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A REMARK ON SLUTSKY'S THEOREM

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1. Introduction and Notation.

In Theorem 1 of the paper by [BEKSY] a generalisation of a theorem of Slutsky is used. In this note I will present a necessary and sufficient condition that assures that whenever X_n is a sequence of random variables that converges in probability to some random variable X, then for each Borel function f we also have that $f(X_n)$ tends to $f(X)$ in probability. The abstract way of formulating the result has the advantage that it shows how to decompose the problem. The key result is the Dunford-Pettis characterisation of relatively weakly compact subsets of the space L^1. Because of this immediate relationship I believe that the result is known. However I could not find a reference.
In the sequel $(\Omega, \mathcal{A}, \mathbb{P})$ is a fixed probability space and $(E, \mathcal{E})$ is a measurable space. The sequence $(X_n)_{n\geq 1}$ denotes a sequence of measurable functions of Ω into E. Also X denotes a measurable function of Ω into E. The distributions (image measures) of X_n, resp. X are denoted by μ_n, resp. μ.
The symbol $\mathcal{H}$, a subset of the space of measurable functions from E into $\mathbb{R}$, denotes the set which consists of those functions g such that $g(X_n)$ tends to $g(X)$ in probability. It is clear that $\mathcal{H}$ satifies some stability properties. First of all it is clear that $\mathcal{H}$ is a vector space stable for multiplication, i.e. an algebra. Also if $\phi\colon \mathbb{R} \to \mathbb{R}$ is continuous and $f \in \mathcal{H}$, then $\phi(f) \in \mathcal{H}$. It follows that for each $m \geq 0$ and $f \in \mathcal{H}$, the truncation f^m of f is also in $\mathcal{H}$, f^m is defined as $f^m(x) = f(x)$ if $|f(x)| \leq m$, $f^m(x) = m$ if $f(x) > m$ and $f^m(x) = -m$ if $f(x) < -m$. Conversely if all the truncations f^m are in $\mathcal{H}$, then also $f \in \mathcal{H}$. It is also obvious that $\mathcal{H}$ is closed for uniform convergence.
Let $M(E, \mathcal{E})$, M for short, be the space of all signed measures defined on the space $(E, \mathcal{E})$. A subset K of $M(E, \mathcal{E})$ is said to be relatively weakly compact if it is relatively weakly compact for the weak topology (i.e. $\sigma(M, M^*)$) on M. The Dunford-Pettis theorem states that K is relatively weakly compact if and only if there is a probability measure $\lambda \in M$ such that every element $\nu \in K$ is absolutely continuous with respect to λ and such that the set $\left\{\frac{d\nu}{d\lambda} \mid \nu \in K\right\}$ of Radon-Nikodym derivatives, is uniformly integrable in $L^1(\lambda)$. For information on weak compactness and related topics I refer to [G], last chapter.

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Theorem 1. *Let us assume that the set $(\mu_n)_{n\geq 1}$ of distributions of X_n is relatively weakly compact. If $(f_k)_{k\geq 1}$ is a sequence of functions in $\mathcal{H}$ that converges pointwise to a function f then also $f \in \mathcal{H}$, i.e. $\mathcal{H}$ is stable for taking pointwise convergent limits.*

Proof. Let $K = \{\mu_n \mid n \geq 1\} \cup \{\mu\}$. Clearly K is relatively weakly compact. Because of the stability properties of $\mathcal{H}$, we may and do assume that the sequence f_k is uniformly bounded, e.g. for each k, we have $|f_k| \leq 1$. Since the measures in K have uniformly integrable RN derivatives, we immediately obtain that $\sup_{\nu\in K} \int_E |f - f_k| \, d\nu$ tends to zero. For given $\epsilon > 0$ we now take k_0 big enough to assure that $\sup_{\nu\in K} \int_E |f - f_{k_0}| \, d\nu < \epsilon$. Now we take n_0 so that for $n \geq n_0$, $\int_\Omega |f_{k_0}(X_n) - f_{k_0}(X)| \, d\mathbb{P} < \epsilon$. For $n \geq n_0$ we then have $\int_\Omega |f(X_n) - f(X)| \, d\mathbb{P} < 3\epsilon$. This reasoning shows that $f(X_n)$ tends to $f(X)$ in $L^1(\mathbb{P})$ and hence in probability. □

By a standard argument on monotone classes we can now deduce the next theorem, which I give without proof.

Theorem 2. *If the set $(\mu_n)_{n\geq 1}$ of distributions of X_n is relatively weakly compact and if $\mathcal{H} \supset \mathcal{G}$, then $\mathcal{H}$ contains all measurable functions with respect to the sigma algebra $\mathcal{B}$, generated by $\mathcal{G}$.*

In the paper by [BEKSY], the functions X_n take values in a separable metric space S and X_n tend to X in probability. Since in their case, all the X_n have the same distribution, it immediately follows that for every Borel function h on S, we have that $h \circ X_n$ tend to $h \circ X$ in probability. More precisely we have the following.

Theorem 3. *Let S be a metric space and suppose that the sequence of S-valued random variables X_n converges to X in probability. In order that for each Borel measurable function f, the sequence $f(X_n)$ converges to $f(X)$ in probability, it is necessary and sufficient that the sequence of distributions $(\mu_n)_{n\geq 1}$ is relatively weakly compact.*

Proof. The sufficency is dealt with in Theorem 1 and 2 above. The necessity of the weak compactness condition is rather trivial. Suppose that the sequence of distributions, $(\mu_n)_{n\geq 1}$, is not weakly compact. Then there is a bounded measurable function such that $\int g \, d\mu_n$ does not converge to $\int g \, d\mu$. It follows that $g(X_n)$ cannot converge to $g(X)$ in probability. □

If in the previous theorem we replace convergence in probability by convergence almost surely, then the statement is wrong. To see this we will give a counterexample. We start with the circle $\mathbb{T} = \mathbb{R}/\mathbb{Z}$ equipped with the usual normalised Lebesgue measure m. Let O be an open subset of $\mathbb{T}$ such that $m(O) < 1/2$ and such that O is dense in $\mathbb{T}$. I will construct a sequence X_n, defined on some probability space, such that X_n converges to a random variable X almost surely. All the variables will be distributed uniformly on $\mathbb{T}$, i.e. $\mu_n = m$ for all n. However it will turn out that the almost sure covergence of $\mathbf{1}_O(X_n)$ to $\mathbf{1}_O(X)$ is false.
The construction goes as follows. For each $\delta > 0$ and $x \in \mathbb{T}$, we put $g(\delta, x) = \frac{1}{2\delta} m(O \cap I_\delta^x)$, where I_δ^x is the symmetric interval around x with length 2δ. Since

O is dense we obtain that $g(\delta, x) > 0$ for all $x \in \mathbb{T}$ and all $\delta > 0$. It is now easy to find integers $(k_l)_{l \geq 1}$ such that for almost every $x \in \mathbb{T}$ we have that

$$\sum_{l \geq 1} k_l \, g\left(x, \frac{1}{l+1}\right) = \infty.$$

To construct the variables X_n, we need a sequence of independent variables $(V_n)_{n \geq 1}$, uniformly distributed on $[-1, 1]$. The variable X is taken to be independent of the sequence V_n and to have a distribution equal to m. Let us put $K_0 = 0$ and $K_{l+1} = K_l + k_{l+1}$. For each n, $K_l < n \leq K_{l+1}$, we define $X_n = X + \frac{1}{l+1} V_n$. The distributions of the X_n are easily seen to be equal to m. Since for almost every $x \in \mathbb{T}$ we have that

$$\sum_n \mathbb{P}[X_n \in O \mid X = x] = \sum_{l \geq 1} k_l \, g\left(x, \frac{1}{l+1}\right) = \infty,$$

it follows from independence and the Borel Cantelli lemma that for almost every $\omega \in X^{-1}(O^c)$, $X_n(\omega) \in O$ infinitely often. The construction of the counterexample is therefore complete.

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