SÉMINAIRE DE PROBABILITÉS (STRASBOURG)

BHASKARAN RAJEEV

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Séminaire de probabilités (Strasbourg), tome 30 (1996), p. 261-287

<http://www.numdam.org/item?id=SPS_1996__30__261_0>

FIRST ORDER CALCULUS AND LAST ENTRANCE TIMES

B. Rajeev
Stat-Math Division
Indian Statistical Institute
203, B.T. Road, Calcutta - 700 035
INDIA.

Introduction. Let (X_t) be a continuous local martingale, $X_0 = 0$ a.s., $\langle X \rangle$ the quadratic variation process, $L(t,0)$ its local time at zero. Let $\tau_t = \max\{s < t : X_s = 0\}$ and (h_t) a locally bounded previsible process. The point of departure in 'First Order Calculus' (see [10], page 241) is an interesting path property of (semi) martingales given by the so called Balayage formula

$$h_{\tau_t} X_t = \int_0^t h_{\tau_s} \, dX_s \qquad (\#)$$

This says that $h_{\tau_t} X_t$ is a continuous local martingale and then it is not too difficult to show that its local time at zero is $\int_0^t |h_s| dL(s,0)$. This is in analogy with the usual second order (Ito) stochastic calculus where the local martingale $\int_0^t h(s) dX_s$ has the quadratic variation $\int_0^t h^2(s) d\langle X \rangle_s$. Actually eqn. $(\#)$ holds in more generality. It holds whenever X is an arbitrary semi-martingale, h a locally bounded previsible process, $\tau_t = \max\{s < t : s \in H\}$, where H is a random closed optional set with $X_t(\omega) = 0$ for $(t,\omega) \in H$. After its first appearance in Azema and Yor [1], it was later studied extensively in a series of papers [4], [6], [7], [11] and [12]. In particular, conditions on both the set H and the process h can be further relaxed (see [4] for the first case, [6] and [11] for the second).

Consider now an equivalent formulation of the above result. Let $\sigma_t = \max\{s \leq t : s \in H\}$. The condition $X_t(\omega) = 0$, $(t,\omega) \in H$ is equivalent to $X_{\sigma_t} \equiv 0$ because $\sigma_t(\omega) = t$ iff

$(t,\omega) \in H$. Observe further that, as a consequence $X_t \equiv (X-X_\sigma)(t)$ $(= X_t - X_{\sigma_t})$. We can write eqn. ($\#$) in terms of $(X-X_\sigma)$ rather than X. What happens if we drop the requirement $X_\sigma \equiv 0$? Note that in any case we have $(X-X_\sigma)(t) = 0$ for $t \in H$ and eqn. ($\#$) would hold for $(X-X_\sigma)$ provided we can show that $(X-X_\sigma)$ is a semi-martingale or equivalently that X_σ is a semi-martingale. In analogy with case of measures, we can think of $(X-X_\sigma)$ as the H-balayage of X. We now give some examples where the condition $X_{\sigma_t} \equiv 0$ fails, but X_σ is in fact a semi-martingale.

The original motivation behind this work is the following example : Let (X_t) be a continuous semi-martingale, $a < b$ and assume for simplicity that $X_0 \notin (a,b)$ a.s. Let $A = \{(s,\omega) : X_s(\omega) \in (a,b)\}$, $H = A^c$ and σ_t as above. In this case $X_{\sigma_t} \not\equiv 0$ and it was shown in [9], that $(X-X_\sigma)$ is a semi-martingale and its decomposition obtained. A typical feature of the process $(X-X_\sigma)$ is clearly reflected in the above example : $(X-X_\sigma)$ is no longer a continuous process. However we do have the following for all $t \geq 0$,

$$\sum_{s\leq t} |\Delta(X-X_\sigma)(s)| = \sum_{s\leq t} |\Delta X_\sigma(s)| = (b-a) \times \text{number of crossings of } (a,b) \text{ by } X \text{ in time } t < \infty \quad \text{a.s.}$$

Another interesting example is given by $A = \{(s,\omega) : X_s(\omega) > a\}$, $X_0 < a$ a.s., $H = A^c$, $\sigma_t = \max\{s \leq t : s \in A^c = H\}$. Here too $X_{\sigma_t} \not\equiv 0$. The process $(X-X_\sigma)$ is however continuous and it is easily seen that $(X-X_\sigma)(t) = (X_t-a)^+$. In this case the semi-martingale decomposition for $(X-X_\sigma)$ is just the Tanaka formula.

To return to the question posed in the previous paragraph : Let (X_t) be an arbitrary semi-martingale with respect to a filtration $\mathcal{F}_t$, A an $\mathcal{F}_t$ optional set with open sections $\subset (0,\infty) \times \Omega$, $\sigma_t = \max\{s \leq t : s \in A^c\}$. Let A_r denote the right end points of the intervals of A and $A] = A \cup A_r$. We show that under the condition, $\sum_{s\le t} I_{A]}(s)|\Delta X_\sigma(s)| = \sum_{s\le t} I_{A_r}(s)|\Delta X_\sigma(s)| < \infty$ a.s., $(X-X_\sigma)$ is an $\mathcal{F}_t$ semi-martingale. Further, we obtain a Tanaka-like formula for $(X-X_\sigma)$ involving a continuous process of finite variation, which is a sort of local time of X on A (see Section 2, Theorem 1, Corollary 1). Note that the condition $\sum_{s\le t} I_{A]}(s)\ |\Delta X_\sigma(s)| < \infty$ a.s. generalises the condition $X_{\sigma_t} = 0$. The former condition is also necessary (see Remarks following Theorem 3).

The paper is organised as follows : In section 0, we describe the main ideas of the proof in the case of continuous semi-martingales. Section 1 contains the notations and other preliminaries, Section 2 the statement of the main result and its corollaries and section 3 the proofs. In section 4, the final section, we deduce the usual Tanaka formula for an arbitrary semi-martingale as a consequence of our main results (Theorem 2). We thus give a new proof of Tanaka's formula. We also relate the process of finite variation (or local time of X on A) obtained in Theorem 1 to the local times at zero of the semi-martingales $(X-X_\sigma)$ and $-(X-X_\sigma)$, thus closing the circle of ideas (see Theorem 3).

0. The Case of Continuous Semi-Martingales : In this section we describe our results and sketch the idea of the proof in the case of continuous semi-martingales. We start however with an arbitrary semi-martingale X and bring in the continuity assumption only when it is required. We hope this will give a better understanding of the course we eventually take in section 3 in the proofs of the main results.

Let then (X_t) be a semi-martingale adapted to a filtration $\mathcal{F}_t$, with rcll trajectories. A will denote an $\mathcal{F}_t$ optional set with open sections and for simplicity will be assumed to be contained in $(0,\infty) \times \Omega$. $\sigma_t = \max\{s \leq t : s \in A^c\}$. Then $X_\sigma(t) = X_{\sigma_t}$ is an adapted rcll process. Suppose first that A is a simple optional set with open sections. i.e. $A = \bigcup_{i=1}^{\infty} (\sigma_i, \tau_i)$ where σ_i, τ_i are stop times, $\sigma_i < \tau_i \leq \sigma_{i+1}$ and $\sigma_k \longrightarrow \infty$ as $k \longrightarrow \infty$. Let $A] = \bigcup_{i=1}^{\infty} (\sigma_i, \tau_i]$ and h a locally bounded previsible process. It is easy to see that

$$h(\sigma_{s-})\, I_{A]}(s) = \sum_{k=1}^{\infty} h(\sigma_k) I_{(\sigma_k, \tau_k]}(s)$$

and that $\triangle X_\sigma(s) \quad = \quad (X_{\tau_k} - X_{\sigma_k}), \quad s = \tau_k$

It follows that $h(\sigma_{s-})\, I_{A]}(s)$ is a simple predictable process and that the process $\sum_{s \leq .} I_{A]}(s) \triangle X_\sigma(s)$ is of finite variation on compact intervals. Now from the definition of the stochastic integral we get

$$\sum_{s \leq t} h(\sigma_{s-}) I_{A]}(s) \triangle X_\sigma(s) + h(\sigma_{t-})(X - X_\sigma)(t) = \int_0^t h(\sigma_{s-}) I_{A]}(s)\, dX_s \quad - (0.1)$$

Note that when X is continuous, $\triangle X_\sigma(s) = -\triangle(X - X_\sigma)(s)$.

Taking $h \equiv 1$ in (0.1) we get

$$\sum_{s \leq t} I_{A]}(s) \triangle X_\sigma(s) + (X - X_\sigma)(t) = \int_0^t I_{A]}(s)\, dX_s \qquad - (0.2)$$

It follows that $(X - X_\sigma)$ is a semi-martingale and taking $H = A^c$, $\tau_t = \sigma_{t-}$, it is easy to see that (0.1) and (0.2) imply ($\#$).

In the case of a general optional set with open sections, we assume $\sum_{s \leq t} I_{A]}(s)\, |\triangle X_\sigma(s)| < \infty$ a.s. for all t. To show that $(X - X_\sigma)$ is a semi-martingale, we approximate A by simple optional sets A_n with open sections.

If σ_n are the entrance times for A_n, the idea is to replace (σ, A) in eqn. (0.2) by (σ_n, A_n) and then let $n \to \infty$. To ensure $(X - X_{\sigma_n})(t) \longrightarrow (X - X_\sigma)(t)$, we choose A_n's such that

$A_n \subset A_{n+1}$ and $A = \bigcup_n A_n$. This implies that $\sigma_n(t) \downarrow \sigma(t)$ and $(X-X_{\sigma_n})(t) \longrightarrow (X-X_\sigma)(t)$ pointwise by right continuity of X. To ensure the convergence of the stochastic integral and jump terms in (0.2) a further condition on the A_n's is necessary : We demand that that the right end points of the intervals of A_n be contained in those of A. i.e., $A_{n,r} \subset A_r$ where $A_{n,r} = \{d_t^n , t > 0\}$, $A_r = \{d_t : t > 0\}$, $d_t^n = \inf\{s > t : s \in A_n^c\}$, $d_t = \inf\{s > t : s \in A^c\}$. Note that $A_n] = A_n \cup A_{n,r}$ and $A] \overset{(defn)}{=} A \cup A_r$. Under these conditions on A_n, it is easy to see that $I_{A_n]}(s) \longrightarrow I_{A]}(s)$ pointwise and hence $\int_0^t I_{A_n]}(s)\, dX_s \longrightarrow \int_0^t I_{A]}(s)\, dX_s$ in probability. The jumps $I_{A_n]}(s) \triangle X_{\sigma_n}(s)$ are now 'aligned' with that of $I_{A]}(s) \triangle X_\sigma(s)$ and we can write the jump term in (0.2) as

$$\sum_{s \le t} I_{A_n]}(s) \triangle X_\sigma(s) + \sum_{s \le t} I_{A_n]}(s)(\triangle X_{\sigma_n}(s) - \triangle X_\sigma(s)) \qquad - (0.3)$$

Now under the assumption $\sum_{s \le t} I_{A]}(s)\, |\triangle X_\sigma(s)| < \infty$ a.s. for all t, the first term in (0.3) converges to $\sum_{s \le t} I_{A]}(s) \triangle X_\sigma(s)$. It now follows from (0.2), with (σ,A) replaced by (σ_n, A_n), that as $n \to \infty$, the 2nd term in (0.3) has a limit$(-L(t))$ given by

$$-L(t) = \int_0^t I_{A]}(s)\, dX_s - (X-X_\sigma)(t) - \sum_{s \le t} I_{A]}(s) \triangle X_\sigma(s) \qquad - (0.4)$$

When X is a continuous semi-martingale it is obvious that the RHS of (0.4) defines an adapted continuous process. In fact, it is easy to see from the properties of the stochastic integral, that even in the case of an arbitrary semi-martingale, the RHS of (0.4) defines a continuous adapted process. We still need a crucial result to deduce from (0.4) that $(X-X_\sigma)(t)$ is a semi-martingale viz. that L is of finite variation. But if we now introduce the condition

$$(X-X_\sigma)(t) > 0 \quad t \in A \qquad - (0.5)$$

then since $A_n \subset A$, each of the terms in the second sum in (0.3) is non-positive. As a consequence the second sum in (0.3) is non-increasing and so is its limit $-L(t)$. i.e. $L(t)$ is a non-decreasing process and from (0.4) it follows that $(X-X_\sigma)(t)$ is a semi-martingale with a decomposition given by (0.4).

The existence of sets A_n satisfying 1) $A_n \subset A_{n+1}$ 2) $A = \bigcup_n A_n$ and 3) $A_{n,r} \subset A_r$ is easily shown : If r_n is an enumeration of the rationals we can take $A_n = \bigcup_{i=1}^{n} (r_i, D_{r_i})$. If in addition (0.5) holds, there is a further choice viz. $A_n = \bigcup_{k=1}^{\infty} (\sigma_k^n, \tau_k^n)$ where σ_k^n, τ_k^n are the successive crossing times of $1/n$ and 0 by $(X-X_\sigma)(t)$. In fact in the proofs of the general case we shall use the latter choice, mainly for its geometric appeal. At this point we note that our proof is a generalisation of the 'down-crossing' proof of the Tanaka formula (see [5], [8] and [9]).

Thus far we have proved the following result :
Suppose X - any semi-martingale, A an optional set with open sections such that $\sum_{s\le t} I_{A]}(s)\, |\Delta X_\sigma(s)| < \infty$ a.s. for all t.

Suppose in addition that (0.5) holds. Then there exists a continuous, adapted increasing process $L(t)$ such that

$$\sum_{s\le t} I_{A]}(s)\, \Delta X_\sigma(s) + (X-X_\sigma)(t) = \int_0^t I_{A]}(s)\, dX_s + L(t) \qquad - (0.6)$$

We now demonstrate how condition (0.5) may be dropped in the case when X is continuous. Introduce the sets $A_u = \{(s,\omega) : (X-X_\sigma)(s,\omega) > 0\}$, $A_d = \{(s,\omega) : (X-X_\sigma)(s,\omega) < 0\}$ observe that A_u and A_d are optional sets with open sections (because X is continuous), contained in A. Let σ_u and σ_d

be entrance times for A_u and A_d respectively. Obviously $(X-X_{\sigma_u})$ and A_u satisfy (0.5). Further because X is continuous, $\sum_{s\le t} |\triangle X_{\sigma_u}(s)| = \sum_{s\le t} |\triangle X_\sigma(s)\, I_{\{s:\triangle X_\sigma(s)>0\}}(s)| < \infty$

a.s. for all t. Then by the above result, there exists a continuous adapted non-decreasing process L_u such that (0.6) holds for $(X-X_{\sigma_u})$, A_u and L_u. Applying the same arguments to $-X$, there exists a continuous adapted increasing process L_d such that (0.6) holds for $(X-X_{\sigma_d})$, A_d and $-L_d$. Hence,

$$\begin{aligned}(X-X_\sigma)(t) &= (X-X_\sigma)^+(t) - (X-X_\sigma)^-(t)\\ &= (X-X_{\sigma_u})(t) + (X-X_{\sigma_d})(t) \quad (\because X \text{ is continuous})\\ &= \int_0^t I_{A_u]\cup A_d]}(s)\, dX_s + L_u - L_d - \sum_{s\le t} I_{A]}(s)\triangle X_\sigma(s)\end{aligned}$$

Now it can be shown (see remark following Corollary 1 to Theorem 1) that

$$d\,\langle X^c\rangle\,(A] - A_u] \cup A_d]) = 0 \quad \text{a.s.}$$

Let $L(t) = L_u(t) - L_d(t) - \int_0^t I_{A]-A_u] \cup A_d]}(s)\, dV_s$

where $X = M+V$.

L is a continuous adapted process of finite variation such that (0.6) holds for $(X-X_\sigma)$, A and L. In particular, $(X-X_\sigma)$ is a semi-martingale.

1. Preliminaries. We continue with the notation and terminology of Sec. 0. We have the filtered probability space $(\Omega, \mathcal{F}, \mathcal{F}_t, P)$ satisfying usual conditions. (X_t) is a given $\mathcal{F}_t$ semi - martingale, fixed for the

rest of the discussion. $A \subset (0, \infty) \times \Omega$ is an optional set with open sections. If $A(\omega) = \bigcup_i (\alpha_i(\omega), \beta_i(\omega))$, let $A]$ be the previsible set with sections $A](\omega) = \bigcup_i (\alpha_i(\omega), \beta_i(\omega)]$. Let $\sigma_t(\omega) = \max\{ s \leq t : s \in A^c(\omega)\}$, $\max\{\phi\} = 0$. Henceforth we suppress the dependance on ω and write, for example, $t \in A$ instead of $t \in A(\omega)$. For any adapted rcll process (Y_t) we define the adapted rcll processes $(Y_\sigma(t))$ and $((Y - Y_\sigma)(t))$ by $Y_\sigma(t) = Y_{\sigma_t}$ and $(Y - Y_\sigma)(t) = Y_t - Y_{\sigma_t}$. For $h > 0$, let $D_{uh} = \inf\{ s > h : (X - X_\sigma)(s) \leq 0 \}$. The optional set A_u is defined by $A_u = \bigcup_h (h, D_{uh})$. Let $\sigma_u(t) = \max\{ s \leq t : s \in A_u^c \}$. We similarly define $D_{dh} = \inf\{ s > h : (X - X_\sigma)(s) \geq 0\}$, $A_d = \bigcup_h (h, D_{dh})$ and $\sigma_d(t) = \max\{ s \leq t : s \in A_d^c\}$. Here the suffix u stands for 'up' and d for 'down'. In particular A_u (A_d) is the set of times in A at which the process X is above (below) X_σ. Let $A_{ud} = A_u \cup A_d$. Let A^r (A^ℓ) denote the set of right (respectively left) end points of the intervals of A. $A_u^r, A_u^\ell, A_d^r, A_d^\ell$ have similar meanings relative to the sets A_u and A_d. X^c as usual denotes the continuous martingale part of X. I_A will denote the indicator function of the set A.

Fix $n \geq 1$. Define $\tau_0^n \equiv 0$ and let

$$\sigma_k^n = \inf\{ s > \tau_{k-1}^n, \ (X - X_\sigma)(s) \geq 1/n \} \quad k = 1,2,\ldots$$

$$\tau_k^n = \inf\{ s > \sigma_k^n, \ (X - X_\sigma)(s) \leq 0 \} \quad k = 1,2,\ldots$$

where we take $\inf\{\phi\} = \infty$. For $n \geq 1$, let $A_n = \bigcup_{k=1}^{\infty} (\sigma_k^n, \tau_k^n)$, $n \geq 1$ and $\sigma_n(t) = \max\{ s \leq t : s \in A_n^c \}$. It is easy to see that $A_n \subset A_{n+1}$ which implies that $\sigma_{n+1}(\cdot) \leq \sigma_n(\cdot)$.

Proposition 1. a) i) $A_u \subset \{s : (X-X_\sigma)(s) > 0\} \subset A$

ii) $A_d \subset \{s : (X-X_\sigma)(s) < 0\} \subset A$

b) i) $A_u = \bigcup_{n=1}^{\infty} A_n$

c) i) $A] - A_{ud}] \subset \{s \in A : X_{s-} = X_\sigma(s)\}$

ii) $(A]-A_{ud}]) \cap \{s : \Delta X_s \neq 0\} \subset A_u^\ell \cup A_d^\ell$.

Proof. a) i). It is easy to see that for $t \in A_u$, $(X-X_\sigma)(t) > 0$. The second inclusion in a) i) follows from the observation that $(X-X_\sigma)(t) = 0$ for $t \notin A$. The proof of a) ii) is similar.

b) It is obvious that $(\sigma_k^n, \tau_k^n) \subset A_u$ for $n \geq 1$, $k \geq 1$. Hence $\bigcup_n A_n \subset A_n$. To see the reverse inclusion suppose $s \in A_n$ and $s \notin A_n$ for all $n \geq 1$. Thus for all $n \geq 1$, there exists k_n such that $s \in [\tau_{k_n-1}^n, \sigma_{k_n}^n]$. In fact there exists n_o such that for all $n \geq n_o$, $s = \sigma_{k_n}^n$: otherwise $s \in [\tau_{k_n-1}^n, \sigma_{k_n}^n)$ for infinitely many n. Hence $(X - X_\sigma)(s) < \frac{1}{n_k}$ for $n_k \to \infty$. This implies $(X - X_\sigma)(s) \leq 0$, contradicting the first inclusion in a) i). Since A_u has open sections, there exist α, β, $\alpha < \beta$ such that $(\alpha,\beta) \subset A_u$, $\alpha < s = \sigma_{k_n}^n < \beta$ for all $n > n_o$. But $s = \sigma_{k_n}^n$ for all $n > n_o$ implies, from the definition of σ_k^n's that $(X-X_\sigma)(s-) \leq 0$. In particular there exists $s_o \in (\alpha,\beta)$ such that $(X-X_\sigma)(s_o) \leq 0$, which contradicts a) i).

c) i) If $s \in A]$ and $s \notin A_{ud}]$ then there exist sequences s_n, t_n, $s_n \uparrow s$, $t_n \uparrow s$, such that

$$X_{s_n} \leq X_\sigma(s_n) = X_\sigma(s) \quad \text{and} \quad X_{t_n} \geq X_\sigma(t_n) = X_\sigma(s).$$

It follows that $X_{s-} = X_{\sigma_s}$

ii) If $s \in A] - A_{ud}]$ and $\triangle X_s \neq 0$, it follows from c) i) that either $X_s > X_{\sigma_s}$ or $X_s < X_{\sigma_s}$. This, together with $s \notin A_{ud}]$ implies that, in fact $s \in A_u^{\ell} \cup A_d^{\ell} - (A_u^r \cup A_d^r)$.

2. The Main Results. Let $X, A, \sigma, A_u, A_d, A_{ud}$ all be as in section 1. We can now state our main result. All the proofs are deferred to the next section. The following hypothesis is fundamental :

for all t, $\sum_{s \leq t} I_{A]}(s) |\triangle X_\sigma(s)| < \infty$ almost surely - (*).

Note that $I_{A]}(s) \triangle X_\sigma(s) = I_{A^r}(s) \triangle X_\sigma(s)$.

Theorem 1. Suppose (*) holds. Then

a) for every t,

$$\sum_{s \leq t} I_{A]-A_{ud}]}(s) |\triangle X_s| < \infty \text{ almost surely.}$$

b) X_σ and $X - X_\sigma$ are semi-martingales.

c) There exists a unique continuous adapted process of finite variation, denoted by $L(.)$, such that almost surely,

$$\sum_{s \leq t} I_{A]}(s) \triangle X_\sigma(s) - \sum_{s \leq t} I_{A]-A_{ud}]}(s) \triangle X_s + (X - X_\sigma)(t)$$

$$= \int_0^t I_{A_{ud}]}(s)\, dX_s + L(t) \qquad (1)$$

for all $t \geq 0$.

d) If in addition to (*), $(X - X_\sigma)(t) \geq 0$ for all t, almost surely, then $L(.)$ is an increasing process.

(e) The process $L(.)$ is supported on the complement of the set $A_{ud}]$. In other words, almost surely,

$$\int_0^t I_{A_{ud}]}(s)\, dL(s) = 0$$

for all $t \geq 0$.

Eqn. (1) can be reduced to a simpler and more elegant form under an additional 'hypothesis' which we now state

almost surely, $\int_0^\infty I_{A]-A_{ud}]}(s) \langle X^c \rangle_s = 0$ — (**)

<u>Corollary 1</u>. Suppose (*) and (**) hold. Then there exists a continuous adapted process of finite variation, denoted by $L'(.)$, such that almost surely,

$$\sum_{s \leq t} I_{A]}(s) \triangle X_\sigma(s) + (X - X_\sigma)(t) = \int_0^t I_{A]}(s) dX_s + L'(t) \quad (2)$$

for all $t \geq 0$. Moreover, almost surely,

$$\int_0^t I_{A]}(s)\, dL'(s) = 0$$

for all $t \geq 0$.

<u>Remark.</u> The condition (**) is actually redundant i.e. it is always satisfied for any semi-martingale X and any optional set with open sections. This is a consequence of the occupation density formula. We indicate a proof of this below. In particular we recover the results of [9] from Corollary. 1. However for an arbitrary semi-martingale, there are certain sets for which we can prove (**) holds without using the occupation density formula. We take this approach in section (4) to prove the Tanaka formula.

To prove (**) is actually redundant, we observe that as a consequence of the occupation density formula, the following holds :

almost surely, $d\langle X^c\rangle\{s : X_{s-} = a\} = 0$ for every $a \in \mathbb{R}$.

From Propn. 1 c) we have

$$A] - A_{ud}] \subset \{s : X_{s-} = X_{\sigma_s}\} \cap A$$

Since the set $\{X_{\sigma_s}, s \in A\}$ is countable, (**) follows.

3. <u>The Proofs</u>. The proof of Theorem 1 is rather long. For simplicity and to keep track of the main ideas, we break it up into different steps which we state as lemmas. The proof leans heavily on the notion of the stochastic integral and its various properties. We refer to [2], Chapter VIII for these. A few steps involve a construction in terms of A_u, σ_u etc. and then a symmetrical construction for A_d, σ_d etc. We prove only the former case. We recall the sets A_n of proposition (1) which play a key role in the proof.

<u>Lemma 1</u>. For every n, there exists an increasing adapted, purely discontinuous process $\eta_n(.), \eta_n(0) = 0$, for which the following equation holds : almost surely,

$$-\eta_n(t) + \sum_{k=1}^{\infty} I_{A^c \cap [0,t]}(\tau_k^n)\, \Delta X_\sigma(\tau_k^n) + (X - X_{\sigma_n})(t)$$

$$= \int_0^t I_{A_n]}(s)\, dX_s \qquad (3)$$

for all $t \geq 0$.

<u>Lemma 2</u>. Suppose (*) holds. Then there exist increasing adapted processes $\eta_u(.), \eta_d(.), \eta_u(0) \equiv \eta_d(0) \equiv 0$ and satisfying : almost surely,

$$-\eta_u(t) + \sum_{s\le t} I_{A_u^r \cap A^c}(s)\, \Delta X_\sigma(s) + (X - X_{\sigma_u})(t)$$

$$= \int_0^t I_{A_u]}(s)\, dX_s \qquad (4)$$

$$\eta_d(t) + \sum_{s\le t} I_{A_d^r \cap A^c}(s)\, \Delta X_\sigma(s) + (X - X_{\sigma_d})(t)$$

$$= \int_0^t I_{A_d]}(s)\, dX_s \qquad (5)$$

Lemma 3. Suppose (*) holds. Then for every $t > 0$,

$\sum_{s\le t} (I_{A_u]}(s)|\Delta X_{\sigma_u}(s)| + I_{A_d]}(s)|\Delta X_{\sigma_d}(s)|) < \infty$ almost surely

and there exists continuous adapted increasing processes $L_u(\cdot), L_d(\cdot), L_u(0) \equiv L_d(0) \equiv 0$ and satisfying : almost surely,

$$\sum_{s\le t} I_{A_u]}(s)\, \Delta X_{\sigma_u}(s) + (X - X_{\sigma_u})(t) = \int_0^t I_{A_u]}(s) dX_s + L_u(t) \qquad (6)$$

$$\sum_{s\le t} I_{A_d]}(s)\, \Delta X_{\sigma_d}(s) + (X - X_{\sigma_d})\, (t) = \int_0^t I_{A_d]}(s) dX_s - L_d(t) \qquad (7)$$

Lemma 4. Suppose (*) holds. Then, for every t,

$\sum_{s\le t} (|\Delta(X_{\sigma_u} - X_\sigma)^+(s)| + |\Delta(X_{\sigma_d} - X_\sigma)^-(s)|) < \infty$ almost surely

and we have almost surely,

$$(X - X_\sigma)^+(t) = (X - X_{\sigma_u})(t) + \sum_{s\le t} \Delta(X_{\sigma_u} - X_\sigma)^+(s) \qquad (8)$$

$$(X - X_\sigma)^-(t) = -(X - X_{\sigma_d})(t) + \sum_{s\le t} \Delta(X_{\sigma_d} - X_\sigma)^-(s) \qquad (9)$$

for all $t \ge 0$.

Proof of Lemma 1. From the definition of the stochastic integral, we have

$$\int_0^t I_{A_n}(s)dX_s = \sum_{k=1}^{\infty} X(\tau_k^n \wedge t) - X(\sigma_k^n \wedge t)$$

$$= \sum_{k=1}^{\infty} I_{[0,t]}(\tau_k^n)(X(\tau_k^n) - X(\sigma_k^n)) + (X-X_{\sigma_n})(t)$$

$$= \sum_{k=1}^{\infty} I_{[0,t]\cap A}(\tau_k^n)(X(\tau_k^n) - X(\sigma_k^n))$$

$$+ \sum_{k=1}^{\infty} I_{[0,t]\cap A^c}(\tau_k^n)(X(\tau_k^n) - X(\sigma_k^n))$$

$$+ (X-X_{\sigma_n})(t) \tag{10}$$

Note that for $\tau_k^n \in A$, $X(\tau_k^n) - X(\sigma_k^n) \le 0$. For $\tau_k^n \in A^c$ we can write

$$X(\tau_k^n) - X(\sigma_k^n) = X(\tau_k^n) - X_\sigma(\tau_k^n) + X_\sigma(\tau_k^n) - X(\sigma_k^n)$$

$$= \Delta X_\sigma(\tau_k^n) + X_\sigma(\tau_k^n) - X(\sigma_k^n)$$

where $X_\sigma(\tau_k^n) - X(\sigma_k^n) \le 0$ for $\tau_k^n \in A^c$.

Define $$\eta_n(t) = - \sum_{k=1}^{\infty} I_{A\cap[0,t]}(\tau_k^n)(X(\tau_k^n) - X(\sigma_k^n))$$

$$- \sum_{k=1}^{\infty} I_{A^c\cap[0,t]}(\tau_k^n)(X_\sigma(\tau_k^n) - X(\sigma_k^n))$$

$\eta_n(.)$ is an adapted, increasing, purely discontinuous process and $\eta_n(0) = 0$. Eqn. (3) is obtained from eqn. (10) by rewriting it in terms of η_n. This completes the proof of Lemma 1.

Proof of Lemma 2. From $A_u = \bigcup_{n=1}^{\infty} A_n$ (Propn. 1 b)), condition (*) and the well known properties of the stochastic Integral, it follows that the three terms in eqn. (3) viz.

$$\int_0^{\cdot} I_{A_n]}(s)dX_s, \quad \sum_{k=1}^{\infty} I_{A^c \cap [0,\cdot]}(\tau_k^n) \Delta X_\sigma(\tau_k^n), \quad (X-X_{\sigma_n})(\cdot)$$

Converge in probability to

$$\int_0^{\cdot} I_{A_u]}(s)dX_s, \quad \sum_{s\leq\cdot} I_{A_u^r \cap A^c}(s) \Delta X_\sigma(s), \quad (X-X_{\sigma_u})(\cdot)$$

respectively. It follows that the fourth term in eqn. (2) viz. $\eta_n(\cdot)$ converges to an adapted increasing process $\eta_u(\cdot)$ and obviously eqn. (4) holds. The existence of η_d and eqn. (5) follows by applying the previous argument to $-X$.

Proof of Lemma 3. We break up η_u into its continuous part and its purely discontinuous part denoting them by L_u and ℓ_u respectively. From eqn. (4), equating jumps on either side at time s we get,

$$-\Delta\ell_u(s) + I_{A_u^r \cap A^c}(s)\Delta X_\sigma(s) + \Delta X(s) - \Delta X_{\sigma_u}(s)$$
$$= I_{A_u]}(s)\Delta X_s \tag{11}$$

If $s \notin A_u]$, $\Delta X(s) = \Delta X_{\sigma_u}(s)$, $I_{A_u^r \cap A^c}(s) = 0$

and it follows that $\Delta\ell_u(s) = 0$.

If $s \in A_u]$, but $s \notin A_u^r$, then $\Delta X_{\sigma_u}(s) = 0$ and again $\Delta\ell_u(s) = 0$.

If $s \in A_u]$ and $s \in A_u^r \cap A$ then $\Delta\ell_u(s) = -\Delta X_{\sigma_u}(s)$.

If $s \in A_u]$ and $s \in A_u^r \cap A^c$ then $\Delta\ell_u(s) - \Delta X_\sigma(s) = -\Delta X_{\sigma_u}(s)$.

Note that $A_u^r \cap A^c \subset A^r \subset A]$. In particular, it follows that for every $t \geq 0$,

$$\sum_{s\leq t} I_{A_u]}(s)|\Delta X_{\sigma_u}(s)| \leq \sum_{s\leq t}|\Delta\ell_u(s)| + \sum_{s\leq t} I_{A]}(s)|\Delta X_\sigma(s)|$$

$$< \infty \quad \text{almost surely.}$$

Moreover,

$$\sum_{s\leq t} I_{A_u]}(s)\,\Delta X_{\sigma_u}(s) = -\Big(\sum_{s\leq t}\Delta \ell_u(s) - \sum_{s\leq t} I_{A_u^r\cap A^c}(s)\,\Delta X_\sigma(s)\Big)$$

$$= -\ell_u(t) + \sum_{s\leq t} I_{A_u^r\cap A^c}(s)\,\Delta X_\sigma(s) \qquad (12)$$

Also

$$\eta_u(t) = L_u(t) + \ell_u(t) \qquad (13)$$

Eqn. (6) now follows from eqns. (4), (11), (12) and (13). Eqn. (7) is proved in a similar manner.

<u>Proof of Lemma 4</u>. We note that $\Delta(X_{\sigma_u}-X_\sigma)^+(s) \neq 0$ iff $s \in A_u^r \cup A_u^\ell$ i.e. when s is a left or right end point of A_u. Moreover the jumps of $(X_{\sigma_u}-X_\sigma)^+$ at successive end points of A_u are equal but of opposite sign. Hence

$$\sum_{s\leq t} |\Delta(X_{\sigma_u}-X_\sigma)^+(s)| \leq 2 \sum_{s\leq t} I_{A_u^r}(s)\,|\Delta(X_{\sigma_u}-X_\sigma)^+(s)|$$

$$+ |\Delta(X_{\sigma_u}-X_\sigma)^+(t)|$$

and

$$\sum_{s\leq t} I_{A_u^r}(s)\,|\Delta(X_{\sigma_u}-X_\sigma)^+(s)| = \sum_{s\leq t} I_{A_u^r\cap A}(s)\,|\Delta(X_{\sigma_u}-X_\sigma)^+(s)|$$

$$+ \sum_{s\leq t} I_{A_u^r\cap A^c}(s)\,|\Delta(X_{\sigma_u}-X_\sigma)^+(s)|$$

$$\leq \sum_{s\leq t} I_{A_u^r\cap A}(s)\,|\Delta X_{\sigma_u}(s)|$$

$$+ \sum_{s\leq t} I_{A_u^r\cap A^c}\big(|\Delta X_{\sigma_u}(s)| + |\Delta X_\sigma(s)|\big)$$

$$= \sum_{s\leq t} I_{A_u]}(s)\,|\Delta X_{\sigma_u}(s)|$$

$$+ \sum_{s\leq t} I_{A_u^r \cap A^c}(s) | \triangle X_\sigma(s) |$$

$$< \infty \quad \text{almost surely}$$

because of condition (*) and lemma 3. It is easy to see that

$$(X-X_\sigma)^+(t) = (X-X_{\sigma_u})(t) + (X_{\sigma_u}-X_\sigma)^+(t) \tag{14}$$

and that

$$(X_{\sigma_u}-X_\sigma)^+(t) = \sum_{s\leq t} \triangle (X_{\sigma_u}-X_\sigma)^+(s) \tag{15}$$

Eqn. (8) follows from eqns. (14) and (15). Eqn. (9) is proved in a similar manner.

<u>Proof of Theorem 1</u>. a) Using proposition 1 c) it is easy to see that

$$\sum_{s\leq t} I_{A]-A_{ud}]}(s) |\triangle X_s| \leq \sum_{s\leq t} (|\triangle (X_{\sigma_u}-X_\sigma)^+(s)| + |\triangle (X_{\sigma_d}-X_\sigma)^-(s)|)$$

$$< \infty \quad \text{almost surely}$$

This proves a). The proof of b) is immediate from Lemmas (3), (4) and the identity $x = x^+ - x^-$.

Proof of c) : To begin with we note the following pathwise identity viz. almost surely,

$$- \sum_{s\leq t} I_{A_u]}(s) \triangle X_{\sigma_u}(s) + \sum_{s\leq t} \triangle (X_{\sigma_u}-X_\sigma)^+ (s)$$

$$- \sum_{s\leq t} \triangle (X_{\sigma_d}-X_\sigma)^-(s) - \sum_{s\leq t} I_{A_d]}(s) \triangle X_{\sigma_d}(s)$$

$$= - \sum_{s\leq t} I_{A]}(s) \triangle X_\sigma(s) + \sum_{s\leq t} I_{A]-A_{ud}]}(s) \triangle X_s \tag{16}$$

The identity is proved by verifying it separately at each $s \in (A_u^r \cup A_d^r) \cap A$, $s \in (A_u^r \cup A_d^r) \cap A^c$, $s \in (A_u^\ell \cup A_d^\ell) \cap A$. We then have using eqns. (6), (7), (8), (9) and (16),

$$(X - X_\sigma)(t) = (X-X_\sigma)^+(t) + (X-X_\sigma)^-(t)$$

$$= (X-X_{\sigma_u})(t) + \sum_{s\leq t} \triangle (X_{\sigma_u}-X_\sigma)^+(s) + (X-X_{\sigma_d})(t)$$

$$- \sum_{s\leq t} \triangle (X_{\sigma_d}-X_\sigma)^-(s)$$

$$= \int_0^t I_{A_{ud}]}(s)dX_s + (L_u-L_d)(t) - \sum_{s\leq t} I_{A]}(s)\triangle X_\sigma(s)$$

$$+ \sum_{s\leq t} I_{A]-A_{ud}]}(s) \triangle X_s$$

which is eqn. (1) with $L(t) = L_u(t) - L_d(t)$. That L is unique is obvious and the proof of c) is complete.

To prove d) we note that when $(X-X_\sigma)(t) \geq 0$ for all t, almost surely then $A_d = \phi$ almost surely and $L_d \equiv 0$ almost surely.

To prove e) recall that $A_{ud} = (\bigcup_{h>0}(h,D_{uh}))\cup(\bigcup_{h>0}(h,D_{dh}))$ where

$$D_{uh} = \inf \{ s > h : (X-X_\sigma)(s) \leq 0 \} \quad \text{and}$$

$$D_{dh} = \inf \{ s > h : (X-X_\sigma)(s) \geq 0 \} .$$

From eqn. (1) and well known properties of stochastic integrals it follows that almost surely

$$L(t \wedge D_{uh}) - L(t\wedge h) = 0$$

and $$L(t\wedge D_{dh}) - L(t\wedge h) = 0$$

for all $t \geq 0$. This proves e).

Proof of Corollary 1. We first observe that the process $\int_0^\cdot I_{A]-A_{ud}]}(s)\, dX_s - \sum_{s\leq \cdot} I_{A]-A_{ud}]}(s) \triangle X_s$ is a continuous semi-martingale. The condition (**) ensures that martingale part is identically zero. Denoting this process by V(.), eqn. (2) follows from eqn. (1) with $L'(.) = L(.) - V(.)$. It follows from eqn. (2), in the same way that Theorem 1 e) follows from eqn. (1) that $L'(.)$ is supported outside A] .

4. <u>The Tanaka Formula</u>. The conditions (*) and (**) are satisfied by all semi-martingales for certain choices of the set A. The tanaka formula is a result of this situation.

Let (X_t) be a semi-martingale. For $h > 0$, let

$$D_{1h} = \inf \left\{ s > h : X_s \leq 0 \right\}$$
$$D_{2h} = \inf \left\{ s > h : X_s \geq 0 \right\}$$

Let

$$A_i = \bigcup_{h>0} (h, D_{ih}) \qquad i = 1,2$$

and $\qquad \sigma_i(t) = \max \left\{ s \leq t : s \in A_i^c \right\}, \qquad i = 1,2.$

Let $A_{1u}, A_{2u}, A_{1d}, A_{2d}$ be defined for A_1 and A_2 as A_u, A_d were defined for A in section 1. Let

$$A_{iud} = A_{iu} \cup A_{id} \qquad i = 1,2.$$

<u>Lemma 5</u>. For any semi-martingale X, condition (*) and (**) holds for the sets A_1 and A_2.

<u>Proof</u>. It is easy to verify the following inequality : For every $t \geq 0$, and $i = 1,2$

$$\sum_{s \leq t} I_{A_i]}(s) \; | \triangle X_{\sigma_i}(s) | \leq \sum_{s \leq t} I_{(-\infty,0]}(X_{s-})(X_s)^+ + I_{[0,\infty)}(X_{s-})(X_s)^-$$

It is well known that the RHS sum is finite almost surely (see [2]). This proves (*) for A_1 and A_2.

We now prove (**). Firstly it is easy to see that

$$(X_t)^+ - (X_o)^+ = (X-X_{\sigma_1})(t) + \sum_{s\leq t} \triangle(X_{\sigma_1})^+(s) \tag{17}$$

$$(X_t)^- - (X_o)^- = -(X-X_{\sigma_2})(t) + \sum_{s\leq t} \triangle(X_{\sigma_2})^-(s)$$

Thus $X^c = (X-X_{\sigma_1})^c + (X-X_{\sigma_2})^c$ and from eqn. (1) applied to $(X-X_{\sigma_1})$ and $(X-X_{\sigma_2})$ we get

$$\langle X^c\rangle = \int_0^t I_{A_{1ud}]\cup A_{2ud}]}(s)\, d\langle X^c\rangle_s$$

Condition (**) for A_1 and A_2 follows from the above equation.

Lemma 6. The process L', occurring in the decomposition of the semi-martingale $X-X_{\sigma_1}$ given by eqn. (2), is an increasing process.

Proof. Eqn. (2) applied to $X - X_{\sigma_1}$ gives

$$\sum_{s\leq t} I_{A_1]}(s)\triangle X_{\sigma_1}(s) + (X-X_{\sigma_1})(t) = \int_0^t I_{A_1]}(s)dX_s + L'(t) \tag{18}$$

Let $A_{11} = \{(t,\omega) : X_{\sigma_1}(t-) > 0,\ X_{\sigma_1}(t) > 0\}$. Then A_{11} is an optional set with open sections and $A_{11} \subset A_1$. Let $A_{12} = A_1 - A_{11}$ so that $A_1 = A_{11} \cup A_{12}$. Note that A_{12} is also an optional set with open sections. Let σ_{11} and σ_{12} be the entrance times for A_{11} and A_{12}. Then

$$(X-X_{\sigma_1})(t) = (X-X_{\sigma_{11}})(t) + (X-X_{\sigma_{12}})(t) \tag{19}$$

Also,

$$\sum_{s\leq t} I_{A_{11}]}(s)\ |\triangle X_{\sigma_{11}}(s)| + I_{A_{12}]}(s)\ |\triangle X_{\sigma_{12}}(s)|$$

$$= \sum_{s\leq t} I_{A_1]}(s)\ |\triangle X_{\sigma_1}(s)| < \infty$$

almost surely by lemma 5. Hence (*) holds for A_{11} and A_{12}. Also $A_{12ud} \equiv A_{12u} \equiv A_{12}$ and $(X-X_{\sigma_{12}})(.) \geq 0$. Hence from Theorem 1 c) and d) we get a continuous adapted increasing process $L_{12}(.)$ such that

$$\sum_{s\leq t} I_{A_{12}]}(s)\triangle X_{\sigma_{12}}(s) + (X-X_{\sigma_{12}})(t) = \int_0^t I_{A_{12}]}(s)dX_s + L_{12}(t) \qquad (20)$$

Let $A_{11n} = \{(t,\omega) : X_{\sigma_1}(t-) > 1/n,\ X_{\sigma_1}(t) > 1/n\}$ $n = 1,2,\ldots$. Then A_{11n} are optional sets with open sections, $A_{11n} \subset A_{11n+1}$ and $A_{11} = \bigcup_{n=1}^{\infty} A_{11n}$. Let σ_{11n} be the entrance times for A_{11n}. Then $\sigma_{11n}(t) \uparrow \sigma_{11}(t)$ as $n \to \infty$. We also have

$$\sum_{s\leq t} I_{A_{11n}]}(s)\ \triangle X_{\sigma_{11n}}(s) + (X-X_{\sigma_{11n}})(t) = \int_0^t I_{A_{11n}]}(s)dX_s$$

Letting $n \to \infty$ we get,

$$\sum_{s\leq t} I_{A_{11}]}(s)\triangle X_{\sigma_{11}}(s) + (X-X_{\sigma_{11}})(t) = \int_0^t I_{A_{11}]}(s)dX_s \qquad (21)$$

Comparing eqn. (18) with eqns. (19), (20) and (21) it follows that $L'(.) \equiv L_{12}(.)$, which is an increasing process.

<u>Theorem 2</u>. For every $a \in \mathbb{R}$, there exists a continuous increasing adapted $L(.,a)$ such that almost surely,

$$(X_t-a)^+ = (X_o-a)^+ + \int_0^t I_{(a,\infty)}(X_{s-})dX_s + \sum_{s\leq t} I_{(-\infty,a]}(X_{s-})(X_s-a)^+$$
$$+ I_{(a,\infty)}(X_{s-})(X_s-a)^- + \frac{1}{2} L(t,a) \qquad (22)$$

<u>Proof</u>. We first prove the case $a = 0$. Eqns. (18) and (17) gives

$$(X_t)^+ - (X_o)^+ = \int_0^t I_{A_1]}(s)dX_s - \sum_{s\leq t} I_{A_1]}(s) \Delta X_{\sigma_1}(s) + L'(t)$$

$$+ \sum_{s\leq t} \Delta(X_{\sigma_1})^+ (s) \qquad (23)$$

where $L'(.)$ is a continuous increasing process by lemma 6.
It is easy to see that

$$A_1] = \{s : X_{s-} > 0\} \cup B \qquad (24)$$

where B is a scanty previsible set given by

$$B = \{s \in A_1 : X_{s-} = 0,\ X_s > 0\} \cup \{D_{1h} : X_{D_{1h}-} = 0,\ X_{D_{1h}} \leq 0,\ h > 0\}$$

It is also easy to see that for all $t \geq 0$,

$$\sum_{s\leq t} I_B(s)|\Delta X_s| \leq \sum_{s\leq t} I_{(-\infty,0]}(X_{s-})(X_s)^+ + I_{[0,\infty)}(X_{s-})(X_s)^-$$

$$< \infty \quad \text{almost surely.}$$

It follows from [3], page 378, that

$$\int_0^t I_B(s)dX_s = \sum_{s\leq t} I_B(s) \Delta X_s \qquad (25)$$

Hence from eqns. (23), (24) and (25) we get

$$(X_t)^+ - (X_o)^+ = \int_0^t I_{(0,\infty)}(X_{s-})dX_s + \int_0^t I_B(s)dX_s - \sum_{s\leq t} I_{A_1]}(s) \Delta X_{\sigma_1}(s)$$

$$+ \sum_{s\leq t} \Delta(X_{\sigma_1})^+ (s) + L'(t)$$

$$= \int_0^t I_{(0,\infty)}(X_{s-})dX_s + \sum_{s\leq t} I_B(s) \Delta X_s$$

$$- \sum_{s\leq t} I_{A_1]}(s) \Delta X_{\sigma_1}(s) + \sum_{s\leq t} \Delta(X_{\sigma_1})^+ (s) + L'(t)$$

$$(26)$$

Using the definition of B following eqn. (24) and noting that the jumps of $s \longrightarrow (X_{\sigma_1})^+(s)$ occur at the end points of A, we can write

$$\sum_{s\le t} I_B(s)\triangle X_s - \sum_{s\le t} I_{A_1]}(s)\triangle X_{\sigma_1}(s) + \sum_{s\le t}\triangle(X_{\sigma_1})^+(s)$$

$$= \sum_{s\le t} I_{A_1\cap\{s:X_{s-}=0,X_s>0\}}(s)\triangle X_s + \sum_{s\le t} I_{A_1^\ell}(s)\triangle(X_{\sigma_1})^+(s)$$

$$+ \sum_{s\le t} I_{A_1^r}(s)\triangle(X_{\sigma_1}^+)(s)$$

$$+ \sum_{s\le t} I_{\{D_{1h}:\, X_{D_{1h}-}=0,\ X_{D_{1h}}\le 0,\ h>0\}}(s)\triangle X_s$$

$$- \sum_{s\le t} I_{A_1]}(s)\triangle X_{\sigma_1}(s) \tag{27}$$

It is easy to see that

$$\sum_{s\le t} I_{A_1\cap\{s:\ X_{s-}=0,\ X_s>0\}}(s)\triangle X_s + \sum_{s\le t} I_{A_1^\ell}(s)\triangle(X_{\sigma_1})^+(s)$$

$$= \sum_{s\le t} I_{(-\infty,0]}(X_{s-})(X_s)^+ \tag{28}$$

and that

$$\sum_{s\le t} I_{\{D_{1h}:\ X_{D_{1h}-}=0,\ X_{D_{1h}}\le 0\}}(s)\triangle X_s + \sum_{s\le t} I_{A_1^r}(s)\triangle(X_{\sigma_1})^+(s)$$

$$- \sum_{s\le t} I_{A_1]}(s)\triangle X_{\sigma_1}(s) = \sum_{s\le t} I_{(0,\infty)}(X_{s-})(X_s)^- \tag{29}$$

Eqn. (22) for the case $a = 0$, now follows from eqns. (26), (27), (28) and (29) with $L(.,0) = 2L'(.)$. The case $a \neq 0$ is proved by considering the semi-martingale $X - a$.

We now return to our original set up. X is an arbitrary semi-martingale, A an optional random set with open sections and (X,A) satisfying (*). A_u, σ_u, A_d, σ_d all are as defined in section 1. L_u and L_d the increasing processes occuring in the decomposition of $(X-X_{\sigma_u})$ and $(X-X_{\sigma_d})$ respectively (see section 3, lemma 3).

<u>Theorem 3</u> The increasing processes $2L_u$ and $2L_d$ are the local times at zero of the semi-martingales $(X-X_\sigma)$ and $-(X-X_\sigma)$ respectively.

<u>Proof</u>. The proof is similar in spirit to the proof of Theorem 2. We shall do the computations only for L_u, the case L_d being similar. We first observe that

$$\left\{s : (X-X_\sigma)(s-) > 0\right\} = A_u] - B \qquad - (30)$$

where B is a scanty previsible set given by

$$B = B_1 \cup B_2$$

$$B_1 = \left\{s \in A_u : (X-X_\sigma)(s-) = 0,\ (X-X_\sigma)(s) > 0\right\}$$

$$B_2 = \left\{s \in A_u^r : (X-X_\sigma)(s-) = 0,\ (X-X_\sigma)(s) \leq 0\right\}$$

We also see that by the result of C.S. Chou [2], applied to the semi-martingale $(X-X_\sigma)$ that,

$$\sum_{s\leq t} I_B(s)\ |\Delta(X-X_\sigma)(s)| \leq \sum_{s\leq t} (I_{(-\infty,0]}(X-X_\sigma)(s-)(X-X_\sigma)^+(s)$$

$$+ I_{[0,\infty)}(X-X_\sigma)(s-)(X-X_\sigma)^-(s)) < \infty \quad \text{a.s.}$$

Now applying the Tanaka formula (22) to the semi-martingale $(X-X_\sigma)^+$ and using eqn. (30) and Cor.1 of Theorem 1 to expand the stochastic integral, we get

$$(X-X_\sigma)^+(t) = \int_0^t I_{A_u]}(s)dX_s + I_1(t) + I_2(t) + I_3(t) + \tfrac{1}{2} L(t,0) \qquad - (31)$$

where $I_1(t) = -\sum_{s\leq t} I_{B_1}(s)\Delta(X-X_\sigma)(s) + \sum_{s\leq t} I_{(-\infty,0]}(X-X_\sigma)(s-)(X-X_\sigma)^+(s)$

$$I_2(t) = -\sum_{s\leq t} I_{B_2}(s)\Delta(X-X_\sigma)(s) + \sum_{s\leq t} I_{(0,\infty)}(X-X_\sigma)(s-)(X-X_\sigma)^-(s)$$

$$I_3(t) = -\sum_{s\leq t} I_{A_u]\cap A}(s)\triangle X_\sigma(s)$$

On the other hand by Lemma 4 and Lemma 3 we get

$$(X-X_\sigma)^+(t) = (X-X_{\sigma_u})(t) + \sum_{s\leq t}\triangle(X_{\sigma_u}-X_\sigma)^+(s)$$

$$= \int_0^t I_{A_u]}(s)dX_s + L_u(t) - \sum_{s\leq t} I_{A_u]}(s)\triangle X_{\sigma_u}(s)$$

$$+ \sum_{s\leq t}\triangle(X_{\sigma_u}-X_\sigma)^+(s)$$

$$= \int_0^t I_{A_u]}(s)\, dX_s$$

$$+ \sum_{s\leq t} I_A(s)\, I_{\{s\,:\ (X_{\sigma_u}-X_\sigma)^+(s)\, >\, 0\}}(s)\, \triangle(X_{\sigma_u}-X_\sigma)^+(s)$$

$$- \sum_{s\leq t} I_{A_u]\cap A}(s)\triangle X_{\sigma_u}$$

$$+ \sum_{s\leq t} I_A(s)\, I_{\{s:\ (X_{\sigma_u}-X_\sigma)^+(s)\, <\, 0\}}(s)\, \triangle(X_{\sigma_u}-X_\sigma)^+(s)$$

$$- \sum_{s\leq t} I_{A_u]\cap A^c}(s)\, \triangle X_{\sigma_u}$$

$$+ \sum_{s\leq t} I_{A^c}(s)\, \triangle(X_{\sigma_u}-X_\sigma)^+(s) + L_u(t) \qquad -\ (32)$$

Comparing eqns. (31) and (32) it is a matter of verification to see that the second term in the RHS of (32) equals $I_1(t)$, the 3rd and 4th terms equal $I_2(t)$ and the 5th and 6th terms equal $I_3(t)$. The result follows.

Remarks : 1) One can improve the statement regarding support of the process of finite variation $L'(.)$, occurring in Cor. 1 of Theorem 1 at no extra cost. In fact the same proof shows that $\text{Supp } L' \subset (A \cup A^+)^c$ where $A^+ = \bigcup_h (h, D_h^+)$ where $D_h^+ = \inf\{s > h : s \in A\}$.

2) If $X_t = X_o + M_t + V_t$ where (M_t) is a local martingale and V is of finite variation, then it is easy to see that

$$\sum_{s\leq t} \Delta V_\sigma(s) + (V-V_\sigma)(t) = \int_0^t I_{A]}(s)\, dV_s$$

Hence from Cor.1 we get

$$\sum_{s\leq t} I_{A]}(s)\, \Delta M_\sigma(s) + (M-M_\sigma)(t) = \int_0^t I_{A]}(s)\, dM_s + L'(t)$$

3) Let X and A be as in Theorem 1. The condition (*) is also necessary for $(X-X_\sigma)$ to be a semi-martingale. For suppose $(X-X_\sigma)$ and hence X_σ is also a semi-martingale. Let A_n be the approximations of A_u as in Proposition 1. We note that X_σ is constant across the intervals of A_u and jumps only at those end points of A_u which are also end points of A. Now it is easy to see using $A_u = \bigcup_n A_n$, that

$$\sum_{s\leq t} (\Delta X_\sigma(s))^+ I_{A]}(s) = \int_0^t I_{A_u]}(s)\, dX_\sigma(s) < \infty \quad \text{a.s.}$$

Similarly $\sum_{s\leq t} (\Delta X_\sigma)^-(s)\, I_{A]}(s) < \infty$ a.s. and condition (*) holds.

Acknowledgements. The author would like to thank Prof. P.A. Meyer for some useful discussions. He would also like to thank Prof. M. Yor for discussions on 'First Order Calculus' and the 'Balayage formula'.

References

[1] J. Azema and M. Yor, En guise d'introduction, Asterisque 52-53, 1978.

[2] Chou Ching Sung, Demonstration Simple D'un resultat Sur le Temps Local, Sem. de Prob XIII, 1979, LN 721, Springer Verlag.

[3] C. Delacherie and P.A. Meyer, Probabilities and Potential B North Holland, 1982.

[4] N. el. Karoui, Temps Locaux et balayage des semi-martingales, Sem. de Prob. XIII, Lecture notes in Mathematics 721, Springer Verlag, page 443.

[5] N. el. Karoui, Sur le montees des semi-martingales, Asterisque 52-53, 1978.

[6] P.A. Meyer, C. Stricker and M. Yor, Sur une formule de la theorie des balayage, Sem. de Prob. XIII, Lecture notes in Mathematics 721, Springer Verlag, p.478.

[7] P.A. Meyer, Construction de quasi-martingale s annulant sur un ensemble donne, Sem. de Prob. XIII, Lecture notes in Mathematics 721, Springer Verlag, p.488.

[8] B. Rajeev, Crossings of Brownian motion : A semi-martingale approach, Sankhyā Series A, Vol.51, 1989.

[9] B. Rajeev, Semi-martingales associated with crossings, Sem. de Prob. XXIV, Lecture notes in Maths. 1426, 1989.

[10] D. Revuz and M. Yor, Continuous Martingales and Brownian Motion, Springer Verlag, 1991.

[11] C. Stricker, Semi-martingales et valeur absolue, Sem. de Prob. XIII, Lecture notes in Maths. 721, Springer Verlag, p.472.

[12] M. Yor, Sur le balayage des semi-martingales continues, Sem. de Prob. XIII, Lecture notes in Maths. 721, Springer Verlag, p.453.