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The Hypercontractivity of Ornstein–Uhlenbeck Semigroups with Drift, Revisited*

Sheng-Wu He and Jia-Gang Wang

1. In Qian, He[3] the hypercontractivity of Ornstein– Uhlenbeck semigroup with drift was established in the framework of white noise analysis. Let $(S) \subset (L^2) \subset (S)^*$ be the Gel'fand's triple over white noise space $(S'(\boldsymbol{R}), \mathcal{B}(S'(\boldsymbol{R})), \mu)$. Let H be a strictly positive self-adjoint operator in $L^2(\boldsymbol{R})$. Then

$$P_t^H \varphi(x) = \int_{S'(\boldsymbol{R})} \varphi(e^{-tH}x + \sqrt{1 - e^{-2tH}}y)\mu(dy), \ \varphi \in (S), t \geq 0,$$

determines a diffusion semigroup in (L^p), $p \geq 1$, called the Ornstein–Uhlenbeck semigroup with drift operator H. It was shown that if

$$\alpha = \inf_{0 \neq \xi \in S(\boldsymbol{R})} \frac{(H\xi,\, H\xi)}{(H\xi,\, \xi)} > 0, \tag{1.1}$$

then (P_t^H) is hypercontractive : for any $p \geq 1$, $q(t) = 1 + (p-1)e^{2\alpha t}$ and nonnegative $f \in (L^p)$,

$$\|P_t^H f\|_{q(t)} \leq \|f\|_p.$$

The proof there was based on Bakry–Emery's local criterion for hypercontractivity by computing Bakry–Emery's curvature of the semigroup $(P_t^H)_{t \geq 0}$. In this note we shall point out that Neveu's probabilistic proof ([2]) remains available for the Ornstein–Uhlenbeck semigroup with drift. After recalling Neveu's result, a simple proof for the hypercontractivity of the semigroup $(P_t^H)_{t \geq 0}$ is given.

The following theorem is indeed extracted from Neveu[2].

Theorem 1. *Let $\{X_t,\, t \in T,\, Y_s,\, s \in S\}$ be a Gaussian process, where T and S are two arbitrary index sets. Assume $\forall t_i \in T,\ s_j \in S,\ a_i,\, b_j \in \boldsymbol{R},\ i = 1, \cdots, n,\ j = 1, \cdots, m;\ n,\, m \geq 1$*

$$|\rho(\sum_{i=1}^n a_i X_{t_i}, \sum_{j=1}^m b_j Y_{s_j})| \leq r, \tag{1.2}$$

$$p > 1, \qquad q > 1, \qquad (p-1)(q-1) \geq r^2. \tag{1.3}$$

Then for any $\sigma\{X_t,\, t \in T\}$-measurable random variable ξ and $\sigma\{Y_s,\, s \in S\}$-measurable random variable η

$$\mathsf{E}|\xi\eta| \leq \|\xi\|_p \|\eta\|_q. \tag{1.4}$$

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2. Now we turn to Ornstein-Uhlenbeck semigroup.

Let $S(\boldsymbol{R})$ be the Schwartz space of rapidly decreasing functions on $\boldsymbol{R}$ and $S'(\boldsymbol{R})$ be its dual space. There exists a unique probability measure μ on $\mathcal{B}(S'(\boldsymbol{R}))$, the σ-field generated by cylinder sets, such that

$$\int_{S'(\boldsymbol{R})} e^{i\langle x,\xi\rangle}\mu(dx) = \exp\Big\{-\frac{1}{2}|\xi|_2^2\Big\}, \quad \xi \in S(\boldsymbol{R}).$$

The measure μ is called the white noise measure, and the probability space $(S'(\boldsymbol{R}), \mathcal{B}(S'(\boldsymbol{R})), \mu)$ is called the white noise space, which is our basic probability space. Let $(S), (L^2)$ and (S^*) be the spaces of test functionals, square-integrable functionals and generalized functionals (or Hida distributions) over $(S'(\boldsymbol{R}), \mathcal{B}(S'(\boldsymbol{R})), \mu)$ respectively. A brief introduction to white noise analysis is given in [3]. More materials on white noise analysis may be refered to Hida et al.[1] and Yan[4].

Let H be a strictly positive self-adjoint operator in $L^2(\boldsymbol{R})$. Set

$$M_t = e^{-tH}, \qquad T_t = \sqrt{1-e^{-2tH}} = \sqrt{1-M_{2t}}, \quad t \geq 0. \tag{2.1}$$

The following assumptions are made:

(H_1) $S(\boldsymbol{R}) \subset \mathcal{D}(H)$ and H is a continuous mapping from $S(\boldsymbol{R})$ into itself.

(H_2) $\forall t > 0$ M_t and T_t are continuous operators from $S(\boldsymbol{R})$ into itself.

Then M_t and T_t, $t > 0$, can be extended onto $S'(\boldsymbol{R}) : \forall x \in S'(\boldsymbol{R})$, $\xi \in S(\boldsymbol{R})$,

$$\langle M_t x, \xi\rangle = \langle x, M_t\xi\rangle, \quad \langle T_t x, \xi\rangle = \langle x, T_t\xi\rangle. \tag{2.2}$$

Now for all $t \geq 0, x \in S'(\boldsymbol{R})$ and $\varphi \in (S)$ define

$$P_t^H\varphi(x) = \int \varphi(M_t x + T_t y)\mu(dy). \tag{2.3}$$

Let $\Gamma(e^{-tH}) = \Gamma(M_t)$ be the second quantization of M_t. Then the Ornstein - Uhlenbeck semigroup with drift H

$$P_t^H = \Gamma(e^{-tH}) = e^{-td\Gamma(H)}, \qquad t \geq 0, \tag{2.4}$$

is a semigroup with infinitesimal operator $-d\Gamma(H)$, where $d\Gamma(H)$ is a self-adjoint operator in (L^2):

$$d\Gamma(H) = \sum_{n=1}^{\infty} \oplus\{\underbrace{H\otimes I\otimes\cdots\otimes I}_{n \text{ terms}} + \underbrace{I\otimes H\otimes I\otimes\cdots\otimes I}_{n \text{ terms}} + \cdots + \underbrace{I\otimes\cdots\otimes H}_{n \text{ terms}}\}.$$

The properties of Ornstein-Uhlenbeck semigroup may be refered to [3].

Theorem 2. *Assume*

$$\beta = \inf_{0\neq\xi\in\mathcal{D}(H)} \frac{(H\xi,\,\xi)}{(\xi,\,\xi)} > 0. \tag{2.5}$$

Then for any $p \geq 1$, $q(t) = 1 + (p-1)e^{2\beta t}$, $t \geq 0$ *and* $f \in (L^p)$ *with* $f \geq 0$ *we have*

$$\|P_t^H f\|_{q(t)} \leq \|f\|_p. \tag{2.6}$$

Proof. Let $\varphi,\,\psi \in (S)$. Then

$$\langle P_t^H \varphi,\,\psi\rangle = \iint \varphi(M_t x + T_t y)\psi(x)\mu(dx)\mu(dy). \tag{2.7}$$

Now take

$$(\Omega,\,\mathcal{F},\,\mathsf{P}) = (S'(\boldsymbol{R}) \times S'(\boldsymbol{R}),\,\mathcal{B}(S'(\boldsymbol{R})) \times \mathcal{B}(S'(\boldsymbol{R})),\,\mu \times \mu).$$

We shall discuss on the probability space $(\Omega,\,\mathcal{F},\,\mathsf{P})$. Put

$$X_\xi(x,y) = \langle \xi,\,x\rangle, \qquad \xi \in S(\boldsymbol{R}),$$
$$Y_\eta(x,y) = \langle \eta,\,M_t x + T_t y\rangle, \quad \eta \in S(\boldsymbol{R}).$$

It is not difficult to see that $\{X_\xi,\,\xi \in S(\boldsymbol{R})\}$ and $\{Y_\eta,\,\eta \in S(\boldsymbol{R})\}$ are jointly normally distributed, and $\forall\xi,\,\eta \in S(\boldsymbol{R})$

$$\mathsf{E}X_\xi = 0, \quad \mathsf{E}Y_\eta = 0,$$
$$\mathsf{E}X_\xi^2 = \|\xi\|_2^2, \quad \mathsf{E}Y_\eta^2 = \|M_t\eta\|_2^2 + \|T_t\eta\|_2^2 = \|\eta\|_2^2.$$

If $\|\xi\|_2 = \|\eta\|_2 = 1$, then

$$|\mathsf{E}X_\xi Y_\eta| = |\langle \xi,\,M_t\eta\rangle| = |\langle e^{-tH}\xi,\,\eta\rangle| \leq e^{-\beta t} = r.$$

Thus $\forall\xi,\,\eta \in S(\boldsymbol{R})$

$$|\rho(X_\xi,\,Y_\eta)| \leq r.$$

Noting that $S(\boldsymbol{R})$ is a linear space and X_ξ, Y_η are linear in ξ, η respectively, $\{X_\xi,\,\xi \in S(\boldsymbol{R})\}$ and $\{Y_\eta,\,\eta \in S(\boldsymbol{R})\}$ satisfy the condition (1.2). Denote by $\overline{q}(t)$ the conjugate index of $q(t)$:

$$\overline{q}(t) = 1 + \frac{1}{p-1}e^{-2\beta t}.$$

Then we have

$$(p-1)(\overline{q}(t) - 1) = e^{-2\beta t} = r^2.$$

Noting that $\psi(x)$ and $\varphi(M_t x + T_t y)$ are measurable with respect to $\{X_\xi,\,\xi \in S(\boldsymbol{R})\}$ and $\{Y_\eta,\,\eta \in S(\boldsymbol{R})\}$ respectively, from (2.7) and Theorem 1 we get

$$|\langle P_t^H \varphi,\,\psi\rangle| \leq \|\varphi\|_p \|\psi\|_{\overline{q}(t)}, \quad \forall\varphi,\,\psi \in (S).$$

Hence

$$\|P_t^H \varphi\|_{q(t)} \leq \|\varphi\|_p.$$

By the density of (S) in (L^p), (2.6) follows immediately. □

By Cauchy - Schwarz inequality for all $\xi \in S(\boldsymbol{R})$ we have

$$\frac{(H\xi,\,H\xi)}{(H\xi,\,\xi)} \geq \frac{(H\xi,\,\xi)}{(\xi,\,\xi)}.$$

Hence $\alpha \geq \beta$, and Theorem 2 is weaker than the result in [3]. In the cases when $\alpha = \beta$, we arrive at the same conclusion as in [3].

Lemma 1. *Let $\beta > 0$. Then*

$$\beta = \inf_{0 \neq \xi \in \mathcal{D}(H)} \frac{(H\xi,\, H\xi)}{(H\xi,\, \xi)}. \tag{2.8}$$

Proof. Denote by γ the right side of (2.8). Obviously, we have $\gamma \geq \beta$. Let $\{E_l,\, l > 0\}$ be the spectral system of H. $\forall \epsilon > 0$, take $0 \neq \xi \in \mathcal{D}(H)$ such that $\xi = (E_{\beta+\epsilon} - E_{\beta-0})\xi$. Then

$$\frac{(H\xi,\, H\xi)}{(H\xi,\, \xi)} = \frac{\int_{[\beta,\beta+\epsilon]} l^2 d(E_l\xi,\, \xi)}{\int_{[\beta,\beta+\epsilon]} l d(E_l\xi,\, \xi)} \leq \frac{(\beta+\epsilon)^2}{\beta}.$$

Hence

$$\gamma \leq \frac{(\beta+\epsilon)^2}{\beta}.$$

Letting $\epsilon \to 0$ yields $\gamma \leq \beta$, and (2.8) follows. □

Theorem 3. *Let $\beta > 0$. If $HS(\boldsymbol{R})$ is dense in $L^2(\boldsymbol{R})$, then*

$$\alpha = \beta. \tag{2.9}$$

Proof. By Lemma 1, it suffices to show

$$\inf_{0 \neq \xi \in S(\boldsymbol{R})} \frac{(H\xi,\, H\xi)}{(H\xi,\, \xi)} = \inf_{0 \neq \xi \in \mathcal{D}(H)} \frac{(H\xi,\, H\xi)}{(H\xi,\, \xi)}. \tag{2.10}$$

Under our assumption, H^{-1} is well defined on $HS(\boldsymbol{R})$, and indeed can be extended as a bounded positive self-adjoint operator on $L^2(\boldsymbol{R})$. Noting that

$$\frac{(H\xi,\, H\xi)}{(H\xi, \xi)} = \left[\frac{(H^{-1}H\xi,\, H\xi)}{(H\xi,\, H\xi)}\right]^{-1},$$

(2.10) is equivalent to

$$\sup_{0 \neq \eta \in HS(\boldsymbol{R})} \frac{(H^{-1}\eta,\, \eta)}{(\eta,\, \eta)} = \sup_{0 \neq \eta} \frac{(H^{-1}\eta,\, \eta)}{(\eta,\, \eta)}. \tag{2.11}$$

Then (2.10) follows from the density of $HS(\boldsymbol{R})$ in $L^2(\boldsymbol{R})$. □

Remark. If H is a continuous operator in $L^2(\boldsymbol{R})$, i.e. H is a bounded self-adjoint operator, then $HS(\boldsymbol{R})$ is dense in $L^2(\boldsymbol{R})$. In fact, in this case $HS(\boldsymbol{R})$ is dense in the range of H. But the range of H is dense in $L^2(\boldsymbol{R})$, since H is a strictly positive self-adjoint operator in $L^2(\boldsymbol{R})$.

It is also easy to see that (2.9) holds for $H = A$, the self-adjoint extension of the harmonic oscillator (cf. [3])

$$-\frac{d^2}{dx^2} + (x^2 + 1),$$

since the system of the eigenfunctions A is contained in $S(\boldsymbol{R})$, and forms an orthogonal base of $L^2(\boldsymbol{R})$.

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