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A martingale proof of the Khinchin iterated logarithm law for Wiener processes

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Let $w_t$ be a $d$-dimensional Wiener process. Our goal here is to give a martingale proof of the following celebrated Khinchin log log law:

**Theorem 1.** With probability one

$$\limsup_{t \to \infty} \frac{|w_t|}{\sqrt{2t \log \log t}} = 1.$$ 

The "standard" proof of this result can be found in many places (cf. for instance [1]). Our martingale proof is based on formula (4), which might be interesting by itself. In particular, this formula allows us to give a rather short proof of the Kolmogorov–Petrovskii criterion allowing one to recognize when a given function $\alpha(t)$ is an "upper" or "lower" function.

Let $\alpha(t)$ be a strictly positive continuously differentiable function on $[0, \infty)$. For $t > 0$, $x \in \mathbb{R}^d$ define

$$h(t, x) = \frac{1}{t^{d/2}} e^{x^2/(2t)}, \quad \beta(t) = \frac{1}{t^{d/2}} e^{\alpha^2(t)/(2t)}, \quad \gamma(t) = \left( \frac{1}{\beta(t)} \right),$$

$$\tau(t) = \inf \{ s \geq t : |w_s| \geq \alpha(s) \}.$$

**Lemma 2.** For any $\varepsilon > 0$ the process

$$m_\varepsilon(t) := h(t + \varepsilon, w_t) \beta^{-1}(t + \varepsilon) - \int_0^t h(r + \varepsilon, w_r) \gamma(r + \varepsilon) \, dr$$

is a martingale.

To prove the lemma it suffices to observe that for $0 \leq s \leq t$ we have

$$\beta^{-1}(t + \varepsilon) - \beta^{-1}(s + \varepsilon) = \int_s^t \gamma(r + \varepsilon) \, dr,$$

then take a normal $(0, (t + \varepsilon)I)$ variable $\xi$ independent of $w$ and notice that

$$E\{ h(t + \varepsilon, w_t) | \mathcal{F}_s \} = E h(t + \varepsilon, x + \xi \left( \frac{t - s}{t + \varepsilon} \right)^{1/2} | x \left( \frac{t - s}{t + \varepsilon} \right)^{1/2}$$

$$= \frac{1}{(2\pi)^{d/2}(t + \varepsilon)^d} \int_{\mathbb{R}^d} \exp - \frac{1}{2(t + \varepsilon)} [w_s + y \left( \frac{t - s}{t + \varepsilon} \right)^{1/2}]^2 - |y|^2 \, dy,$$

$$= \frac{1}{2(t + \varepsilon)} [w_s + y \left( \frac{t - s}{t + \varepsilon} \right)^{1/2}]^2 - |y|^2 =$$

$$\frac{1}{2(s + \varepsilon)} |w_s|^2 - \frac{1}{2(t + \varepsilon)} \left( \frac{s + \varepsilon}{t + \varepsilon} \right)^{1/2} + w_t \left( \frac{t - s}{s + \varepsilon} \right)^{1/2},$$

$$E\{ h(t + \varepsilon, w_t) | \mathcal{F}_s \} =$$

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Corollary 3. If \(0 < s < T < \infty\), then
\[
P\{s < \tau(s) \leq T\} + \frac{E}{\beta(T)} \int_0^T h(r, w_r) \gamma(r) \mathbf{1}(r > T) dr =
\]
\[
\frac{\alpha^d(s)}{\beta(s) s^d} \kappa_d + E \int_s^T h(r, w_r) \gamma(r) \mathbf{1}(r > T) dr,
\]
where \(\kappa_d\) is \((2\pi)^{-d/2}\) times the volume of the unit sphere.

Indeed, \(E m_x(s) I_{(s)s > T} = E m_x(\tau(s) \land T) I_{(s)s > T}\), which means that
\[
E \frac{h(s + \varepsilon, w_s)}{\beta(s + \varepsilon)} I_{r > T} + E \int_s^T h(r + \varepsilon, w_r) \gamma(r + \varepsilon) dr =
\]
\[
E \frac{h(\tau(s) \land T + \varepsilon, w_{\tau(s) \land T})}{\beta(\tau(s) \land T + \varepsilon)} I_{(s)s > T}.
\]

It remains only to let here \(\varepsilon \downarrow 0\) and apply the dominated convergence theorem along with the observations that for \(s < r \leq \tau(s) \land T\),
\[
E \frac{h(s, w_s)}{\beta(s)} I_{r > T} = E \frac{h(s, w_s)}{\beta(s)} I_{r > T} =
\]
\[
\frac{1}{\beta(s) s^d (2\pi)^{d/2}} \int_{|x| < \alpha(s)} dx = \frac{\alpha^d(s)}{\beta(s) s^d} \kappa_d = \kappa_d \left(\frac{\alpha^2(s)}{s}\right)^{d/2} \exp\left[-\frac{\alpha^2(s)}{2s}\right],
\]
\[
\frac{h(\tau(s) \land T, w_{\tau(s) \land T})}{\beta(\tau(s) \land T)} I_{(s)s > T} = \frac{1}{\beta(T)} h(T, w_T) I_{(s)s > T} +
\]
\[
\frac{1}{\beta(\tau(s))} h(\tau(s), w_{\tau(s)}) I_{T \geq \tau(s)} > T = \frac{1}{\beta(T)} h(T, w_T) I_{(s)s > T} + I_{T \geq \tau(s)} > T.
\]

By letting \(T \to \infty\) in (1) and by applying the monotone convergence theorem, relations (2) and the fact that \(\{\tau(s) > T\} \subset \{|w_r| < \alpha(T)\}\) we immediately get the first assertion of the following lemma

Lemma 4. Define \(f(t) = \kappa_d \left(\frac{\alpha^2(t)}{t}\right)^{d/2} \exp\left[-\frac{\alpha^2(t)}{2t}\right], \) let \(\gamma(t) \geq 0\) for large \(t\) and let
\[
\lim_{t \to \infty} \frac{\alpha(t)}{\sqrt{t}} = \infty \quad \text{(lim \(t \to \infty\) \(f(t) = 0\)).}
\]

Then for any \(s > 0\)
\[
P\{s < \tau(s) < \infty\} = f(s) + \int_s^\infty Eh(r, w_r) \gamma(r) I_{r > T} dr;
\]
\[
\int_0^\infty r^{-d/2} \gamma(r) dr = \infty \implies \lim_{t \to \infty} \sup\{|w_t| - \alpha(t)| \geq 0 \text{ (a.s.)};
\]
\[
\int_0^\infty \alpha^d(r) r^{-d/2} \gamma(r) dr < \infty \implies \lim_{t \to \infty} \sup\{|w_t| - \alpha(t)| \leq 0 \text{ (a.s.).}
\]
Proof. To prove (5) notice that $h(r, w_r) \geq r^{-d/2}$, so that
\[
\int_s^\infty E h(r, w_r) \gamma(r) I_{(s) > r} \, dr \geq \int_s^\infty r^{-d/2} \gamma(r) \, dr P\{\tau(s) = \infty\},
\]
and under the condition in (5) we have $P\{\tau(s) = \infty\} = 0$. It remains only to observe that
\[
\{\omega : \limsup_{t \to \infty} |w_t| - \alpha(t) < 0\} \subset \bigcup_{n=1}^{\infty} \{\omega : \tau(n) = \infty\}.
\]
To prove (6) we use (2) and that $\{\tau(s) > r\} \subset \{ |w_s| < \alpha(r)\}$ if $r > s$. Then
\[
\int_s^\infty E h(r, w_r) \gamma(r) I_{(s) > r} \, dr \leq \kappa_d \int_s^\infty \alpha^d(r) r^{-d} \gamma(r) \, dr,
\]
and from (4) we see that under the condition in (6), $P\{s < \tau(s) < \infty\} \to 0$ as $s \to \infty$. Finally, for any $s > 0$
\[
\{\omega : \limsup_{t \to \infty} |w_t| - \alpha(t) > 0\} \subset \{\omega : |w_s| \geq \alpha(s)\} \cup \{\omega : s < \tau(s) < \infty\},
\]
\[
P\{\limsup_{t \to \infty} |w_t| - \alpha(t) > 0\} \leq \lim_{s \to \infty} P\{|w_s| \geq \alpha(s)\} = \lim_{s \to \infty} P\{|w_s| \geq \alpha(s) / \sqrt{s}\} = 0.
\]
The lemma is proved.

Proof of Theorem 1. Take $\epsilon \in [0, 1)$ and define $\alpha(t) = ((1 + \epsilon)2t \log \log t)^{1/2}$ if $t \geq 10$ and for $t < 10$ define $\alpha(t)$ in any way just to get a positive differentiable function on $[0, \infty)$. Then $\beta(t) = t^{-d/2}(\log t)^{1+\epsilon}$, $\alpha(t)/\sqrt{t} \to \infty$ for any $\epsilon$, and as easy to check
\[
\int_s^\infty r^{-d/2} d \frac{1}{\beta(r)} = \infty \text{ if } \epsilon = 0, \int_s^\infty \alpha^d(r) r^{-d} d \frac{1}{\beta(r)} < \infty \text{ if } \epsilon > 0;
\]
from Lemma 4 it follows that
\[
\limsup_{t \to \infty} \frac{|w_t|}{(2t \log \log t)^{1/2}} \geq 1, \limsup_{t \to \infty} \frac{|w_t|}{(2t \log \log t)^{1/2}} \leq 1 + \epsilon \text{ (a.s.) if } \epsilon > 0.
\]
The theorem is proved.

Next observe that
\[
\int_0^\infty \frac{\alpha^d(r)}{r^d} \gamma(r) \, dr = \int_0^\infty \frac{\alpha^d(r)}{r^d} d\left(\frac{r^{d/2}}{2r^2} \exp\left[-\frac{\alpha^2(r)}{2r}\right]\right) = \frac{d}{2} \int_0^\infty \frac{\alpha^d(r)}{r^{d/2}} \exp\left[-\frac{\alpha^2(r)}{2r}\right] \frac{dr}{r} - \frac{1}{2} \int_0^\infty \frac{\alpha^d(r)}{r^{d/2}} \exp\left[-\frac{\alpha^2(r)}{2r}\right] d^{2}(r) \frac{dr}{r},
\]
and under condition (3) the integral
\[
\int_0^\infty \frac{\alpha^d(r)}{r^{d/2}} \exp\left[-\frac{\alpha^2(r)}{2r}\right] d^{2}(r) \frac{dr}{r} = \int_0^\infty \alpha^{d/2} e^{-z^2/2} \, dz
\]
is finite, so that the integrals
\[
\int_0^\infty \frac{\alpha^d(r)}{r^d} \gamma(r) \, dr, \int_0^\infty \frac{\alpha^d(r)}{r^{d/2}} \exp\left[-\frac{\alpha^2(r)}{2r}\right] \frac{dr}{r}
\]
converge or diverge simultaneously.

Now we see that the statement (6) in Lemma 4 implies the second statement in the following Kolmogorov-Petrovskii criterion.
Theorem 5. Assume that $t^{d/2} \exp \left( -\alpha^2(t)/(2t) \right)$ and $\alpha^{-1}(t)$ increase for large $t$, and assume (3). Then

$$\int_0^\infty \frac{\alpha^d(r)}{r^{d/2}} \exp \left[ -\frac{\alpha^2(r)}{2r} \right] \frac{dr}{r} = \infty \quad \implies \quad \limsup_{t \to \infty} \frac{|w_t|}{\alpha(t)} \geq 1 \quad \text{(a.s.)}; \quad (7)$$

$$\int_0^\infty \frac{\alpha^d(r)}{r^{d/2}} \exp \left[ -\frac{\alpha^2(r)}{2r} \right] \frac{dr}{r} < \infty \quad \implies \quad \limsup_{t \to \infty} \frac{|w_t|}{\alpha(t)} \leq 1 \quad \text{(a.s.)}. \quad (8)$$

Proof. We only need to prove (7), and to do this we come back to statement (4) of Lemma 4 but analyze it slightly more carefully. Notice that for any given $r > 0$ the process $w_t - w_t t/r, t \in [0, r]$, and the random vector $w_r$ are independent. Therefore for $r > s$

$$Eh(r, w_r) I_{r \leq r} = \frac{1}{(2\pi r)^{d/2}} \int_{|x| < \alpha(r)} e^{-|x|^2/(2r)} h(r, x) P\{r(s) > r|w_r = x\} \, dx =$$

$$\frac{1}{(2\pi r)^{d/2}} \int_{|x| < \alpha(r)} P\{r(s) > r|w_r = x\} \, dx =$$

$$\frac{1}{(2\pi r)^{d/2}} \int_{|x| < \alpha(r)} P\{\sup_{t \in [s, r]} |w_t - w_r + \frac{t}{r}\alpha^{-1}(r)| < 1\} \, dx =$$

$$\frac{\alpha^d(r)}{(2\pi r)^{d/2}} \int_{|x| < 1} P\{\sup_{t \in [s, r]} |w_t - w_r + \frac{t}{r}\alpha(r)x|\alpha^{-1}(r) < 1\} \, dx.$$  

From (4) it now follows that if $\int_0^\infty \alpha^d(r) r^{-d} \gamma(r) \, dr = \infty$, then for any $s > 0$

$$\liminf_{r \to \infty} \int_{|x| < 1} P\{\sup_{t \in [s, r]} |w_t - w_r + \frac{t}{r}\alpha(r)x|\alpha^{-1}(r) < 1\} \, dx = 0. \quad (8)$$

Now observe that for $\varepsilon \in (0, 1)$, $|x| \leq \varepsilon$ and $s$ so large that $\alpha^{-1}(t)$ increases for $t \geq s$, we have $(t/r)\alpha(r)|x|\alpha^{-1}(t) \leq \varepsilon$,

$$P\{\sup_{t \in [s, r]} |w_t - w_r + \frac{t}{r}\alpha(r)x|\alpha^{-1}(r) < 1\} \geq$$

$$P\{\sup_{t \in [s, r]} |w_t - w_r + \frac{t}{r}\alpha(r)x|\alpha^{-1}(r) < 1 - \varepsilon\} \geq$$

$$P\{\sup_{t \in [s, r]} |w_t - w_r + \frac{t}{r}\alpha(r)x|\alpha^{-1}(r) < 1 - \varepsilon, |w_r| \alpha^{-1}(r) \leq \varepsilon\} \geq$$

$$P\{\sup_{t \in [s, r]} |w_t| < 1 - 2\varepsilon\} - P\{|w_r| > \varepsilon \alpha(r)\}.$$  

Since $\alpha(r)/\sqrt{r} \to \infty$, the last probability tends to zero as $r \to \infty$, and from (8) it follows that for any $\varepsilon \in (0, 1)$ we have

$$P\{\sup_{t \in [s, r]} |w_t| < 1 - 2\varepsilon\} \leq \lim_{r \to \infty} P\{\sup_{t \in [s, r]} |w_t| < 1 - 2\varepsilon\} = 0.$$  

This obviously yields the first assertion of our theorem, which is thus proved.

Remark 6. From the zero-one law it follows easily that for one-dimensional Wiener process $B_t$

$$\limsup_{t \to \infty} \frac{B_t}{\alpha(t)} \leq (\geq) 1 \quad \text{(a.s.)} \implies \limsup_{t \to \infty} \frac{|w_t|}{\alpha(t)} \leq (\geq) 1 \quad \text{(a.s.)}.$$
Therefore, consideration of arbitrary $d$ does not yield any advantage, though it actually might happen that for $d = 1$ the integral in (7) converges and, say for $d = 2$ diverges. In this case $\lim \sup$ in (7) simply equals 1 (a. s.).

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References


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