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DIFFEOMORPHISMS OF THE CIRCLE AND THE BASED STOCHASTIC LOOP SPACE

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Let $L_x(M)$ be the based loop space of a compact manifold M, that is, the space of all continuous applications γ from the circle $S^1 = \mathbb{R}/\mathbb{Z}$ to the manifold M such that $\gamma_0 = x$.

Let α be a diffeomorphism of S^1 fixing the point $0 \in S^1$. We get an application $\tilde{\alpha} : L_x(M) \to L_x(M)$ by putting:

$$\tilde{\alpha}(\gamma) = \gamma \circ \alpha \,.$$

In other words, the space of diffeomorphisms of the circle acts on the based loop space by reparametrization of the loop.

If we put on M a Riemannian structure, we can consider the law P_x of the Brownian bridge; it is a measure on $L_x(M)$. $\tilde{\alpha}$ does not preserve P_x; more precisely, P_x and its image by $\tilde{\alpha}$ are in general singular with respect to each other. So the group of diffeomorphisms of the circle does not act on the Brownian bridge.

The purpose of this paper is to make diffeomorphisms act on a space of stochastic loops by using the theory of quasi-invariant measures on the group of diffeomorphisms of the circle due to [Sh], [K] and [M.M].

Construction of the probabilistic model.

Let $(B_s)_{0 \leqslant s \leqslant 1}$ be a Brownian motion. Using B, we can define a (random) diffeomorphism of $[0,1]$ by

$$\phi(B)_t = \frac{\int_0^t \exp B_s \, ds}{\int_0^1 \exp B_s \, ds} \,. \tag{1}$$

The law of $\phi(B)$ is a measure $dP(\phi)$ on the set of increasing diffeomorphisms of $[0,1]$. (See [K], [M.M].)

Consider the Brownian bridge measure dP_x on $L_x(M)$. We introduce another measure on $L_x(M)$ which takes care of all possible orientation-preserving reparametrizations of the loops. Denoting by $F(\gamma_{s_1}, ..., \gamma_{s_r})$ a bounded cylindrical function on $L_x(M)$, we put

$$E_{tot}[F] = E_{P(\phi)}\left[E_{P_x}[F(\gamma_{\phi_{s_1}^{-1}}, ..., \gamma_{\phi_{s_r}^{-1}})]\right] \,. \tag{2}$$

In other words, considering the Brownian bridge γ and the family of all possible time-changes ϕ, we work on all time-changed Brownian bridges $\gamma_{\phi_s^{-1}}$, and we average in γ and in ϕ. From a loop $\gamma_{\phi_s^{-1}}$, the associated clock ϕ_s^{-1} can be recovered via the (Riemannian) quadratic variation $|\phi_s'^{-1}|Id$ of the path.

The action of $\tilde{\alpha}$ can be seen as the transformation $\phi \mapsto \alpha^{-1} \circ \phi$ at the level of the time-change.

In the sequel, we will denote by $\mathrm{Diff}_0^{\infty,+}(S^1)$ the group of all diffeomorphisms of the circle preserving the orientation and the origin $0 \in S^1$.

Infinitesimal action of $\mathrm{Diff}_0^{\infty,+}(S^1)$.

The bijection given by (1) can be seen at another level:

$$B_t = \mathrm{Log}\, \phi_t' - \mathrm{Log}\, \phi_0' \;;$$

considering the diffeomorphism $\alpha^{-1}\phi_t = \phi_{\alpha,t}$, so that

$$\phi_{\alpha,t}' = (\alpha^{-1})'(\phi_t)\phi_t' \,,$$

we get

$$\begin{aligned} \mathrm{Log}\, \phi_{\alpha,t}' - \mathrm{Log}\, \phi_{\alpha,0}' &= \mathrm{Log}\, \phi_t' - \mathrm{Log}\, \phi_0' + \mathrm{Log}\, (\alpha^{-1})'(\phi_t) - \mathrm{Log}\, (\alpha^{-1})'(0) \\ &= B_t + u_{\alpha,t} \,. \end{aligned} \tag{3}$$

Let $(k_t)_{t \in S^1}$ be a smooth vector field on S^1 such that $k_0 = 0$. Putting

$$\alpha_{\lambda,t} = [\exp \lambda k]_t \,,$$

formula (3) can be written for $\alpha_{\lambda,t}$. Differentiating it with respect to λ at $\lambda = 0$, we get infinitesimally a vector field (see [N]) on the Brownian motion B, of the form $j(\phi_t) - j(0)$ for some smooth function j on $[0,1]$. This vector field is anticipative and has finite energy; denote it by $H\big(k(\phi)\big)$ and consider the derivative D_H associated with H:

$$D_H \phi_t = \frac{\int_0^t (\exp B_s)\, H_s\, ds}{\int_0^1 \exp B_s\, ds} - \frac{\int_0^t \exp B_s\, ds}{\int_0^1 \exp B_s\, ds} \, \frac{\int_0^1 (\exp B_s)\, H_s\, ds}{\int_0^1 \exp B_s\, ds} \,. \tag{4}$$

Its kernels are bounded; therefore $H\big(k(\phi)\big)$ is bounded in all Sobolev spaces of first order (see [N]). Hence, we get the following integration by parts formula (see [N]):

$$E_B\big[<dF, H\big(k(\phi)\big)>\big] = E_B\big[F \ \mathrm{div}\, H\big(k(\phi)\big)\big] \,, \tag{5}$$

where $\mathrm{div}\, H\big(k(\phi)\big)$ is bounded in all L^p-spaces.

Consider now all functionals on $L_x(M)$ that are linear combinations of products of functionals of the type $\tilde{F} = \int_0^1 F(\gamma_s) h_s\, ds$, where h is deterministic and smooth on S^1 and where F is bounded. These functionals form a dense linear subspace K of L^2_{tot}.

Let us compute $\tilde{F}\big(\tilde{\alpha}(\gamma)\big)$:

$$\tilde{F}\big(\tilde{\alpha}(\gamma)\big) = \int_0^1 F(\gamma_{\alpha_s}) h_s\, ds = \int_0^1 F(\gamma_s) h_{\alpha_s^{-1}}\, d\alpha_s^{-1} \,.$$

In particular, if $\alpha_s = [\exp \lambda k]_s$, $\tilde{F}\big(\tilde{\alpha}(\gamma)\big)$ is differentiable in λ; the differential can be seen as the action of a formal vector field $k.\gamma'$ defined on $L_x(M)$.

This formal vector field can be considered as an operator H_k from the vector space K to L^2_{tot}. By (5) and the Fubini theorem in (2), we get for $F \in K$:

$$E_{tot}[<dF, H_k>] \leqslant C \, \|F\|_{L^2_{tot}} \; .$$

We deduce, since K is dense in L^2_{tot}:

THEOREM: The formal vector field $H_k = k_.\gamma'_.$ can be seen as a densely defined, closable operator on L^2_{tot}.

REMARK: In [L], we have defined the "formal vector field" $c'_.$ over γ, by using the rotational invariance of the B.H.K. measure on the free loop space. It is the generator of a periodic group of isometries, hence it is antisymmetric, therefore closable. But here we do not have a Haar measure on the space of diffeomorphisms of the circle, so we cannot expect to obtain the "vector field" $k_.\gamma'_.$ as some antisymmetric operator on $L_x(M)$ for a suitable measure. We only have a quasi-invariant measure on the space of diffeomorphisms of the circle; and this allows to define the adjoint to the formal vector field $k_.\gamma'_.$ by integration by parts. This leads to the next chapter.

Global action of $\mathrm{Diff}_0^{\infty,+}(S^1)$.

Let us come back to the transformation (3) and suppose that α is not too far from identity in the C^3 sense. From (4), the finite-variation norm of Du_α is smaller than $c < 1$. Therefore $Id + Du_\alpha$ is a bijection from the space of curves with an L^1-derivative to itself. And if $(Id + Du_\alpha)H$ has finite energy, H has finite energy too. This shows that $Id + Du_\alpha$ is a bijection from the Cameron-Martin space to itself. On the other hand, the transformation acts bijectively on Brownian paths, for it is already bijective at the level of the diffeomorphisms of the circle. Therefore the law of $(Id + u_\alpha)(B_.)$ is equivalent to the law of B (see [N]).

To get rid of the restriction that α is close to identity, it suffices to notice that the space of C^3-diffeomorphisms is arcwise-connected and to express every diffeomorphism α as a product of finitely many diffeomorphisms close to identity. Thus the law of $\alpha^{-1} \circ \phi$ is equivalent to the law of ϕ.

Therefore the mapping $\tilde{\alpha}$ is an absolutely continuous transformation of the loop space endowed with the probability P_{tot}. Denote by J_α the quasi-invariance density.

THEOREM: The mapping

$$\psi_\alpha : F \mapsto J_\alpha^{1/2} \, F\big(\tilde{\alpha}(\gamma)\big)$$

is a unitary transformation of L^2_{tot}. And $\alpha \mapsto \psi_\alpha$ is a unitary representation of $\mathrm{Diff}_0^{\infty,+}(S^1)$ over L^2_{tot}.

A proof can be found in [K] and other proofs of the quasi-invariance formula in [K] or in [M.M].

Chen forms.

For almost all $\gamma \in L_x(M)$, the parallel transport τ_s along γ is defined (using the Levi-Civita connection) and compatible with the time-changing.

Let $\Omega_.(M)$ be the space of all forms on M with degree larger than 1; elements of $\Omega_.(M)^n$ will be denoted $\tilde{\omega} = (\omega_1, ..., \omega_n)$. On $\Omega_.(M)^n$ it is possible to define the stochastic Chen form

$$\sigma(\tilde{\omega}) = \int_{0<s_1<...<s_n<1} \omega_1(d\gamma_{s_1}, \, .\,) \wedge ... \wedge \omega_n(d\gamma_{s_n}, \, .\,)$$

(see [J.L]). The tangent space to $L_x(M)$ at γ, called $T_\gamma L_x(M)$, is the space of all vector fields along γ having the form $\tau_s H_s$, where H has finite energy and verifies $H_0 = H_1 = 0$ (see [B], [J.L]). $T_\gamma L_x(M)$ is endowed with the norm $\int_0^1 \|H'_s\|^2 ds$. Every $\alpha \in \mathrm{Diff}_0^{\infty,+}(S^1)$ acts as an operator from $T_\gamma L_x(M)$ to $T_{\tilde{\alpha}(\gamma)} L_x(M)$ by time-changing. This operator is continuous, but is not an isometry. To make it an isometry, we change the Hilbert structure by

$$\|H\|_\gamma^2 = \int_0^1 \|H'_s\|^2 \left(\tfrac{d}{ds}<\gamma,\gamma>_s\right)^{-1} ds \, .$$

This formula makes sense because the quadratic variation $<\gamma,\gamma>_s Id$ of of γ has finite energy; it yelds the isometry property because the quadratic variation of γ behaves nicely under time-changing.

Moreover, noticing that stochastic integration too is compatible with time-changing, we get the following stochastic analogue to a remark of [G.J.P]:

THEOREM: $\sigma(\tilde{\omega})$ is invariant under the action of $\mathrm{Diff}_0^{\infty,+}(S^1)$.

REMARK: Both these results, isometry and invariance, extend to the case of the free loop space. This can be done by using the measure introduced by [M.M] on the full space of diffeomorphisms of the circle.

Bibliography.

[B] Bismut J.M.: Large deviations and the Malliavin calculus. Progress in Math. 45, Birkhäuser (1984).

[G.J.P] Getzler E., Jones J.D.S., Petrack S.: Differential forms on loop spaces and the cyclic bar complex. Topology 30, 339–373 (1971).

[J.L] Jones J.D.S., Léandre R.: L^p Chen forms on loop spaces. In "Stochastic Analysis", M. T. Barlow and N. H. Bingham editors, 104–162, Cambridge University Press (1991).

[K] Kosyak A.V.: Irreducible regular gaussian representations of the group of the interval and circle diffeomorphism. J. Funct. Anal. 125, 493–547 (1994).

[L] Léandre R.: Invariant Sobolev Calculus on the free loop space. To appear in Acta Applicandae Mathematicae.

[M.M] Malliavin M.P., Malliavin P.: An infinitesimally quasi-invariant measure on the group of diffeomorphisms of the circle. In "Integration on loop groups", Publication de l'Université Paris VI (1990).

[N] Nualart D.: The Malliavin calculus and related topics. Springer (1995).

[Sh] Shavgulidze E.T.: Distributions on infinite-dimensional spaces and second quantization in string theories II. In "V$^{\text{th}}$ International Vilnius Conference in Probability and Mathematical Statistics", 359-360 (1989).

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