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## The Hypercontractivity of Ornstein–Uhlenbeck Semigroups with Drift, Revisited\*

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1. In Qian, He[3] the hypercontractivity of Ornstein–Uhlenbeck semigroup with drift was established in the framework of white noise analysis. Let  $(S) \subset (L^2) \subset (S)^*$  be the Gel'fand's triple over white noise space  $(S'(\mathbf{R}), \mathcal{B}(S'(\mathbf{R})), \mu)$ . Let  $H$  be a strictly positive self-adjoint operator in  $L^2(\mathbf{R})$ . Then

$$P_t^H \varphi(x) = \int_{S'(\mathbf{R})} \varphi(e^{-tH}x + \sqrt{1 - e^{-2tH}}y) \mu(dy), \quad \varphi \in (S), t \geq 0,$$

determines a diffusion semigroup in  $(L^p)$ ,  $p \geq 1$ , called the Ornstein–Uhlenbeck semigroup with drift operator  $H$ . It was shown that if

$$\alpha = \inf_{0 \neq \xi \in S(\mathbf{R})} \frac{(H\xi, H\xi)}{(H\xi, \xi)} > 0, \quad (1.1)$$

then  $(P_t^H)$  is hypercontractive : for any  $p \geq 1$ ,  $q(t) = 1 + (p-1)e^{2\alpha t}$  and nonnegative  $f \in (L^p)$ ,

$$\|P_t^H f\|_{q(t)} \leq \|f\|_p.$$

The proof there was based on Bakry–Emery's local criterion for hypercontractivity by computing Bakry–Emery's curvature of the semigroup  $(P_t^H)_{t \geq 0}$ . In this note we shall point out that Neveu's probabilistic proof ([2]) remains available for the Ornstein–Uhlenbeck semigroup with drift. After recalling Neveu's result, a simple proof for the hypercontractivity of the semigroup  $(P_t^H)_{t \geq 0}$  is given.

The following theorem is indeed extracted from Neveu[2].

**Theorem 1.** *Let  $\{X_t, t \in T, Y_s, s \in S\}$  be a Gaussian process, where  $T$  and  $S$  are two arbitrary index sets. Assume  $\forall t_i \in T, s_j \in S, a_i, b_j \in \mathbf{R}, i = 1, \dots, n, j = 1, \dots, m; n, m \geq 1$*

$$|\rho(\sum_{i=1}^n a_i X_{t_i}, \sum_{j=1}^m b_j Y_{s_j})| \leq r, \quad (1.2)$$

$$p > 1, \quad q > 1, \quad (p-1)(q-1) \geq r^2. \quad (1.3)$$

*Then for any  $\sigma\{X_t, t \in T\}$ -measurable random variable  $\xi$  and  $\sigma\{Y_s, s \in S\}$ -measurable random variable  $\eta$*

$$E|\xi\eta| \leq \|\xi\|_p \|\eta\|_q. \quad (1.4)$$

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2. Now we turn to Ornstein-Uhlenbeck semigroup.

Let  $S(\mathbf{R})$  be the Schwartz space of rapidly decreasing functions on  $\mathbf{R}$  and  $S'(\mathbf{R})$  be its dual space. There exists a unique probability measure  $\mu$  on  $\mathcal{B}(S'(\mathbf{R}))$ , the  $\sigma$ -field generated by cylinder sets, such that

$$\int_{S'(\mathbf{R})} e^{i\langle x, \xi \rangle} \mu(dx) = \exp\left\{-\frac{1}{2}|\xi|_2^2\right\}, \quad \xi \in S(\mathbf{R}).$$

The measure  $\mu$  is called the white noise measure, and the probability space  $(S'(\mathbf{R}), \mathcal{B}(S'(\mathbf{R})), \mu)$  is called the white noise space, which is our basic probability space. Let  $(S), (L^2)$  and  $(S^*)$  be the spaces of test functionals, square-integrable functionals and generalized functionals (or Hida distributions) over  $(S'(\mathbf{R}), \mathcal{B}(S'(\mathbf{R})), \mu)$  respectively. A brief introduction to white noise analysis is given in [3]. More materials on white noise analysis may be referred to Hida et al.[1] and Yan[4].

Let  $H$  be a strictly positive self-adjoint operator in  $L^2(\mathbf{R})$ . Set

$$M_t = e^{-tH}, \quad T_t = \sqrt{1 - e^{-2tH}} = \sqrt{1 - M_{2t}}, \quad t \geq 0. \quad (2.1)$$

The following assumptions are made:

(H<sub>1</sub>)  $S(\mathbf{R}) \subset \mathcal{D}(H)$  and  $H$  is a continuous mapping from  $S(\mathbf{R})$  into itself.

(H<sub>2</sub>)  $\forall t > 0$   $M_t$  and  $T_t$  are continuous operators from  $S(\mathbf{R})$  into itself.

Then  $M_t$  and  $T_t$ ,  $t > 0$ , can be extended onto  $S'(\mathbf{R}) : \forall x \in S'(\mathbf{R}), \xi \in S(\mathbf{R})$ ,

$$\langle M_t x, \xi \rangle = \langle x, M_t \xi \rangle, \quad \langle T_t x, \xi \rangle = \langle x, T_t \xi \rangle. \quad (2.2)$$

Now for all  $t \geq 0$ ,  $x \in S'(\mathbf{R})$  and  $\varphi \in (S)$  define

$$P_t^H \varphi(x) = \int \varphi(M_t x + T_t y) \mu(dy). \quad (2.3)$$

Let  $\Gamma(e^{-tH}) = \Gamma(M_t)$  be the second quantization of  $M_t$ . Then the Ornstein - Uhlenbeck semigroup with drift  $H$

$$P_t^H = \Gamma(e^{-tH}) = e^{-td\Gamma(H)}, \quad t \geq 0, \quad (2.4)$$

is a semigroup with infinitesimal operator  $-d\Gamma(H)$ , where  $d\Gamma(H)$  is a self-adjoint operator in  $(L^2)$ :

$$d\Gamma(H) = \sum_{n=1}^{\infty} \oplus \left\{ \underbrace{H \otimes I \otimes \cdots \otimes I}_{n \text{ terms}} + \underbrace{I \otimes H \otimes I \otimes \cdots \otimes I}_{n \text{ terms}} + \cdots + \underbrace{I \otimes \cdots \otimes H}_{n \text{ terms}} \right\}.$$

The properties of Ornstein-Uhlenbeck semigroup may be referred to [3].

**Theorem 2.** Assume

$$\beta = \inf_{0 \neq \xi \in \mathcal{D}(H)} \frac{(H\xi, \xi)}{(\xi, \xi)} > 0. \quad (2.5)$$

Then for any  $p \geq 1$ ,  $q(t) = 1 + (p-1)e^{2\beta t}$ ,  $t \geq 0$  and  $f \in (L^p)$  with  $f \geq 0$  we have

$$\|P_t^H f\|_{q(t)} \leq \|f\|_p. \quad (2.6)$$

*Proof.* Let  $\varphi, \psi \in (S)$ . Then

$$\langle P_t^H \varphi, \psi \rangle = \iint \varphi(M_t x + T_t y) \psi(x) \mu(dx) \mu(dy). \quad (2.7)$$

Now take

$$(\Omega, \mathcal{F}, \mathbf{P}) = (S'(\mathbf{R}) \times S'(\mathbf{R}), \mathcal{B}(S'(\mathbf{R})) \times \mathcal{B}(S'(\mathbf{R})), \mu \times \mu).$$

We shall discuss on the probability space  $(\Omega, \mathcal{F}, \mathbf{P})$ . Put

$$\begin{aligned} X_\xi(x, y) &= \langle \xi, x \rangle, & \xi \in S(\mathbf{R}), \\ Y_\eta(x, y) &= \langle \eta, M_t x + T_t y \rangle, & \eta \in S(\mathbf{R}). \end{aligned}$$

It is not difficult to see that  $\{X_\xi, \xi \in S(\mathbf{R})\}$  and  $\{Y_\eta, \eta \in S(\mathbf{R})\}$  are jointly normally distributed, and  $\forall \xi, \eta \in S(\mathbf{R})$

$$\mathbf{E}X_\xi = 0, \quad \mathbf{E}Y_\eta = 0,$$

$$\mathbf{E}X_\xi^2 = \|\xi\|_2^2, \quad \mathbf{E}Y_\eta^2 = \|M_t \eta\|_2^2 + \|T_t \eta\|_2^2 = \|\eta\|_2^2.$$

If  $\|\xi\|_2 = \|\eta\|_2 = 1$ , then

$$|\mathbf{E}X_\xi Y_\eta| = |\langle \xi, M_t \eta \rangle| = |\langle e^{-tH} \xi, \eta \rangle| \leq e^{-\beta t} = r.$$

Thus  $\forall \xi, \eta \in S(\mathbf{R})$

$$|\rho(X_\xi, Y_\eta)| \leq r.$$

Noting that  $S(\mathbf{R})$  is a linear space and  $X_\xi, Y_\eta$  are linear in  $\xi, \eta$  respectively,  $\{X_\xi, \xi \in S(\mathbf{R})\}$  and  $\{Y_\eta, \eta \in S(\mathbf{R})\}$  satisfy the condition (1.2). Denote by  $\bar{q}(t)$  the conjugate index of  $q(t)$ :

$$\bar{q}(t) = 1 + \frac{1}{p-1} e^{-2\beta t}.$$

Then we have

$$(p-1)(\bar{q}(t) - 1) = e^{-2\beta t} = r^2.$$

Noting that  $\psi(x)$  and  $\varphi(M_t x + T_t y)$  are measurable with respect to  $\{X_\xi, \xi \in S(\mathbf{R})\}$  and  $\{Y_\eta, \eta \in S(\mathbf{R})\}$  respectively, from (2.7) and Theorem 1 we get

$$|\langle P_t^H \varphi, \psi \rangle| \leq \|\varphi\|_p \|\psi\|_{\bar{q}(t)}, \quad \forall \varphi, \psi \in (S).$$

Hence

$$\|P_t^H \varphi\|_{q(t)} \leq \|\varphi\|_p.$$

By the density of  $(S)$  in  $(L^p)$ , (2.6) follows immediately.  $\square$

By Cauchy - Schwarz inequality for all  $\xi \in S(\mathbf{R})$  we have

$$\frac{(H\xi, H\xi)}{(H\xi, \xi)} \geq \frac{(H\xi, \xi)}{(\xi, \xi)}.$$

Hence  $\alpha \geq \beta$ , and Theorem 2 is weaker than the result in [3]. In the cases when  $\alpha = \beta$ , we arrive at the same conclusion as in [3].

**Lemma 1.** *Let  $\beta > 0$ . Then*

$$\beta = \inf_{0 \neq \xi \in \mathcal{D}(H)} \frac{(H\xi, H\xi)}{(H\xi, \xi)}. \quad (2.8)$$

*Proof.* Denote by  $\gamma$  the right side of (2.8). Obviously, we have  $\gamma \geq \beta$ . Let  $\{E_l, l > 0\}$  be the spectral system of  $H$ .  $\forall \epsilon > 0$ , take  $0 \neq \xi \in \mathcal{D}(H)$  such that  $\xi = (E_{\beta+\epsilon} - E_{\beta-0})\xi$ . Then

$$\frac{(H\xi, H\xi)}{(H\xi, \xi)} = \frac{\int_{[\beta, \beta+\epsilon]} l^2 d(E_l \xi, \xi)}{\int_{[\beta, \beta+\epsilon]} l d(E_l \xi, \xi)} \leq \frac{(\beta + \epsilon)^2}{\beta}.$$

Hence

$$\gamma \leq \frac{(\beta + \epsilon)^2}{\beta}.$$

Letting  $\epsilon \rightarrow 0$  yields  $\gamma \leq \beta$ , and (2.8) follows.  $\square$

**Theorem 3.** *Let  $\beta > 0$ . If  $HS(\mathbf{R})$  is dense in  $L^2(\mathbf{R})$ , then*

$$\alpha = \beta. \quad (2.9)$$

*Proof.* By Lemma 1, it suffices to show

$$\inf_{0 \neq \xi \in S(\mathbf{R})} \frac{(H\xi, H\xi)}{(H\xi, \xi)} = \inf_{0 \neq \xi \in \mathcal{D}(H)} \frac{(H\xi, H\xi)}{(H\xi, \xi)}. \quad (2.10)$$

Under our assumption,  $H^{-1}$  is well defined on  $HS(\mathbf{R})$ , and indeed can be extended as a bounded positive self-adjoint operator on  $L^2(\mathbf{R})$ . Noting that

$$\frac{(H\xi, H\xi)}{(H\xi, \xi)} = \left[ \frac{(H^{-1}H\xi, H\xi)}{(H\xi, H\xi)} \right]^{-1},$$

(2.10) is equivalent to

$$\sup_{0 \neq \eta \in HS(\mathbf{R})} \frac{(H^{-1}\eta, \eta)}{(\eta, \eta)} = \sup_{0 \neq \eta} \frac{(H^{-1}\eta, \eta)}{(\eta, \eta)}. \quad (2.11)$$

Then (2.10) follows from the density of  $HS(\mathbf{R})$  in  $L^2(\mathbf{R})$ .  $\square$

**Remark.** If  $H$  is a continuous operator in  $L^2(\mathbf{R})$ , i.e.  $H$  is a bounded self-adjoint operator, then  $HS(\mathbf{R})$  is dense in  $L^2(\mathbf{R})$ . In fact, in this case  $HS(\mathbf{R})$  is dense in the range of  $H$ . But the range of  $H$  is dense in  $L^2(\mathbf{R})$ , since  $H$  is a strictly positive self-adjoint operator in  $L^2(\mathbf{R})$ .

It is also easy to see that (2.9) holds for  $H = A$ , the self-adjoint extension of the harmonic oscillator (cf. [3])

$$-\frac{d^2}{dx^2} + (x^2 + 1),$$

since the system of the eigenfunctions  $A$  is contained in  $S(\mathbf{R})$ , and forms an orthogonal base of  $L^2(\mathbf{R})$ .

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