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## **Integration by parts and Cameron-Martin formulas for the free path space of a compact riemannian manifold**

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INTEGRATION BY PARTS AND  
CAMERON–MARTIN FORMULAS  
FOR THE FREE PATH SPACE OF A  
COMPACT RIEMANNIAN MANIFOLD

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## 1. Introduction

Let  $(h_s : s \geq 0)$  denote an absolutely continuous function with values in  $\mathbb{R}^m$  whose derivative is square-integrable:

$$\int_0^\infty |\dot{h}_s|^2 ds < \infty.$$

The Cameron–Martin formula states that if  $(x_s : s \geq 0)$  is a Brownian motion in  $\mathbb{R}^m$ , starting from 0, then, *provided also*  $h_0 = 0$ , the law of  $(x_s + h_s : s \geq 0)$  is absolutely continuous with respect to that of  $(x_s : s \geq 0)$  with density

$$\rho_s = \exp \left\{ \int_0^\infty \langle \dot{h}_s, dx_s \rangle - \frac{1}{2} \int_0^\infty |\dot{h}_s|^2 ds \right\}.$$

In fact if one randomizes the starting point  $x_0$  according to Lebesgue measure, then the formula remains valid without the assumption  $h_0 = 0$ . Thus we obtain a Cameron–Martin formula for the free path space of  $\mathbb{R}^m$ . For suitable functions  $F$  on the path space, the expectation

$$\mathbb{E} \left[ F(x + th) \exp \left\{ - \int_0^\infty \langle t\dot{h}_s, dx_s \rangle - \frac{1}{2} \int_0^\infty |t\dot{h}_s|^2 ds \right\} \right]$$

does not depend on  $t$ . So on differentiating in  $t$  at 0 we obtain an integration by parts formula:

$$\mathbb{E}[D_h F(x)] = \mathbb{E}[F(x) \int_0^\infty \langle \dot{h}_s, dx_s \rangle].$$

This may be regarded as the infinitesimal form of the Cameron–Martin formula.

In this note we shall discuss Cameron–Martin and integration-by-parts formulas for the free path space of a compact Riemannian manifold. The case of paths with a fixed starting point has already been thoroughly discussed: see [D],[H]. The results we obtain are at a technical level simple corollaries of results in [L2] or [N2]. The emphasis here is rather on the efficient calculation of densities and divergences for flows and vector fields. The integration by parts formula is proved first in §2, by a direct argument based on the methods of [L2]. Then in §3 we use the main result of [N2] to establish independently a corresponding Cameron–Martin formula. From here we can recover the integration by parts formula by differentiating.

Integration by parts formulas of a similar type are proved in [L1],[L2],[LR] by using a mixture of small time asymptotics and developments of Bismut’s formula [B],[EL],[N1]. They rely deeply on the identity between the tangent spaces to path space used by Bismut [B] and Jones and Léandre [JL]. Such integration by parts formulas are also known for free twisted loops: see [LR]. In this case we do not yet have a corresponding Cameron–Martin formula.

We would like to thank David Elworthy for his warm hospitality during the Warwick Symposium on Stochastic Analysis and Related Topics 1994/5, where this work was done.

## 2. An integration by parts formula on the free path space

Let  $\Omega$  denote the set of continuous paths  $(x_s : s \geq 0)$  with values in a compact Riemannian manifold  $M$ . Let  $X$  denote a vector field on  $\Omega$ , thus  $X(x) = (X_s(x) : s \geq 0)$  where  $X_s(x)$  belongs to the tangent space to  $M$  at  $x_s$ . We shall investigate the relationship between  $X$  and the equilibrium Wiener measure on  $\Omega$ :

$$\mathbb{P}(dx) = \int_M \mathbb{P}^{x_0}(dx) dx_0$$

where  $\mathbb{P}^{x_0}$  denotes the law of Brownian motion in  $M$  starting from  $x_0$  and  $dx_0$  denotes the normalized Riemannian volume. Let us consider the vector field  $X$  given by

$$X_s(x) = \tau_s \sum_{i=1}^m h_s^i X_i(x_0)$$

where, for  $i = 1, \dots, m$ ,  $X_i$  is a  $C^2$  vector field over  $M$  and  $\tau_s$  is the parallel transport from  $x_0$  to  $x_s$ .

Conditional on  $x_0$ , we define a Brownian motion  $b_s$  in  $T_{x_0}M$  by  $b_0 = 0$  and

$$\partial b_s = \tau_s^{-1} \partial x_s$$

where  $\partial$  denotes the Stratonovich differential.

For each  $s \geq 0$  denote by  $e_s : \Omega \rightarrow M$  the evaluation map  $e_s(x) = x_s$ . We consider the pullback  $T_s$  by  $e_s$  of the tangent bundle  $TM$  equipped with the pullback of the Levi–Civita connection of  $M$ . Then formally  $\partial x_s$  is a section of  $T_s$ ,  $\partial b_s$  is a section of  $T_0$ ,  $\tau_s$  is a section of  $T_s \otimes (T_0)^*$ , and we have

$$\nabla_X \partial x_s = (\nabla_X \tau_s) \partial b_s + \tau_s \nabla_X \partial b_s.$$

This formula is justified in [L2] at (4.65). We know also by [L2] (see (4.64), (3.87)) that

$$\begin{aligned}\nabla_X \tau_s &= \tau_s \int_0^s \tau_r^{-1} R(\partial x_r, X_r) \tau_r, \\ \nabla_X \partial x_s &= \tau_s \sum_{i=1}^m \dot{h}_s^i X_i(x_0) \partial s\end{aligned}$$

so

$$\nabla_X \partial b_s = \sum_{i=1}^m \dot{h}_s^i X_i(x_0) \partial s - \left( \int_0^s \tau_r^{-1} R(\partial x_r, X_r) \tau_r \right) \partial b_s.$$

Hence we obtain for the Itô differential

$$\nabla_X db_s = \sum_{i=1}^m \dot{h}_s^i X_i(x_0) ds - \frac{1}{2} \tau_s^{-1} \text{Ricci}(X_s) ds - \left( \int_0^s \tau_r^{-1} R(\partial x_r, X_r) \tau_r \right) db_s.$$

We compute now the action of  $X$  on a test functional

$$F = \sum_n \int_{0 < s_1 < \dots < s_n} H(s_1, \dots, s_n; x_0) db_{s_1} \dots db_{s_n}$$

where the sum in  $n$  is finite. Here  $H$  is a cotensor in  $T_{x_0}M$ . We have

$$\begin{aligned}\langle dF, X \rangle &= \sum_n \int_{0 < s_1 < \dots < s_n} \sum_j H(s_1, \dots, s_n; x_0) db_{s_1} \dots \nabla_X db_{s_j} \dots db_{s_n} \\ &\quad + \sum_n \int_{0 < s_1 < \dots < s_n} \nabla_{X_0} H(s_1, \dots, s_n; x_0) db_{s_1} \dots db_{s_n},\end{aligned}$$

so

$$\begin{aligned}\mathbb{E}\langle dF, X \rangle &= \int_M dx_0 \mathbb{E}^{x_0} \left( \sum_n \int_{0 < s_1 < \dots < s_n} H(s_1, \dots, s_n; x_0) db_{s_1} \dots db_{s_{n-1}} (\theta_{s_n} ds_n) \right) \\ &\quad + \int_M X_0 f(x_0) dx_0\end{aligned}$$

where  $f(x_0) = \mathbb{E}^{x_0}(F)$  and

$$\theta_s = \tau_s^{-1} (D/\partial s - \frac{1}{2} \text{Ricci}) X_s.$$

Here  $D/\partial s$  denotes covariant differentiation along  $x_s$ . In the first term we used the fact that integrals in  $db_s$  vanish under the expectation. In the second we used the fact that the Fock space structure, being derived from the metric, is preserved by the Levi-Civita connection. Let us define

$$\text{div } X(x) = \text{div } X_0(x_0) + \int_0^\infty \left\langle \left( \frac{D}{\partial s} - \frac{1}{2} \text{Ricci} \right) X_s(x), dx_s \right\rangle.$$

We have shown:

**Theorem 2.1.** *We have*

$$\mathbb{E}(dF, X) = \mathbb{E}(F \operatorname{div} X).$$

### 3. A Cameron–Martin formula on the free path space

Recall that  $\Omega$  denotes the set of continuous paths  $(x_s : s \geq 0)$  with values in  $M$ . Let  $X$  denote a vector field on  $\Omega$ , thus  $X(x) = (X_s(x) : s \geq 0)$  with  $X_s(x) \in T_x M$ . Our object now is to compute the image of the equilibrium Wiener measure  $\mathbb{P}$  under the flow determined by  $X$ .

We begin with a rough argument from which some technical points are missing. Later, in order to fill these gaps we shall specialize our choice of vector field  $X$ , which may obscure the simplicity of the basic argument. Let us assume that  $X_s$  is previsible, and that  $DX_s/\partial s$  exists for almost all  $s$ , and is square-integrable. Let us assume also that for  $\mathbb{P}$ -almost all  $x_0 \in \Omega$ , we can integrate  $X$  to a flow in  $\Omega$

$$\dot{x}_t = X(x_t). \quad (1)$$

Here we use  $t$  to parametrize a family of paths  $x_t = (x_{st} : s \geq 0)$ . Let us suppose that  $x_{st}$  is a two-parameter semimartingale in the sense of [N2], then the two-parameter stochastic calculus provides a means to compute the law of  $x_t$  when  $x_0$  has law  $\mathbb{P}$ .

We may rewrite (1) in differential form

$$\partial_t x_{st} = X_{st} \partial t$$

where  $X_{st} = X_s(x_t)$ . Recall that we write  $d_s$  and  $\partial_s$  for the Itô and Stratonovich differentials in  $s$ ; we also write  $D_s$  for the covariant Stratonovich differential corresponding to the Levi–Civita connection. Then

$$D_s \partial_t x_{st} = \left( \frac{D}{\partial s} X_{st} \right) \partial s \partial t.$$

Let us introduce a lift  $v_{st}$  of  $x_{st}$  to the bundle  $OM$  of orthonormal frames in  $TM$ , choosing  $v_{00}$  arbitrarily and imposing the following horizontality conditions:

$$D_s v_{s0} = 0, \quad D_t v_{st} = 0,$$

which determine  $v_{st}$  uniquely, given  $v_{00}$ . In addition we introduce two further processes,  $q_{st}$  in  $TM$  over  $x_{st}$ , and  $b_{st}$  in  $\mathbb{R}^n$ , by the equations

$$\begin{aligned} D_t q_{st} &= \left( \frac{D}{\partial s} - \frac{1}{2} \operatorname{Ricci} \right) X_{st} \partial t, \quad q_{s0} = 0, \\ d_s b_{st} &= v_{st}^{-1} (d_s x_{st} - q_{st} ds), \quad b_{0t} = 0. \end{aligned}$$

Since  $x_{s0}$  is a Brownian motion in  $M$ , it follows that  $b_{s0}$  is a Brownian motion in  $\mathbb{R}^n$ . Since our connection is torsion-free,

$$D_s \partial_t x_{st} = D_t \partial_s x_{st},$$

hence

$$\partial_t(\partial_s b_{st} \otimes \partial_s b_{st}) = v_{st}^{-1} D_t(\partial_s x_{st} \otimes \partial_s x_{st}) = 0$$

and so

$$\partial_s b_{st} \otimes \partial_s b_{st} = \partial_s b_{s0} \otimes \partial_s b_{s0} = \sum_{i=1}^n e_i \otimes e_i \partial s,$$

where  $e_i$  runs over the standard basis in  $\mathbb{R}^n$ . We recall the basic identity ([N2], (2.38))

$$D_t \partial_s x_{st} = D_t d_s x_{st} + \frac{1}{2} R(\partial_t x_{st}, \partial_s x_{st}) \partial_s x_{st},$$

where  $R$  denotes the curvature. But we have identified the quadratic variation in  $s$  of  $x_{st}$  as the trace, so

$$R(\partial_t x_{st}, \partial_s x_{st}) \partial_s x_{st} = \text{Ricci}(\partial_t x_{st}) \partial s.$$

Hence

$$\begin{aligned} D_t d_s x_{st} &= D_t \partial_s x_{st} - \frac{1}{2} \text{Ricci}(\partial_t x_{st}) \partial s \\ &= \left( \frac{D}{\partial s} - \frac{1}{2} \text{Ricci} \right) X_{st} \partial s \partial t \\ &= D_t q_{st} \partial s, \end{aligned}$$

and

$$\partial_t d_s b_{st} = v_{st}^{-1} (D_t d_s x_{st} - D_t q_{st} ds) = 0.$$

Therefore  $b_{st} = b_{s0}$  for all  $t$ , and  $(x_{st} : s \geq 0)$  is a Brownian motion in  $M$  with drift  $q_{st}$ .

So far we have ignored what is happening to the starting point, but that is very simple. Previsibility forces  $X_0(x)$  to be a function of the starting point  $x_0$  alone, giving us a vector field on  $M$ , which we again denote  $X_0$ . Then  $x_{0t}$  obeys the autonomous equation

$$\partial_t x_{0t} = X_0(x_{0t}) \partial t.$$

If we assume that  $X_0$  is  $C^1$  say, then the law of  $x_{0t}$  is given by

$$0 = \frac{\partial}{\partial t} \mathbb{E} \left[ f(x_{0t}) \exp \left\{ - \int_0^t \text{div} X_0(x_{0r}) dr \right\} \right].$$

On the other hand, conditional on  $x_{0t}$ , the law of  $(x_{st} : s \geq 0)$  is absolutely continuous with respect to  $\mathbb{P}^{x_{0t}}$ , at least on compact  $s$ -intervals, with density given by the Cameron–Martin formula. We have

$$\begin{aligned} & \frac{\partial}{\partial t} \left\{ \int_0^\infty \langle q_{st}, d_s x_{st} \rangle - \frac{1}{2} \int_0^\infty |q_{st}|^2 ds \right\} \\ &= \int_0^\infty \left\langle \frac{D}{\partial t} q_{st}, d_s x_{st} \right\rangle \\ &= \int_0^\infty \left\langle \left( \frac{D}{\partial s} - \frac{1}{2} \text{Ricci} \right) X_{st}, d_s x_{st} \right\rangle. \end{aligned}$$

Hence the law of  $x_t = (x_{st} : s \geq 0)$  is given by

$$0 = \frac{\partial}{\partial t} \mathbb{E} \left[ F(x_t) \exp \left\{ - \int_0^t \operatorname{div} X(x_\tau) d\tau \right\} \right]$$

for all bounded Borel functions  $F$  on  $\Omega$ , where

$$\operatorname{div} X(x) = \operatorname{div} X_0(x_0) + \int_0^\infty \left\langle \left( \frac{D}{\partial s} - \frac{1}{2} \operatorname{Ricci} \right) X_s(x), dx_s \right\rangle.$$

This is our Cameron–Martin formula for the free path space. For suitably smooth functions  $F$  we can evaluate the derivative at  $t = 0$  to recover the integration by parts formula of §2

$$\mathbb{E} \langle dF, X \rangle = \mathbb{E} [F \operatorname{div} X].$$

That concludes our rough argument.

The only serious gap in the above argument is the need to establish the existence of a flow for our vector field  $X$ , within the class of two-parameter semimartingales. Obviously, something in the nature of a Lipschitz condition looks desirable. But truly Lipschitz functions on  $\Omega$  form an overly restricted class. We shall not attempt to find natural conditions on the vector field  $X$ , but restrict attention to the case already considered in §2. Let there be given  $C^2$  vector fields  $X_1, \dots, X_m$  on  $M$ , together with an absolutely continuous function  $h_s = (h_s^1, \dots, h_s^m)$  in  $\mathbb{R}^m$  satisfying

$$\int_0^\infty |h_s|^2 ds < \infty. \quad (2)$$

Then for  $\mathbb{P}$ -almost all  $x \in \Omega$  and all  $s \geq 0$  we can define  $X_s(x) \in T_x M$  by

$$X_s(x) = \tau_s \sum_{i=1}^m h_s^i X_i(x_0) \quad (3)$$

where  $\tau_s$  denotes parallel translation  $T_{x_0} M \rightarrow T_x M$  along  $x$ .

We state a special case of ([N2], Theorem 3.2.6) suited to our present needs.

**Theorem 3.1.** *Let  $M$  be a  $C^4$  compact Riemannian manifold with Levi–Civita connection. Let*

$$\beta : OM \rightarrow TM \otimes T^*M$$

*be a  $C^2$  map of the fibres. Suppose we are given regular semimartingale boundary values  $(x_{s0} : s \geq 0)$  and  $(x_{0t} : t \geq 0)$  in  $M$  together with  $u_{00} = v_{00} \in O_{x_{00}} M$ . Then there exist unique two-parameter semimartingales  $x_{st}$  in  $M$  and  $u_{st}, v_{st}$  in  $OM$  over  $x_{st}$  such that  $u_{s0} = v_{s0}$  and  $u_{0t} = v_{0t}$ , and satisfying*

$$\begin{aligned} D_s \partial_t x_{st} &= \beta(u_{st}) \partial_s x_{st} \partial t, \\ D_s u_{st} &= 0, \\ D_t v_{st} &= 0. \end{aligned}$$

We make some explanatory remarks. In this context regularity of the boundary values means uniformly Lipschitz quadratic variation and finite variation part. The auxiliary processes  $u_{st}$  and  $v_{st}$  are lifts of  $x_{st}$  in  $OM$ , which agree and are horizontal on the  $s$  and  $t$ -axes; then  $u_{st}$  is made horizontal along  $(x_{st} : s \geq 0)$  for each  $t \geq 0$ , whereas  $v_{st}$  is horizontal along  $(x_{st} : t \geq 0)$  for each  $s \geq 0$ . Parallel translation along  $(x_{st} : s \geq 0)$  is then given by

$$\tau_{st} = u_{st}u_{0t}^{-1}.$$

The process  $v_{st}$  already appeared above in analysing the law of  $x_{st}$ .

In order to apply Theorem 3.1 to our present problem, we first integrate the  $C^2$  vector field

$$X_0(x_0) = \sum_{i=1}^m h_0^i X_i(x_0)$$

which governs the autonomous motion of the base point

$$\dot{x}_{0t} = X_0(x_{0t}).$$

We denote by  $u_{0t}$  the horizontal lift along  $x_{0t}$  starting from  $u_{00}$ , and set

$$(k_t)_i = u_{0t}^{-1} X_i(x_{0t}),$$

$k_t$  taking values in  $(\mathbb{R}^m)^*$ . The flow equation

$$\dot{x}_t = X(x_t) \tag{3}$$

is then equivalent to the system of two-parameter hyperbolic equations

$$D_s \partial_t x_{st} = u_{st} k_t (\partial h_s) \partial t,$$

$$D_s u_{st} = 0,$$

$$D_t v_{st} = 0.$$

In the case where  $h_s$  has bounded derivative and so is regular, we can now appeal to Theorem 3.1, applied to the augmented process  $\tilde{x}_{st} = (x_{st}, h_{st}, k_{st})$  in  $M \times \mathbb{R}^m \times (\mathbb{R}^m)^*$ , with  $h_{st} = h_s$  and  $k_{st} = k_t$ , satisfying

$$D_s \partial_t h_{st} = 0, \quad D_s \partial_t k_{st} = 0.$$

Hence (3) has a unique solution, which is a two-parameter semimartingale. One can then pass to the case of general  $h_s$  by a time-change argument in  $s$ , as in ([N2], §4.2). Thus we obtain

**Theorem 3.2.** *Let  $x_0 = (x_{s0} : s \geq 0)$  be a Brownian motion in  $M$ . Then there exists a unique two-parameter semimartingale  $(x_{st} : s \geq 0, t \geq 0)$  satisfying*

$$\partial_t x_{st} = \tau_{st} \sum_{i=1}^m h_s^i X_i(x_0) \partial t.$$

The calculation of the law of  $x_{st}$  made above is now justified. The presence of the Ricci term in the drift means that (2) is not sufficient to make the law of  $x_t = (x_{st} : s \geq 0)$  absolutely continuous with respect to  $\mathbb{P}$ , unless one restricts to compact  $s$ -intervals. The combination of (2) and

$$\int_0^\infty |h_s|^2 ds < \infty \tag{4}$$

is of course sufficient. We summarize our conclusions.



**Theorem 3.3.** *Let  $X$  be defined  $\mathbb{P}$ -almost everywhere on  $\Omega$  by*

$$X_s(x) = \tau_s \sum_{i=1}^m h_s^i X_i(x_0)$$

where  $h_s$  satisfies (2) and (4) and  $X_1, \dots, X_m$  are  $C^2$  vector fields on  $M$ . There exists a unique two-parameter semimartingale  $(x_{st} : s \geq 0, t \geq 0)$  such that the path-valued process  $x_t = (x_{st} : s \geq 0)$  satisfies

- (i)  $x_0 = x$ ;
- (ii)  $x_t$  has law absolutely continuous with respect to  $\mathbb{P}$  for all  $t \geq 0$ ;
- (iii)  $\dot{x}_t = X(x_t)$ .

Moreover for every bounded Borel function  $F$  on  $\Omega$  we have

$$0 = \frac{\partial}{\partial t} \mathbb{E}[F(x_t) \exp\{-\int_0^t \operatorname{div} X(x_\tau) d\tau\}]$$

where

$$\operatorname{div} X(x) = \operatorname{div} X_0(x_0) + \int_0^\infty \left\langle \left( \frac{D}{\partial s} - \frac{1}{2} \operatorname{Ricci} \right) X_s(x), dx_s \right\rangle.$$

Finally for every smooth cylinder function  $F(x) = f(x_{s_1}, \dots, x_{s_k})$  we have the integration by parts formula

$$\mathbb{E}\langle dF, X \rangle = \mathbb{E}[F \operatorname{div} X]$$

where

$$\langle dF, X \rangle(x) = \sum_{j=1}^k \langle d_j f(x_{s_1}, \dots, x_{s_k}), X_{s_j}(x) \rangle.$$

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