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Hirsch's Integral Test for the Iterated Brownian Motion

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ABSTRACT. We present an analogue of Hirsch's integral test to decide whether a function belongs to the lower class of the supremum process of an iterated Brownian motion.

1. Introduction and main statement

Consider $B^+ = (B^+(t), t \geq 0)$, $B^- = (B^-(t), t \geq 0)$ and $B = (B_t, t \geq 0)$ three independent linear Brownian motions started from 0. The process $X = (X_t, t \geq 0)$ given by

$$X_t = \begin{cases} B^+(B_t) & \text{if } B_t \geq 0 \\ B^-(-B_t) & \text{if } B_t < 0 \end{cases}$$

is called an iterated Brownian motion. The study of its sample path behaviour has motivated numerous works in the recent years; see the bibliography. Many results in that field are analogues of well-known almost sure properties of the standard Brownian motion, which are originally due to Chung, Khintchine, Kolmogorov, Strassen ... The purpose of this note is to present such an analogue of Hirsch's integral test, that is to determine the lower functions of the supremum process of X,

$$\overline{X}_t = \sup\{X_s : 0 \leq s \leq t\} \qquad (t \geq 0).$$

In this direction, the lower functions of the increasing process

$$M_t = \sup\{X_s : 0 \leq s \leq t \text{ and } B_s \geq 0\} = \sup\{B^+(B_s \vee 0), 0 \leq s \leq t\}$$

have been characterized in Bertoin (1996) as follows: If $f : (0, \infty) \to (0, \infty)$ is an increasing function, then $\liminf_{t\to\infty} M_t/f(t) = 0$ or ∞ a.s. according as the integral $\int_1^\infty f(t)t^{-5/4}dt$ diverges or converges. More precisely, this follows readily from the observation that the right-continuous inverse of M is a stable subordinator of index $1/4$ and an application of Khintchine's test for the upper functions of stable processes.

Plainly, the inequality $M \leq \overline{X}$ can then be used to deduce some information on $\overline{X}$; however this does not suffice to characterize the lower functions of $\overline{X}$.

Theorem. *Let $f : (0, \infty) \to (0, \infty)$ be an increasing function. Then*

$$\liminf_{t\to\infty} \overline{X}_t \, f(t) \, t^{-1/4} = 0 \text{ or } \infty \qquad \text{a.s.}$$

according as the integral

$$\int_1^\infty \frac{dt}{tf(t)^2}$$

diverges or converges.

This shows that the asymptotic behaviours of M and $\overline{X}$ differ; a feature that is perhaps a priori not obvious. For instance, one has

$$\liminf_{t\to\infty} M_t t^{-1/4} \log t = 0 \quad , \quad \liminf_{t\to\infty} \overline{X}_t t^{-1/4} \log t = \infty .$$

A related phenomenon in connection with Strassen's theorem has been pointed out recently by Csáki, Földes and Révész (1995).

The Theorem will be proven in the next section. Though we shall not give any precise statement, we also mention that it has a small time analogue.

2. Proof of the Theorem

To start with, we introduce some notation. We consider the supremum processes S^+, S^-, S and I, of B^+, B^-, B and $-B$, respectively. It is immediately noticed that

$$\overline{X}_t = S^+(S_t) \vee S^-(I_t) , \qquad \text{for all } t \geq 0 . \tag{1}$$

Lemma 1. *There is a constant $c > 0$ such that for every $a > 0$:*

$$I\!P\left(\overline{X}_1 < a\right) \leq ca^2 .$$

Proof. By (1) and the scaling property, we have

$$\begin{aligned} I\!P\left(\overline{X}_1 < a\right) &= I\!P\left(S^+(S_1) < a, S^-(I_1) < a\right) \\ &= I\!P\left(S_1^+ < a/\sqrt{S_1}, S_1^- < a/\sqrt{I_1}\right) \\ &\leq \frac{2a^2}{\pi} I\!E\left(1/\sqrt{S_1 I_1}\right) \end{aligned}$$

(because S_1^+ and S_1^- can be viewed as the absolute values of two independent normal variables). All that is needed now is to check that the expectation in the last displayed formula, is finite.

We present two approaches. First, the joint law of (S_1, I_1) is given on page 342 in Feller (1971), from which several lines of elementary computation enable us to conclude that $I\!E\left(1/\sqrt{S_1 I_1}\right) < \infty$. Alternatively, in order to avoid theta functions in the joint law of (S_1, I_1), we can instead use the following elegant formula (see Pitman and Yor (1993)):

$$I\!P\left(\sqrt{T}\, S_1 < x;\ \sqrt{T}\, I_1 < y\right) = 1 - \frac{\sinh x + \sinh y}{\sinh(x+y)}, \qquad x > 0,\, y > 0,$$

where T is an exponential variable with $I\!E(T) = 2$, independent of B. This yields

$$\begin{aligned} I\!E\Big(\frac{1}{\sqrt{T}}\Big)\, I\!E\Big(\frac{1}{\sqrt{S_1 I_1}}\Big) &= I\!E\Big(\frac{1}{(\sqrt{T}\, S_1)^{1/2}} \frac{1}{(\sqrt{T}\, I_1)^{1/2}}\Big) \\ &= \int_0^\infty dx \int_0^\infty dy\, \frac{1}{\sqrt{xy}}\, \frac{\partial^2}{\partial x\, \partial y}\Big(-\frac{\sinh x + \sinh y}{\sinh(x+y)}\Big), \end{aligned}$$

which is easily seen to be finite. ◇

We now prove the easy part of the Theorem:

Proof of the Theorem, first part. By Lemma 1 and the scaling property, we have for every integer n

$$\mathbb{P}\left(\overline{X}_{2^n} < \frac{2^{(n+1)/4}}{f(2^n)}\right) = \mathbb{P}\left(\overline{X}_1 < \frac{2^{1/4}}{f(2^n)}\right) \leq cf(2^n)^{-2}.$$

If the integral in the Theorem converges, then so does the series $\sum f(2^n)^{-2}$. Hence

$$\overline{X}_{2^n} \geq \frac{2^{(n+1)/4}}{f(2^n)} \qquad \text{for every sufficiently large } n \text{, a.s.}$$

and by a standard argument of monotonicity,

$$\liminf_{t\to\infty} \overline{X}_t \, f(t) \, t^{-1/4} \geq 1 \qquad \text{a.s.}$$

Because the integral in the test remains finite if we replace f by εf for any $\varepsilon > 0$, we conclude that the liminf above must be infinite. ◇

Next, we establish a zero-one law for the supremum of the iterated Brownian motion.

Lemma 2. *Let $g : [0,\infty) \to [0,\infty)$ be a measurable function. The event*

$$\{\overline{X}_t < g(t) \text{ infinitely often as } t \to \infty\}$$

has probability zero or one.

Proof. The argument relies on the Hewitt-Savage's zero-one law. For every integer n, let ${}^nB$ be the increment (process) of B on the time-interval $[n, n+1]$:

$${}^nB = (B_{n+t} - B_n, 0 \leq t \leq 1) \, .$$

The increments ${}^nB^+$ and ${}^nB^-$ are defined analogously. The random variables (with values in a space of paths) $({}^nB, {}^nB^+, {}^nB^-)$, $n \in \mathbb{N}$, are i.i.d. One can clearly recover B, B^+ and B^- from the sequence of their increments.

Consider a finite permutation Σ on $\mathbb{N}$, i.e. for some $N > 0$, one has $\Sigma(n) = n$ for all $n \geq N$. Denote by ${}^\Sigma B$ the Brownian motion obtained by the permutation of the increments of B, that is the increment of ${}^\Sigma B$ on the time-interval $[n, n+1]$ is ${}^{\Sigma(n)}B$. Define similarly ${}^\Sigma B^+$ and ${}^\Sigma B^-$ by the permutation of the increments of B^+ and B^-, respectively. Finally, denote by ${}^\Sigma X$ the resulting iterated Brownian motion.

By construction, we have

$$B_t = {}^\Sigma B_t \quad , \quad B_t^+ = {}^\Sigma B_t^+ \quad , \quad B_t^- = {}^\Sigma B_t^- \qquad \text{whenever } t \geq N+1 \, .$$

Put

$$\mu = \sum_{n=0}^{N} \left(\max_{0\leq t\leq 1} |{}^nB_t^+| + \max_{0\leq t\leq 1} |{}^nB_t^-| \right) ,$$

so that we have a fortiori $B_t^{\pm} =^{\Sigma} B_t^{\pm}$ whenever $|B_t^{\pm}| \geq \mu$. As a consequence, we see that ${}^{\Sigma}X_t = X_t$ provided that $|X_t| \geq \mu$ and $t \geq N+1$. Because the increasing process $\overline{X}$ tends to ∞, we deduce that the asymptotic events $\{\overline{X}_t < g(t)$ infinitely often as $t \to \infty\}$ and $\{{}^{\Sigma}\overline{X}_t < g(t)$ infinitely often as $t \to \infty\}$ coincide, where ${}^{\Sigma}\overline{X}$ stands for the supremum process of ${}^{\Sigma}X$. In conclusion, the zero-one law of Hewitt-Savage applies. $\diamond$

To establish the converse part of the Theorem, we denote by σ^+ and σ^- the right-continuous inverses of S^+ and S^-, respectively:

$$\sigma_t^{\pm} = \inf\{s : S_s^{\pm} > t\} \qquad (t \geq 0).$$

We also denote the inverse local time of B at level 0 by τ; so that σ^+, σ^- and τ are three independent stable subordinators with index 1/2. We consider the sequence of events

$$\begin{aligned} E_n = \Big\{ & 2^n < S_{\tau(2^n)} < \sigma^+\left(\frac{\tau(2^n)^{1/4}}{f(2^n)}\right) < S_{\tau(2^{n+1})} < 34 \cdot 2^n \,; \\ & 2^n < I_{\tau(2^n)} < \sigma^-\left(\frac{\tau(2^n)^{1/4}}{f(2^n)}\right) < I_{\tau(2^{n+1})} < 34 \cdot 2^n \,;\, 2^{2n} < \tau(2^n) < 2^{2n+1} \Big\}, \end{aligned}$$

where f is an increasing function. Aiming at applying a well-known extension of the Borel-Cantelli lemma, we first establish the following:

Lemma 3. *The series* $\sum \mathbb{P}(E_n)$ *diverges whenever* $\int_1^\infty dt/tf(t)^2 = \infty$.

Proof. By the scaling property, we can rewrite $\mathbb{P}(E_n)$ first as

$$\begin{aligned} \mathbb{P}\Big(& 1 < S_{\tau(1)} < \sigma^+\left(\frac{\tau(1)^{1/4}}{f(2^n)}\right) < S_{\tau(2)} < 34 \,; \\ & 1 < I_{\tau(1)} < \sigma^-\left(\frac{\tau(1)^{1/4}}{f(2^n)}\right) < I_{\tau(2)} < 34 \,;\, 1 < \tau(1) < 2 \Big) \end{aligned}$$

and then as

$$\begin{aligned} \mathbb{P}\Big(& 1 < S_{\tau(1)} \,;\, S^+\left(S_{\tau(1)}\right) < \frac{\tau(1)^{1/4}}{f(2^n)} < S^+\left(S_{\tau(2)}\right) \,;\, S_{\tau(2)} < 34 \,; \\ & 1 < I_{\tau(1)} \,;\, S^-\left(I_{\tau(1)}\right) < \frac{\tau(1)^{1/4}}{f(2^n)} < S^-\left(I_{\tau(2)}\right) \,;\, I_{\tau(2)} < 34 \,;\, 1 < \tau(1) < 2 \Big). \end{aligned}$$

The latter quantity is bounded from below by

$$\begin{aligned} \mathbb{P}\Big(& 1 < S_{\tau(1)} < 2 \,;\, S^+(2) < \frac{1}{f(2^n)} \,;\, S^+(33) > \frac{2}{f(2^n)} \,;\, 33 < \sup_{\tau(1) \leq t \leq \tau(2)} B_t < 34 \,; \\ & 1 < I_{\tau(1)} < 2 \,;\, S^-(2) < \frac{1}{f(2^n)} \,;\, S^-(33) > \frac{2}{f(2^n)} \,;\, -34 < \inf_{\tau(1) \leq t \leq \tau(2)} B_t < -33 \,; \\ & 1 < \tau(1) < 2 \Big). \end{aligned}$$

Applying the strong Markov property and using the independence of B, B^+ and B^-, this probability is given by the product

$$\mathbb{P}\left(S^+(2) < \frac{1}{f(2^n)}\,;\, S^+(33) > \frac{2}{f(2^n)}\right) \mathbb{P}\left(S^-(2) < \frac{1}{f(2^n)}\,;\, S^-(33) > \frac{2}{f(2^n)}\right)$$
$$\mathbb{P}\left(1 < S_{\tau(1)} < 2\,;\, 1 < I_{\tau(1)} < 2\,;\, 1 < \tau(1) < 2\right) \mathbb{P}\left(33 < S_{\tau(1)} < 34\,;\, 33 < I_{\tau(1)} < 34\right)$$
$$= C_1 \left[\mathbb{P}\left(S_2 < \frac{1}{f(2^n)}\,;\, S_{33} > \frac{2}{f(2^n)}\right)\right]^2 .$$

Thus, if $f(\infty) < \infty$, then the claim of Lemma 3 is clear. Otherwise, we have on the one hand for every sufficiently large n:

$$\mathbb{P}\left(S_2 < \frac{1}{f(2^n)}\right) \geq \frac{1}{2\sqrt{\pi} f(2^n)} . \tag{2}$$

On the other hand, an inequality observed by Csáki (1978) on page 210 yields

$$\mathbb{P}\left(S_2 < \frac{1}{f(2^n)}\,;\, S_{33} \leq \frac{2}{f(2^n)}\right) \leq \frac{1}{\sqrt{2\pi} f(2^n)^2} + \sqrt{\frac{8}{33\pi}} \frac{1}{f(2^n)} . \tag{3}$$

We deduce from (2) and (3) that

$$\mathbb{P}(E_n) \geq C_1 \left[\frac{1}{2\sqrt{\pi} f(2^n)} - \frac{1}{\sqrt{2\pi} f(2^n)^2} - \sqrt{\frac{8}{33\pi}} \frac{1}{f(2^n)}\right]^2 \geq \frac{C_2}{f(2^n)^2} . \tag{4}$$

The divergence of the integral $\int_1^\infty dt/tf(t)^2$ combined with the monotonicity of f thus ensures that $\sum \mathbb{P}(E_n) = \infty$. ◇

Lemma 4. *There is a finite constant C_3 such that*

$$\mathbb{P}(E_m \cap E_n) \leq C_3 \mathbb{P}(E_m) \mathbb{P}(E_n) \qquad \text{provided that } |m-n| \geq 7.$$

Proof. Suppose that $m \leq n-7$, so $2^n > 68 \cdot 2^m$. The probability $\mathbb{P}(E_m \cap E_n)$ is bounded from above by

$$\begin{aligned}\mathbb{P}\Big(E_m\,;\, &\sigma^+\left(\frac{\tau(2^n)^{1/4}}{f(2^n)}\right) - \sigma^+\left(\frac{\tau(2^m)^{1/4}}{f(2^m)}\right) > S_{\tau(2^n)} - S_{\tau(2^{m+1})}\,;\\ &\sigma^-\left(\frac{\tau(2^n)^{1/4}}{f(2^n)}\right) - \sigma^-\left(\frac{\tau(2^m)^{1/4}}{f(2^m)}\right) > I_{\tau(2^n)} - I_{\tau(2^{m+1})}\,;\\ &S_{\tau(2^n)} > 2^n\,;\, S_{\tau(2^{m+1})} < 34 \cdot 2^m\,;\, I_{\tau(2^n)} > 2^n\,;\, I_{\tau(2^{m+1})} < 34 \cdot 2^m\,;\\ &\tau(2^m) < 2^{2m+1}\,;\, 2^{2n} < \tau(2^n) < 2^{2n+1}\Big) .\end{aligned}$$

In turn, this is less than or equal to

$$\begin{aligned}\mathbb{P}\Big(E_m\,;\, &\sigma^+\left(\frac{\tau(2^n)^{1/4}}{f(2^n)}\right) - \sigma^+\left(\frac{\tau(2^m)^{1/4}}{f(2^m)}\right) > \frac{1}{2} \sup_{\tau(2^m) \leq t \leq \tau(2^n)} B_t\,;\\ &\sigma^-\left(\frac{\tau(2^n)^{1/4}}{f(2^n)}\right) - \sigma^-\left(\frac{\tau(2^m)^{1/4}}{f(2^m)}\right) > -\frac{1}{2} \inf_{\tau(2^m) \leq t \leq \tau(2^n)} B_t\,;\\ &(2^n - 2^m)^2 < \tau(2^n) - \tau(2^m) < 2(2^n - 2^m)^2\Big) .\end{aligned}$$

The inequality

$$\frac{\tau(2^n)^{1/4}}{f(2^n)} - \frac{\tau(2^m)^{1/4}}{f(2^m)} \leq \frac{\tau(2^n)^{1/4} - \tau(2^m)^{1/4}}{f(2^n)} \leq \frac{(\tau(2^n) - \tau(2^m))^{1/4}}{f(2^n)}$$

and the fact that τ, σ^+ and σ^- have independent and homogeneous increments then entail:

$$\begin{aligned} \mathbb{P}(E_m \cap E_m) \leq \mathbb{P}(E_m)\,\mathbb{P}\Big(\sigma^+\left(\frac{\tau(2^n-2^m)^{1/4}}{f(2^n)}\right) &> \frac{1}{2} S_{\tau(2^n-2^m)}\,; \\ \sigma^-\left(\frac{\tau(2^n-2^m)^{1/4}}{f(2^n)}\right) &> \frac{1}{2} I_{\tau(2^n-2^m)}\,; \\ (2^n-2^m)^2 < \tau(2^n-2^m) &< 2(2^n-2^m)^2\Big). \end{aligned}$$

Then using the scaling property, we can bound the right-hand-side by

$$\begin{aligned} &\mathbb{P}(E_m)\mathbb{P}\left(\sigma^+\left(\frac{\tau(1)^{1/4}}{f(2^n)}\right) > \frac{1}{2}S_{\tau(1)}\,;\, \sigma^-\left(\frac{\tau(1)^{1/4}}{f(2^n)}\right) > \frac{1}{2}I_{\tau(1)}\,;\, 1 < \tau(1) < 2\right) \\ \leq &\mathbb{P}(E_m)\mathbb{P}\left(\sigma^+\left(\frac{2}{f(2^n)}\right) > \frac{1}{2}S_1\,;\, \sigma^-\left(\frac{2}{f(2^n)}\right) > \frac{1}{2}I_1\right) \\ = &\mathbb{P}(E_m)\mathbb{P}\left(S^+\left(\frac{1}{2}S_1\right) < \frac{2}{f(2^n)}\,;\, S^-\left(\frac{1}{2}I_1\right) < \frac{2}{f(2^n)}\right) \\ \leq &\frac{8}{f(2^n)^2}\mathbb{P}(E_m)\mathbb{E}\left(1/\sqrt{S_1 I_1}\right) \end{aligned}$$

We have seen in the proof of Lemma 1 that $\mathbb{E}\left(1/\sqrt{S_1 I_1}\right) < \infty$, and Lemma 4 now follows from (4). $\diamond$

We are now able to complete the proof of the Theorem.

Proof of the Theorem, second part. Suppose that the integral in the Theorem diverges. By Lemmata 3 and 4 and an extension of the Borel-Cantelli lemma [see e.g. Spitzer (1964) on page 317], we know that $\mathbb{P}(\limsup_n E_n) > 0$. This implies that

$$\mathbb{P}\left(S^+\left(S_{\tau(t)}\right) < \frac{\tau(t)^{1/4}}{f(t)}\,;\, S^+\left(S_{\tau(t)}\right) < \frac{\tau(t)^{1/4}}{f(t)} \text{ i.o. as } t \to \infty\right) > 0.$$

Using (1) and the well-known fact that $\lim_{t\to\infty} \tau(t)/t^3 = 0$ a.s., we deduce that the probability of the event $\{\overline{X}_t < t^{1/4}/f(t^{1/3})$ infinitely often as $t \to \infty\}$ is positive, and hence must be one by virtue of Lemma 2. We thus have

$$\liminf_{t\to\infty} \overline{X}_t f(t^{1/3}) t^{-1/4} \leq 1 \qquad \text{a.s.}$$

The equivalence

$$\int_1^\infty \frac{dt}{tf(t)^2} = \infty \iff \int_1^\infty \frac{dt}{tf(t^3)^2} = \infty$$

shows that we have also

$$\liminf_{t\to\infty} \overline{X}_t f(t) t^{-1/4} \leq 1 \qquad \text{a.s.}$$

Finally, the integral test is unchanged when one replaces f by kf for any $k > 0$, and we conclude that the liminf above is zero a.s. ◇

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