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The gap between the past supremum and the future infimum of a transient Bessel process

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Summary. This paper consists of some remarks on an earlier paper of the author with T.M. Lewis and W.V. Li regarding some almost sure path properties of the future infima of transient Bessel processes. In particular, we study the a.s. asymptotic discrepancy between $\sup_{s\leq t} X_s$ and $\inf_{s\geq t} X_s$.

1. Introduction. Let $\{X_t; t \geq 0\}$ denote a transient Bessel process starting at zero. This means that X is a positive diffusion with a (strong) infinitesimal generator given by the following:

$$\mathcal{L}f(x) = \frac{1}{2}f''(x) + \frac{(d-1)}{2x}f'(x).$$

By transience, we must have $d > 2$ and this condition is assumed throughout this article without further mention. Moreover, the domain of the above generator is exactly all real–valued functions, f, which are twice continuously differentiable on $(0,\infty)$ and have the following boundary behavior: $|f(\varepsilon) - f(0)| = o(\varepsilon^{2-d})$, as $\varepsilon \to 0$.

Let $I_t \triangleq \inf_{s\geq t} X_s$ and $M_t \triangleq \sup_{s\leq t} X_s$. Future infima processes such as I occur quite naturally in a variety of situations: see Aldous [A] for an application to random walks on trees. Furthermore, when $d = 3$, I appears quite naturally in Pitman's theorem: $2I_t - X_t$ is a Brownian motion. For the latter, see Revuz and Yor [RY, Thm. VI.(3.5), p. 234]; a surprising extension to all $d > 2$ appears in Chapter 12 of Yor [Y]. General extensions of Pitman's theorem to transient diffusions appear in the work of Saisho and Tanemura [ST].

In an earlier paper ([KLL]), together with T.M. Lewis and W.V. Li, we proved results about the asymptotic behavior of the process I with respect to the process X.

This, in turn, gives some information on the nature of transience of the underlying Bessel process X. Motivated by a question of K. Burdzy, this note is concerned with the size of the "gap" between the processes I and M. More precisely, we offer the following results:

Theorem 1.1. *Suppose* $\varphi : \mathbb{R}^1_+ \mapsto (0,\infty)$ *is increasing with* $\lim_{t\to\infty}\varphi(t) = \infty$. *Suppose further that* $t \mapsto \varphi(t)$ *is slowly varying and that*

$$\int_1^\infty \frac{dt}{t\varphi(t)} = \infty.$$

Then with probability one,

$$\liminf_{t\to\infty} \varphi(t)\cdot\left(1 - \frac{I_t}{M_t}\right) = 0.$$

Theorem 1.1 has a "converse" which is the following:

Proposition 1.2. *Suppose* $\varphi : \mathbb{R}^1_+ \mapsto (0,\infty)$ *is increasing with* $\lim_{t\to\infty}\varphi(t) = \infty$. *If*

$$\int_1^\infty \frac{dt}{t\varphi(t)} = \infty,$$

then with probability one,

$$\limsup_{t\to\infty} \varphi(t)\cdot\left(1 - \frac{I_t}{M_t}\right) = \infty.$$

We next mention a related result which has to do with the difference between M_t and I_t instead of their ratio.

Theorem 1.3. *Suppose* $\varphi : \mathbb{R}^1_+ \mapsto (0,\infty)$ *is decreasing. Suppose further that* $t \mapsto \varphi(t)$ *is slowly varying and*

$$\int_1^\infty \varphi(t)\frac{dt}{t} = \infty.$$

Then with probability one,

$$\liminf_{t\to\infty} \frac{M_t - I_t}{\varphi(t)} = 0.$$

An obvious consequence of Theorem 1.1 is that almost surely,

$$\liminf_{t\to\infty} (\ln t \ln\ln t)\cdot\left(1 - I_t\cdot M_t^{-1}\right) = 0,$$

and correspondingly by Theorem 1.3,

$$\liminf_{t\to\infty} \big(\ln t \ln\ln t\big)\cdot\big(M_t - I_t\big) = 0.$$

It may at first seem surprising that the above rates are independent of the dimension, $d > 2$; but this is not so, since we believe the results above are far from being sharp. To illustrate the problem, we point out that our techniques cannot establish that almost surely,

$$\liminf_{t\to\infty} f(t)\cdot\big(1 - I_t \cdot M_t^{-1}\big) = \infty,$$

even for a function such as $f(x) = \exp(e^x)$, which we believe ought to do the job. Thus it is important to find a more robust method of handling this gap.

The above results are partial attempts at estimating the size of the gap when it is small; the gap is small when $I_t \simeq M_t$. Below we provide the following theorem which gives a complete characterization of the size of the gap when it is large, i.e., when I_t is much smaller then M_t.

Theorem 1.4. *Suppose $\psi : (0,\infty) \mapsto (0,\infty)$ is decreasing to zero and is slowly varying. Then with probability one,*

$$\liminf_{t\to\infty} \frac{I_t}{M_t\psi(t)} = \begin{cases} 0, & \text{if } J(\psi) = \infty \\ \infty & \text{if } J(\psi) < \infty, \end{cases}$$

where

$$J(\psi) \triangleq \int_1^\infty \big(\psi(t)\big)^{d-2}\frac{dt}{t}.$$

Therefore, almost surely,

$$\liminf_{t\to\infty} \big(\ln t\big)^a \frac{I_t}{M_t} = \begin{cases} 0, & \text{if } a \le \frac{1}{d-2} \\ \infty, & \text{if } a > \frac{1}{d-2} \end{cases}.$$

From now on, $\{\mathcal{F}_t; t \ge 0\}$ denotes the natural filtration of the process X and for any measurable $A \subseteq C([0,\infty))$, $\mathbb{P}^x(A)$ is a nice version of the probability of A conditional on $\{\omega : X_0 = x\}$. Unimportant finite positive constants are denoted as K_0 and K and their value may vary from line to line.

I wish to thank Chris Burdzy for introducing me to this problem as well as for interesting conversations on this topic. Also many thanks are due to Yuval Peres, Russ Lyons, Marc Yor and an anonymous referee for several useful suggestions.

2. The Proofs. Define the first hitting times, $\sigma(t)$, by the following:

$$\sigma(t) \triangleq \inf\{s > 0 : X_s = t\}.$$

Supposing that $\psi : \mathbb{R}^1_+ \mapsto (0,1)$ increases to one as $t \to \infty$, let us define measurable events, $E(t) = E_\psi(t)$ by

$$E(t) \triangleq \{\omega : \ I_{\sigma(t)} \geq t\psi(t)\}. \tag{2.1}$$

Lemma 2.1. *In the above notation,*

$$\mathbb{P}\big(E(t)\big) = 1 - \psi^{d-2}(t).$$

Proof. First condition on $\mathcal{F}_{\sigma(t)}$ and then use the gambler's ruin problem for X, using the fact that X^{2-d} is a continuous martingale. □

Lemma 2.2. *Let ψ be as in the statement of Lemma 2.1 and let $E(t)$ be defined by (2.1). Suppose that $t > s > 0$ are such that $t\psi(t) \geq s$. Then*

$$\mathbb{P}\big(E(t) \cap E(s)\big) = \mathbb{P}\big(E(t)\big)\mathbb{P}\big(E(s)\big) \times \frac{1}{1 - \big(s\psi(s)/t\big)^{d-2}}.$$

Proof. By a gambler's ruin calculation,

$$\begin{aligned}\mathbb{P}^s\big(\sigma(t) < \sigma(s\psi(s))\big) &= \frac{\big(s\psi(s)\big)^{2-d} - s^{2-d}}{\big(s\psi(s)\big)^{2-d} - t^{2-d}} \\ &= \frac{1 - \psi^{d-2}(s)}{1 - \big(s\psi(s)/t\big)^{d-2}} \\ &= \mathbb{P}\big(E(s)\big) \times \frac{1}{1 - \big(s\psi(s)/t\big)^{d-2}}, \end{aligned} \tag{2.2}$$

by Lemma 2.1. On the other hand, by another gambler's ruin calculation,

$$\begin{aligned}\mathbb{P}^t\big(\sigma(t\psi(t)) = \infty\big) &= 1 - \psi^{d-2}(t) \\ &= \mathbb{P}\big(E(t)\big), \end{aligned} \tag{2.3}$$

by Lemma 2.1. By the strong Markov property,

$$\mathbb{P}\big(E(s) \cap E(t)\big) = \mathbb{P}^s\big(\sigma(t) < \sigma(s\psi(s))\big) \cdot \mathbb{P}^t\big(\sigma(t\psi(t)) = \infty\big).$$

The lemma follows from (2.2) and (2.3). □

Proof of Theorem 1.1. Without loss of generality, we shall assume that $\varphi(t) \geq 1$ for all $t > 0$. Otherwise, we let $t' \triangleq \inf\{s : \varphi(s) \geq 1\}$ and shift everything by t'. With this in mind, fix $\varepsilon \in (0,1)$ and define $\psi(t) \triangleq (1 - \varepsilon\varphi^{-1}(t))$. (Note that $\psi(t) \geq 1 - \varepsilon$ since $\varphi \geq 1$.) Also define

$$t_n \triangleq e^n \qquad \text{and} \qquad \psi_n \triangleq \psi(t_n).$$

Recalling the definition of $E(t)$ from (2.1), it follows from Lemma 2.1 that as $n \to \infty$, $P(E(t_n)) \sim \varepsilon(d-2) \cdot \varphi^{-1}(t_n)$. In particular, there exists some constant, K, such that for all $n \geq 1$, $\mathbb{P}(E(t_n)) \geq K/\varphi(t_n)$. Hence,

$$\begin{aligned} \sum_{n=0}^{\infty} \mathbb{P}(E(t_n)) &\geq K \sum_{n=0}^{\infty} \int_{t_n}^{t_{n+1}} \frac{dt}{(t_{n+1} - t_n)\varphi(t_n)} \\ &= \frac{K}{e-1} \sum_{n=0}^{\infty} \int_{t_n}^{t_{n+1}} \frac{dt}{t_n \varphi(t_n)} \\ &\geq \frac{K}{e-1} \int_1^{\infty} \frac{dt}{t\varphi(t)} = \infty, \end{aligned} \tag{2.4}$$

by assumption. On the other hand, since $\varphi(t_k) \to \infty$, for all integers k and all integers n large enough,

$$\frac{t_{n+k}\psi_{n+k}}{t_n} = e^k\big(1 - \varepsilon/\varphi(t_{n+k})\big) \geq 1,$$

since $\varepsilon \in (0,1)$. This means that Lemma 2.2 applies. More precisely by Lemma 2.2, for all $k \geq 1$ and all n large,

$$\begin{aligned} \mathbb{P}(E(t_n) \cap E(t_{n+k})) &= \mathbb{P}(E(t_n))\mathbb{P}(E(t_{n+k})) \times \frac{1}{1 - \big(t_n\psi_k/t_{n+k}\big)^{d-2}} \\ &\leq \big(1 - e^{2-d}\big)^{-1}\, \mathbb{P}(E(t_n))\mathbb{P}(E(t_{n+k})), \end{aligned}$$

since $d > 2$ and $\psi_k \leq 1$ for all $k \geq 1$. By the Kochen–Stone lemma ([KS]), (2.4) and the above together imply that

$$\mathbb{P}\big(E(t_n),\ \text{i.o.}\big) \geq \big(1 - e^{2-d}\big).$$

By Kolmogorov's 0–1 law, $\mathbb{P}\big(E(t_n),\ \text{i.o.}\big) = 1$. In other words, with probability one,

$$I_{\sigma(t_n)} \geq t_n\psi(t_n), \qquad \text{i.o.}$$

Hence for every $\varepsilon \in (0,1)$, almost surely,

$$\varphi(t_n) \cdot \left(1 - \frac{I_{\sigma(t_n)}}{t_n}\right) \leq \varepsilon, \qquad \text{i.o.}$$

Since $t \mapsto M_t$ is a.s. continuous, $M_{\sigma(t)} = t$. Substituting M_t for t,

$$\liminf_{n\to\infty} \varphi(M_{t_n}) \cdot \left(1 - \frac{I_{t_n}}{M_{t_n}}\right) = 0, \qquad \text{a.s.} \tag{2.5}$$

It, therefore, remains to prove that for some $K > 0$,

$$\liminf_{n\to\infty} \frac{\varphi(M_{t_n})}{\varphi(t_{n+1})} \geq \frac{1}{K}, \qquad \text{a.s.} \tag{2.6}$$

For if we proved (2.6), (2.5) implies the following:

$$\begin{aligned}
\liminf_{t\to\infty} \varphi(t) \cdot \left(1 - \frac{I_t}{M_t}\right) &\leq \liminf_{n\to\infty} \varphi(t_n) \cdot \left(1 - \frac{I_{t_n}}{M_{t_n}}\right) \\
&= \liminf_{n\to\infty} \varphi(e^{-1} t_{n+1}) \cdot \left(1 - \frac{I_{t_n}}{M_{t_n}}\right) \\
&\leq K_0 \liminf_{n\to\infty} \varphi(t_{n+1}) \cdot \left(1 - \frac{I_{t_n}}{M_{t_n}}\right) \\
&\leq K_0 K \cdot \liminf_{n\to\infty} \varphi(M_{t_n}) \cdot \left(1 - \frac{I_{t_n}}{M_{t_n}}\right) = 0,
\end{aligned}$$

for some $K_0 > 1$, since φ is slowly varying. As this proves Theorem 1.1, it suffices to prove (2.6). It is pointed out to us by Marc Yor that (2.6) can be obtained as a consequence of Chung's law of the iterated logarithm. We shall provide a direct proof for the sake of completeness.

Recall that for any $c > 0$ there exists some $K = K(c) > 0$ and $T(c) > 0$, so that for all $t \geq T(c)$, $\varphi(t^c) \geq K^{-1}\varphi(t)$. Since φ is increasing, by standard calculations for any $c \in (1/2, 1)$, and all $t_n \geq T(c)$,

$$\begin{aligned}
\mathbb{P}\big(\varphi(M_{t_n}) \leq K^{-1}\varphi(t_{n+1})\big) &\leq \mathbb{P}\big(M_{t_n} \leq t_{n+1}^c\big) \\
&\leq \left(\frac{t_n^{1/2}}{t_{n+1}^c}\right) \\
&\leq e^c \cdot \exp\big(-n(c - 0.5)\big),
\end{aligned}$$

which sums. An application of the easy half of the Borel–Cantelli lemma proves (2.6) and hence finishes the proof of the theorem. □

Proof of Proposition 1.2. By considering $\varphi_0(t) \triangleq \varphi(t) \wedge (\ln t)$, we might as well assume that $\varphi(t) \geq \ln t$ for all $t > 1$. By Theorem 4.1 (2) of Khoshnevisan et al. [KLL],

$$\limsup_{t\to\infty} \frac{X_t - I_t}{\sqrt{2t \ln \ln t}} = 1, \qquad \text{a.s.}$$

Hence for any $\varepsilon > 0$,

$$\frac{M_t - I_t}{\sqrt{2t \ln \ln t}} \geq (1-\varepsilon), \qquad \text{i.o., a.s.} \tag{2.7}$$

By the usual law of the iterated logarithm for Bessel processes (see Revuz and Yor [RY, Ex. XI.(1.20), p. 419], for instance),

$$\limsup_{t \to \infty} \frac{M_t}{\sqrt{2t \ln \ln t}} = 1,$$

almost surely. Hence almost surely, $M_t \leq \sqrt{2t \ln t}$, eventually. By (2.7), the above immediately implies Proposition 1.2. □

Proof of Theorem 1.3. The proof of Theorem 1.3 is very similar to that of Theorem 1.1; the main difference is the choice of the subsequence along which one can use the Borel–Cantelli lemma. To this end, define

$$\psi(t) \triangleq 1 - \frac{\varphi(t)}{t}.$$

Recalling (2.1), we see from Lemma 2.1 that

$$\begin{aligned} \mathbb{P}\big(E(n)\big) &= 1 - \left(1 - \frac{\varphi(n)}{n}\right)^{d-2} \\ &\sim (d-2)\frac{\varphi(n)}{n}. \end{aligned} \tag{2.8}$$

Therefore,

$$\begin{aligned} \sum_n \mathbb{P}\big(E(n)\big) &\geq K \sum_n \frac{\varphi(n)}{n} \\ &\geq K \int_1^\infty \varphi(t) \frac{dt}{t} = \infty, \end{aligned} \tag{2.9}$$

by assumption. It now follows from Lemma 2.2 and (2.8) that

$$\begin{aligned} \mathbb{P}\big(E(n) \cap E(n+k)\big) &= \mathbb{P}\big(E(n)\big)\mathbb{P}\big(E(n+k)\big)\left[1 - \left(\frac{n - \varphi(n)}{n+k}\right)^{d-2}\right]^{-1} \\ &\sim (d-2)^3 \frac{\varphi(n)}{n} \cdot \frac{\varphi(n+k)}{n+k} \cdot \frac{n+k}{k + \varphi(n)} \\ &\leq (d-2)^3 \frac{\varphi(n)}{n} \cdot \frac{\varphi(n+k)}{k} \\ &\leq (d-2)^3 \frac{\varphi(n)}{n} \cdot \frac{\varphi(k)}{k} \\ &\sim (d-2)\mathbb{P}\big(E(n)\big)\mathbb{P}\big(E(k)\big), \end{aligned}$$

where the penultimate line follows from the fact that $t \mapsto \varphi(t)$ is assumed to be decreasing. This development implies the existence of some $\varepsilon_N \to 0$, such that

$$\begin{aligned}\sum_{n=1}^{N}\sum_{k=1}^{N-n} \mathbb{P}\big(E(n)\cap E(n+k)\big) &\leq (d-2)(1+\varepsilon_N)\sum_{n=1}^{N}\sum_{k=1}^{N-n} \mathbb{P}\big(E(n)\big)\mathbb{P}\big(E(k)\big) \\ &\leq (d-2)(1+\varepsilon_N)\Big(\sum_{n=1}^{N}\mathbb{P}\big(E(n)\big)\Big)^2. \end{aligned} \tag{2.10}$$

By the lemma of Kochen and Stone [KS], (2.9) and (2.10) together imply

$$\mathbb{P}\big(E(n), \text{ i.o. }\big) \geq (d-2)^{-1}.$$

Hence, by Kolmogorov's 0–1 law, $\mathbb{P}\big(E(n), \text{ i.o. }\big) = 1$. Since $M_{\sigma(n)} = n$, we have shown that with probability one,

$$M_t - I_t \leq \varphi(M_t), \text{ i.o..}$$

By Chung's law of the iterated logarithm, for each $\varepsilon > 0$, almost surely we have: $M_t \geq t^{1/2-\varepsilon}$, eventually. By the assumed properties of φ, for each $K > 1$,

$$M_t - I_t \leq K \cdot \varphi(t), \text{ i.o.}$$

Replacing $\varphi(\cdot)$ by $\varepsilon\varphi(\cdot)$ and letting $\varepsilon \to 0$, the result follows. □

Proof of Theorem 1.4. Let $t_n \triangleq e^n$, and $\psi_n \triangleq \psi(t_n)$. Since $\psi(t) \to 0$ as $t \to \infty$, we might as well assume that $\psi(t) < 1$. We begin with the elementary observation that $J(\psi) < \infty$ if and only if $\sum_n \psi_n^{d-2} < \infty$. Arguing as in Lemma 2.1, there exists some $K > 1$ such that for all $n \geq 1$,

$$\mathbb{P}\big(I_{\sigma(t_n)} < t_{n+1}\psi_{n+1}\big) \leq K\psi_{n+1}^{d-2}.$$

Hence, if $J(\psi) < \infty$, then by the Borel–Cantelli lemma,

$$I_{\sigma(t_n)} > t_{n+1}\psi_{n+1}, \qquad \text{eventually,}$$

almost surely. Now suppose t is large. Then there exists some large n such that $t_n \leq t \leq t_{n+1}$. Since $t \mapsto I_{\sigma(t)}$ and $t \mapsto t\psi(t)$ are both increasing, it follows that with probability one, $I_{\sigma(t)} > t\psi(t)$, eventually. Now if $J(\psi)$ converges, $J(K\psi)$ also converges no matter the value of $K > 0$. Hence applying the above to $K \cdot \psi(t)$ instead, it follows that with probability one: $I_{\sigma(t)} > K \cdot \psi(t)$, eventually. In other words, we have argued that if $J(\psi) < \infty$, then almost surely,

$$\lim_{t\to\infty} \frac{I_t}{M_t\psi(M_t)} = \infty. \tag{2.11}$$

Now supppose $J(\psi) = \infty$ and define,

$$E_n \triangleq \{\omega : \ I_{\sigma(t_n)} < t_n \psi_n\}.$$

As in Lemma 2.1, it follows that there exists some $K > 1$ such that for all $n \geq 1$,

$$K^{-1}\psi_n^{d-2} \leq \mathbb{P}(E_n) \leq K\psi_n^{d-2}.$$

Since $J(\psi) < \infty$ if and only if $\sum_n \psi_n^{d-2} < \infty$, we have shown that $\sum_n \mathbb{P}(E_n) = \infty$. Suppose we could prove the following:

$$\liminf_{N\to\infty} \frac{\sum_{n=1}^{N}\sum_{k=1}^{N-n} \mathbb{P}(E_n \cap E_{n+k})}{\left(\sum_{n=1}^{N} \mathbb{P}(E_n)\right)^2} < \infty. \tag{2.12}$$

By the Kochen–Stone lemma ([KS]) and Kolmogorov's 0–1 law, it would then follow that almost surely, $I_{\sigma(t)} \leq t\psi(t)$, infinitely often. However, $J(\psi)$ diverges if and only if $J(\varepsilon\psi)$ diverges, for any choice of $\varepsilon > 0$. Hence, applying the above discussion to $t \mapsto \varepsilon\psi(t)$, we see that $J(\psi) = \infty$ implies the following:

$$\liminf_{t\to\infty} \frac{I_t}{M_t\psi(M_t)} = 0, \qquad \text{a.s.}$$

Together with (2.11), this shows that if we could prove (2.12), then we have shown the following:

$$\liminf_{t\to\infty} \frac{I_t}{M_t\psi(M_t)} = \begin{cases} 0, & \text{if } J(\psi) = \infty \\ \infty, & \text{if } J(\psi) < \infty \end{cases}. \tag{2.13}$$

Supposing (2.12) for the time being, let us see how (2.13) implies Theorem 1.4. For any $\theta > 0$, let $\psi_\theta(t) \triangleq \psi(t^\theta)$. Note that ψ_θ satisfies the conditions of the theorem if and only if ψ does. Moreover, $J(\psi_\theta) = \theta^{-1}J(\psi)$, and hence $J(\psi_\theta) = \infty$ if and only if $J(\psi) = \infty$.

Suppose $J(\psi) = \infty$. By (2.13), for any $\theta > 2$,

$$\liminf_{t\to\infty} \frac{I_t}{M_t\psi(M_t^\theta)} = 0, \qquad \text{a.s.}$$

Refining the proof of (2.6) (alternatively, using Chung's LIL), $M_t^\theta \geq t$, eventually, a.s.. Since $t \mapsto \psi(t)$ is decreasing, it follows that

$$\text{if} \quad J(\psi) = \infty \quad \text{then} \quad \liminf_{t\to\infty} \frac{I_t}{M_t\psi(t)} = 0, \qquad \text{a.s.} \tag{2.14}$$

On the other hand, if $J(\psi) < \infty$, then $J(\psi_\theta) < \infty$ for $\theta \in (0,2)$. Applying (2.13) to J_θ for such a θ, it follows that

$$\liminf_{t\to\infty} \frac{I_t}{M_t \psi(M_t^\theta)} = \infty, \qquad \text{a.s.}$$

Another argument (e.g, the LIL or a refinement of the argument leading to (2.6)) shows that almost surely: $M_t^\theta \le t$, eventually. Therefore we have shown that (a.s.),

$$\text{if} \qquad J(\psi) < \infty \qquad \text{then} \qquad \liminf_{t\to\infty} \frac{I_t}{M_t \psi(t)} = \infty.$$

This and (2.14) together prove the theorem. It therefore remains to prove (2.12).

In the course of the proof of (2.12), there are potentially two seperate cases to consider: (1) when k is so large that $t_{n+k}\psi_{n+k} \in [t_n, t_{n+k}]$, and (2) when $t_{n+k}\psi_{n+k} \in [t_n\psi_n, t_n]$. Both estimates follow the guidelines of the proof of Lemma 2.2 and one gets the same estimates (modulo some constant multiples) in both cases. Therefore, we shall be content to handle case (1) only. In this case, by the gambler's ruin problem (cf. Lemma 2.1) and the strong Markov property, for all $n, k \ge 1$,

$$\begin{aligned}
\mathbb{P}(E_n \cap E_{n+k}) &= \mathbb{P}\big(I_{\sigma(t_n)} < t_n\psi_n, I_{\sigma(t_{n+k})} \le t_n\psi_n\big) \\
&\quad + \mathbb{P}\big(I_{\sigma(t_n)} < t_n\psi_n \ , \ t_n\psi_n < I_{\sigma(t_{n+k})} < t_{n+k}\psi_{n+k}\big) \\
&= \left(\frac{t_n\psi_n}{t_{n+k}}\right)^{d-2} + \mathbb{P}^{t_n}\big(\sigma(t_n\psi_n) < \sigma(t_{n+k})\big) \cdot \mathbb{P}^{t_{n+k}}\big(\sigma(t_{n+k}\psi_{n+k}) < \infty\big) \\
&\qquad \cdot \mathbb{P}^{t_{n+k}\psi_{n+k}}\big(\sigma(t_n\psi_n) = \infty\big) \\
&\triangleq T_1 + T_2.
\end{aligned}$$

Evidently, there exists some $K > 0$ such that for all $n \ge 1$, $T_1 \le K e^{-k(d-2)} \mathbb{P}(E_n)$. Likewise, T_2 is estimated as follows,

$$\begin{aligned}
T_2 &= \frac{t_{n+k}^{d-2} - t_n^{d-2}}{t_{n+k}^{d-2} - (t_n\psi_n)^{d-2}} \psi_n^{d-2} \psi_{n+k}^{d-2} \big(1 - (t_n\psi_n)^{d-2} \psi_{n+k}^{2-d}\big) \\
&\le 2\psi_n^{d-2}\psi_{n+k}^{d-2} \\
&\le K\mathbb{P}(E_n)\mathbb{P}(E_{n+k}).
\end{aligned}$$

Hence there exists some $K > 1$ such that for all k and n satisfying case (1), we have $\mathbb{P}(E_n \cap E_{n+k}) \le K \cdot \mathbb{P}(E_n)\big(1 + \mathbb{P}(E_{n+k})\big)$. Since a similar estimate holds for k and n satisfying case (2), (2.12) follows from the fact that $\sum_n \mathbb{P}(E_n) = \infty$. This proves Theorem 1.4. □

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