $H^\infty$, and $BMO$

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1 Introduction

In [1] C. Dellacherie, P.A. Meyer and M. Yor proved that $L^\infty$ is neither closed nor dense in $BMO$, except in trivial cases (i.e. if the underlying filtration is constant). The same is true for $H^\infty$ (c.f. [3] section 2.6 and [5]). So one may ask, whether it is possible to find a martingale $X \in BMO$, which has a best approximation in $L^\infty$ resp. in $H^\infty$, i.e.

$$\inf_{Z \in L^\infty} \|X - Z\|_{BMO} = \|X - \tilde{Z}\|_{BMO} \text{ for some } \tilde{Z} \in L^\infty$$

resp.

$$\inf_{Z \in H^\infty} \|X - Z\|_{BMO} = \|X - \tilde{Z}\|_{BMO} \text{ for some } \tilde{Z} \in H^\infty.$$ 

It is easy to see that this is equivalent to the question: does there exist a martingale $X \in BMO$ s.t.

$$\inf_{Z \in L^\infty} \|X - Z\|_{BMO} = \|X\|_{BMO}$$

resp.

$$\inf_{Z \in H^\infty} \|X - Z\|_{BMO} = \|X\|_{BMO}.$$ 

holds? R. Durrett poses this problem for $L^\infty$ in [2], p. 214, and he conjectures a solution for $X$. We show in this paper that a discrete time analogue of Durrett’s example works, but in continuous time it does not. In the case of $H^\infty$ we provide a class of processes (including Durrett’s example), for which $\tilde{Z} = 0$ is indeed the best approximation in $H^\infty$. Note that for the negative result in $L^\infty$ we work with the norm $\| \cdot \|_{BMO_1}$, as the problem was posed by Durrett in this way. For the positive result in $H^\infty$ we use $\| \cdot \|_{BMO_2}$, which seems to be more natural in this case.

2 Notations and Preliminaries

We denote by $BMO$ the space of continuous martingales $X$ on a given stochastic basis $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \geq 0}, P)$, satisfying the "usual conditions" of completeness and right continuity, for which the the following equivalent norms are finite

$$\|X\|_{BMO_1} = \sup_T \{\|E[|X_\infty - X_T|]\|_{\mathcal{F}_T}\} = \sup_T \left\{\left(\frac{E[|X_\infty - X_T|]}{P[T < \infty]}\right)^{1/2}\right\}.$$
\[ ||X||_{BMO_2} = \sup_T \{ ||E[(X_{\infty} - X_T)^2|\mathcal{F}_T]|\infty \} = \sup_T \left\{ \left( \frac{E[(X_{\infty} - X_T)^2]}{P[T < \infty]} \right)^{\frac{1}{2}} \right\} = \sup_T \{ ||E[(X_{\infty} - (X)_T)|\mathcal{F}_T]|\infty \}. \]

Here \( T \) runs through all stopping times. In the present context \( H^\infty \) denotes the space of continuous martingales \( M \) on \((\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \geq 0}, P)\) s.t.

\[ ||M||_{H^\infty} = \text{ess sup} \langle M \rangle^\infty < \infty \]

holds. We also use the following standard notation. If \( M \) is a martingale and \( T \) a stopping time, we denote by \( M^T \) the martingale stopped at time \( T \), i.e.

\[ M^T_t = M_{t \wedge T} \]

and by \( ^TM \) the martingale started at time \( T \), i.e.

\[ ^TM = M - M^T. \]

The next easy lemma is maybe folklore, but for the convenience of the reader we provide a proof.

**Lemma 2.1** Let \( X \) be in \( BMO \) and \( R \) an arbitrary stopping time. Then we have

\[ ||R^X||_{BMO_2} \leq ||X||_{BMO_2}. \]

**Proof:** We prove that \( ||\int H dX||_{BMO_2} \leq ||X||_{BMO_2} \), if \( H \) is previsible with \( |H| \leq 1 \), which immediately implies the assertion of the lemma.

\[ ||\int H dX||_{BMO_2} = \sup_T \{ ||E[\int_0^\infty H^2 d(X)|\mathcal{F}_T]|\infty \} = \sup_T \{ ||E[\int_T^\infty d(X)|\mathcal{F}_T]|\infty \} = \sup_T \{ ||E[(X_{\infty} - (X)_T)|\mathcal{F}_T]|\infty \} = ||X||_{BMO_2}^2. \]

\[ \square \]

### 3 The case \( L^\infty \) - a discrete time example

We give in this section an example of a discrete-time process, for which \( \hat{Z} = 0 \) is indeed the best approximation in \( L^\infty \), if we use the space \( bmo_1 \) (c.f. [4]) as an analogue to \( BMO_1 \) in the continuous-time setting. Let \( (W_n)_{n=0}^\infty \) be a standard random walk on \((\Omega, \mathcal{F}, (\mathcal{F}_n)_{n=0}^\infty, P)\) with natural filtration, i.e. \( P[\{\Delta W_n = 1\}] = P[\{\Delta W_n = -1\}] = \frac{1}{2} \) and \( W_0 = 0 \). Let \( T \) be the stopping time \( T = \inf\{n|\Delta W_n = -1\} \), and \( B_n = W_i^n - W_T^n \). This is a discrete-time analogue of the continuous martingale, which we consider in section 4, and which was suggested by Durrett in [2].

Denoting the \( bmo_1 \)-norm by \( || \cdot ||_* \), an easy calculation gives

\[ ||B||_* = \sup_S ||E[|B_{\infty} - B_S||\mathcal{F}_S]|\infty = \sup_{k \in N_0} ||E[|B_{\infty} - B_S||B_S = k]|\infty = \sum_{r=-1}^\infty |r|^2^{-r(r+2)} = 1, \]

\[ \square \]
where the supremum is taken over all stopping times $S$ and $N_0$ denotes the set \{0,1,2,...\}. We denote by $L_\infty (\Omega, \mathcal{F}, (\mathcal{F}_n)_{n=0}^\infty, P)$ the space of all bounded martingales with respect to the given filtration. Our claim is

**Proposition 3.1**

$$\inf_{Z \in L_\infty} \|B - Z\|_* = 1,$$

i.e. $\hat{Z} = 0$ is the best approximation in $L_\infty$ of $B$.

**Proof:** We shall show that assuming the existence of a $Z \in L_\infty$, which fulfills $\|B - Z\|_* = \alpha < 1$, leads to a contradiction.

As the definition of the bmo$_1$-norm is invariant with respect to an additive constant, we assume that $Z \geq 0$ holds. Furthermore the function $f(t) = \|B - tZ\|_*$ is a continuous convex function with $f(0) = 1$ and $f(1) = \alpha$. Therefore we may assume w.l.o.g. that $\|Z\|_\infty < \frac{1}{4}$ holds, and we remain with

$$Z_\infty = a_k \quad \text{on } C_k \text{ for } k = -1,0,1,...,$$

where

$$0 \leq a_k \leq \frac{1}{4} \quad (1)$$

holds, and the atoms $C_k$ are defined by $C_k = \{B_\infty = k\}$. Since the filtration is given by

$$\mathcal{F}_n = \{\mathcal{C}_{-1}, \mathcal{C}_0, ..., \mathcal{C}_{n-2}, (\mathcal{C}_{n-1} \cup \mathcal{C}_n \cup ...\} \quad n = 0,1,2,...,$$

and $P[C_k] = 2^{-(k+2)}$, one can easily calculate $Z_n = E[Z_\infty | \mathcal{F}_n]$ for $n = 0,1,2,...$

$$Z_n = a_k \quad \text{on } C_k \text{ for } k = -1,0,1,...,n-2$$

$$Z_n = \sum_{r=n-1}^\infty a_r 2^{-(r+2-n)} =: \gamma_{n-1} \quad \text{on } (\mathcal{C}_{n-1} \cup \mathcal{C}_n \cup ...)$$

Hence we get for $n = 0,1,2,...$

$$B_\infty - B_n = Z_\infty - Z_n = 0 \quad \text{on } C_{-1} \cup C_0 \cup ... \cup C_{n-2},$$

dep.

$$B_\infty - B_n = s - n \quad \text{on } C_s \text{ for } s = n-1,n,n+1,...$$

and

$$Z_\infty - Z_n = a_s - \gamma_{n-1} \quad \text{on } C_s \text{ for } s = n-1,n,n+1,...$$

As the supremum over all stopping times in the definition of the bmo$_1$-norm can be replaced by a supremum over all fixed times $n$, we calculate $E[|B_\infty - B_n - Z_\infty + Z_n| | \mathcal{F}_n]$, which is 0 on $C_{-1}, C_0, ..., C_{n-2}$ and

$$\sum_{s=-1}^\infty (s - a_{n+s} + \gamma_{n-1}) 2^{-(s+2)}$$

on $C_{n-1} \cup C_n \cup ...$. 
Using eq. (1) and our assumption \( \| B - Z \|_\infty = \alpha < 1 \), we conclude that

\[
(a_{n-1} - \gamma_{n-1}) \frac{1}{2} + |a_n - \gamma_{n-1}| \frac{1}{2} + \sum_{s=n}^{\infty} (-a_{n+s} + \gamma_{n-1}) 2^{-(s+2)} \leq -\rho := \alpha - 1
\]

has to hold for \( n = 0, 1, 2, ... \). We now distinguish two cases.

**Case 1:** \( a_n + \gamma_{n-1} \geq 0 \)

A simple calculation gives

\[
a_{n-1} \leq \gamma_{n-1} - \rho.
\]

**Case 2:** \( a_n + \gamma_{n-1} < 0 \)

In this case we get \( -\gamma_{n-1} + \frac{2}{3}a_{n-1} + \frac{1}{3}a_n \leq -\frac{2}{3}\rho \). This inequality and our assumption in case 2 allow us to conclude that \( a_{n-1} < a_n \) has to hold, and we finally get

\[
a_{n-1} < \gamma_{n-1} - \frac{2}{3}\rho.
\]

Denoting now \( \sigma = \frac{2}{3}\rho > 0 \), we can combine case 1 and case 2, which yields

\[
-a_{n-1} + \gamma_{n-1} > \sigma \quad n = 0, 1, 2, ...
\]

or

\[
-\frac{1}{2}a_{n-1} + \sum_{s=n}^{\infty} a_s 2^{-(s-n+2)} > \sigma \quad n = 0, 1, 2, ...
\]

Defining \( A = \sup_{s=-1,0,...} a_s \), implies the existence of an \( M \), s.t. \( a_M > A - \sigma \), and we infer that

\[
-\frac{1}{2}a_M + \sum_{s=M+1}^{\infty} a_s 2^{-(s-M+1)} < \frac{\sigma}{2}
\]

holds, which is a contradiction to eq. (2).

\[\Box\]

### 4 The case \( L^\infty \) - a continuous time example

In contrast to the discrete case it seems to be not so easy to find a martingale in \( BMO \) in continuous time, which has a best approximation \( \hat{Z} = 0 \) in \( L^\infty \). It is shown in this section that the - in some sense - natural guess of Durrett [2] of a martingale, which is quasi-stationary, in a sense to be defined later, does not work. However, it will be shown in section 5 that this quasistationarity is sufficient to guarantee a best approximation \( \hat{Z} = 0 \) in \( L^\infty \).

Let \( (W_t)_{t \geq 0} \) be a standard Brownian motion on \((\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \geq 0}, P)\). As in [2] we define \( R_0 = 0, R_n = \inf\{t > R_{n-1} : |W_t - W_{R_{n-1}}| > 1\} \), \( N = \inf\{n : W_{R_n} - W_{R_{n-1}} = -1\} \) and finally \( X_t = W_{t \wedge N} \). The following formula is valid for \( \alpha \in (-1, 1) \) (c.f. [2], p. 208)

\[
\|X\|_{BMO_1} = \sup_{T} \|E[|X_\infty - X_T|]\|_\infty = \sup_{a \in (-1, 1)} E[|X_\infty - X_T| | X_T = a] = \sup_{a \in (-1, 1)} 1(\alpha, 1)(1 - a^2) + 1(0, 1)(a + 1)(2 - a) = \frac{9}{8}.
\]

Our claim is now
Proposition 4.1

\[ \inf_{z \in L^\infty} \|X - Z\|_{BMO} < \frac{9}{8}, \]

where \( L^\infty = L^\infty(\Omega, \mathcal{F}, \mathcal{F}, \mathbb{P}) \) is the space of continuous bounded martingales.

This answers negatively the question posed by Durrett in Ex. 1 of sect. 7.7 in [2].

**Proof:** In order to prove the proposition some further notation is needed. We define

\[
A_r = \{ \omega : X_\infty = r \} \quad r = -1, 0, 1, \ldots
\]

\[
S_n = \inf \{ t > R_{n-1} : X_t - X_{R_{n-1}} = \frac{1}{2} \} \quad n = 1, 2, 3, \ldots,
\]

where we use the convention \( \inf 0 = \infty \). Furthermore we need

\[
A_r^+ = A_r \cap \{ S_{r+2} = \infty \}
\]

\[
A_r^- = A_r \cap \{ S_{r+2} < \infty \}, \quad r = -1, 0, 1, \ldots
\]

and finally

\[
M_t = \left\{ \begin{array}{ll}
0 & R_{n-1} \leq t < S_n \\
1 & S_n \leq t < R_n,
\end{array} \right. \quad n = 1, 2, 3, \ldots
\]

The process \( M \) indicates, whether \( X \) has reached the value \( X_{R_{n-1}} + \frac{1}{2} \) in the stochastic interval \([R_{n-1}, R_n]\) or not, and is essential for the calculation of the conditional expectations occurring in the sequel.

A straightforward but lengthy application of the optional stopping theorem yields the following table of conditional probabilities, which we will need later on.

<table>
<thead>
<tr>
<th>( r = 0, 1, 2, \ldots )</th>
<th>(-1 &lt; a \leq \frac{1}{2})</th>
<th>( \frac{1}{2} &lt; a &lt; 1)</th>
</tr>
</thead>
<tbody>
<tr>
<td>( P[A_{r+1}^+ X_T = a, M_T] )</td>
<td>( \frac{1-a}{3}(1 - M_T) )</td>
<td>0</td>
</tr>
<tr>
<td>( P[A_{r+1}^- X_T = a, M_T] )</td>
<td>( \frac{a+1}{6}(1 - M_T) + \frac{1-a}{2} M_T )</td>
<td>( \frac{1-a}{2} )</td>
</tr>
<tr>
<td>( P[A_r^+ X_T = a, M_T] )</td>
<td>( \frac{1+a}{3} )</td>
<td>( \frac{1+a}{3} )</td>
</tr>
<tr>
<td>( P[A_r^- X_T = a, M_T] )</td>
<td>( \frac{1+a}{12} )</td>
<td>( \frac{1+a}{12} )</td>
</tr>
</tbody>
</table>

We define now a bounded continuous martingale \( Z \), which gives a better approximation of \( X \) than the trivial approximation \( Z = 0 \):

\[
Z_\infty = \delta 1_{|X_{R_{n-1}}| \leq A_r^+}
\]

\[
Z_t = E[Z_\infty | \mathcal{F}_t]
\]

with \( \delta > 0 \). This yields

\[
Z_T = \delta P[S_N = \infty | \mathcal{F}_T].
\]

Again it suffices to consider \( a \in (-1, 1) \). Since \( \|Z\|_{BMO} \leq \delta \) holds, we only have to show that

\[
E[|X_\infty - X_T - Z_\infty + Z_T||X_T = a, M_T] < E[|X_\infty - X_T||X_T = a] = \frac{(a + 1)(2 - a)}{2}
\]

holds for \( a \in [\frac{1}{4}, \frac{3}{4}] \) uniformly in \( M_T \), and then to choose \( \delta \) small enough.
Using again the optional stopping theorem, an easy calculation gives the following table.

<table>
<thead>
<tr>
<th>( r = -1, 0, 1, \ldots )</th>
<th>( X_\infty )</th>
<th>( X_T )</th>
<th>( Z_\infty )</th>
<th>( Z_T )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( A_1^2 )</td>
<td>( r )</td>
<td>( a )</td>
<td>( \delta )</td>
<td>( \delta )</td>
</tr>
<tr>
<td>( A_2^2 )</td>
<td>( r )</td>
<td>( a )</td>
<td>( 0 )</td>
<td>( \frac{a+1}{3} )</td>
</tr>
</tbody>
</table>

Putting things together we arrive - after a lot of algebra - at

\[
E[|X_\infty - X_T - Z_\infty + Z_T||X_T = a, M_T|] =
\begin{cases}
\frac{(a+1)/2 - a}{2} + \frac{\delta}{6}(MT(a^2 - 1) + (1 - MT)(-a^2 - a)) & \text{if } 1/4 \leq a \leq 1/2 \\
\frac{(a+1)/2 - a}{2} + \frac{\delta}{6}(a^2 - 1) & \text{if } 1/2 \leq a \leq 3/4,
\end{cases}
\]

which clearly proves our assertion. \( \square \)

5 The case \( H^\infty \)

In this section we introduce a class of processes for which \( \hat{Z} = 0 \) is indeed the best approximation in \( H^\infty \). This class includes also the example of section 4. We start with definitions.

**Definition 5.1** Let \( X \) be in \( BMO \). Then we call a stopping time \( T \) proper for \( X \), if

\[ P[T(X)_T < (X)_{\infty}] > 0. \]

**Definition 5.2** A process \( X \) in \( BMO \) has the property QS (quasi-stationary), if for each proper stopping time \( T \) for \( X \), we can find another proper stopping time \( S \sim T \) P-a.s. for \( X \), s.t. \( S \sim X \) hold. Here \( \sim \) stands for equality in law.

Our next lemma shows that - not very surprisingly - for QS processes the \( BMO \)-norm "does not decline", no matter when the process is started.

**Lemma 5.1** Let \( X \) be in \( BMO \) with the property QS. Then for all proper stopping times \( R \) we have

\[ ||R\hat{X}||_{BMO_2} = ||X||_{BMO_2}. \]

**Proof:** Let \( U \) be a proper stopping time s.t. \( U \geq R \) P-a.s. and \( U\hat{X}1_{\{U \neq 0\}}/P[\{U \neq 0\}] \sim X \) hold. We get

\[ ||R\hat{X}||_{BMO_2} = \sup_T E[(R\hat{X}_T^\infty - R\hat{X}_T^2)] \geq E[(X_\infty - X_T^2)] \sup_{T \geq R} E[|X_\infty - X_T|^2] \geq P[T < \infty] \sup_{T \geq R} \frac{E[(X_\infty - X_T)^2]}{P[T < \infty]} = ||X||_{BMO_2}^2. \]

The reverse inequality follows from Lemma 2.1. \( \square \)

Using a result proved by W. Schachermayer in [5], which characterizes the distance of a given martingale to \( H^\infty \) in \( ||\cdot||_{BMO_2} \), we get our final result.

**Theorem 5.1** Let \( X \) be in \( BMO \) with the property QS. Then we have

\[ \inf_{\hat{Z} \in H^\infty} ||X - Z||_{BMO_2} = ||X||_{BMO_2}. \]
Proof: Assuming the contrary, namely

$$\inf_{Z \in H^\infty} \|X - Z\|_{BMO_2} < \|X\|_{BMO_2},$$

yields, by applying Theorem 1.1 of [5], a finite increasing sequence of stopping times

$$0 = T_0 \leq T_1 \leq \ldots \leq T_N \leq T_{N+1} = \infty$$

s.t.

$$||T_n X_{T_{n+1}}||_{BMO_2} < ||X||_{BMO_2} \quad n = 0, \ldots, N$$

(Without loss of generality we may assume that $T_N$ is a proper stopping time for $X$.) In particular we find

$$||T_n X||_{BMO_2} < ||X||_{BMO_2},$$

which is a contradiction to Lemma 5.1. □

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References


