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WEI-AN ZHENG

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Meyer's Topology and Brownian motion in a composite medium

Weian Zheng
Department of mathematics
University of California, Irvine, CA 92717, USA

Résumé— On associe au problème de propagation de la chaleur dans un milieu composite un processus de diffusion qui est une semimartingale. On étudie surtout le problème de Stefan.

1 Introduction

Let's first consider one dimensional case. When we consider heat transfer on an infinite rod, we use real line $(-\infty, \infty)$ to replace the rod. Suppose that $-\infty = x_0 < x_1 < ...x_n < x_{n+1} = \infty$ are $n+2$ points such that each interval $I_i = (x_i, x_{i+1})$ is made of one material. Then the temperature $u(t,x)$ satisfies the equation (see [12] [17])

$$a_i \frac{\partial^2}{\partial x^2} u(t,x) = 2\frac{\partial}{\partial t} u(t,x) \qquad x_i < x < x_{i+1}$$

subjected to the boundary condition

$$k_{i-1} \frac{\partial}{\partial x_-} u(t,x_i) = k_i \frac{\partial}{\partial x_+} u(t,x_i) \qquad i = 1, ..., n,$$

and the initial condition

$$u(x,0) = \xi_i(x), \qquad x_i < x < x_{i+1}$$

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where we use $\frac{\partial}{\partial x_-}$ ($\frac{\partial}{\partial x_+}$) to denote the left (resp. right) derivative. k_i is the thermal conductivity and a_i is the thermal diffusivity of the material of which I_i is made. To compare with the engineering literature ([12] [17]), we put a constant factor 2 in (1) and thereafter for the convenience of probabilists. In fact, the standard Gaussian density satisfies (1) when $a_i \equiv 1$.

The above boundary problem may be formulated in terms of Dirichlet forms (see [11], [1], [2] for examples). Let $\{A_i\}_i$ be a collection of disjoint simply connected open sets (made of different materials) in R^d and $\{\overline{A}_i\}_i$ are their closures respectively. Suppose $\bigcup_i \overline{A}_i = R^d$. Let a_i and k_i be the termal diffusivity and the thermal conductivity of the material of which A_i is made. Denote

$$a(x) = \sum_{i=0}^{n} a_i I_{A_i}(x), \qquad b(x) = \sum_{i=0}^{n} \frac{k_i}{a_i} I_{A_i}(x). \tag{1}$$

We call $b_i = k_i a_i^{-1}$ the intrinsic thermal conductivity. Then the temperature $u(x,t)$ satisfies the heat equation

$$\frac{\partial}{\partial t} u = \frac{1}{2} \sum_j b^{-1}(x) \frac{\partial}{\partial x_j} [a(x) b(x) \frac{\partial}{\partial x_j} u].$$

It is well known that there is a symmetric diffusion process $\{X_t\}_t$ with generator (see [6])

$$\mathcal{L} f = \frac{1}{2} b^{-1}(x) \sum_{j=1}^{d} \frac{\partial}{\partial x_j} [b(x) a(x) \frac{\partial}{\partial x_j} f(x)] \tag{2}$$

such that X_t has $u(x,t)$ as its density function with respect to $b(x)dx$. Generally speaking $\{X_t\}_t$ is just a Dirichlet process. However we will prove in Section 2 that $\{X_t\}_t$ is a semimartingale when the complements $\{(A_i)^c\}$ have locally finite lower Minkowski contents [21] [3]. More precisely the following Skorohod type of decomposition holds:

$$X_t = \int_0^t \sqrt{a(X_s)} dB_s + L_t \tag{3}$$

where B_t is a standard Brownian motion and L_t is a process of bounded variation supported only on the boundaries $\bigcup_i \partial A_i$. A difference between the ordinary reflecting Brownian motion and the process constructed here is the latter may cross the boundaries $\bigcup_i \partial A_i$.

Then we show in Section 4 that there is a martingale process associated to Stefan's moving boundary problem. The moving boundary is the set of all the discontinuous points of the clock of that martingale and the density function of that martingale is related to the enthalpy. We hope that further studies will enable us to get more information about the free boundary.

Meyer's pseudo-path topology for weak convergence is the major tool for proving the above diffusion process is a semimartingale in Section 2. Let's recall the latter here for the readers' convenience. Given a sequence of semimartingales

$$X_t^{(n)} = X_0^{(n)} + M_t^{(n)} + A_t^{(n)}, \qquad t \in [0, T]$$

where $\{\{M_t^{(n)}\}_t\}_n$ are martingales with 0-initial values and $\{\{A_t^{(n)}\}_t\}_n$ are processes of bounded variation such that

$$\sup_n E\{|X_0^{(n)}| + |M_T^{(n)}| + Var_{[0,T]}[A^{(n)}]\} < \infty. \tag{4}$$

Then their laws are tight on $D[0, T]$ under pseudo-path topology. Moreover, any of their weak limit is still a semimartingale [16] [10].

2 In a fixed composite medium

In [21], we introduced a condition (C.1) to the boundary of a domain, under which we proved reflecting Brownian motion in that domain is a semimartingale. Z.Chen [3] independently proved the same result under the condition that the domain has finite lower Minkowski content. It is easy to see that if we allow to take any subsequence instead of the special sequence in (C.1), then the finite lower Minkowski condition is equivalent to (C.1). So let us recall the definition of Minkowski content here. Let $m(.)$ be the Lebesgue measure. Denote for each bounded set F,

$$F_r = \{x \in R^d, \ 0 < dis(x, F) \leq r\}.$$

We say that a set F has locally finite lower Minkowski content if

$$\liminf_{r \to 0} \frac{m(F_r \bigcap \{x, \ |x| < n\})}{r} < \infty \tag{5}$$

for each fixed n.

Theorem 1 *If each bounded set only intersects a finite number of $\{A_i\}_i$ and if all $(A_i)^c$ has finite lower Minkowski content, then X_t with generator (2) is a semimartingale with the decomposition*

$$X_t = \int_0^t \sqrt{a(X_s)} dB_s + L_t$$

where L_t is a process of bounded variation. Moreover, L_t is supported only on $\bigcup_i \partial A_i$.

Proof. Without losing generality, we assume that each A_i is bounded and $r = \{\frac{1}{m}\}_m$ gives the lim inf in (5). Let $\phi_n(x)$ be a sequence of C^1 functions such that

1) $\phi_n(x) = 1$ when $|x| < n$ and $\phi_n(x) = 0$ when $|x| > n+1$;

2) $0 \leq \phi_n(x) \leq 1$ and $\sup_n |\frac{\partial}{\partial x}\phi_n(x)| \leq 2$.

Denote by $\delta_i(x)$ the Stein's regularized distance function to A_i (see Lemma 2.1 of [21]). Take a decreasing function $f_m(r) \in C^\infty$ for each integer m such that

$$f_m(0) = 1, \qquad f_m(s) = 0 \quad (\forall s \geq \frac{1}{m}).$$

Let

$$\tilde{a}_{m,n}(x) = (1 - \phi_{n-1}(x)) + \phi_n(x) \sum_i a_i f_m(\delta_i(x))$$

and

$$\tilde{b}_{m,n}(x) = (1 - \phi_{n-1}(x)) + \phi_n(x) \sum_i \frac{k_i}{a_i} f_m(\delta_i(x)).$$

Then $\tilde{a}_{m,n}(x)$ and $\tilde{b}_{m,n}(x)$ are differentiable. Denote by $X_s^{(m,n)}$ the diffusion process associated to the Dirichlet form

$$\mathcal{E}_{m,n}(f,g) = \frac{1}{2}\int (\frac{\partial}{\partial x} f)\tilde{a}_{m,n}(x)\tilde{b}_{m,n}(x)\frac{\partial}{\partial x} g(x)dx$$

on $L_2(R^d, \tilde{b}_{m,n}(x)dx)$. By [13], we know that $\{\{X_s^{(m,n)}\}\}_m$ converge weakly to the diffusion $\{X_s^{(n)}\}$ associated to the Dirichlet form

$$\begin{aligned} & \mathcal{E}_n(f,g) \\ = & \frac{1}{2}\int (\frac{\partial}{\partial x} f)(1 - \phi_{n-1}(x) + \phi_n(x)a(x))(1 - \phi_{n-1}(x) + \phi_n(x)b(x))\frac{\partial}{\partial x} g(x)dx. \end{aligned}$$

on $L_2(R^d, (1 - \phi_{n-1}(x) + \phi_n(x)b(x))dx)$. On the other hand,

$$dX_t^{(m,n)} = dM_t^{m,n} + \frac{1}{2}\tilde{b}_{m,n}^{-1}(X_t^{(m,n)})\frac{\partial}{\partial x}[\tilde{a}_{m,n}(X_t^{(m,n)})\tilde{b}_{m,n}(X_t^{(m,n)})]dt$$

where $\{M^{m,n}\}$ are martingales with bounded quadratic variations and the drift parts satisfy the inequality:

$$\begin{aligned} & E\{\int_0^T |\frac{1}{2}b_{m,n}^{-1}(X_s^{(m,n)})\frac{\partial}{\partial x}[\tilde{a}_{m,n}^{-1}(X_t^{(m,n)})b_{m,n}(X_s^{(m,n)})]|ds\} \\ \leq \ & C_{1,n}T\{\int |b_{m,n}^{-1}(x)\frac{\partial}{\partial x}b_{m,n}(x)|dx + \int |\frac{\partial}{\partial x}a_{m,n}(x)|dx\} \\ \leq \ & C_{2,n}T \end{aligned} \qquad (6)$$

where $C_{1,n}$ and $C_{2,n}$ are constants independent of m and T. The last inequality is from Lemma 2.2 of [21] and the remark we gave before the description of this theorem. Therefore from (4) we know the laws of $\{\{X_t^{(m,n)}\}_t\}_m$ form a tight sequence under Meyer's pseudo-path topology on $D[0,T]$ and any limit process is still a semimartingale. Thus $\{X_t^{(n)}\}_t$ is a continuous semimartingale. Since $X_t^{(n)} = X_t^{(n+1)}$ before they hit the ball $\{x,\ |x| < n\}$, we get (3) when $n \to \infty$. As $\{X_t\}_t$ is just ordinary Brownian motion while it stays away from $\bigcup_i \partial A_i$, we get the last conclusion of the theorem. ∎

3 Regularizing enthalpy

Let's consider the case where some phase transitions are involved. Suppose there are $n+1$ possible phases with their thermal diffusivities $\{a_i\}_{i=0,\dots,n}$ and intrinsic thermal conductivities $\{b_i\}_{i=0,\dots,n}$ respectively. Suppose $\{u_i\}_{i=1,\dots,n}$ are the fusion temperatures between the $(i-1)$-th state and the i-th state and suppose $u_1 = 0$. Denote

$$b(u) = b_0 I_{\{u=0\}} + \sum_{i=1}^{n-1} b_i I_{(u_i,\ u_{i+1}]}(u) + b_n I_{(u_n,\ \infty)}(u)$$

and

$$a(u) = a_0 I_{\{u=0\}} + \sum_{i=1}^{n-1} a_i I_{(u_i,\ u_{i+1}]}(u) + a_n I_{(u_n,\ \infty)}(u).$$

Let $L_i > 0$ be the latent heat of fusion at temperature u_i. Then an enthalpy function is defined by

$$H(u) = \int_0^u b(v)dv + \sum_{\{i,\ u_i<u\}} L_i, \tag{7}$$

and the temperature $u(x,t)$ satisfies the following equation in the weak sense:

$$\frac{\partial}{\partial t} H(u(x,t)) = \frac{1}{2}\sum_j \frac{\partial}{\partial x_j}[b(u(x,t))a(u(x,t))\frac{\partial}{\partial x_j}u(x,t)] \tag{8}$$

with the initial condition $u(x,0) = u_0(x)$ (see [5], [9], for example).

Withou losing generality, we will assume $a(u)b(u) \equiv 1$. In fact, we may always realize that assumption by changing the variable u to

$$v(x,t) = \int_0^{u(x,t)} a(\xi)b(\xi)d\xi.$$

See (p.497, [11]) for details. Thus (8) becomes

$$\frac{\partial}{\partial t}H(u(x,t)) = \frac{1}{2}\sum_j \frac{\partial^2}{\partial x_j^2}u(x,t). \tag{9}$$

Since $H(.)$ is a function with jumps, (9) should be understood in the sense of distribution. Now let us regularize it. Denote by $J_{m,v}(u)$ the regularizing sequence of the δ-function at $v+\frac{1}{2m}$ such that $J_{m,v}(u) \in C_0^\infty[v, v+\frac{1}{m}]$ and $\int J_{m,v}(u)du = 1$. Let

$$b_m(u) = b_0 + \sum_{i=1}^n (b_i - b_{i-1}) \int_0^u J_{m,u_i}(v)dv.$$

Denote $L_m(u) = \sum_{i=1}^n L_i J_{m,u_i}(u)$ and $H_m(u) = \int_0^u (b_m(v) + L_m(v))dv$. Then $b_m(u)$ and $H_m(u)$ are smooth functions tending to $b(u)$ and $H(u)$ respectively on their continuous points. Thus (8) is regularized to

$$(b_m(u(x,t)) + L_m(u(x,t)))\frac{\partial}{\partial t}u(x,t) = \frac{1}{2}\sum_j \frac{\partial^2}{\partial x_j^2}u(x,t)\ . \tag{10}$$

Denote $p_m(u) = u^{-1}H_m(u)$ and $\overline{a}_m(u) = p_m^{-1}(u)$. Then the above equation becomes Fokker-Planck equation (see Lemma 1 of [23]):

$$u\frac{\partial}{\partial t}p_m(u) + p_m(u)\frac{\partial}{\partial t}u = \frac{1}{2}\sum_j \frac{\partial^2}{\partial x_j^2}u. \tag{11}$$

We also give a restriction on the initial value $u^{(0)}(x)$ through the following assumption. We assume that there is a sequence of functions $u_m^{(0)}(x) \to u^{(0)}(x)$, *a.e.* such that 1) $\int u_m^{(0)}(x)p_m(u_m^{(0)}(x))dx = 1$; 2) $u_m^{(0)}(x)p_m(u_m^{(0)}(x))$ are uniformly bounded in m.

Let $u_m(x,t)$ be the solution to (11) with the initial condition $u_m(x,0) = u_m^{(0)}(x)$. Then it is not difficult to see that for each fixed m, $u_m(x,t)p_m(u_m(x,t))$ is the density function with respect to Lebesgue measure of a Markov diffusion process with the following decomposition:

$$X_{m,t} - X_{m,0} = \int_0^t \sqrt{\overline{a}_m(u_m(X_{m,s},s))}dW_{m,s} \tag{12}$$

where $\{W_{m,s}\}_s$ is a d-dimensional Brownian motion (see [8]). Since $\overline{a}_m$ is bounded, the laws of $\{X_m\}$ are tight and any limit process is still a continuous martingale ([22])

4 Towards Stefan's problem

Now let us consider in more details the limit process. By (12) and Ito's formula,

$$\begin{aligned}
& E[u_m(X_{m,t},t)p_m(u_m(X_{m,t},t))] - E[u_m(X_{m,0},0)p_m(u_m(X_{m,0},0))] \\
= & \int_0^t E\{\frac{\partial}{\partial s}(u_m(X_{m,s},s)p_m(u_m(X_{m,s},s)))\}ds \\
+ & \sum_j \int_0^t E\{\frac{1}{2\,p_m(u_m(X_{m,s}))}\frac{\partial^2}{\partial x_j^2}[u_m(X_{m,s},s)p_m(u(X_{m,s},s))]\}ds.
\end{aligned}$$

That is,

$$\begin{aligned}
& E[u_m(X_{m,t},t)p_m(u_m(X_{m,t},t))] - E[u_m(X_{m,0},0)p_m(u_m(X_{m,0},0))] \\
= & \frac{1}{2}\sum_j \int_0^t \int u_m \frac{\partial^2}{\partial x_j^2}[u_m p_m(u_m)]dxds \\
+ & \int_0^t \int u_m p_m(u_m)\frac{\partial}{\partial s}(u_m p_m(u_m))dxds \\
= & \frac{1}{2}\sum_j \int_0^t \int u_m \frac{\partial^2}{\partial x_j^2}[u_m p_m(u_m)]dxds \\
+ & \frac{1}{2}\sum_j \int_0^t \int u_m p_m(u_m)\frac{\partial^2}{\partial x_j^2}u_m dxds \\
= & -\sum_j \int_0^t \int (b_m(u_m) + L(u_m))\frac{\partial}{\partial x_j}u_m \frac{\partial}{\partial x_j}u_m dxds.
\end{aligned}$$

Since $E[u_m(X_{m,t},t)p_m(u_m(X_{m,t},t))] \geq 0$, $b(u_m) > 0$ and $L(u_m) \geq 0$, we deduce

$$\sup_m\{\sum_j \int_0^t \int |\frac{\partial}{\partial x_j}u_m|^2 dxds\} \leq \sup_{m,x}\{u_m^{(0)}(x)p_m(u_m^{(0)}(x))\} < \infty. \tag{13}$$

Define on $[0,T] \times R^d$ a Hilbert space $\mathcal{H}$ with the norm

$$\| f \| = \sqrt{\sum_{j=1}^d \int \int_0^T |\frac{\partial}{\partial x_j}f(x,s)|^2 ds\,dx + \int \int_0^T |f(x,s)|^2 ds\,dx.}$$

Then from (13), $u_m(.,.)$ is contained in a bounded ball in $\mathcal{H}$. Since $\mathcal{H}$ is reflexive, the bounded ball in $\mathcal{H}$ is weakly compact. So we can find a weakly convergent subsequece still denoted as $u_m(x,t)$ such that u_m converge weakly to some $u \in \mathcal{H}$. Furthermore, it is standard to find an almosl everywhere convergent subsequenceon of $\{u_m\}_m$ in the space-time (see [7] and [23] for details) and denote by $u(x,t)$ their limit. Thus we conclude our discussion with the following

Theorem 2 *There is a martingale diffusion process*

$$X_t = X_0 + \int_0^t \sqrt{uH^{-1}(u(X_s, s))}dW_s$$

with the enthalpy $H(x,t)$ as its density function with respect to Lebesgue measure. In the above formula, W_t is a standard d-dimensional Brownian motion. The generator of X_t may be formally written as

$$\frac{1}{2}uH^{-1}(u(x,t))\sum_{j=1}^{d}\frac{\partial^2}{\partial x_j^2}.$$

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