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# A REMARK ON SLUTSKY'S THEOREM

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## 1. Introduction and Notation.

In Theorem 1 of the paper by [BEKSY] a generalisation of a theorem of Slutsky is used. In this note I will present a necessary and sufficient condition that assures that whenever  $X_n$  is a sequence of random variables that converges in probability to some random variable  $X$ , then for each Borel function  $f$  we also have that  $f(X_n)$  tends to  $f(X)$  in probability. The abstract way of formulating the result has the advantage that it shows how to decompose the problem. The key result is the Dunford-Pettis characterisation of relatively weakly compact subsets of the space  $L^1$ . Because of this immediate relationship I believe that the result is known. However I could not find a reference.

In the sequel  $(\Omega, \mathcal{A}, \mathbb{P})$  is a fixed probability space and  $(E, \mathcal{E})$  is a measurable space. The sequence  $(X_n)_{n \geq 1}$  denotes a sequence of measurable functions of  $\Omega$  into  $E$ . Also  $X$  denotes a measurable function of  $\Omega$  into  $E$ . The distributions (image measures) of  $X_n$ , resp.  $X$  are denoted by  $\mu_n$ , resp.  $\mu$ .

The symbol  $\mathcal{H}$ , a subset of the space of measurable functions from  $E$  into  $\mathbb{R}$ , denotes the set which consists of those functions  $g$  such that  $g(X_n)$  tends to  $g(X)$  in probability. It is clear that  $\mathcal{H}$  satisfies some stability properties. First of all it is clear that  $\mathcal{H}$  is a vector space stable for multiplication, i.e. an algebra. Also if  $\phi: \mathbb{R} \rightarrow \mathbb{R}$  is continuous and  $f \in \mathcal{H}$ , then  $\phi(f) \in \mathcal{H}$ . It follows that for each  $m \geq 0$  and  $f \in \mathcal{H}$ , the truncation  $f^m$  of  $f$  is also in  $\mathcal{H}$ ,  $f^m$  is defined as  $f^m(x) = f(x)$  if  $|f(x)| \leq m$ ,  $f^m(x) = m$  if  $f(x) > m$  and  $f^m(x) = -m$  if  $f(x) < -m$ . Conversely if all the truncations  $f^m$  are in  $\mathcal{H}$ , then also  $f \in \mathcal{H}$ . It is also obvious that  $\mathcal{H}$  is closed for uniform convergence.

Let  $M(E, \mathcal{E})$ ,  $M$  for short, be the space of all signed measures defined on the space  $(E, \mathcal{E})$ . A subset  $K$  of  $M(E, \mathcal{E})$  is said to be relatively weakly compact if it is relatively weakly compact for the weak topology (i.e.  $\sigma(M, M^*)$ ) on  $M$ . The Dunford-Pettis theorem states that  $K$  is relatively weakly compact if and only if there is a probability measure  $\lambda \in M$  such that every element  $\nu \in K$  is absolutely continuous with respect to  $\lambda$  and such that the set  $\{\frac{d\nu}{d\lambda} \mid \nu \in K\}$  of Radon-Nikodym derivatives, is uniformly integrable in  $L^1(\lambda)$ . For information on weak compactness and related topics I refer to [G], last chapter.

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**Theorem 1.** *Let us assume that the set  $(\mu_n)_{n \geq 1}$  of distributions of  $X_n$  is relatively weakly compact. If  $(f_k)_{k \geq 1}$  is a sequence of functions in  $\mathcal{H}$  that converges pointwise to a function  $f$  then also  $f \in \mathcal{H}$ , i.e.  $\mathcal{H}$  is stable for taking pointwise convergent limits.*

*Proof.* Let  $K = \{\mu_n \mid n \geq 1\} \cup \{\mu\}$ . Clearly  $K$  is relatively weakly compact. Because of the stability properties of  $\mathcal{H}$ , we may and do assume that the sequence  $f_k$  is uniformly bounded, e.g. for each  $k$ , we have  $|f_k| \leq 1$ . Since the measures in  $K$  have uniformly integrable RN derivatives, we immediately obtain that  $\sup_{\nu \in K} \int_E |f - f_k| d\nu$  tends to zero. For given  $\epsilon > 0$  we now take  $k_0$  big enough to assure that  $\sup_{\nu \in K} \int_E |f - f_{k_0}| d\nu < \epsilon$ . Now we take  $n_0$  so that for  $n \geq n_0$ ,  $\int_\Omega |f_{k_0}(X_n) - f_{k_0}(X)| d\mathbb{P} < \epsilon$ . For  $n \geq n_0$  we then have  $\int_\Omega |f(X_n) - f(X)| d\mathbb{P} < 3\epsilon$ . This reasoning shows that  $f(X_n)$  tends to  $f(X)$  in  $L^1(\mathbb{P})$  and hence in probability.  $\square$

By a standard argument on monotone classes we can now deduce the next theorem, which I give without proof.

**Theorem 2.** *If the set  $(\mu_n)_{n \geq 1}$  of distributions of  $X_n$  is relatively weakly compact and if  $\mathcal{H} \supset \mathcal{G}$ , then  $\mathcal{H}$  contains all measurable functions with respect to the sigma algebra  $\mathcal{B}$ , generated by  $\mathcal{G}$ .*

In the paper by [BEKSY], the functions  $X_n$  take values in a separable metric space  $S$  and  $X_n$  tend to  $X$  in probability. Since in their case, all the  $X_n$  have the same distribution, it immediately follows that for every Borel function  $h$  on  $S$ , we have that  $h \circ X_n$  tend to  $h \circ X$  in probability. More precisely we have the following.

**Theorem 3.** *Let  $S$  be a metric space and suppose that the sequence of  $S$ -valued random variables  $X_n$  converges to  $X$  in probability. In order that for each Borel measurable function  $f$ , the sequence  $f(X_n)$  converges to  $f(X)$  in probability, it is necessary and sufficient that the sequence of distributions  $(\mu_n)_{n \geq 1}$  is relatively weakly compact.*

*Proof.* The sufficiency is dealt with in Theorem 1 and 2 above. The necessity of the weak compactness condition is rather trivial. Suppose that the sequence of distributions,  $(\mu_n)_{n \geq 1}$ , is not weakly compact. Then there is a bounded measurable function such that  $\int g d\mu_n$  does not converge to  $\int g d\mu$ . It follows that  $g(X_n)$  cannot converge to  $g(X)$  in probability.  $\square$

If in the previous theorem we replace convergence in probability by convergence almost surely, then the statement is wrong. To see this we will give a counterexample. We start with the circle  $\mathbb{T} = \mathbb{R}/\mathbb{Z}$  equipped with the usual normalised Lebesgue measure  $m$ . Let  $O$  be an open subset of  $\mathbb{T}$  such that  $m(O) < 1/2$  and such that  $O$  is dense in  $\mathbb{T}$ . I will construct a sequence  $X_n$ , defined on some probability space, such that  $X_n$  converges to a random variable  $X$  almost surely. All the variables will be distributed uniformly on  $\mathbb{T}$ , i.e.  $\mu_n = m$  for all  $n$ . However it will turn out that the almost sure convergence of  $\mathbf{1}_O(X_n)$  to  $\mathbf{1}_O(X)$  is false. The construction goes as follows. For each  $\delta > 0$  and  $x \in \mathbb{T}$ , we put  $g(\delta, x) = \frac{1}{2\delta} m(O \cap I_\delta^x)$ , where  $I_\delta^x$  is the symmetric interval around  $x$  with length  $2\delta$ . Since

$O$  is dense we obtain that  $g(\delta, x) > 0$  for all  $x \in \mathbb{T}$  and all  $\delta > 0$ . It is now easy to find integers  $(k_l)_{l \geq 1}$  such that for almost every  $x \in \mathbb{T}$  we have that

$$\sum_{l \geq 1} k_l g\left(x, \frac{1}{l+1}\right) = \infty.$$

To construct the variables  $X_n$ , we need a sequence of independent variables  $(V_n)_{n \geq 1}$ , uniformly distributed on  $[-1, 1]$ . The variable  $X$  is taken to be independent of the sequence  $V_n$  and to have a distribution equal to  $m$ . Let us put  $K_0 = 0$  and  $K_{l+1} = K_l + k_{l+1}$ . For each  $n$ ,  $K_l < n \leq K_{l+1}$ , we define  $X_n = X + \frac{1}{l+1} V_n$ . The distributions of the  $X_n$  are easily seen to be equal to  $m$ . Since for almost every  $x \in \mathbb{T}$  we have that

$$\sum_n \mathbb{P}[X_n \in O \mid X = x] = \sum_{l \geq 1} k_l g\left(x, \frac{1}{l+1}\right) = \infty,$$

it follows from independence and the Borel Cantelli lemma that for almost every  $\omega \in X^{-1}(O^c)$ ,  $X_n(\omega) \in O$  infinitely often. The construction of the counterexample is therefore complete.

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#### REFERENCES

- [BEKSY] M.T. Barlow, M.Émery, F.B. Knight, S. Song, M. Yor, *Autour d'un théorème de Tsirelson sur des filtrations browniennes et non browniennes*, Séminaire de Probabilités XXXII (1998).
- [G] A. Grothendieck, *EVT*, Publicação da Sociedade de Matemática e S. Paulo (1954).

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