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Some Remarks on Pitman's Theorem

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Abstract — Pitman constructs $\mathrm{BES}^3(0)$ as $2M-X$ where X is $\mathrm{BM}^1(0)$ and $M_t = \sup_{r\leq t} X_r$. Equivalently, $X-2J$ is $\mathrm{BM}^1(0)$ when X is $\mathrm{BES}^3(0)$ and $J_t = \inf_{r\geq t} X_r$. Now the fact that $X-2J$ gives a local martingale may be extended to a general result for linear diffusions. In particular, if X is a linear diffusion, we introduce a general class of nontrivial transformations φ such that $Z = \varphi(X, J)$ is a local martingale.

In his fundamental paper [6] Pitman constructed $\mathrm{BES}^3(0)$ as $2M - X$ where X is $\mathrm{BM}^1(0)$ and $M_t = \sup_{r\leq t} X_r$ using random walk approximations. This result immediately leads to the path decompositions for BES^3 obtained by Williams [16], so it was of natural interest whether the path transformation $2M - X$ gives again a diffusion when X is a linear diffusion and $M_t = \sup_{r\leq t} X_r$ is defined as above. Yet, this $2M - X$ property just holds for a few more types of diffusions including BM with constant drift and may not be extended to a wider class of diffusions as was proved in Rogers-Pitman [9], Rogers [8], see also Salminen [10].

Apart from diffusions on the real line, further generalizations of Pitman's Theorem in different directions were obtained by Tanaka [13], [14] for random walks, Bertoin [1], [2] for certain Lévy processes, and Biane [3] for the planar Brownian motion in a cone. Recently, the case of a general class of diffusions was taken up again by Saisho and Tanemura [11] using solutions of Skorokhod type SDEs with globally Lipschitz continuous coefficients.

Now Pitman's result amounts to the same as saying that $X - 2J$ is a $\mathrm{BM}^1(0)$ when X is $\mathrm{BES}^3(0)$ and $J_t = \inf_{r\geq t} X_r$. In particular, the process $X-2J$ is observed to be a local martingale, and this fact was recently extended by Yor [17] to a general result about diffusions on the real line, hereby using techniques from the theory of enlargement of filtrations initiated by Jeulin [4], [5]. For instance, let $(X_t)_{t\geq 0}$ be a transient diffusion in $]0,\infty[$ which satisfies the Itô type SDE

$$dX = d\beta + b(X)\,dt$$

where $b\colon]0,\infty[\to \mathbb{R}$ allows uniqueness in law for this equation. Further assume that a scale function s can be chosen to satisfy $s(]0,\infty[) =]-\infty,0[$. Then, by Theorem 12.7 in Yor's book [17], the process

$$Z = -\frac{1}{s(X)} + \frac{2}{s(J)}$$

is a local martingale where $J_t = \inf_{r\geq t} X_r$. In the particular case of BES^δ, these martingales were studied in depth by Takaoka [15] thus revealing some interesting relations and differences to Brownian motion. In this paper, using elementary methods, we construct local martingales of the type $Z = \varphi(X, J)$ for linear diffusions and a whole class of transformations φ.

Consider an open interval $E =]c_1, c_2[\subseteq \mathbb{R}$ with $-\infty \leq c_1 < c_2 \leq \infty$ and let $\sigma, b \colon E \to \mathbb{R}$ be continuous functions with $\sigma > 0$. By $\bar{E} = E \cup \{c_1, c_2\}$ we denote the two-point-compactification of E. Let (X, β) denote a weak solution of the Itô type SDE

$$(*) \qquad dX = \sigma(X)\, d\beta + b(X)\, dt$$

in the following sense: Both X and β are adapted processes defined on a filtered probability space $(\Omega, \mathscr{F}, P; (\mathscr{F}_t))$ satisfying the usual conditions such that β is a linear $(\mathscr{F}_t)$-BM, X is an $\bar{E}$-valued continuous adapted process, and such that for any $f \in C_c^2(E)$ (compactly supported C^2 functions on E)

$$d\,f(X) = (f'\sigma)(X)\, d\beta + (Lf)(X)\, dt$$

holds. Hereby, $L = \frac{1}{2}\sigma^2 D^2 + bD$ denotes the generator of X. Note that X is allowed to launch from the boundary and hit the boundary in possibly finite life time

$$\zeta = \inf\{t > 0 \colon X_t \in \{c_1, c_2\}\}$$

which gives a predictable stopping time. We suppose $\zeta > 0$. Clearly, the solutions of the equation $(*)$ become uncontrolled when reaching the boundary, yet there is uniqueness in law up to the life time as may be proved by changing scale and speed. Using the same method we obtain general existence of solutions when starting from a point within E whereas coming in from the boundary requires the drift to be enough singular there.

Further we pick a function $s \in C^2(E)$ with $Ls = 0$ and $s' > 0$, called *scale function*. We suppose $s(E) =]-\infty, 0[$ which enforces $X_0 < c_2$ a.s. and the process X to be transient, namely $X_t \to c_2$ as $t \to \zeta$, a.s.

Associated with X we consider the continuous increasing process $J_t = \inf_{r \geq t} X_r^\zeta$ valued in $\bar{E}$, representing the absolute minimum of X past time t. Here, $X_r^\zeta = X_{\zeta \wedge r}$. In order to get both X and J adapted, we introduce the filtration $(\mathscr{G}_t)$ defined as the usual augmentation of $\mathscr{G}_t^0 = \sigma(X_r, J_r : r \leq t)$, $t \geq 0$. Then the main result is as follows.

THEOREM. *Let X denote a solution of the equation $(*)$ with positive life time ζ and scale function s as described above. Pick $\mu \in C^1(E)$ and consider $\varphi \colon E \times E \to \mathbb{R}$,*

$$\varphi(x, y) = \left(\frac{s^2 \mu'}{s'}\right)(y) - \frac{1}{s(x)} \left(\frac{s^2 (\mu s)'}{s'}\right)(y).$$

Then the process $Z = \varphi(X, J)$ is a local martingale with respect to $(\mathscr{G}_t)$ on $]0, \zeta[$.

We postpone the proof. Note that on $]0, \zeta[$ both the processes X and J take their values in E, hence Z is defined. The assertion of the theorem is to be understood in the sense of Sharpe [12] as follows. Let Ω_0 denote the set where $Z_0 = \lim_{t \searrow 0} Z_t$ exists in $\mathbb{R}$. Then there is a sequence (R_n) of $(\mathscr{G}_t)$-stopping times with $0 \leq R_n < \zeta$, $R_n \searrow 0$, and $\{R_n = 0\} \subseteq \Omega_0$ such that $(Z_{R_n + t})_{t < \zeta - R_n}$ is a local martingale over $(\mathscr{G}_{R_n + t})$, for each n. The reason for this technical statement is that X may start from the boundary c_1 and then Z_0 is not defined. Nevertheless, conditionally on Ω_0, the process $(Z_t)_{0 \leq t < \zeta}$ gives a local martingale over $(\mathscr{G}_t)$ in the usual sense, cf. [12].

Before entering the proof of our theorem we present some corollaries and discuss the connections with the previous results mentioned above.

First of all, taking $\mu = -1/s^2$ and $\mu = 1$ in the theorem, we obtain Yor's result as well as a multiplicative version thereof.

COROLLARY 1. *Both the processes*

$$-\frac{1}{s(X)} + \frac{2}{s(J)}, \quad -\frac{s^2(J)}{s(X)}$$

are local martingales with respect to $(\mathscr{G}_t)$ *on* $]0, \zeta[$.

Considering $Z = -1/s(X) + 2/s(J)$ it is easily seen that $\lim_{t \searrow 0} Z_t$ exists a.s. in $\mathbb{R}$, hence the process Z is a local martingale even on the left-closed interval $[0, \zeta[$ with quadratic variation

$$\langle Z, Z\rangle = \int (s'\sigma/s^2)^2(X)\, dt.$$

Thus, we may integrate $(s^2/s'\sigma)(X)\, 1_{]0,\zeta[}$ with respect to Z and the process

$$B = \int (s^2/s'\sigma)(X)\, dZ$$

is a BM on $[0, \zeta[$ which will be referred to in the next corollary. We further introduce the 'dual' coefficients $\sigma^* = \sigma$ and $b^* = b + \sigma^2 s'/s$ so that $L^* = \frac{1}{2}(\sigma^*)^2 D^2 + b^* D = s^{-1} L s$. Then $s^* = -1/s$ serves as a scale function for L^* and $b = b^* + (\sigma^*)^2 (s^*)'/s^*$ which shows the dual character of this construction.

COROLLARY 2. *On* $]0, \zeta[$ *we have* $d(X - 2J) = \sigma^*(X)\, dB + b^*(X)\, dt$.

Proof. This is an application of Itô's formula to $\varphi(x) = s^{-1}(-1/x)$ and the $]0, \infty[$-valued semimartingale $-1/s(X) = Z - 2/s(J)$ on $]0, \zeta[$ where $Z = -1/s(X) + 2/s(J)$ as in Corollary 1. Hereby, the function s^{-1} denotes the inverse of s. Note that the measure dJ is carried by the set $\{t \colon X_t = J_t\}$. □

Now, if X is a $\mathrm{BES}^3(0)$, we obtain $X - 2J = B$ which is Pitman's Theorem. More generally, let X be a Bessel process of index $\nu > 0$ on $]0, \infty[$ starting from 0, then the corollary says

$$X_t = B_t + 2J_t + (-\nu + \tfrac{1}{2}) \int_0^t \frac{dr}{X_r}$$

which is also stated in Corollary 12.7.1 in Yor's book [17].

Further, if both σ^* and b^* are constant functions, the process $X - 2J$ is a diffusion again which goes with [8]. But due to [9], in case of σ^* or b^* being nonconstant, $X - 2J$ is not a diffusion any longer which may be at least heuristically apparent by the corollary.

The contents of the corollary are also closely related to the results of Saisho and Tanemura [11]. Namely, putting $Y = X - J$ and $K = J$, the corollary may be stated as

$$dY = \sigma^*(Y + K)\, dB + b^*(Y + K)\, dt + dK$$

where Y is adapted, continuous, nonnegative and K is adapted, continuous, and nondecreasing with $dK = 1_{\{0\}}(Y)\, dK$. We further know from the beginning that the process $X = Y + K$ is a diffusion with generator L. This is in accord with the main part of [11] where it is proved, in the particular case of both σ^* and b^* being globally Lipschitz continuous functions on $\mathbb{R}$, that the above SDE of Skorokhod type can be solved uniquely by a pair (Y, K) and that the process $Y + K$ is a diffusion with generator L.

Further, we recall the first part of Williams' result on path decomposition of diffusions [16] which fits into the result of the corollary. Suppose $X_0 = e \in E$, then there is a.s. a unique time $T < \zeta$ with $X_T = J_0$ and, conditionally on $J_0 = c$, the process $(X_t)_{t<T}$ is a diffusion with generator L^* on $]c, c_2[$.

We finally remark that the theorem may be related to a result of Azema and Yor, cf. Revuz-Yor [7] Chap. VI, §4. Indeed, considering the particular case of X a $\mathrm{BES}^3(0)$ on $]0,\infty[$ with $s(x) = -1/x$ reveals $\mu'(J)+X\,(\mu s)'(J)$ to be a local martingale. Consequently, by Pitman's Theorem, the process $\mu'(S) + (2S - B)\,(\mu s)'(S) = (-\mu s)(S) - (S - B)\,(-\mu s)'(S)$ is a local martingale where B denotes a $\mathrm{BM}^1(0)$ and $S_t = \sup_{r\leq t} B_r$.

Now we turn to the proof of the theorem which is prepared by three lemmas.

LEMMA 1. *Consider the kernel K in $\bar{E}$ defined by*

$$K(e,\,\cdot\,) = \left\{ \begin{array}{cl} -s(e)\,s'(t)\,s(t)^{-2}\,1_{]c_1,e[}(t)\,dt & \text{for } e \in E \\ \delta_e & \text{for } e = c_i \end{array} \right\}.$$

Then $K\big(X_0(\,\cdot\,),\,\cdot\,\big)$ is a conditional distribution for $J_0 = \inf_{t\leq\zeta} X_t$ given $\mathscr{F}_0$.

Proof. For every $x \in \bar{E}$ we need to show $P\{J_0 \leq x\,|\,A\} = E\big[K\big(X_0, [c_1, x]\big)\,\big|\,A\big]$ for all $A \in \mathscr{F}_0$ with $P(A) > 0$. This may further be reduced to proving $P\{J_0 \leq x\} = E\big[K\big(X_0, [c_1, x]\big)\big]$ where $X_0 > x$ a.s. But, using the optional stopping theorem, we easily compute

$$P\{J_0 \leq x\} = \frac{E[s(X_0)]}{s(x)} = E\big[K\big(X_0, [c_1, x]\big)\big]\,,$$

where the last equation stems from $K\big(e, [c_1, x]\big) = s(e)/s(x)$ for all $e \in\,]x, c_2]$. □

LEMMA 2. *Let $\tau < \zeta$ be a $(\mathscr{F}_t)$-stopping time with $X_\tau \leq y$ a.s. for some $y \in E$. Then, for any $z \in E$, the random variable $Z_\tau 1_{\{J_\tau > z\}}$ is integrable and the equation*

$$\mu(z)\big[s(X_\tau) - s(z)\big]\,1_{\{X_\tau > z\}} = E\big[Z_\tau 1_{\{J_\tau > z\}}\big|\mathscr{F}_\tau\big] \quad \text{a.s.}$$

holds.

Proof. Since the pair $(X_{\tau+t}, \beta_{\tau+t})$ over $(\mathscr{F}_{\tau+t})$ is a solution to $(*)$, too, we may assume $\tau = 0$ without loss of generality. On $\{J_0 > z\}$ we clearly have $z < J_0 \leq X_0 \leq y$ a.s., hence all integrands in the sequel are integrable and we get

$$\begin{aligned} E\big[Z_0 1_{\{J_0>z\}}\big|\mathscr{F}_0\big] = {} & E\left[\left(\frac{s^2\mu'}{s'}\right)(J_0)\,1_{\{J_0>z\}}\middle|\mathscr{F}_0\right] \\ & - \frac{1}{s(X_0)}\,E\left[\left(\frac{s^2(\mu s)'}{s'}\right)(J_0)\,1_{\{J_0>z\}}\middle|\mathscr{F}_0\right]. \end{aligned}$$

By virtue of Lemma 1 these conditional expectations may be explicitely computed:

$$\begin{aligned} E\left[\left(\frac{s^2\mu'}{s'}\right)(J_0)\,1_{\{J_0>z\}}\middle|\mathscr{F}_0\right] &= -s(X_0)\,1_{\{X_0>z\}} \int_{]z,X_0[} \left(\frac{s^2\mu'}{s'}\cdot\frac{s'}{s^2}\right)(t)\,dt \\ &= -s(X_0)\big[\mu(X_0) - \mu(z)\big]\,1_{\{X_0>z\}}\,. \end{aligned}$$

So we finally get to

$$\begin{aligned} E\big[Z_0 1_{\{J_0>z\}}\big|\mathscr{F}_0\big] &= -s(X_0)\big[\mu(X_0)-\mu(z)\big]\,1_{\{X_0>z\}} \\ &\qquad + \frac{1}{s(X_0)}s(X_0)\big[(\mu s)(X_0)-(\mu s)(z)\big]\,1_{\{X_0>z\}} \\ &= \mu(z)\big[s(X_0)-s(z)\big]\,1_{\{X_0>z\}} \end{aligned}$$

which is the desired result. □

LEMMA 3. *Let $R \le \rho \le \tau \le S < \zeta$ be stopping times with respect to $(\mathscr{F}_t)$ such that $X^S_{R+t} \le y$ a.s. on $\{S > R\}$ for some $y \in E$. Then, for any $z \in E$, both $Z_\rho 1_{\{S>R,J_\rho>z\}}$ and $Z_\tau 1_{\{S>R,J_\rho>z\}}$ are integrable and*

$$E\big[Z_\rho 1_{\{S>R,J_\rho>z\}}\big|\mathscr{F}_\rho\big] = E\big[Z_\tau 1_{\{S>R,J_\rho>z\}}\big|\mathscr{F}_\rho\big] \quad \text{a.s.}$$

holds.

Proof. We may assume $\rho = 0$ and $S > R$ a.s. without loss of generality. On $\{J_0 > z\}$ we have $z < J_0 \le J_\tau \le X_\tau \le y$ a.s., hence $Z_0 1_{\{J_0>z\}}$ and $Z_\tau 1_{\{J_0>z\}}$ are integrable. Next we put $\tau_z = \inf\{t \ge 0 : X_t \le z\}$ and compute step by step

$$\begin{aligned} E\big[Z_\tau 1_{\{J_0>z\}}\big|\mathscr{F}_0\big] &= E\big[Z_\tau 1_{\{\inf_{t\le\tau} X_t>z\}} 1_{\{J_\tau>z\}}\big|\mathscr{F}_0\big] \\ &= E\big[1_{\{\inf_{t\le\tau} X_t>z\}} E\big[Z_\tau 1_{\{J_\tau>z\}}\big|\mathscr{F}_\tau\big]\big|\mathscr{F}_0\big] \\ &= E\big[1_{\{\inf_{t\le\tau} X_t>z\}} \mu(z)\big[s(X_\tau)-s(z)\big]\big|\mathscr{F}_0\big] \\ &= E\big[1_{\{X_0>z\}} \mu(z)\big[s(X_{\tau\wedge\tau_z})-s(z)\big]\big|\mathscr{F}_0\big] \\ &= 1_{\{X_0>z\}} \mu(z)\big[s(X_0)-s(z)\big] \\ &= E\big[Z_0 1_{\{J_0>z\}}\big|\mathscr{F}_0\big], \end{aligned}$$

where in the last but one equation we made use of the fact that $s(X^{\tau\wedge\tau_z})\,1_{\{X_0>z\}}$ is a bounded martingale with respect to $(\mathscr{F}_t)$. □

Proof (of the theorem). To begin with, we fix points $x < e < y$ in E and consider the first entry times $R = \inf\{t : X_t \ge e\}$ and $S = \inf\{t \ge R : X_t \ge y\}$. Then the map $T = S\,1_{\{J_R>x\}}$ clearly is a stopping time with respect to $(\mathscr{G}^0_{R+t})$, and we shall prove that the process $(Z^T_{R+t} 1_{\{T>R\}})$ is a martingale over $(\mathscr{G}^0_{R+t})$. Namely, we have $X^S_{R+t} \le y$ a.s. on $\{S > R\}$ and for any $t \ge 0$, putting $\tau = (R+t) \wedge S$, the random variable

$$Z^T_{R+t} 1_{\{T>R\}} = Z_\tau 1_{\{S>R,J_R>x\}}$$

is integrable by Lemma 3. The same lemma, for any $z \in E$, $r \le t$ and considering the times $\rho = (R+r) \wedge S$, $\tau = (R+t) \wedge S$, yields

$$\begin{aligned} E\big[Z_\tau 1_{\{S>R,J_R>x\}}\,1_{\{J_\rho>z\}}\big|\mathscr{F}_\rho\big] &= 1_{\{\inf_{R\le t\le\rho} X_t>x\}}\, E\big[Z_\tau 1_{\{S>R,J_\rho>x\vee z\}}\big|\mathscr{F}_\rho\big] \\ &= 1_{\{\inf_{R\le t\le\rho} X_t>x\}}\, E\big[Z_\rho 1_{\{S>R,J_\rho>x\vee z\}}\big|\mathscr{F}_\rho\big] \\ &= E\big[Z_\rho 1_{\{S>R,J_R>x\}}\,1_{\{J_\rho>z\}}\big|\mathscr{F}_\rho\big] \quad \text{a.s.} \end{aligned}$$

But, using $J^\rho_t = \inf_{r\ge t} X^\rho_r \wedge J_\rho$, we have $\mathscr{G}^0_\rho = \mathscr{F}^X_\rho \vee \sigma(J_\rho) \subseteq \mathscr{F}_\rho \vee \sigma(J_\rho)$, hence

$$\begin{aligned} Z^T_{R+r} 1_{\{T>R\}} &= E\big[Z^T_{R+t} 1_{\{T>R\}}\big|\mathscr{G}^0_\rho\big] \\ &= E\big[Z^T_{R+t} 1_{\{T>R\}}\big|\mathscr{G}^0_{R+r}\big] \quad \text{a.s.} \end{aligned}$$

which is the assertion on the martingale property.

Now letting $x \searrow c_1$ and $y \nearrow c_2$ the process (Z_{R+t}) is seen to be a local martingale over $(\mathscr{G}_{R+t})$, and the proof is completed by letting $e \searrow c_1$. $\square$

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